# ON CERTAIN RIEMANNIAN MANIFOLDS WITH POSITIVE RICCI CURVATURE

A Dissertation presented

by

Yoe Itokawa

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August, 1982

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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The dissertation is accepted by the Graduate School

Dean of the Graduate School

Abstract of the Dissertation

On Certain Riemannian Manifolds With

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It is an interesting and important problem in modern geometry to understand the meaning of Ricci curvature on a Riemannian manifold. In this paper, we give some topological restrictions for a manifold of positive Ricci curvature to have prescribed ranges for certain other geometric invariants. Specifically, we show that if the volume is large relative to an arbitrarily given upper bound on the sectional curvature or if the injectivity radius is large, the manifold has to be essentially a sphere. In addition, the

extremal value for the diameter is studied to give a new direct proof and some applications of the Cheng-Toponogov theorem. The paper includes a fairly comprehensive account on the history of the positive curvature problem.

Table of contents

## Acknowledgements

... And now we come to the first difficulty of our paper. The problem is that if I were to mention every person to whom I am indebted in association with my dissertation study, the Acknowledgements pages might turn out longer than the paper itself!

First and foremost, I would like to thank my advisor and mentor, Professor Detlef Gromoll, but the human language is much too inadequate to express enough gratitude. Whatever worth the present paper might have, I owe it all to him, as he has been my guiding light, not only for this research, but throughout my graduate career. An equal share of gratitude also goes to Suzan Gromoll for all her graciousness. At the same time, I owe much appology to her as well as to Hans Christian, Heidi, and Stefan for disrupting their family moments so often by calling Detlef to my needs.

If I lived my life twice, I can never repay what I owe to Professors Jeff Cheeger and H. Blaine Lawson. Their encouragements, assistance, and help in times of need are too numerous to detail here, but they are deeply appreciated.

Professor Michio Kuga and Professor Takushiro Ochiai introduced me to the Mathematical Society of Japan. Joining the Society was one of the most worthwhile things that I have done during my study. I must also thank Professor Kuga for instilling in me much general knowledge and philosophy, which, most of all, made the mathematical study more "fun".

Some of the possibilities, impossibilities, plausibilities, and implausibilities in real analysis, the subject to which I feel a little awe, have been indicated to me by Professors Raoul Doss, Ronald Douglas and Daryl Geller.

Professor John A. Thorpe should not go unthanked for helping me "survive" my stay at Stony Brook, and the same appreciation goes to Professor Paul G. Kumpel.

I would also like to express heartfelt gratitude to Professors Anthony Phillips and (the one and only) William Cassidy Fox for moral support, general chit-chatting, and, most of all, for training me to take abuses.

Professor Eugene Zaustinsky made the first siren-call to differential geometry in my undergraduate years. Logically, he should be condemned for this action, but, for some unknown reason, I thank him.

If it were not for the wonderful experience I had in the High-school Research Club, I would not have majored in mathematics in the first place. In this context, I am still grateful to Mr. Harvey Goodman and Ms. Elizabeth Sibley of Francis Lewis High School .

If good mathematical ideas only develop in a healthy mathematical environment, I could not have hoped for a better one. I would like to thank all of my fellow graduate students at Stony Brook for creating it, especially, Mike Anderson, Rodney Carr, Dan Cass, Ira Moscowitz, Bill Rosenthal, John Hurley, and Perry Susskind. Eric Schweitzer and Alan Horwitz are responsible for helping my car help me get to school.

How could I ever forget Ms. Lucille Meci, the woman who not only gets us out of administrative difficulties every time, but also relieves the pain involved. How could any of us graduate students enjoy our study at all, if it were not for her delightful presence! We could all thank her a thousand times for just existing in the Universe, and we still would not have thanked her enough.

None of us graduate students, as well as the rest of the Mathematics Department would be able to do anything without the patience and dedication of our secretarial staff. During my stay as a graduate student, these fantastic people, to whom I am much indebted, included: Ms.'s Claire Briones, Joann Debis, Dolores Marge, Toni Nagy, Florence O'Brien, Marianne Oliveri, Vicki Parker, and Stella Shivers.

It may connote an unintended sarcasm to some, when I say that the one thing that helped me the most during my graduate study was a prayer, but I do mean quite sincerely that it was the key to my overcoming some moments of personal difficulty. I am grateful to my fellow parishoners at the Catholic Campus Parish, especially, Rev. Robert Smith, Rev. Vincent Rush, Mike Quinn, Elaina and Lewis Alvarez-Gaume, Don and late Mimi Goodman, Bro. Jean-Marie Goodman, Lou Mc Whirter, Don and Dot Murch, Bonnie and Steve Paysen, and last but far from the least, Ann and Bill Powers, for teaching me how to pray. Emma Frigerio has been a good friend of mine both in the Parish and the Department.

I would be an ingrate if I did not thank many of my relatives here. Without the understanding of my grandmother and my brother, my study would have

been impossible, and without the underlying of my parents, I would have been impossible.

Finally, I am also grateful to the State University of New York for funding my support.

To all of the above and possibly 200 others, may God's blessings be upon them forever and ever!

## 0. Historical introduction

Riemannian geometry has its beginning in an attempt at the unified foundation and classification of non-Euclidean geometries, which were, in the middle of the 19-th century, still in the process of gaining public recognition. As the single most important computational tool to distinguish the local geometries of various non-Euclidean spaces, G. F. B. Riemann already emphasized the role of a certain biquadratic form which he called "the curvature", or, equivalently, a function on 2-dimensional directions through each point which is now called the sectional curvature. Yet, in those days, the concept of manifolds was not fully established, and the study of local properties was all that could have been hoped for by Riemann's methods.

Around the turn of the century, dramatic changes took place. On one hand, H. Poincaré, G. Cantor, et al. began a completely new branch of mathematics which we now know as topology, and it became possible to understand Riemann's concept of manifolds properly as portions of more global spacial configurations. On the other hand, W. Blaschke and others' startling revelation that local properties of curves and surfaces often influence and at times even determine the shapes of these figures in the large brought about the birth of the study of global differential geometry. It was then only a natural course that people started asking the question: "To what extent is the global geometry of a Riemannian manifold M determined by its curvature quantities?"

The first breakthrough in this direction was probably the following solution to the problem raised by W. K. Clifford, F. Klein, W. K. J. Killing:

Theorem 1 (H. Hopf, 1925): Let M be a complete simply connected Riemannian manifold of dimension n with a constant sectional curvature, $\kappa$. Then, depending on whether $\kappa$ is:

(i) positive,

(ii) negative, or

(iii) zero;

M is:

(i) homothetic to the n-dimensional Euclidean sphere, $S^n$,

(ii) homothetic to the n-dimensional hyperbolic space, $H^n$, of J. Bolyai, C. F. Gauss, and N. I. Lobachevsky;

(iii) isometric to the Euclidean space, $\mathbb{R}^n$.

The first result for the variable curvature case was

Theorem 2 (J. Hadamard, É. Cartan, 1928): If M is complete, simply connected, and of non-negative sectional curvature, then M is diffeomorphic to $\mathbb{R}^n$.

Then, in 1951, a paper appeared which was to give a decisive influence on global Riemannian geometry that is felt for the rest of the history.

Theorem 3 (H. Rauch, [R]): If M is a complete simply connected manifold whose sectional curvature, K, lies in the range

$$\delta \leq K < \kappa$$

for some $0 < \delta < \kappa$, then M is homeomorphic to $S^n$. Note that the $\delta$ in this theorem is just a normalization constant and may well be assumed to be 1 by applying a homothetic change of metric, if necessary. The value of $\kappa/\delta$ originally given by Rauch was $\doteqdot 4/3$. This was later improved by M. Berger [Be2] and W. Klingenberg [K2] to

$$1 \leq K < 4 \quad .$$

Note that these theorems are, in some sense, generalizations of Theorem 1. The assumption on the range of the curvature is weakened from single values to intervals with the consequences that the manifolds are no longer isometric to the classical spaces, but they still retain the topological types. Thus, a distinct philosophy in Riemannian geometry was recognized. "The perturbation principle" states that if a fixed value for a geometric invariant determines the space, then values sufficiently close to it will still result in manifolds that are somehow topologically similar to the distinguished ones.

In the years following, many generalizations and strengthenings of Theorem 3 appeared. First, Berger studied the case when the range of the curvature is precisely [1,4].

Theorem 4 [Be1], [Be2]: If a complete simply connected manifold M of even dimension has sectional curvature in the range

$$1 \leqq K \leqq 4 \quad ,$$

then M is either homeomorphic to $S^n$ or isometric to a Riemannian symmetric space of compact type with rank 1.

The latter spaces are called CROSS and are completely classified. They are, besides spheres:

$\mathbb{C}P^n := U(n+1)/U(n) \times U(1)$

$\mathbb{H}P^n := sp(n+1)/sp(n) \times sp(1)$

$(\text{Cayley})P^n := F_4/spin(9)$ .

In [K3], Klingenberg conjectured that the assumption on the parity of dimension in the last theorem may not be necessary. This was indeed verified by J. Cheeger and D. Gromoll in 1972 (see [CG2]).

There also have been attempts to strengthen the topological similarity in the conclusion of Theorem 3. This was prompted by the discovery of J. Milnor in the period 1956 - 1960 that in each dimension $\geq 7$, there are compact differentiable manifolds that are homeomorphic, but not diffeomorphic to $S^n$. After the pioneering works of E. Calabi (unpublished, 1966), Gromoll [Gl], and Y. Shikata [Sh], the works of M. Sugimoto an K. Shiohama [SuS] and of K. Grove, H. S. Im-Hoff, H. Karcher, and E. Ruh: [Ru], [GrK], [GrKR], [IR] have established

<u>Theorem</u> 5: A complete Riemannian manifold, whether simply connected or not, whose sectional curvature ranges in

$$1 \leq K < 1.47$$

is diffeomorphic to some manifold of constant curvature 1.

The spaces of constant sectional curvature 1 are called the spherical spaceforms and have been classified completely by J. Wolf [Wo] previously.

Yet another way in which Theorem 3 has been generalized is to replace the assumption on the curvature by a weaker hypothesis. Among the first of these attempts was

Theorem 6 (Berger, 1962): Suppose that the sectional curvature, K, and diameter, d, of M satisfy

$$1 \leq K \quad ,$$

$$\pi/2 < d \quad .$$

Then, M has the homotopy type of $S^n$.

In fact, if the sectional curvature of M lies in $[1,4)$, then [K1] shows that $\pi/2 < d$. Therefore, by the generalized Poincaré conjecture of S. Smale (1961), in dimensions $\geq 5$, this implies Theorem 3 and is stronger. Later, Grove and Shiohama [GS] found a more direct argument which obtains homeomorphism in all dimensions $\geq 2$ without the use of Poincare conjecture. Finally, Gromoll and Grove [GG1], [GG2] (confer also [S], [SaS] for preliminary results) proved

Theorem 7: If the sectional curvature and diameter o M are in the range

$$1 \leq K \quad ,$$

$$\pi/2 \leq d \quad .$$

then M is either homeomorphic to $S^n$ or isometric to one of the following:

a CROSS ,

a spherical spaceform with its fundamental group having a fully reducible representation in $O(n + 1)$, or

a quotient of $\mathbb{C}P^{n/2}$ where $n \equiv 2 \pmod 4$ by an isometric $\mathbb{Z}_2$-action.

There are also results of a more generalized nature.

Theorem 8 (Cheeger [C]): In each dimension n, given any $1 < \kappa$ and $0 < v$, there are only finitely many homeomorphism classes of manifolds that admit Riemannian metrics whose sectional curvatures are in the range

$$1 \leqq K \leqq \kappa$$

and the volume, vol, satisfies

$$v < \text{vol}.$$

In fact, it is easily seen from the result in [K1] that if n is even and $0 < K \leqq \kappa$, then there is an *a priori* lower bound, computable in terms of n and $\kappa$, for the volume. Hence, Theorem 8 has

Corollary 0.1: For n even, given any $1 < \kappa$, there are only finitely many homeomorphism classes of n-dimensional manifolds that admit Riemannian structures with

$$1 \leqq K \leqq \kappa .$$

A weaker version of the above is due independently to A. Weinstein [W]. For n odd, Corollary 0.1 is no longer true. In fact, for each $n \geqq 3$ odd, there is a topologically distinct manifold of constant curvature 1 for each prime integer, while for $n = 7$, there is an infinite family of homotopically distinct simply connected manifolds all of whose curvatures lie in the range

$$1 \leqq K < 1073/16$$

as shown by H.-M. Huang [Hu] using examples of N. Wallach. However, very recently, we have

<u>Theorem</u> 9 (M. Gromov [Gm]): For any n, there are only finitely many rational homology types of manifolds that admit Riemannian structures of positive curvature.

One of the stimulating influences for the vigourous development of Riemannian geometry in this century was A. Einstein's liberal use of it in his theory of general relativity (1918). Playing a majour role in his equation for the gravitational potential is the sum of n - 1 sectional curvature terms containing one tangential direction. The same quantity had already been studied in 1904 by G. Ricci-Curbastro for purely geometric considerations and is now called the Ricci curvature.

The first result on the influences of Ricci curvature on the global geometry of a Riemannian manifold is still considered by many to be one of the most beautiful theorems in mathematics:

Theorem 10 (S. Myers, 1941): If the Ricci curvature of M is in the range

$$n - 1 \leqq \mathrm{Ric} \quad ,$$

then M is compact, has diameter

$$d \leqq \pi ,$$

and its fundamental group is finite.
Hence, by the topological theorem of Hurewicz, the first Betti number of M, $\beta_1 = 0$. The last assertion has been extended by S. Bochner (1946) to

Theorem 11: If M is a compact Riemannian manifold whose Ricci curvature is non-negative, and if its first Betti number $\beta_1 \neq 0$, then there are $\beta_1$ linearly independent globally defined parallel vector fields on M. In particular, $\beta_1 \leqq n$.
In fact the last theorem as well as more informations on the structure of the fundamental group can be recovered from

Theorem 12 (Cheeger and Gromoll [CG1]): If M is compact and has non-negative Ricci curvature, then the universal covering space of M splits isometrically as

$$\tilde{M} = N \times \mathbb{R}^k$$

where N is compact and of dimension $n - k$.

There have been other directions to sharpen Theorem 10. In 1963, R. Bishop showed

Theorem 13: With the Ricci curvature of M as in the condition of Theorem 10, if d is the diameter of M, then the volume of M is no larger than the volume of a metric ball of radius d in $S^n$. Moreover, if the volume of M equals that of $S^n$, then M is isometric to it.

The second half of this theorem has been generalized by S. Y. Cheng [Cn2] to

Theorem 14: If the Ricci curvature of M is in the range

$$n - 1 \leq \mathrm{Ric} \quad ,$$

then M has diameter $\pi$ if and only if M is isometric to $S^n$.

The last theorem was known earlier to V. A. Toponogov [T] in the case the sectional curvature of M was $\geq 1$.

There are other results on Ricci curvature that are certainly striking but of more specialized natures. For example, if M has the structure of a Kähler manifold, the celebrated solution by S. T. Yau [Y1], [Y2] to the problem of Calabi provides to a large extent the complete description of the Ricci curvature. In dimension 3, the problem on positive Ricci curvature seems completely settled. For open manifolds, we have

<u>Theorem</u> 15 (R. Schoen and Yau [ScY]): A complete but non-compact 3-dimensional manifold with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$.

Note that the analogous result is true in arbitrary dimension if the Ricci curvature is positive by the work of Gromoll and W. Meyer [GM]. On the other hand, the compact case has been dealt with by

<u>Theorem</u> 16 (A. Hamilton, 1982): A Riemannian metric on a compact 3-dimensional manifold whose Ricci curvature is positive can be smoothly deformed to the metric of a spherical spaceform.

In general, however, our knowledge on the Ricci curvature still seems meager. For instance, previously, no perturbation result for Ricci curvature on a $C^{\infty}$-manifold of arbitrary dimension seems to have been known. It has been proposed:

<u>Problem</u> A (Yau): Is there a constant $\varkappa > 1$ such that, if the Ricci curvature of M is in the range

$$n - 1 \leqq \mathrm{Ric} \leqq (n - 1)\varkappa \quad ,$$

then M admits an Einstein metric, i.e., a metric of constant Ricci curvature?

The solution to the above seems to require new tools in partial differential equations that are not yet available. In view of Theorem 14, we might hope for a more geometric solution to the following

Problem B: Is there a constant $\rho > 0$, depending only on the dimension n, such that if M has Ricci curvature

$$n - 1 \leq \mathrm{Ric}$$

and diameter

$$\pi - \rho < d \quad ,$$

then M is, in some way, topologically similar to $S^n$? If the sectional curvature has the range

$$1 \leq K \quad ,$$

then Theorem 6 states that $\rho = \pi/2$ provides the solution independent of n. However, in the Ricci curvature case, the dependence on the dimension, at least, is inevitable as the following example shows. Let M be the Riemannian product

$$M := S^j(\sqrt{(j-1)/(j+k-1)}) \times S^k(\sqrt{(k-1)/(j+k-1)})$$

where we have written $S^i(r)$ for the i-dimensional sphere of radius r in $\mathbb{R}^{i+1}$. Then, calculations show that M has an Einstein metric with

$$\mathrm{Ric} \equiv j + k - 1$$

and

$$d = \pi\left(\frac{j + k - 2}{j + k - 1}\right)^{\frac{1}{2}} .$$

Clearly, d approaches $\pi$ as $j + k$ increases to $\infty$.

Although Problem B, as it stands, is still open, the purpose of the present paper is to answer some related questions. Specifically, we show that there is a pinching on volume that forces a manifold M of positive Ricci curvature to have the homotopy type of $S^n$. Although, at the present, the pinching constant itself depends on a sectional curvature information, we can give some pinching constant for an arbitrarily given upper bound. For a stronger geometric invariant, the injectivity radius, we can prove the existence of a pinching constant, depending only on the dimension, that makes M homeomorphic to $S^n$. If we allow the constant again to depend on the upper bound for the sectional curvature, then the pinching on injectivity radius can be weakened to that at a single point.

The last fact deserves additional comment. A manifold M is said to have the Blaschke property at a point $p \in M$, if the injectivity radius at p is equal to the diameter as measured from p. For example, any of the CROSS's satisfies the condition at an arbitrary point. On the other hand, the classification by É. Cartan (1927) already shows that they are the only symmetric spaces with the Blaschke property at any point. The following famous conjecture is now usually called the Blaschke conjecture, even though the present form is due to A. Besse [Bs], H. Nakagawa and Shiohama [NS], and possibly several others:

Problem C: If a Riemannian manifold M satisfies the Blaschke property at all of its points, is it then isometric to one of the CROSS's?

The best partial solution so far seems to be that pointed Blaschke manifolds have the cohomology types of the CROSS's, although R. Bott [Bo], Nakagawa [Nk], and L. Bérard-Bérgery [B] have shown that this conclusion holds under much weaker hypotheses. Recently, O. Durumeric [D] has studied the problem from a different viewpoint. He investigated the case when the injectivity radius is $\varepsilon$-close to the diameter. Such a manifold is said to have the $\varepsilon$-Blaschke property at p and can have arbitrary topology already in dimension 2 without any further assumptions. However:

Theorem 17 (Durumeric, 1982): Given any $\delta$, a lower bound for the sectional curvature of M, there is an $\varepsilon$, depending only on $\delta$, such that if M satisfies the $\varepsilon$-Blaschke property at all $p \in M$, then M is either simply connected or has the homotopy type of the real projective space $\mathbb{R}P^n$.

Notice that by virtue of Theorem 10, manifolds with

$$n - 1 \leq Ric$$

and large injectivity radii automatically have the almost Blaschke property. In view of this, we might pose

Problem D: Is there an $\varepsilon > 0$ such that if a manifold M with

$$n - 1 \leq Ric$$

satisfies the $\varepsilon$-Blaschke property at some $p \in M$, then M share some topology with one of the CROSS's?

In this paper, we shall state our results more precisely as well as establish our notations and discuss some standard preliminaries in Section 1. Then, in Section 2, we describe our main tool. It is a criterion by which a manifold can be covered by two balls of prescribed radii and is essentially a refinement of a work used previouly by Gromov in [Gm]. In Section 3, the result on the injectivity radius will be proven and some applications will be given. In Section 4, we shall show how our geometric methods can be used to give a new direct proof of Theorem 13.

We will also give additional applications in the same section. For the Main theorem. we shall require some measure-theoretic analysis as well as standard theorems of purely topological nature. These will occupy the last two sections, Section 5 and Section 6, of our paper.

## 1. Notations and statements of the results

In this paper, M will always denote a connected $C^{\infty}$-manifold of dimension n with a fixed Riemannian metric $\langle,\rangle$. The tangent bundle of M is TM and its fibre over $p \in M$, $T_pM$, while the cotangent bundle is $T^*M$. We shall refer to $C^{\infty}$-sections of TM and $T^*M$ simply as *vector fields* and *Pfaffian forms*. The spaces of all vector fields and Pfaffian forms are denoted respectively $\mathfrak{X}_M$ and $\mathfrak{D}_M$. The metric induces the *covariant derivation of Levi-Civita*,

$$\nabla: \mathfrak{X}_M \to \mathfrak{X}_M \otimes \mathfrak{D}_M \quad ,$$

where the tensor product is taken over the ring $C^{\infty}_M$. Customarily, we write $_YX$ for $X(Y)$. The *Riemann-Christoffel tensor* is

$$R(X,Y;Z) := -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$$

The so-called curvature identities amount to the fact that the adjoint of R in the fourth index may be viewed as a section in the bundle

$$\Lambda^2 T^*M \text{ (symmetric product) } \Lambda^2 T^*M \quad .$$

Thus, R is equivalent to a quadratic form

$$K: \Lambda^2 TM \to \mathbb{R}$$

called the *Riemannian curvature*. The Riemannian curvature evaluated on simple elements of unit length is the *sectional curvature*. Viewed this way, it is evident that the sectional curvature determines the Riemann-Christoffel tensor.

For a fixed $v \in T_pM$, let us consider the function $K_v$ defined on $v^{\perp}$, the orthogonal complement of $\mathbb{R}v$ in $T_pM$, by

$$K_v(x) = K(x \wedge v) \quad .$$

This function is quadratic, and so, in the presence of a metric, has real eigenvalues $\rho_1(v) \leq \dots \leq \rho_{n-1}(v)$. These eigenvalues are the critical values of the function $K_v$ on the unit vectors in $v^{\perp}$ and are called the *principal sectional curvatures* of the direction v. For $2 \leq m \leq n - 1$, the (partial) trace

$$\mathrm{Ric}_m(v) := \rho_1(v) + \dots + \rho_m(v)$$

is called the *m-th partial Ricci curvature*. The (n - 1)st partial Ricci curvature is simply the *Ricci curvature* and is denoted Ric(v).

Let $c: [\alpha, \beta] \to M$ be a piecewise $C^{\infty}$ curve. We denote by L(c) the length of c,

$$L(c) := \int_{\alpha}^{\beta} |\dot{c}(s)|ds$$

where, as usual, $\dot{c}(s) := T_sc(d/ds)$ and $|\ |$ is the norm. We consider the pull-back bundle $c^*(TM)$ over $[\alpha, \beta]$. The piecewise-$C^{\infty}$-sections of this bundle are the vector fields along c. We shall write $\mathscr{X}_c$ for the space of all vector fields along c. The covariant derivation of Levi-Civita induces a natural derivation from $\mathscr{X}_c \to \mathscr{X}_c$ which we denote by a dot over the argument. In $\mathscr{X}_c$, there is a special subspace

$$\mathscr{X}''_c := \{X \in \mathscr{X}_c \mid \langle X,\dot{c}\rangle \equiv 0,\ X(\alpha) = X(\beta) = 0\} \quad .$$

The fields in $\mathscr{X}''_c$ may be viewed as the directional derivative operators at c for the parametrization-invariant functionals on the space of all curves joining $c(\alpha)$ and $c(\beta)$.

Now, assume that $c\colon\ [0,1] \to M$ is a geodesic. Unless otherwise specified, all of our geodesics will be parametrized by the arclength. The second derivative of the length functional is, for $W,\ X \in \mathscr{X}''_c$,

$$W\cdot X\cdot L(c) = \int_0^1 \langle\dot{W},\dot{X}\rangle - R(\dot{c}\wedge W,\dot{c}\wedge X) \quad .$$

If $0 < r \leq 1$, the symmetric bilinear form

$$\begin{aligned} I_0^r(W,X) &:= \int_0^r \langle\dot{W},\dot{X}\rangle - R(\dot{c}\wedge W,\dot{c}\wedge X) \\ &= \langle\dot{W},X\rangle\Big|_0^r - \int_0^r \langle\ddot{W},X\rangle + R(\dot{c}\wedge W,\dot{c}\wedge X) \end{aligned}$$

is called the index form of c on $[0,r]$. If c has the minimal length among neighbouring curves joining $c(\alpha)$ and $c(\beta)$, $I_0^r$ is positive definite for all $r < 1$. Moreover, for all $r \leq 1$, $I_0^r(Y,Y)$ attains a minimum among all vector fields along $c|[0,r]$ with the same with the same boundary values or with the same von-Neumann conditions if and only if $Y$ satisfies the *Jacobi equation*

$$\ddot{Y} + R(\dot{c}, Y; \dot{c}) = 0 \quad .$$

Let us now view $M$ as a metric space. If $p$, $q \in M$, then $d(p,q)$ is the distance between them. If $p \in M$, for $r > 0$, we set

$$B(r;p) := \{q \in M \mid d(p,q) < r\} \quad ,$$
$$D(r;p) := \{q \in M \mid d(p,q) < r\} \quad ,$$
$$S(r;p) := \{q \in M \mid d(p,q) < r\} \quad .$$

We give $T_pM$ the metric structure of an Euclidean space and set

$$\mathfrak{B}_{r,p} := \{u \in T_pM \mid |u| < r\} \quad ,$$
$$\mathfrak{D}_{r,p} := \{u \in T_pM \mid |u| < r\} \quad ,$$
$$\mathfrak{S}_{r,p} := \{u \in T_pM \mid |u| < r\} \quad .$$

In case $r = 1$, we usually suppress this fact from the notation; i. e., $\mathfrak{B}_p := \mathfrak{B}_{r,p}$, etc.

The classical theorem of H. Hopf and W. Rinow (1931) states that $M$ is complete as a metric space if and only if, for any $v \in TM$, there is a geodesic $c_v$ in $M$ with $\dot{c}_v(0) = v/|v|$ and $L(c_v) = |v|$; i.e., the *exponential map*, exp, is defined on all of $TM$. In such a case, for $u \in \mathfrak{S}_p$, we define

$$\mathscr{c}(u) := \sup\{r > 0 \mid d(x, \exp ru) = r\} \quad .$$

we have:

the injectivity radius at p: $i(p) := \inf\{\vartheta(u) \mid u \in \mathfrak{S}_p\}$;

the diameter at p: $d(p) := \sup\{\vartheta(u) \mid u \in \mathfrak{S}_p\}$ ;

the injectivity radius of M: $i_M := \inf\{i(p) \mid p \in M\}$ ;

the diameter of M: $d_M := \sup\{d(p) \mid p \in M\}$ .

The following calassical theorem provides a method for estimating the injectivity radius:

Proposition 1.1 (W. Klingenberg [K1]): If M is a complete manifold with K bounded from above by some constant $\varkappa > 0$, and if there is some point $p \in M$ such that

$$i(p) < \pi/\sqrt{\varkappa} \quad ,$$

then there is a simple closed geodesic loop at p of length $2i(p)$. In particular, if M is compact and

$$i_M < \pi/\sqrt{\varkappa} \quad ,$$

then there is a periodic geodesic in M of fundamental period $2i_M$.

One can see that M is compact if and only if $d(p)$ is finite at some p.

It is well-known that the function $\vartheta(u)$ is continuous on the unit sphere bundle $\mathfrak{S}M := \{u \in TM \mid |u| = 1\}$, and that the set

$$\mathfrak{N}_p := \{v \in T_pM \mid |v| < \vartheta(v/|v|)\}$$

is star-shaped with respect to the origin. The boundary of the last set

$$\mathscr{C}_p := \partial \mathscr{N}_p$$

is the tangential cut locus of p. The cut locus in M will be denoted

$$C(p) := \exp(\mathscr{N}_p) \quad .$$

For $p \in M$, the exponential map restricted to $T_pM$ is denoted $\exp_p$. If $v \in T_pM$, the space $T_vT_pM$ can be canonically identified with $T_pM$. Thus, we get a linear map

$$T_v\exp_p\colon \; T_pM \to T_{\exp v}M$$

given by the differential. It is well-known that for $u \in \mathscr{C}_p$, a map $X\colon \; \mathbb{R} \to T_pM$ is an affine map if and only if $Y(s) := T_{su}\exp_p(X(s))$ satisfies the Jacobi equation. The norm of the map $T_v\exp_p$ is estimated by the following famous

Proposition 1.2 (Rauch comparison theorem): Let $v \in \mathscr{N}_p$. Then, if the sectional curvature of M satisfies

$$\delta \leqq K(\sigma) \leqq \kappa$$

for all simple $\sigma$ of unit length in $\Lambda^2TM$ containing $T_v\exp v$ as a factor. Then, for any $u \in T_pM$,

$$\frac{\sin\sqrt{\delta}\,|v|}{\sqrt{\delta}\,|v|}\,|u| \geq |T_v\exp u| \geq \frac{\sin\sqrt{\kappa}\,|v|}{\sqrt{\kappa}\,|v|}\,|u| \quad .$$

Here, we do not assume that $\sqrt{\delta}$ and $\sqrt{\kappa}$ are real, but we use the complex extension of sine if either $\delta$ or

$\kappa$ happens to be negative. Note, however, that $\sin\sqrt{\delta}/\sqrt{\delta}$ is always real, so that the inequality makes sense.

The map

$$x := \exp_p^{-1}: \quad \exp(\mathcal{N}_p) \longrightarrow T_pM$$

provides a coördinate chart for the complement of C(p) called the normal coördinate system. If $U \in M$ is a Borel set, we have

$$\mathrm{vol}_U = \int_{\exp^{-1}(U) \cap \mathcal{N}_p} \det(T\exp_p)\, dx^1 \ldots dx^n \quad .$$

The square of $\det(T\exp_p)$ is called the Gramian:

$$Gr_p^{\frac{1}{2}}(v) := |\det(T_v \exp_p)| \quad .$$

In this paper, we shall employ techniques from the comparison theory. Given any $\delta \in \mathbb{R}$, we shall consider $\hat{M}$, the simply connected Riemannian manifold of dimension n and the constant sectional curvature $\delta$. By Hopf's theorem (Theorem 1), $\hat{M}$ is unique up to isometry, and:

$\hat{M} = S^n(1/\sqrt{\delta}) := \{x \in \mathbb{R}^{n+1} \mid |x| = 1/\sqrt{\delta}\}$ with the metric induced from the standard Euclidean metric of $\mathbb{R}^{n+1}$, if $\delta > 0$;

$\hat{M} = \mathbb{R}^n$ with the natural inner product, if $\delta = 0$; and

$\hat{M} = H^n(1/\sqrt{-\delta}) := \{x \in \mathbb{R}^n \mid |x| < -1/\delta\}$ with the metric

$$\langle \hat{v}, \hat{w} \rangle := \frac{4\langle \hat{v}, \hat{w} \rangle_{\text{Euclidean}}}{1 + \delta |\hat{p}|^2_{\text{Euclidean}}}$$

for $\hat{v}$, $\hat{w} \in T_{\hat{p}}\hat{M}$, if $\delta < 0$.

Sometimes, we consider $\hat{M}$ as a pointed space with a specified origin. However, as these spaces are homogenious, it matters not which point is chosen. The metric figures in $\hat{M}$, centered at the origin will be denoted:

$$\hat{B}(r),\ \hat{D}(r),\ \hat{S}(r),\ \hat{\mathfrak{B}}_r,\ \hat{\mathfrak{D}}_r,\ \hat{\mathfrak{S}}_r \quad .$$

The Gramian in $\hat{M}$ is given by

$$\hat{Gr}^{\frac{1}{2}}(\hat{v}) = \frac{\sin^{n-1}(\sqrt{\delta}|\hat{v}|)}{(\sqrt{\delta}|\hat{v}|)^{n-1}} \quad .$$

Now, we are ready to state the main results of our paper.

<u>Theorem</u> 18 (Main theorem): Suppose that the Ricci curvature is in the range

$$n - 1 \leq \text{Ric}$$

as a quadratic form relative to $|\ |^2$. Then, given whatever upper bound $\kappa$ for $K$, there is a constant $0 < \nu$, depending only on $n$ and $\kappa$, such that if

$$(1 - \nu)\text{vol}_{S^n} < \text{vol}_M \quad ,$$

then $M$ has the homotopy type of $S^n$.

Using some of the same techniques, we shall also obtain

<u>Theorem</u> 19: Let $M$ satisfy

$$n - 1 \leq \mathrm{Ric} .$$

Then, there is a constant $0 < \rho$, depending only on $n$, such that if

$$\pi - \rho < i_M ,$$

then M is homeomorphic to $S^n$.

Theorem 20: Given any $\kappa$, an upper bound for K, there is a $\rho$, depending only on n and $\kappa$, such that if there is a point $p \in M$ where

$$\pi - \rho < i(p) ,$$

then M is homeomorphic to $S^n$.

If $f\colon M \to \mathbb{R}$ is of class $C^\infty$, we denote the Hessian form of f by $\mathrm{hess}_f$,

$$\mathrm{hess}_f(W,X) := \langle \nabla_W \mathrm{grad}\ f, X\rangle ,$$

where, as usual, grad f is the gradient of f. We take the convention that the Laplace operator acting on functions is

$$\Delta f := -\mathrm{trace}\ \mathrm{hess}_f .$$

The next formula is useful:

Lemma 1.3 (A. Lichnerowicz [L]): If $f \in C^\infty_M$, then

$$-\tfrac{1}{2}\Delta|\mathrm{grad}\ f|^2 = |\mathrm{hess}_f|^2 - \langle \mathrm{grad}\ f, \mathrm{grad}\ \Delta f\rangle + \mathrm{Ric}(\mathrm{grad}\ f).$$

Let $\lambda$ be the first non-0 eigenvalue of $\Delta$, and suppose that g is one of the corresponding eigenfunctions. The set $g^{-1}(0)$ is called the nodal set. The following is classical:

Proposition 1.4 (R. Courant): The set $\neg g^{-1}(0)$ has exactly two components.

It is well-known that the first eigenvalue of $S^n$ is n. In Section 4, we shall give a new geometric proof to the following celebrated

Theorem 21 (M. Obata [O]): If M has

$$n - 1 \leqq \mathrm{Ric} \quad ,$$

then $\lambda = n$, only if M is isometric to $S^n$.

## 2. Volume estimates and covering lemmas

In the rest of the paper, we assume that M is complete.

<u>Lemma</u> 2.1 (R. Bishop): Let $p \in M$ and $u \in \mathcal{O}_p$. Set $c(s) := \exp su$ for $0 \leq s \leq \mathcal{J}(u)$. Suppose that the Ricci curvature of M satisfies

$$(n-1)\delta \leq \mathrm{Ric}(\dot{c}(s))$$

for all s and some fixed $\delta \in \mathbb{R}$. Then, the function

$$s \longmapsto \frac{\mathrm{Gr}_p^{\frac{1}{2}}(su)}{\{\sin\sqrt{\delta}\, s/(\sqrt{\delta}\, s)\}^{n-1}}$$

is monotonically non-increasing.

<u>Proof</u>: Let $v_1, \ldots, v_{n-1}$ be any basis for $u^{\perp} \subset T_pM$. Set

$$Y_j(s) := T_{su}\exp_p sv_j \; ; \; j = 1, \ldots, n-1 \; .$$

Then, by the discussion in Section 1, $Y_j \in \mathcal{X}_c$ satisfy the Jacobi equation, and

$$\mathrm{Gr}_p^{\frac{1}{2}}(su) = \frac{|Y_1(s) \wedge \ldots \wedge Y_{n-1}(s)|}{s^{n-1}|v_1 \wedge \ldots \wedge v_{n-1}|}$$

Now, fix a $t \in (0, \mathcal{J}(u))$. Since $T_{tu}\exp_p$ is an isomorphism, there is a set $v_1', \ldots, v_{n-1}$ so that $Y_1(t), \ldots, Y_{n-1}(t)$ are orthonormal at $c(t)$. Then,

$$\frac{d}{ds}s^{n-1}\mathrm{Gr}_p^{\frac{1}{2}}(su)_{|t}$$

$$= \frac{d}{ds}\left\{\frac{|Y_1(s) \wedge \cdots \wedge Y_{n-1}(s)|}{|V_1 \wedge \cdots \wedge V_{n-1}|}\right\}_{|t}$$

$$= \frac{\sum \left\{\begin{matrix}\langle Y_1(t) \wedge \cdots \wedge \dot{Y}_j(t) \wedge \cdots \wedge Y_{n-1}(t), \\ Y_1(t) \wedge \cdots \wedge Y_{n-1}(t)\rangle\end{matrix}\right\}}{|Y_1(t) \wedge \cdots \wedge Y_{n-1}(t)|\,|V_1 \wedge \cdots \wedge V_{n-1}|}$$

$$= \frac{\sum \langle \dot{Y}_j(t), Y_j(t)\rangle}{|V_1 \wedge \cdots \wedge V_{n-1}|} ,$$

so that

$$\frac{\frac{d}{ds} s^{n-1} Gr_p^{\frac{1}{2}}(su)_{|t}}{t^{n-1} Gr_p^{\frac{1}{2}}(tu)} = \sum \langle \dot{Y}_j(t), Y_j(t)\rangle .$$

Now, extend $Y_1(t), \ldots, Y_{n-1}(t)$ to a parallel orthonormal frame field $E_1(s), \ldots, E_{n-1}(s)$ along $c$. Set

$$V_j(s) := \frac{\sin\sqrt{\delta}\, s}{\sin\sqrt{\delta}\, t} E_j(s) .$$

Since $Y_j$'s satisfy the Jacobi equation and vanish at $s = 0$, while $V_j$'s have the same boundary values on $[0,t]$,

$$\langle \dot{Y}_j(t), Y_j(t)\rangle = I_0^t(Y_j, Y_j) \leqq I_0^t(V_j, V_j) .$$

Therefore,

$$\frac{\frac{d}{ds} s^{n-1} Gr_p^{\frac{1}{2}}(su)_{|t}}{t^{n-1} Gr_p^{\frac{1}{2}}(tu)} \leqq \sum I_0^t(V_j, V_j)$$

$$= \sum \int_0^t \langle \dot{V}_j(s), V_j(s) \rangle - K(V_j(s) \wedge \dot{c}(s))\, ds$$

$$= \sum \int_0^t \frac{\cos^2\sqrt{\delta}\, s}{\sin^2\sqrt{\delta}\, t} - K(E_j(s) \wedge \dot{c}(s)) \frac{\sin^2\sqrt{\delta}\, s}{\sin^2\sqrt{\delta}\, t}\, ds$$

$$= \int_0^t (n-1) \frac{\cos^2\sqrt{\delta}\, s}{\sin^2\sqrt{\delta}\, t} - \mathrm{Ric}(\dot{c}(s)) \frac{\sin^2\sqrt{\delta}\, s}{\sin^2\sqrt{\delta}\, t}\, ds .$$

By our assumption on the Ricci curvature, the last integral is

$$\leqq (n-1) \int_0^t \frac{\cos^2\sqrt{\delta}\, s}{\sin^2\sqrt{\delta}\, t} - \delta \frac{\sin^2\sqrt{\delta}\, s}{\sin^2\sqrt{\delta}\, t}\, ds .$$

This integral evaluates to

$$(n-1) \frac{\sqrt{\delta}\ \cos\sqrt{\delta}\, t \cdot \sin\sqrt{\delta}\, t}{\sin^2\sqrt{\delta}\, t}$$

$$= \frac{(n-1)\ \sin^{n-2}\sqrt{\delta}\, t\ \sqrt{\delta}\ \cos\sqrt{\delta}\, t}{\sin^{n-1}\ \ t}$$

$$= \frac{\frac{d}{ds}\sin^{n-1}\sqrt{\delta}\, s\,|_t}{\sin^{n-1}\sqrt{\delta}\, t} .$$

That is,

$$\frac{\frac{d}{ds} s^{n-1} \mathrm{Gr}_p^{\frac{1}{2}}(su)\,|_t}{t^{n-1}\mathrm{Gr}_p^{\frac{1}{2}}(tu)} \leqq \frac{\frac{d}{ds}\sin^{n-1}\sqrt{\delta}\, s\,|_t}{\sin^{n-1}\sqrt{\delta}\, t} .$$

Multiplying both the numerator and the denominator of the right side by $(1/\sqrt{\delta})^{n-1}$ to make them each real and cross-multiplying, we obtain

$$\frac{\sin^{n-1}\sqrt{\delta}\,t}{(\sqrt{\delta})^{n-1}} \cdot \frac{d}{ds} s^{n-1} Gr_p^{\frac{1}{2}}(su)\Big|_t$$

$$- \frac{d}{ds}\left\{\frac{\sin^{n-1}\sqrt{\delta}\,s}{(\sqrt{\delta})^{n-1}}\right\}\Big|_t \cdot t^{n-1} Gr_p^{\frac{1}{2}}(tu) \leqq 0 \quad .$$

Note that the left side of the inequality above is the numerator of the expression

$$\frac{d}{ds}\left\{\frac{s^{n-1} Gr_p^{\frac{1}{2}}(su)}{\sin^{n-1}\sqrt{\delta}\,s \;/\; \sqrt{\delta}^{\,n-1}}\right\}\Big|_t \quad .$$

But, since t was chosen arbitrarily, we conclude that

$$\frac{d}{ds}\,\frac{s^{n-1} Gr_p^{\frac{1}{2}}(su)}{\sin^{n-1}\sqrt{\delta}\,s \;/\; \sqrt{\delta}^{\,n-1}} \leqq 0$$

for all s.

q.e.d.

Noting in the above that, if the function in the lemma is a constant, $V_j$'s must satisfy the Jacobi equation, we can derive Bishop's theorem 13. However, the following observation is much stronger:

<u>Proposition</u> 2.2 (M. Gromov [Gm]): Let $p \in M$. Suppose that M satisfies, for all $u \in \mathfrak{S}_p$ and $s \in (0, \vartheta(u))$,

$$(n-1)\delta \leqq Ric(\dot{c}_u(s)) \quad ,$$

where $c_u(s) := \exp su$ and $\delta$ is some real number. Let $\hat{M}$ be the model space of constant sectional curvature as described in Section 1. Then, for all $0 < r \leqq R$,

$$\frac{vol_{B(R;p) \sim B(r;p)}}{vol_{B(r;p)}} \leqq \frac{vol_{\hat{B}(R) \sim \hat{B}(r)}}{vol_{\hat{B}(r)}} \quad .$$

<u>Proof</u>: Define $\bar{r}(u) := \min\{r, \mathscr{S}(u)\}$. Then,

$$\mathrm{vol}_{B(r;p)} = \int_{\mathfrak{S}_p}\int_0^{\bar{r}(u)} Gr_p^{\frac{1}{2}}(su)s^{n-1}\, dsdu \quad ;$$

$$\mathrm{vol}_{B(R;p) \sim B(r;p)} = \int_{\mathfrak{S}_p}\int_{\bar{r}(u)}^{\bar{R}(u)} Gr_p^{\frac{1}{2}}(su)s^{n-1}\, dsdu \quad ;$$

$$\mathrm{vol}_{\hat{B}(r)} = \int_{S^{n-1}}\int_0^r \frac{\sin^{n-1}\sqrt{\delta}\, s}{\sqrt{\delta}^{\,n-1}}\, dsdu \quad ;$$

$$\mathrm{vol}_{\hat{B}(R) \sim \hat{B}(r)} = \int_{S^{n-1}}\int_r^R \frac{\sin^{n-1}\sqrt{\delta}\, s}{\sqrt{\delta}^{\,n-1}}\, dsdu \quad .$$

Therefore, our desired result follows immediately from the following intergration trick:

<u>Lemma</u> 2.3: Let f, g: $[0,R] \longrightarrow \mathbb{R}$ be integrable functions. If f/g is a monotonically non-increasing function, then, for any $0 < r \leqq R$,

$$\int_r^R f(s)\, ds \Big/ \int_0^r f(s)\, ds \leqq \int_r^R g(s)\, ds \Big/ \int_0^r g(s)\, ds \quad .$$

<u>Proof</u>: It suffices to show that

$$\int_r^R f(s)\, ds \int_0^r g(t)\, dt \leqq \int_r^R g(s)\, ds \int_0^r f(t)\, dt \quad .$$

But, by assumption, $f(s)/g(s) \leqq f(r)/g(r)$ for all $r \leqq s$ and $f(r)/g(r) \leqq f(t)/g(t)$ for all $t \leqq r$. Thus,

$$\int_r^R f(s)\, ds \int_0^r g(t)\, dt = \int_r^R \frac{f(s)}{g(s)} g(s)\, ds \int_0^r g(t)\, dt$$

$$\leqq \frac{f(r)}{g(r)} \int_r^R g(s)\, ds \int_0^r g(t)\, dt$$

$$\int_r^R g(s)\, ds \int_0^r \frac{f(t)}{g(t)} g(t)\, dt = \int_r^R g(s)\, ds \int_0^r f(t)\, dt \quad .$$

q.e.d.

Corollary 2.4 (Bishop-Gromov volume comparison theorem): Under the same situation as Proposition 2.2,

$$\frac{\mathrm{vol}_{B(R;p)}}{\mathrm{vol}_{B(r;p)}} \leq \frac{\mathrm{vol}_{\hat{B}(R)}}{\mathrm{vol}_{\hat{B}(r)}} .$$

Proof: Note that since $B(R;p) \supset B(r;p)$,

$$\mathrm{vol}_{B(R;p) \sim B(r;p)} = \mathrm{vol}_{B(R;p)} - \mathrm{vol}_{B(r;p)} ,$$

and likewise in $\hat{M}$. Therefore, we can add

$$\frac{\mathrm{vol}_{B(r;p)}}{\mathrm{vol}_{B(r;p)}} = 1 = \frac{\mathrm{vol}_{\hat{B}(r)}}{\mathrm{vol}_{\hat{B}(r)}}$$

side by side to the inequality of Proposition 2.2 to obtain the corollary.

q.e.d.

The following particular observation, though simple, does not seem to appear in literature:

Lemma 2.5: Let M be a compact manifold, and suppose that $p \in M$ has the property prescribed in Proposition 2.2. Assume furthermore that

$$(1 - \nu)\mathrm{vol}_{\hat{B}(R)} \leq \mathrm{vol}_M ,$$

where $d(p) \leq R$ and $0 < \nu < 1$ is some real number. Then, for any $0 < r$, we have

$$(1 - \nu)\mathrm{vol}_{\hat{B}(r)} \leq \mathrm{vol}_{B(r;p)} .$$

Proof: Since $d(p) < R$, Corollary 2.4 states that

$$\frac{vol_M}{vol_{B(r;p)}} \leq \frac{vol_{\hat{B}(R)}}{vol_{\hat{B}(r)}} .$$

Therefore,

$$\frac{vol_M}{vol_{\hat{B}(R)}} vol_{\hat{B}(r)} \leq vol_{B(r;p)} .$$

But, by assumption,

$$1 - \nu \leq \frac{vol_M}{vol_{\hat{B}(R)}} .$$

So, the desired inequality obtains. q.e.d.

It is a standard fact that, for any two Riemannian manifolds $M_1$ and $M_2$ and $p_1 \in M_1$, $p_2 \in M_2$,

$$\lim_{r \to 0} \frac{vol_{B_1(r;p_1)}}{vol_{B_2(r;p_2)}} = 1 ,$$

where $B_i$ is the ball in $M_i$; $i = 1, 2$. Using this, Corollary 2.4 implies the following, which actually predates Corollary 2.4:

Corollary 2.6 (Bishop comparison theorem): Still under the hypotheses of Proposition 2.2, for any $0 < R$,

$$vol_{B(R;p)} \leq vol_{\hat{B}(R)} .$$

Proof: Multiply both sides of the inequality in Corollary 2.4 by $vol_{B(r;p)}$ and take the limit as r goes to 0

q.e.d.

Corollary 2.7: Let M be a compact manifold, and let $p \in M$ satisfy the assumptions of Proposition

2.2. Given $0 < U$, let b be the number such that

$$\mathrm{vol}_{\hat{B}(b)} = U .$$

Then, whenever M satisfies

$$U \leqq \mathrm{vol}_M ,$$

we have $b \leqq d(p)$.

Proof: This follows immediately from the fact that $B(d(p);p) = M$.

q.e.d.

Now, assume that

$$n - 1 \leqq \mathrm{Ric} .$$

We wish to find a criterion by which M is covered by two balls of prescribed radii. A first related result seems to be due to Y. Tsukamoto [Ts1], who proved that if $1 \leqq K$ and $p, q \in M$ realize the diameter of M, then $D(\pi/2;p)$ and $D(\pi/2;q)$ cover M. This fact is no longer true, if we assume only that $n - 1 \leqq \mathrm{Ric}$. In fact, let M be the product of spheres described in Section 0. Let us take $j = k$; i.e.,

$$M := S^j(\sqrt{(j-1)/(2j-1)}) \times S^j(\sqrt{(j-1)/(2j-1)}) .$$

Let $p_1$, $q_1$ be a pair of antipodal points in the first factor, and $p_2$, $q_2$ a pair of antipodal points in the second factor. Then, the pair $(p_1,p_2)$ and $(q_1,q_2)$ realizes the diameter

$$\pi\sqrt{(2j-2)/(2j-1)} ,$$

but the point $(q_1,p_2)$ has the distance

$$\pi\sqrt{(j-1)/(2j-1)}$$

from both $(p_1, p_2)$ and $(q_1, q_2)$. Since $j \geq 2$ and

$$\sqrt{(j-1)/(2j-1)} > \sqrt{\tfrac{1}{3}} > \tfrac{1}{2} \quad ,$$

the two balls of the radius $\pi/2$ from $(p_1, p_2)$ and $(q_1, q_2)$ do not cover $(q_1, p_2)$. In [Gm], Gromov uses Corollary 2.4 to obtain an estimate on the number of metric balls $\geq 3$, all of a fixed radius, that are needed to cover a manifold with a given lower bound on the Ricci curvature and an upper bound on the diameter. However, this is not good enough for our purpose.

Now, for $0 \leq \nu < 1$, we define $b(\nu)$ to be the number between 0 and $\pi$, such that

$$(1 - \nu)\mathrm{vol}_{S^n} = \mathrm{vol}_{\hat{B}(b(\nu))} \quad ,$$

where $\hat{B}$ is in $S^n$. Note that $b(\nu)$ increases to $\pi$ as $\nu$ decreases to 0.

<u>Proposition</u> 2.8 (Main covering lemma): Let M be a compact manifold with the Ricci curvature in the range

$$n - 1 \leq \mathrm{Ric} \quad .$$

Let $0 \leq \nu$ be small enough so that

$$2\nu\mathrm{vol}_{S^n} \leq \mathrm{vol}_{\hat{B}(b(\nu))} \quad ,$$

and suppose that

$$(1 - \nu)\mathrm{vol}_S n \leqq \mathrm{vol}_M \quad .$$

Take $p \in M$ and choose $q \in M$ so that

$$d(p,q) = d(p) \quad .$$

Finally, given any $0 \leqq d_1$, let $d_2 := b(\nu) - d_1$. Then, there exists a constant $0 < r$, depending only on $n$, $\nu$, and $d_1$, such that the closed balls $D(d_1 + r;p)$ and $D(d_2 + r;q)$ cover $M$. Moreover, for fixed $n$ and $d_1$, $r$ can be so chosen as to approach 0 as $\nu$ tends to 0.

<u>Proof</u>: By Corollary 2,7, $\bar{b}(\nu) \leqq d(p)$. This implies that

$$B(d_1;p) \cap B(d_2;q) = \emptyset \quad .$$

Hence,

$$\begin{aligned} &\mathrm{vol}_{M \sim B(d_1;p) \sim B(d_2;q)} \\ &= \mathrm{vol}_M - \mathrm{vol}_{B(d_1;p)} - \mathrm{vol}_{B(d_2;q)} \quad . \end{aligned}$$

By setting $R = \pi$ in Corollary 2.5, we have

$$\mathrm{vol}_M \leqq \mathrm{vol}_S n \quad ;$$

while, by Corollary 2.6,

$$\mathrm{vol}_{B(d_1;p)} \geqq (1 - \nu)\mathrm{vol}_{\hat{B}(d_1)}$$

and

$$\mathrm{vol}_{B(d_2;p)} \geqq (1 - \nu)\mathrm{vol}_{\hat{B}(d_2)} \quad .$$

Substituting, we obtain

$$\mathrm{vol}_M \sim B(d_1;p) \sim B(d_2;q)$$
$$\leq \mathrm{vol}_S n - (1-\upsilon)(\mathrm{vol}_{\hat{B}(d_1)} + \mathrm{vol}_{\hat{B}(d_2)}) \quad .$$

Let us call the right side of the above, $\tilde{v}(\upsilon)$. Now, choose r so that

$$(1-\upsilon)\mathrm{vol}_{\hat{B}(r)} = \tilde{v}(\upsilon) \quad .$$

We claim that such an r exists.

To prove this claim, it suffices to prove that

$$\mathrm{vol}_S n \geq \frac{\tilde{v}(\upsilon)}{1-\upsilon} \quad .$$

But,

$$\mathrm{vol}_S n - \frac{\tilde{v}(\upsilon)}{1-\upsilon}$$
$$= \mathrm{vol}_S n - \frac{\mathrm{vol}_S n}{1-\upsilon} + \mathrm{vol}_{\hat{B}(d_1)} + \mathrm{vol}_{\hat{B}(d_2)}$$
$$= \frac{(1-\upsilon)\mathrm{vol}_S n - \mathrm{vol}_S n + (1-\upsilon)(\mathrm{vol}_{\hat{B}(d_1)} + \mathrm{vol}_{\hat{B}(d_2)})}{1-\upsilon}$$
$$= \frac{\mathrm{vol}_{\hat{B}(d_1)} + \mathrm{vol}_{\hat{B}(d_2)} - \upsilon(\mathrm{vol}_{\hat{B}(d_1)} + \mathrm{vol}_{\hat{B}(d_2)} + \mathrm{vol}_S n)}{1-\upsilon} \quad .$$

But since, for one of i = 1 or 2, $d_i \geq b(\upsilon)/2$ and $\mathrm{vol}_{\hat{B}(d_1)} + \mathrm{vol}_{\hat{B}(d_2)} < \mathrm{vol}_S n$, the numerator of the above is

$$> \mathrm{vol}_{\hat{B}(b(\upsilon))} - 2\upsilon\mathrm{vol}_S n \geq 0$$

by assumption.

We now show that this r has the desired property.

Choose an arbitrary $x \in M \sim B(d_1;p) \sim B(d_2;q)$. Using Corollary 2.6 again, we see that

$$\mathrm{vol}_{M \sim B(d_1;p) \sim B(d_2;q)} \leq \tilde{V}(\nu) \leq \mathrm{vol}_{B(r;x)} \quad .$$

This shows that the closed ball $D(r;x)$ cannot be contained completely in the set $\sim(D(d_1;p) \cup D(d_2;q))$; i.e. , we must have either

$$D(r;x) \cap D(d_1;p) \neq \emptyset$$

or

$$D(r;x) \cap D(d_2;q) \neq \emptyset \quad .$$

By the triangle inequality, this implies either

$$d(x,p) \leq d_1 + r$$

or

$$d(x;q) \leq d_2 + r \quad .$$

Therefore,

$$x \in D(d_1 + r;p) \cup D(d_2 + r;q) \quad .$$

By virtue of the way x was chosen, this proves the first statement.

To prove the second statement, choose $\hat{p}$, $\hat{q}$ a pair of antipodal points in $S^n$. Then, for any $0 < d \leq \pi$,

$$D(d;\hat{p}) \cup D(\pi - d;\hat{q}) = S^n \quad .$$

Since $S^n$ is homogeneous, this implies

$$\mathrm{vol}_{\hat{B}(d)} + \mathrm{vol}_{\hat{B}(\pi - d)} = \mathrm{vol}_{S^n} \quad .$$

Now, as $\nu$ tends to 0, $d_1 + d_2 = b(\nu)$ approaches $\pi$.

Thus,

$$\mathrm{vol}_{\hat{B}(d_1)} + \mathrm{vol}_{\hat{B}(d_2)} \to \mathrm{vol}_{S^n} \quad ,$$

and, consequently, $\hat{V}(\nu) \to 0$. This allows r also to go to 0.

q.e.d.

Remark: With a little more care, Proposition 2.8 can be improved in a number of ways. First, by examining the way $\mathrm{vol}_{\hat{B}(d_1)} + \mathrm{vol}_{\hat{B}(d_2)}$ changes as $d_1$ and $d_2$ do, it can be seen that the assumption on can be weakened to

$$\nu \mathrm{vol}_{S^n} \leq \mathrm{vol}_{\hat{B}(\overline{b(\nu)})} \quad .$$

The numerical value of r can be improved somewhat by estimating the various inequalities a little more carefully. Then, there is an $R < \pi$, depending only on n, such that the first statement is true whenever

$$(1 - \nu)\mathrm{vol}_{\hat{B}(R)} \leq \mathrm{vol}_M \quad .$$

Finally, we mention that there is a method to find the r which is independent even of the given $d_1$. However, as none of the subsequent arguments seem to be sharp, we shall not persue these questions further.

For convenience later, we rephrase Proposition 2.8 slightly.

<u>Corollary</u> 2.9 (Covering lemma): Suppose that

$$n - 1 \leq \mathrm{Ric} \quad .$$

Let p, q $\in$ M be such that d(p,q) realizes $d_M$. Then, given any $0 < \delta, \rho$, there is an $\nu$, depending only on n, $\delta$, and $\rho$, such that

$$D(\delta + \rho; p) \cup D(b(\nu) - \delta + \rho; q) = M.$$

## 3. Positively Ricci-curved manifolds with large injectivity radii

In this and all subsequent sections, M is assumed to satisfy

$$n - 1 \leqq \mathrm{Ric}$$

and, accordingly, we shall use $M = S^n$. By the theorem of Myers (Theorem 10), M is then compact and has $d_M \leqq \pi$. Theorem 10, itself, has the following consewuence:

<u>Lemma</u> 3.1: If there is a point $p \in M$ where any geodesic loop at p has length $> \pi$, then M is simply connected.

<u>Proof</u>: If M is not simply cinnected, then there is a minimal geodesic for each element of $\pi_1(M;q)$. To see this, let $\pi$: $\tilde{M} \to M$ be the universal covering space, and take $q_1 \neq q_2 \in \pi^{-1}(q)$. Then, the minimal path from $q_1$ to $q_2$ in $\tilde{M}$ projects to a geodesic loop of length $d(q_1, q_2)$ by $\pi$. On the other hand, if such a geodesic loop representing a non-trivial homotopy class at p has length $> \pi$, then there must be $p_1, p_2 \in \pi^{-1}(p)$ with $d(p_1, p_2) > \pi$; a contradiction.

q.e.d.

<u>Corollary</u> 3.2: If for some $p \in M$,

$$\pi/2 < i(p) \quad ,$$

then M is simply connected.

Proof: This follows immediately as every geodesic loop must contain a cut point. q.e.d

We now prove Theorem 19.

Lemma 3.3: Let $\nu$ be any positive number so small that

$$2\nu \mathrm{vol}_{S^n} < \mathrm{vol}_{B(b(\nu))} \quad ,$$

and suppose that

$$(1 - \nu)\mathrm{vol}_{S^n} \leq \mathrm{vol}_M \quad .$$

Let $p, q \in M$ be such that $d(p,q)$ realizes $d_M$. Then, there is a number $0 < R < \pi$ such that if

$$R < i(p), i(q) \quad ,$$

then M is homeomorphic to $S^n$.

Proof: Let $d_1 = d_2 := b(\nu)/2$. Let r be as in the conclusion of the Main covering lemma. We claim that $R := b(\nu)/2 + r$ is the desired constant. In fact, then

$$D(R;p) \cup D(R;q) = M \quad ,$$

whence $D(R;p) \subset \exp(\mathcal{U}_p)$ and $D(R;q) \subset \exp(\mathcal{U}_q)$. Thus, $D(R;p)$ and $D(R;q)$ are both diffeomorphic to the Euclidean disc

$$D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\} \quad .$$

This implies that $B(R;p)$ is homeomorphic to $\mathbb{R}^n$ and $S(R;q)$ is homeomorphic to $S^{n-1}$. By the Jordan-Brouwer separation theorem, we see that $B(R;q) \sim B(R;p)$ is also homeomorphic to $D^n$. Thus $M$ is written as the union of two homeomorphic copies of $D^n$, joined at their common boundary. Now it is a standard topological fact that such a manifold is homeomorphic to $S^n$.

q.e.d.

Lemma 3.4: Suppose that there is a point $q \in M$ for which

$$\pi/2 < i(q) \quad .$$

Then,

$$\mathrm{vol}_{\hat{B}(i(q))} = \mathrm{vol}_{S^{n-1}}\, i(q) \sin^{n-1} i(q) < \mathrm{vol}_M \quad .$$

Proof: Choose an $u \in \mathfrak{S}_q$. Then,

$$\left(\frac{\sin s}{s}\right)^{n-1} \frac{d}{ds} Gr_q^{\frac{1}{2}}(su) - Gr_q^{\frac{1}{2}}(su) \frac{d}{ds}\left(\frac{\sin s}{s}\right)^{n-1} \leqq 0$$

by Lemma 2.1. On the other hand, since

$$\lim_{s \to 0} \frac{Gr_q^{\frac{1}{2}}(su)}{(\sin s/s)^{n-1}} = 1 \quad ,$$

$(\sin s/s)^{n-1} \geqq Gr_q^{\frac{1}{2}}(su)$, and so

$$\frac{d}{ds} Gr_q^{\frac{1}{2}}(su) - \frac{d}{ds}\left(\frac{\sin s}{s}\right)^{n-1}$$

$$= \frac{d}{ds}\left\{Gr_q^{\frac{1}{2}}(su) - \frac{\sin s}{s}^{\,n-1}\right\} \leqq 0 \quad ;$$

i.e., the function

$$Gr_q^{\frac{1}{2}}(su) - \left(\frac{\sin s}{s}\right)^{n-1}$$

is monotonically non-increasing. Hence, for $0 < s \leqq i(q)$,

$$Gr_q^{\frac{1}{2}}(i(q)u) - \left(\frac{\sin i(q)}{i(q)}\right)^{n-1} \leqq Gr_q^{\frac{1}{2}}(su) - \left(\frac{\sin s}{s}\right)^{n-1} .$$

Since $0 \leqq Gr_q^{\frac{1}{2}}(i(q)u)$,

$$\left(\frac{\sin s}{s}\right)^{n-1} - \left(\frac{\sin i(q)}{i(q)}\right)^{n-1} \leqq Gr_q^{\frac{1}{2}}(su)$$

for all $u \in \mathfrak{S}_q$. Now we integrate both sides with respect to the polar coördinates:

$$\int_{\mathfrak{S}_q}\int_0^{i(q)} \sin^{n-1}s \; dsdu$$

$$- \left(\frac{\sin i(q)}{i(q)}\right)^{n-1}\int_{\mathfrak{S}_q}\int_0^{i(q)} s^{n-1} \; dsdu$$

$$\lneqq \int_{\mathfrak{S}_q}\int_0^{i(q)} Gr_q^{\frac{1}{2}}(su)s^{n-1} \; dsdu \quad ,$$

whence

$$\int_{\mathfrak{S}_q}\int_0^{i(q)} \sin^{n-1}s \; dsdu = \mathrm{vol}_{\hat{B}}(i(q)) \quad ,$$

$$\left(\frac{\sin i(q)}{i(q)}\right)^{n-1}\int_{\mathfrak{S}_q}\int_0^{i(q)} s^{n-1} \; dsdu$$

$$= \sin^{n-1} i(q) \; i(q) \; \mathrm{vol}_{S^{n-1}} \quad ,$$

and

$$\int_{\mathfrak{S}_q}\int_0^{i(q)} Gr_q^{\frac{1}{2}}(su)s^{n-1} \; dsdu$$

$$= \int_{\exp^{-1}(B(i(q);q)_q} Gr_q^{\frac{1}{2}}(v)dv \quad .$$

The right side of the last expression is

$$\mathrm{vol}_{B(i(q);q)} \leqq \mathrm{vol}_M \quad ,$$

thus obtaining the desired inequality. q.e.d.

Proof of Theorem 19: In view of Lemma 3.3, it suffices to show that if $i_M$ is sufficiently large,

$$(1 - \nu)\mathrm{vol}_{S^n} \leqq \mathrm{vol}_M$$

where $2\nu\mathrm{vol}_{S^n} < \mathrm{vol}_{\hat{B}(b(\nu))}$. But this follows from Lemma 3.4, since as $i(q)$ approaches $\pi$,

$$\mathrm{vol}_{\hat{B}(i(q))} \rightsquigarrow \mathrm{vol}_{S^n}$$

and

$$i(q)\sin^{n-1} i(q) \rightsquigarrow 0.$$

q.e.d.

Let us now turn to Theorem 20.

Lemma 3.5: Suppose that there is an upper bound

$$K \leq \varkappa$$

for the sectional curvature of M. Then,

$$\delta \leq K$$

where $\delta := n - 1 - (n - 2)\varkappa$.

Proof: Take an $u \in \mathfrak{S}M$. Note that whenever $\sigma = u \wedge v$, $v \in u^{\perp}$ is of unit length, then

$$\rho_1(u) \leqq K(\sigma) \leqq \rho_{n-1}(u) \leqq \varkappa .$$

Now,

$$n - 1 \leqq \mathrm{Ric}(u) = \rho_1(u) + \dots + \rho_{n-1}(u)$$

$$\leq \rho_1(u) + (n - 2)\varkappa \ .$$

Therefore,

$$n - 1 - (n - 2) \leq \rho_1(u) \ .$$

q.e.d.

The following theorem is already classical:

Proposition 3.6 (J. Cheeger [C]): Let V be an n-dimensional compact manifold of diameter $\leq \pi$, such that

$$\delta \leq K \leq \varkappa \ .$$

Then, given any $\varepsilon \leq \pi/\sqrt{\varkappa}$, there is an U, depending only on n, $\delta$, and $\varepsilon$, such that, whenever

$$U < vol_V \ ,$$

then

$$\varepsilon < i_V \ .$$

After Cheeger's original proof, E. Heintze and H. Karcher [HK] offered a considerable simplification. The following proof is simply a reproduction of the essence of their argument.

Proof: Take $\hat{V} := \mathbb{R}^n$ if $0 \leq \delta$, and $\hat{V} := H^n(1/\sqrt{-\delta})$ if $\delta < 0$. Thanks to Proposition 1.1, it suffices to find an U such that, whenever

$$U < vol_V \ ,$$

then any periodic geodesic in V must have fundamental period $> 2\varepsilon$. For this, let

$$\hat{C} := \{q \in \hat{V} \mid d(\hat{q},\hat{c}) \leq \pi, \text{ where } \hat{c} \text{ is the portion of the } x^1\text{- axis of length } 2\varepsilon\},$$

and set $U := \mathrm{vol}_{\hat{C}}$. We assume that there is a periodic geodesic $c$ in $V$ of fundamental period $\leq 2\varepsilon$ and show that $\mathrm{vol}_V \leq U$. In fact, since $d_V \leq \pi$, the set

$$\{q \in V \mid d(q, c[0,2\varepsilon]) \leq \pi\}$$

is all of $V$. Let $u \in \mathfrak{S}_{c(s)}$ be normal to $\dot{c}(s)$. Let $v_2, \ldots, v_{n-1}$ be a basis for $\dot{c}(s)^{\perp} \cap u^{\perp}$.

Define the Jacobifields along $\gamma(r) := \exp ru$

$$Y(r) := T_{ru}\exp r\dot{c}(s)$$

and

$$Y_j(r) := T_{ru}\exp rv_j \;;\; j = 1, \ldots, n-2 \quad .$$

Then, in the Fermi coordinates,

$$\mathrm{Gr}_{c(s)}{}^{\frac{1}{2}}(ru) = \frac{|Y(r) \wedge Y_1(r) \wedge \ldots \wedge Y_{n-2}(r)|}{r^{n-1}\,|v_1 \wedge \ldots \wedge v_{n-2}|} \quad .$$

Then, there is a choice of $v_1, \ldots, v_{n-2}$ so that $Y(t)$, $Y_1(t), \ldots, Y_{n-2}(t)$ are orthonormal at a fixed $t \in (0,\vartheta(u))$. Extend $Y(t), Y_1(t), \ldots, Y_{n-2}(t)$ to a parallel frame field $E(r), E_1(r), \ldots, E_{n-2}(r)$ along $\gamma$. Set

$$V(r) := \frac{\cos\sqrt{\delta}\,r}{\cos\sqrt{\delta}\,t}E(r)$$

$$V_j(r) := \frac{\sin\sqrt{\delta}\,r}{\cos\sqrt{\delta}\,t}E_j(r)$$

Then, exactly as in the proof of Proposition 2.1, we show that

$$\frac{\frac{d}{ds} r^{n-2} Gr_{c(s)}^{\frac{1}{2}}(ru)_{|t}}{t^{n-2} Gr_{c(s)}^{\frac{1}{2}}(tu)}$$

$$\leq \int_0^t \frac{\sin^2\sqrt{\delta}\, r}{\cos^2\sqrt{\delta}\, t} - K(E(r) \wedge \mathring{\gamma}(r)) \frac{\cos^2\sqrt{\delta}\, r}{\cos^2\sqrt{\delta}\, t}\, dr$$

$$+ \int_0^t \frac{\cos^2\sqrt{\delta}\, r}{\sin^2\sqrt{\delta}\, t} - K(E_j(r) \wedge \mathring{\gamma}(r)) \frac{\sin^2\sqrt{\delta}\, r}{\sin^2\sqrt{\delta}\, t}\, dr$$

$$\leq \int_0^t \frac{\sin^2\sqrt{\delta}\, r}{\cos^2\sqrt{\delta}\, t} - \delta \frac{\cos^2\sqrt{\delta}\, r}{\cos^2\sqrt{\delta}\, t}\, dr$$

$$+ (n-1) \int_0^t \frac{\cos^2\sqrt{\delta}\, r}{\sin^2\sqrt{\delta}\, t} - \delta \frac{\sin^2\sqrt{\delta}\, r}{\sin^2\sqrt{\delta}\, t}\, dr$$

$$= \frac{\frac{d}{dr} \cos\sqrt{\delta}\, r \, \sin^{n-1}\sqrt{\delta}\, r_{\,|t}}{\cos\sqrt{\delta}\, t \, \sin^{n-2}\sqrt{\delta}\, t} .$$

This allows us to conclude that

$$Gr_{c(s)}^{\frac{1}{2}}(ru) \leq \frac{\cos\sqrt{\delta}\, r \, \sin^{n-2}\sqrt{\delta}\, r}{\sqrt{\delta}^{\,n-1}} r^{n-2} ,$$

and, by integration,

$$\mathrm{vol}_V \leq \int_0^{2\varepsilon} \int_{\dot{c}(s)^{\perp} \cap \mathcal{G}_{c(s)}} \int_0^{\vartheta(u)} \widehat{Gr}^{\frac{1}{2}}(r) r^{n-2}\, dsduds$$

$$\leq \int_0^{2\varepsilon} \int_{S^{n-1}} \int_0^{\pi} \widehat{Gr}^{\frac{1}{2}}(r) r^{n-2}\, drduds = \mathrm{vol}_{\hat{C}} = U$$

as required.

q.e.d.

<u>Proof of Theorem 20</u>: Given $K \leq \kappa$, let $\delta := n - 1 - (n-2)\kappa$ and $\varepsilon = \pi/\sqrt{\kappa}$. Then, choose U as

in Proposition 3.6, so that, if

$$U < \mathrm{vol}_M ,$$

then

$$\varepsilon < i_M .$$

Take p, q ∈ M so that $d(p,q) = d_M$. In Corollary 2.9, set $\delta := 2\varepsilon/3$ and $\rho := \varepsilon/3$. Then, there is an $\nu' > 0$, and if

$$(1 - \nu')\mathrm{vol}_{S^n} \leq \mathrm{vol}_M ,$$

then

$$D(\varepsilon;p) \cup D(b(\nu') - \varepsilon/3;q) = M .$$

Now, as in the proof of Theorem 19, Lemma 3.4 implies the existence of $R > 0$ so that, if

$$R < i(q),$$

then

$$\max\{U, (1 - \nu')\mathrm{vol}_{S^n}\} < \mathrm{vol}_M .$$

The rest of the proof can be accomplished exactly as in Theorem 19.

q.e.d.

We give two more applications of Lemma 3.3. In the following famous theorem, we need not assume the condition on the Ricci curvature.

<u>Proposition</u> 3.7 (W. Klingenberg [K1]): Let V be a compact manifold of even dimension. Assume that

$$0 < K \leq \kappa .$$

Then,

$$\pi/\sqrt{\kappa} \leq i_V .$$

Combining the above, whose proof we omit, with Lemma 3.3, we get

Theorem 22: Let M be an even-dimensional manifold with

$$n - 1 \leq \mathrm{Ric}$$

and

$$0 < K \leq 4 - \varepsilon$$

for some $0 < \varepsilon$. Then, there is a constant $0 < \upsilon$, depending only on n and $\varepsilon$, such that, if, in addition,

$$(1 - \upsilon)\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad ,$$

then M is homeomorphic to $S^n$.

Proof: Take $p, q \in M$ with $d(p,q) = d_M$. Then, by Proposition 3.7, $D(\pi/\sqrt{4 - \varepsilon}\,;p)$ and $D(\pi/\sqrt{4 - \varepsilon}\,;q)$ are both diffeomorphic to $D^n$. Now let $r := \pi/\sqrt{4 - \varepsilon} - \pi/2$, and choose $\upsilon$ as in Proposition 2.8, so that these balls cover M. The proof can then be completed as in Lemma 3.3.

q.e.d.

The volume assumption in the above is necessary. In fact, the standard metric (of Fubini-Study) on $\mathbb{C}P^n$ can be normalized so that

$$n - 1 \equiv \mathrm{Ric}$$

and

$$\frac{n-1}{n+2} \leqq K \leqq 4\left(\frac{n-1}{n+2}\right) .$$

For arbitrary dimensions, there is the following recent result of P. Hartman, which we also quote here without the proof:

Proposition 3.8 [H]: Assume that M is a compact manifold such that

$$n-2 \leqq \mathrm{Ric}_{n-2}$$

and

$$K \leqq \kappa < 4 .$$

Then, we have

$$\pi/\sqrt{\kappa} < i_M .$$

Here, $\mathrm{Ric}_{n-2}$ is the lower partial Ricci curvature defined in Section 1. Notice that the condition above implies $n-1 < \mathrm{Ric}$ and is implied, in turn, by

$$n-1 \leqq \mathrm{Ric}$$

and

$$K \leqq \kappa < \frac{4(n-1)}{n+2} .$$

Then, exactly as in Theorem 22, we can prove

Theorem 23: Suppose that we have

$$n-2 \leqq \mathrm{Ric}_{n-2}$$

and

$$K < 4 - \varepsilon$$

for some $0 < \varepsilon$. Then, there is a constant $0 < \nu$, depending on n and $\varepsilon$, such that whenever

$$(1 - \nu)\mathrm{vol}_{S^n} < \mathrm{vol}_M,$$

then M is homeomorphic to $S^n$.

Remark: Using Lemma 3.4, we can assert the existence of some $\varkappa$, depending only on n, such that if

$$n - 1 \leqq \mathrm{Ric}$$

and

$$0 < K < \varkappa$$

in even dimensions, or if

$$n - 2 \leqq \mathrm{Ric}_{n-2}$$

and

$$K < \varkappa$$

in arbitrary dimensions, then M is homeomorphic to $S^n$. However, such a statement can be obtained much more easily by appealing to Lemma 3.5 and Theorem 3. That is to say, if we choose $\varkappa$ so that

$$n - 1 - (n - 2)\varkappa = \frac{1}{4}\varkappa ,$$

then the metric can be renormalized so as to have

$$1 \leqq K < 4$$

by homothecy. Whether the constant obtained by our methods represents any improvement is not immediately clear without an aid of a moderate-speed calculating device.

## 4. The theorem of Cheng-Toponogov

In this section, we show that Theorem 14 of S. Y. Cheng and V. A. Toponogov can be proved more directly by geometric means. Cheng's original proof was based on the estimates for the eigenvalues of the Laplace operator acting on functions, and is related closely to Obata's theorem (Theorem 21). We shall show that after the geometric proof of Theorem 14, Obata's theorem can be deduced from it. Assuming still that

$$n - 1 \leqq Ric \quad ,$$

we have

Lemma 4.1: At any $p \in M$.

$$\frac{vol_{M \setminus B(\pi/2;p)}}{vol_{B(\pi/2;p)}} \leqq 1 \ .$$

Proof: This follows immediately from Proposition 2.2 by setting $R := \pi$ and $r := \pi/2$, where the right side can be calculated explicitly. q.e.d.

Lemma 4.2: Suppose that $d_M = \pi$. Choose $p, q \in M$ so that $d(p,q) = \pi$. Then,

$$vol_{B(\pi/2;p)} + vol_{B(\pi/2;q)} = vol_M \ .$$

Proof: Note that

$$B(\pi/2;q) \subset M - B(\pi/2;p) \quad .$$

Therefore,

$$\frac{\mathrm{vol}_{B(\pi/2;q)}}{\mathrm{vol}_{B(\pi/2;p)}} \leqq \frac{\mathrm{vol}_{M - B(\pi/2;p)}}{\mathrm{vol}_{B(\pi/2;p)}} \leqq 1 \quad .$$

By reversing the roles of p and q, we also have

$$\frac{\mathrm{vol}_{B(\pi/2;p)}}{\mathrm{vol}_{B(\pi/2;q)}} \leqq 1 \quad ;$$

i.e.,

$$\mathrm{vol}_{B(\pi/2;p)} = \mathrm{vol}_{B(\pi/2;q)} \quad .$$

We hence calculate that

$$0 \leqq \frac{\mathrm{vol}_{M - B(\pi/2;p) - B(\pi/2;q)}}{\mathrm{vol}_{B(\pi/2;p)}}$$

$$= \frac{\mathrm{vol}_{M - B(\pi/2;p)} - \mathrm{vol}_{B(\pi/2;q)}}{\mathrm{vol}_{B(\pi/2;p)}}$$

$$= \frac{\mathrm{vol}_{M - B(\pi/2;p)}}{\mathrm{vol}_{B(\pi/2;p)}} - \frac{\mathrm{vol}_{B(\pi/2;q)}}{\mathrm{vol}_{B(\pi/2;p)}} \leqq 1 - 1 = 0 \quad .$$

q.e.d.

Since $M - D(\pi/2;p) - D(\pi/2;q)$ is an open set, Lemma 4.2 has the following

<u>Corollary</u> 4.3: In the same situation as Lemma 4.2, the two balls $D(\pi/2;p)$ and $D(\pi/2;q)$ cover M. Moreover,

$$\partial D(\pi/2;p) = \partial D(\pi/2;q) \quad .$$

Lemma 4.4: If $p, q \in M$ satisfy $d(p,q) = \pi$ and if $c: [0,\pi/2] \to M$ is any minimal geodesic with $c(0) = q$, then $c$ extends to a minimal curve: $[0,\pi] \to M$ such that $c(\pi) = p$.

Proof: Since $c$ is assumed to be minimal on $[0,\pi/2]$, $c(\pi/2) =: x \in \partial D(\pi/2:q) = \partial D(\pi/2;p)$. Therefore, there is a minimal geodesic $c': [0,\pi/2] \to M$ with $c'(0) = p$ and $c'(\pi/2) = x$. Consider the broken geodesic obtained by traversing $c$ followed by $-c'$. This is a (possibly) broken connection from $q$ to $p$ of length $\pi$. But, as $d(p,q) = \pi$, this is, in fact, a smooth minimal geodesic.

q.e.d.

Now, define the following set:

$$\mathfrak{U} := \{u \in \mathcal{S}_q \mid \mathcal{A}(u) > \pi/2\}.$$

Lemma 4.5: If $q \in M$ has $d(q) = \pi$, then $\mathfrak{U}$ is an open and closed set in $\mathcal{S}_q$.

Proof: The fact that $\mathfrak{U}$ is open is clear from the continuity of $\mathcal{A}$ on $\mathcal{S}_q$. To show that it is closed, note that by Lemma 4.5, whenever $u \in \mathfrak{U}$, we have, in fact, $\mathcal{A}(u) = \pi$. Suppose that the sequence $\{u_i\} \subset \mathfrak{U}$ converges to a $v \in \mathcal{S}_q$. Then, by continuity of $\mathcal{A}$ again, $\mathcal{A}(v) = \pi$. Hence, $v \in \mathfrak{U}$.

q.e.d.

Since $\mathcal{S}_q$ is homeomorphic to $S^{n-1}$ and, thus, connected, $\mathfrak{U} = \mathcal{S}_q$; i.e.:

Lemma 4.6: For $q \in M$ with $d(q) = \pi$, we also have $i(q) = \pi$.

Proof of Theorem 14: Let $q$ be as in the above. Note first that $D(\pi; q) = \bar{M}$. Let $c: [0,\pi] \rightarrow M$ be any geodesic with $c(0) = q$. Take an orthonormal basis $v_1, \ldots, v_{n-1}$ for $\dot{c}(0)^{\perp}$ and extend it to a parallel frame field $E_1, \ldots, E_{n-1}$ along $c$. Define

$$X_j(s) := \sin s \, E_j(s) \ ; \ j = 1, \ldots, n-1 \ .$$

Since $i(q) = \pi$,

$$I_0^{\pi}(X_j, X_j) \geqq 0$$

for each $j$. On the other hand, since $X_j(0) = X_j(\pi) = 0$,

$$\sum I_0^{\pi}(X_j, X_j) = \sum -\int_0^{\pi} \langle X_j(s), X_j(s)\rangle + K(X_j(s) \wedge \dot{c}(s)) \, ds$$

$$= \sum \int_0^{\pi} \sin^2 s - K(E_j(s) \wedge \dot{c}(s)) \sin^2 s \, ds$$

$$= \int_0^{\pi} \{(n-1) - \mathrm{Ric}(\dot{c}(s)\} \sin^2 s \, ds \ .$$

By our assumption on the Ricci curvature,

$$\sum I_0^{\pi}(X_j, X_j) \leqq 0.$$

Thus, for each $j$, $I_0^{\pi}(X_j, X_j) = 0$. Again, since $c$ is minimal, $X_j$ must satisfy the Jacobi equation; i.e.,

$$-\sin^2 s \, E_j(s) + K(E_j(s) \wedge \dot{c}(s)) \sin^2 s \, E_j(s) = 0$$

at each $s$. Since $E_j(s) \neq 0$ and $\sin s \neq 0$ on $0 < s < \pi$,

$$1 = K(E_j(s) \wedge \dot{c}(s)) = K_{\dot{c}(s)}(E_j(s))$$

for arbitrary s. Since $K_{\dot c(s)}$ is quadratic, and so depends only on n − 1 independent directions, this implies that $K_{\dot c(s)}(u) = 1$ for all s and all $u \in \dot c(s)^{\perp}$ of unit length.

Now, let $\hat q$ be any point in $S^n$ and $\hat p$ its antipode. Fix an orthogonal transformation

$$\tau: \quad T_{\hat q}S^n \longrightarrow T_q M$$

and define a map $\Phi$: $S^n \setminus \{\hat p\} \longrightarrow B(\pi;q)$ by

$$\Phi(\hat x) := \exp_q \circ \tau \circ \exp_{\hat q}^{-1}(\hat x) \quad .$$

Since $i(q) = d_{S^n} = \pi$, $\Phi$ is a diffeomorphism. Moreover, by Rauch comparison theorem (Proposition 1.2), $K_{\dot c(s)} \equiv 1$ implies that $\Phi$ is, in fact, an isometry. But, as $D(\pi;q) \setminus B(\pi;q) = \{p\}$, by continuity, $\Phi$ is seen to extend to an isometry:

$$S^n \longrightarrow D(\pi;q) = M \quad .$$

q.e.d.

Now, we consider the Laplace operator $\Delta$ acting on $C^{\infty}_M$. We let $\lambda$ be the first non-0 eigenvalue and g the corresponding eigenfunction. First, we prove the already classical

Lemma 4.7 (A. Lichnerowicz): Under our assumption on Ric,

$$n \leqq \lambda \quad ,$$

and the equality is obtained if and only if

$$\text{hess}_g \equiv -g\langle , \rangle \quad .$$

The argument we present here is due to M. Obata [O].

Proof: By Lemma 1.3,

$$-\tfrac{1}{2}\Delta|\text{grad } g|^2 = |\text{hess}_g|^2 - \lambda|\text{grad } g|^2 + \text{Ric}(\text{grad } g) \quad .$$

If we integrate the above over M, since M is compact, the left side integrates to 0 by Stokes' theorem. Thus,

$$\int_M |\text{hess}_g|^2 = \int_M \lambda|\text{grad } g|^2 - \text{Ric}(\text{grad } g) \quad .$$

Also obtainable from Stokes' theorem are

$$\int_M \langle \text{hess}_g, g\langle , \rangle\rangle = -\int_M |\text{grad } g|^2$$

and

$$\int_M (\Delta g)^2 = \int_M \lambda g \Delta g = \int_M \lambda|\text{grad } g|^2 \quad .$$

Now, we compute

$$0 \leqq \int_M |\text{hess}_g + \frac{\lambda}{n} g\langle , \rangle|^2$$

$$= \int_M |\text{hess}_g|^2 + 2\frac{\lambda}{n}\langle \text{hess}_g, g\langle , \rangle\rangle + \frac{\lambda^n}{n^2} g^2 |\langle , \rangle|^2 \quad .$$

Substituting from the formulas above and the fact that $|\langle , \rangle|^2 = n$,

$$0 \leqq \int_M \lambda|\text{grad } g|^2 - \text{Ric}(\text{grad } g)$$

$$- 2\frac{\lambda}{n}|\mathrm{grad}\ g|^2 + \frac{\lambda}{n}|\mathrm{grad}\ g|^2$$

$$= \int_M \frac{n-1}{n}\lambda|\mathrm{grad}\ g|^2 - \mathrm{Ric}(\mathrm{grad}\ g) \quad .$$

Therefore,

$$\frac{n}{n-1}\int_M \mathrm{Ric}(\mathrm{grad}\ g) \Big/ \int_M |\mathrm{grad}\ g|^2 \leq \lambda \quad .$$

Since $(n-1)|\mathrm{grad}\ g|^2 \leq \mathrm{Ric}(\mathrm{grad}\ g)$, we obtain that

$$n \leq \lambda \ ,$$

whence the equality implies equality in all of the above. In particular,

$$|\mathrm{hess}_g + \frac{n}{n} g\langle,\rangle|^2 \equiv 0 \quad ,$$

from which the second part of the lemma follows.

q.e.d.

<u>Proof of Theorem</u> 21: By the preceding lemma,

$$\mathrm{hess}_g = -g\langle,\rangle \quad .$$

Take p, q $\in$ M so that g attains a minimum at p and a maximum at q. Without loss of generality, we assume that $g(q) = 1$. Choose c to be a minimal connection from q to p. Then,

$$\mathrm{hess}_g(\dot{c}(s),\dot{c}(s)) = g(c(s))\langle\dot{c}(s),\dot{c}(s)\rangle$$

or

$$(g \circ c)''(s) = -g \circ c(s) \quad .$$

Since $q = c(0)$ is a maximum for g, we have the initial condition

$$g \circ c(0) = 1$$

$$(g \circ c)'(0) = 0 \quad .$$

We see that

$$g \circ c(s) = \cos s \quad .$$

Thus, the first point where c intersects the nodal set of g is at the parameter value $s = \pi/2$; i.e. at a point x such that $d(q,x) = \pi/2$.

Now, consider $-g$ and repeat the argument. We see that the last point of intersection of c with the nodal set is at y where $d(p,y) = \pi/2$. Therefore, $d(p,q) \geq \pi$. In view of Myers' Theorem 10, $x = y$ and

$$d(p,q) = \pi \quad .$$

Therefore, by Theorem 14, M is isometric to $S^n$.

q.e.d.

We consider two more applications of Theorem 14 which give rigidity for the case when M is non-simply connected.

Theorem 24: If there is a point $p \in M$ such that every simple geodesic loop at p has length $\geqq \pi$, then M is either simply connected or is isometric to $\mathbb{R}P^n$. In particular, if $\pi/2 \leqq i(p)$ for some p, the conclusion follows.

Proof: Assume that M is not simply connected. Let $\pi$: $\tilde{M} \rightarrow M$ be the universal covering space. Then, as in the proof of Lemma 3.1, for each homotopy class in $\pi_1(M;p)$, there is a pair $p_1, p_2 \in \pi^{-1}(p)$ with $d(p_1,p_2) \geq \pi$. By Myers' theorem and Theorem 14, $d(p_1,p_2) = \pi$ and $\tilde{M}$ is isometric to $S^n$. Consequently, M is of constant sectional curvature 1. From the fact that, on $S^n$, for each $q_1$, there is only one $q_2$ with $d(q_1,q_2) = \pi$, we see that $\{p_1, p_2\} = \pi^{-1}(p)$ and that $\pi_1 M \cong \mathbb{Z}_2$. This implies that M is isometic to $S^n$.

q.e.d.

We now present a new proof to the following well-known

Lemma 4.8: Suppose that V is a compact and non-simply connected manifold. Then, every conjugacy class in $\pi_1 V$ is represented by a smooth closed geodesic which has the minimal length among all loops representing that class.

Proof: Let $g \in \pi_1 V$ and $\Gamma$ be the cyclic subgroup generated by g. Then, there is a covering space $\pi$: $\tilde{V} \to V$ and V is isometric to $\tilde{V}/\Gamma$. Let $G \subset \tilde{V}$ be a fundamental domain of the $\Gamma$-action. Then $G^-$ is compact and the function

$$G^- \ni \tilde{q} \mapsto d(q, g\cdot\tilde{q})$$

takes on the minimum value at some $\tilde{p}$. Let $\tilde{c}$ be the minimal geodesic from $\tilde{p}$ to $g\cdot\tilde{p}$. Then, $c := \pi \circ \tilde{c}$

is a geodesic loop in $V$. The fact that $c$ has the minimal length among all loops in the conjugacy class of $g$ follows from our choice combined with the fact that conjugacy classes of $\pi_1 V$ are in natural one-to-one correspondence with the free homotopy lasses of loops in $V$.

To show that $c$ is smoothly closed, it suffices to show that $\dot{c}(0) = \dot{c}(1)$ where $l := L(c)$. Let $q := \tilde{c}(\frac{1}{2}l) \in \tilde{V}$. By minimality, $\exp_q^{-1}$ is injective in a neighborhood of image($\tilde{c}$). By Gauss' Lemma, then, $\dot{\tilde{c}}(0)$ and $\dot{\tilde{c}}(1)$ are orthogonal to $S(\frac{1}{2}l;\tilde{q})$ which is a smoothly embedded hypersurface near $\tilde{p}$ and $g\cdot\tilde{p}$. Since, by choce, $\pi$ is injective on $B(\frac{1}{2}l;\tilde{q})$, $S(\frac{1}{2}l;\pi(\tilde{q}))$ is tangent to itself at $\pi(\tilde{p}) = \pi(g\cdot\tilde{p})$. But, since $T_{\tilde{p}}\pi$ and $T_{g\cdot\tilde{p}}\pi$ are orthogonal transformations, this implies that $\dot{c}(0) = \dot{c}(1)$.

q.e.d.

See [CE], § 5.9 and [Sp] vol. IV, pp.352 ff. for slightly different proofs.

Theorem 25: Suppose that all smooth closed geodesics on $M$ have length $\geq \pi$. Then, the conclusion of Theorem 24 holds.

Proof: Assume that $M$ is not simply connected. Take $e \neq g_1 \in \pi_1 M$, and let $c_1$ be a minimal smooth closed geodesic representing the conjugacy class of

$g_1$. Let $p := c_1(0)$. Suppose that $\pi: \tilde{M} \to M$ is the universal covering space and fix a $\tilde{p} \in \pi^{-1}(p)$. Then, $d(\tilde{p}, g\cdot\tilde{p}) = L(c_1) \geqq \pi$. By the same argument as in Theorem 24, $d(\tilde{p}, g\cdot\tilde{p}) = \pi$ and $\tilde{M}$ is isometric to $S^n$. Now, it suffices to show that $\pi_1 M \cong \mathbb{Z}_2$.

Suppose that there is a $e \neq g_2 \in \pi_1(M;p)$. Then, the minimal geodesic from $\tilde{p}$ to $g_2\cdot\tilde{p}$ in $\tilde{M}$ projects to a geodesic loop $c_2$ by $\pi$ which is longer than a smooth closed geodesic representing the same conjugacy class; i.e.,

$$d(\tilde{p}, g\cdot\tilde{p}) = L(c_2) = \pi \quad .$$

By spherical geometry, this implies that $g_1\cdot\tilde{p} = g_2\cdot\tilde{p}$, and so $g_1 = g_2$. Thus, $\#\pi_1 M = 2$, and our conclusion follows.

q.e.d.

In either Theorem 24 or Theorem 25, if M is simply connected, it is not true necessarily that M has any topology of $S^n$. In fact, if M is the manifold

$$S^j(\sqrt{(j - 1)/(2j - 1)}) \times S^j(\sqrt{(j - 1)/(2j - 1)}) \quad ,$$

which we have already considered in Section 2, then all geodesic loops on M are smoothly closed and have length

$$\geqq \pi\sqrt{(4j - 4)/(2j - 1)} \quad .$$

It is not known, if the conclusion follows if we only assume that all smooth closed geodesics through some single point have length $\geq \pi$.

## 5. Main theorem - part one

### Measure calculations in the tangent space

In this and the following sections, we finally prove our Main theorem (Theorem 18). We let $q \in M$ and consider the Euclidean space $T_qM$. The Lebesgue measure of $T_qM$ is denoted by $\tilde{m}$. First, we prove

Lemma 5.1: Given any $0 < \alpha$, there is a constant $0 < \tilde{\upsilon}$, depending only on n and $\alpha$, such that whenever

$$(1 - \tilde{\upsilon})\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad ,$$

then

$$\tilde{m}(\mathfrak{D}_{\pi, q}) - \alpha < \tilde{m}(\mathfrak{N}_q) \quad .$$

Assume that

$$(1 - \tilde{\upsilon})\mathrm{vol}_{S^n} < \mathrm{vol}_M$$

for some $0 < \tilde{\upsilon} < 1$. Note that Myers' theorem (Theorem 10) implies that $\mathfrak{N}_q \subset \mathfrak{D}_{\pi, q}$. Therefore, it suffices to prove that as $\tilde{\upsilon}$ tends to 0,

$$\tilde{m}(\mathfrak{D}_{\pi, q} \setminus \mathfrak{N}_q) \longrightarrow 0 \quad .$$

Fix some $\hat{o} \in S^n$ as the origin. Choose an orthogonal transformation

$$\tau: \ T_qM \longrightarrow T_{\hat{o}}S^n \quad .$$

Set $\hat{\mathfrak{N}} := \tau(\mathfrak{N}_q) \subset \hat{\mathfrak{D}}_\pi$. Then, by Lemma 2.1,

$$\mathrm{vol}_M = \int_{\mathfrak{N}_q} \mathrm{Gr}_q^{\frac{1}{2}}\, d\tilde{m} \leq \int_{\hat{\mathfrak{M}}} \hat{\mathrm{Gr}}^{\frac{1}{2}}\, d\tilde{m} = \mathrm{vol}_{\exp(\hat{\mathfrak{M}})} \quad .$$

So, our assumption implies that

$$(1 - \tilde{\upsilon})\mathrm{vol}_{S^n} \leq \mathrm{vol}_{\exp(\hat{\mathfrak{M}})} \quad .$$

Since $S^n = \exp(\hat{\mathfrak{D}}_\pi)$, pulling the set-up in $T_{\hat{O}}S^n$,

$$(1 - \tilde{\upsilon})\int_{\hat{\mathfrak{D}}_\pi} \hat{\mathrm{Gr}}^{\frac{1}{2}}\, d\tilde{m} \leq \int_{\hat{\mathfrak{M}}} \hat{\mathrm{Gr}}^{\frac{1}{2}}\, d\tilde{m} \quad ,$$

or, as $\hat{\mathfrak{M}} \subset \hat{\mathfrak{D}}_\pi$, equivalently,

$$\int_{\hat{\mathfrak{D}}_\pi \sim \hat{\mathfrak{M}}} \hat{\mathrm{Gr}}^{\frac{1}{2}}\, d\tilde{m} \leq \tilde{\upsilon}\int_{\hat{\mathfrak{D}}_\pi} \hat{\mathrm{Gr}}^{\frac{1}{2}}\, d\tilde{m} \quad .$$

On the other hand, it is clear that the measures $\int \hat{\mathrm{Gr}}^{\frac{1}{2}}\, d\tilde{m}$ and $\tilde{m}$ are absolutely continuous with respect to each other. That is to say, for some fixed constants $\Gamma, \Gamma' > 0$, depending only on n, we have

$$\Gamma \tilde{m}(\hat{\mathfrak{D}}_\pi \sim \hat{\mathfrak{M}}) \leq \tilde{\upsilon}\Gamma' \tilde{m}(\hat{\mathfrak{D}}_\pi) \quad .$$

Now, apply the isometry $\tau^{-1}$ to get

$$\tilde{m}(\mathfrak{D}_{\pi,q} \sim \mathfrak{N}_q) \leq \tilde{\upsilon}\frac{\Gamma'}{\Gamma}\tilde{m}(\mathfrak{D}_{\pi,q})$$

which proves our assertion.

q.e.d.

For $0 < r < \pi$, we regard the sets $\mathfrak{S}_{r,q} \subset T_qM$ as Riemannian submanifolds with the metric induced from the Euclidean structure of the ambient $T_qM$. Let $m_r$ denote the Carathéodory measure on $\mathfrak{S}_{r,q}$ given by the volume element of this metric. Then, we assert

<u>Lemma</u> 5.2: Given any $0 < r < \pi$ and $0 < \delta$, there is a constant $0 < \tilde{\nu}$, so that whenever

$$(1 - \tilde{\nu})\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad ,$$

then

$$m_r(\mathcal{S}_{r,q} \sim \mathcal{N}_q) < \delta \quad .$$

<u>Proof</u>: For $0 < t \in \mathbb{R}$, let $\eta_t$: $T_qM \to T_qM$ be the homothecy operator

$$\eta_t(u) := tu \quad .$$

Then, the fact that $\mathcal{N}_q$ is star-shaped is rephrased by saying that whenever $r < s$ and $v \in \mathcal{S}_{r,q} \sim \mathcal{N}_q$,

$$\eta_{s/r}(v) \in \mathcal{S}_{s,q} \sim \mathcal{N}_q \quad .$$

Thus,

$$\mathcal{N}_q \subset \mathcal{D}_{\pi,q} \sim \bigcup_{r<s<\pi} \eta_{s/r}(\mathcal{S}_{r,q} \sim \mathcal{N}_q) \quad .$$

On the other hand,

$$\tilde{m}\Big(\bigcup_{r<s<\pi} \eta_{s/r}(\mathcal{S}_{r,q} \sim \mathcal{N}_q)\Big) = \int_r^{\pi} \frac{s}{r} m_r(\mathcal{S}_{r,q} \sim \mathcal{N}_q)\, ds$$
$$= \frac{m_r(\mathcal{S}_{r,q} \sim \mathcal{N}_q)}{2r}(\pi^2 - r^2) \quad .$$

Hence, we see that

$$\tilde{m}(\mathcal{N}_q) \leq \tilde{m}(\mathcal{D}_{\pi,q}) - \frac{(\pi^2 - r^2)}{2r} m_r(\mathcal{S}_{r,q} \sim \mathcal{N}_q) \quad .$$

Now, given $0 < \delta$, set

$$\alpha := \frac{(\pi^2 - r^2)}{2r}\delta$$

Use this $\alpha$ in Lemma 5.1 to find an $\tilde{\upsilon}$ so that whenever

$$(1 - \tilde{\upsilon})\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad ,$$

we have

$$\tilde{m}(\mathcal{D}_{\pi',q}) - \frac{(\pi^2 - r^2)}{2r}\delta < \tilde{m}(\mathcal{N}_q) \quad .$$

We see by adding this to the previous inequality that our conclusion must follow. q.e.d.

We shall find two uses for our lemma above later. For one of them, it is convenient to state it in a slightly more refined form. Let $0 < R < \pi$. The structure of M does not enter explicitly in the next proposition. Therefore, we might as well identify $\mathcal{G}_{R,q}$ with $S^{n-1}(R) \subset \mathbb{R}^n$ by a suitably chosen isometry. Let us denote the canonical distance function and Carathéodory measure of this Riemannian space by $\tilde{d}$ and m, respectively.

<u>Lemma</u> 5.3: Suppose that $0 < \eta$ is any given number. There is a constant $0 < \delta < m(S^{n-1}(R))$ so that for any Borel set $\mathcal{N} \subset S^{n-1}$ with

$$m(\neg\mathcal{N}) < \delta$$

and any $v \in S^{n-1}(R)$,

$$\tilde{d}(v,\mathcal{N}) < \eta \quad .$$

<u>Proof</u>: Let $\tilde{B}(r;v)$ denote the open metric ball in $S^{n-1}(R)$ of radius r and center v. Then, the

function $f(r) := m(\tilde{B}(r;v))$ is independent of the choice of v. In fact, on $[0,\pi R]$, it is given by

$$f(r) = \int_0^r R^{n-1} \sin^{n-2} s/R \, ds \quad .$$

We contend that our assertion follows by choosing $\delta := f(\eta)$. To see this, assume that for some pair $v \in S^{n-1}(R)$, $\mathcal{N} \subset S^{n-1}(R)$ a Borel set,

$$\tilde{d}(v,\mathcal{N}) \geq \eta \quad .$$

Then,

$$\tilde{B}(\eta;v) \subset \neg\mathcal{N} \quad .$$

Thus,

$$\delta = f(\eta) = m(\tilde{B}(\eta;v)) \leq m(\neg\mathcal{N})$$

proving the contrapositive.

q.e.d.

Let $\sigma: \mathcal{G}_{R,q} \to S^{n-1}(R)$ be the aforementioned isometry. Set $\mathcal{N} := \sigma(\mathcal{G}_{R,q} \cap \mathcal{N}_q)$. Then, combining Lemmas 5.2 and 5.3, we immediately get

Corollary 5.4: Let any $0 < R < \pi$ and $0 < \eta < \pi R$ be given. Then, for some constant $0 < \upsilon_1$, depending only on n, R, and $\eta$, if

$$(1 - \upsilon_1)\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad ,$$

then any $v \in \mathcal{G}_{R,q}$ whose length is $< \eta$.

We now wish to transfer the measure computations above down on M. Our main tool is a standard result in ordinary differential equations, which we rewrite in our own context as the next

Lemma 5.5 (Gronwall): Let E be a vector bundle of rank k over the space $[0,R]$ equipped with an Euclidean metric $\langle , \rangle$ and a covariant derivation $\nabla$ which is compatible with the metric. Let $\mathcal{R}: E \to E$ be a fibre-preserving map such that each $\mathcal{R}|E_s$ is Lipschitz continuous with constant $\rho(s)$ with respect to the norm $| \, |$. Assume that $\rho(s)$ is a continuous function and let $\mathfrak{y} \in \Gamma(E)$ satisfy

$$\nabla_{\frac{d}{ds}} \mathfrak{y}(s) = \mathcal{R}(\mathfrak{y}(s))$$

$$\mathfrak{y}(0) =: \mathfrak{y}_0 \neq 0 \quad .$$

Then,

$$|\mathfrak{y}(R)| \leq |\mathfrak{y}_0| e^{\int_0^R \rho(s) ds} \quad .$$

Proof: Let $\mathfrak{v}_1, \ldots, \mathfrak{v}_k$ be an orthonormal parallel frame field for E, which exists since $[0,R]$ is a contractible space. Then, we define an $E_R$-valued integration for $\gamma \in \Gamma(E)$,

$$\int_E \gamma(s) \, ds := \sum_{i=1}^{k} \int_0^R \langle \gamma(s), \mathfrak{v}_i(s) \rangle \, ds \, \mathfrak{v}_i(R)$$

By the Fundamental theorem of calculus, we have

$$\mathfrak{y}(R) = \mathfrak{y}_0 + \int_E \nabla_{\frac{d}{ds}} \mathfrak{y}(s) \, ds = \mathfrak{y}_0 + \int_E \mathcal{R}(\mathfrak{y}(s)) \, ds$$

Using the triangle inequality, we see that

$$|\eta(R)| \leq |\eta_0| + \int_0^R |\Omega(\eta(s))|\, ds$$
$$\leq |\eta_0| + \int_0^R \rho(s)|\eta(s)|\, ds \quad .$$

We define a function $f\colon [0,R] \longrightarrow \mathbb{R}^+$ by

$$f(r) := |\eta_0| + \int_0^r \rho(s)|\eta(s)|\, ds \quad .$$

It suffices, then, to show that $f(R) \leq |\eta_0| e^{\int_0^R \rho(s)ds}$.

Note that $|\eta(r)| \leq f(r)$ for all $0 \leq r \leq R$, while $f$ is differentiable and

$$f'(r) = \rho(r)|\eta(r)| \quad .$$

Hence,

$$\frac{f'(r)}{f(r)} = \frac{\rho(r)|\eta(r)|}{f(r)} \leq \rho(r)$$

or

$$(\log f)'(r) \leq \rho(r)$$

for all $0 \leq r \leq R$. Integrating this, we obtain

$$\log f(R) - \log f(0) \leq \int_0^R \rho(s)\, ds \quad .$$

Since $f(0) = |\eta_0|$,

$$\log f(R) \leq \log |\eta_0| + \int_0^R \rho(s)\, ds$$

which, when integrated, gives the desired inequality.

q.e.d.

We consider the following special case: $V$ is a Riemannian manifold of dimension $n$ and $c\colon [0,R]$

$\to V$ is a geodesic. Let $E := c^*(TV) \oplus c^*(TV)$, the Whitney sum bundle over $[0,R]$ endowed with the product metric $\langle , \rangle$ and the product covariant derivation $\nabla$ of those induced from TV by $c^*$. Define $\pi_1, \pi_2$ to be the projections on $c^*(TV)$ by the first and second factor, respectively. Then, for $\mathscr{Y} \in \Gamma(E)$, $Y := \pi_1 \circ \mathscr{Y} \in \mathscr{X}_c$ satisfies the Jacobi equation if and only if $\mathscr{Y}$ satisfies

$$\nabla_{\frac{d}{ds}} \mathscr{Y}(s) = \mathscr{R}(\mathscr{Y}(s)) \quad ,$$

where

$$\mathscr{R}(\mathscr{Y}(s)) := \pi_2 \circ \mathscr{Y}(s) \oplus R(\dot{c}(s), Y(s); \dot{c}(s)) \quad .$$

This is just the familiar reduction of a second-order system to a first-order system, written in the bundle language.

Lemma: In the situation just described above, suppose that Y satisfies the Jacobi equation and that the sectional curvature of V satisfies

$$\delta \leqq K(\sigma) \leqq \kappa$$

for all $\sigma \in \Lambda^2 TV$ which are simple, have unit length, and contain $\dot{c}(s)$. Then,

$$|Y(R)| \leqq \sqrt{|Y(0)|^2 + |\dot{Y}(0)|^2}\; e^{\nu R}$$

where $\nu := \sqrt{n^2 + (n-1)(n-2)/2\,(\kappa - \delta)^2}$.

<u>Proof</u>: Since clearly, $|Y(R)| \leqq \sqrt{|Y(R)|^2 + |\dot{Y}(R)|^2}$

$= |\mathcal{H}(R)|$, in view of Lemma 5.6, we only need estimate the Lipschitz constant for $\mathcal{R}|E_s$, which is just its norm as a linear operator. But,

$$|\mathcal{R}|E_s| = \sqrt{|\pi_2|^2 + |R_{\dot{c}(s)}|^2} = \sqrt{n^2 + |R_{\dot{c}(s)}|^2} \quad ,$$

where

$$R_{\dot{c}(s)}: \quad c^*(TV) \ni y \mapsto R(\dot{c}(s), y; \dot{c}(s)) \quad .$$

Let $e_1, \ldots, e_{n-1}$ be an orthonormal basis for $\dot{c}(s)^{\perp}$. Then, by a standard identity for symmetric linear forms and the estimates in our assumption,

$$|R_{\dot{c}(s)}|^2 = \sum_{i,j=1}^{n-1} R(\dot{c}(s), e_i; \dot{c}(s), e_j)^2$$

$$= \sum_{i,j=1}^{n-1} \{\tfrac{1}{2}[K(\dot{c}(s) \wedge (e_i + e_j)) - K(\dot{c}(s) \wedge e_i) - K(\dot{c}(s) \wedge e_j)]\}^2$$

$$\leq \frac{(n-1)(n-2)}{2} \tfrac{1}{2}(2\kappa - \delta - \delta)^2$$

$$= \frac{(n-1)(n-2)}{2} (\kappa - \delta)^2 \quad .$$

The remainder of the proof is simply substitution.

q.e.d.

Corollary 5.7: Suppose that a Riemannian manifold V has sectional curvature in the range

$$\delta \leq K(\sigma) \leq \kappa \quad .$$

Then, for any $q \in V$ and $u \in \mathcal{G}_q$ and $0 < r$, we have the following estimate for the Gramian

$$Gr_q^{\frac{1}{2}}(ru) \leqq \left(\frac{e^{\nu r}}{r^2}\right)^{n-1}$$

Proof: Recall form the proof of Lemma 2.1 that

$$Gr_q^{\frac{1}{2}}(ru) = \frac{|Y_1(r) \wedge \cdots \wedge Y_{n-1}(r)|}{r^{n-1}|v_1 \wedge \cdots \wedge v_{n-1}|}$$

$$\leqq \frac{|Y_1(r)| \cdots |Y_{n-1}(r)|}{r^{n-1}|v_1 \wedge \cdots \wedge v_{n-1}|} \quad ,$$

where $v_1, \ldots, v_{n-1} \in ru^{\perp}$ form a basis and $Y_j(s) := T_{su}\exp(sv_j)$; $j = 1, \ldots, n-1$. We may take $v_1, \ldots, v_{n-1}$ to be orthonormal, and then $Y_j$'s have the initial condition

$$Y_j(0) = 0$$
$$\dot{Y}_j(0) = v_j/r \quad .$$

So, for each j,

$$|Y_j(r)| \leqq e^{\nu r}/r$$

by Lemma 5.6.

q.e.d.

Corollary 5.8: Let V be again as in Corollary 5.7 and $q \in V$. Suppose that $v, w \in \mathcal{G}_{R,q}$ can be joined by a great circular arc in $\mathcal{G}_{R,q}$ of length $\leqq \eta$. Let $x := \exp v$, $y := \exp w$. Then,

$$d(x,y) \leqq \frac{\eta}{R} e^{\nu R} \quad .$$

Proof: Let $\gamma$: $[0,1] \to \mathcal{G}_{R,q}$ be the great circular arc from v to w parametrized by arclength, so that $1 \leqq \eta$. Then, $d(x,y) \leqq L(\exp \circ \gamma)$. Let $u_t \in \mathcal{S}_q$ be such that $Ru_t = \gamma(t)$, and let $c_t(s) := \exp su_t$. Then,

$$\dot{\widehat{\exp \circ \gamma}}(t) = Y_t(R) \quad ,$$

where $Y_t \in \mathcal{X}_{c_t}$ is the Jacobi field with the initial condition

$$Y_t(0) = 0$$
$$\dot{Y}_t(0) = \dot{\gamma}(t)/R \quad .$$

Since $|\dot{\gamma}| \equiv 1$, By Lemma 5.6, therefore,

$$|\dot{\widehat{\exp \circ \gamma}}(t)| \leqq \frac{1}{R} e^{\nu R} \quad ,$$

integrating which on $0 \leqq t \leqq 1$, we obtain the required result.

q.e.d.

We now return to M with $n - 1 \leqq Ric$ and state the two main computations of this section.

Proposition 5.9 (Geodesic endpoint lemma): Suppose that $\kappa$ is any given upper bound for the sectional curvature of M. Then, given $0 < R < \pi$ and $0 < \varepsilon$, there is a constant $0 < \nu_1$, depending only on n, $\kappa$, R, and $\varepsilon$, such that whenever

$$(1 - \nu_1)\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad ,$$

$v \in \mathcal{S}_{R,q}$ and $x := \exp v$, then

$$d(x, S(R;q)) < \varepsilon \quad .$$

Proof: Let $\delta := n - 1 - (n - 2)\varkappa$. Then,

$$\delta \leq K \leq \varkappa \quad .$$

Set $\eta := Re^{-\nu R}$ where

$$\nu = \sqrt{n^2 - (n - 1)(n - 2)/2(\varkappa - \delta)^2} \quad .$$

Let $\upsilon_1$ be as in Corollary 5.4 so that any $v \in \mathcal{G}_{R,q}$ can be joined to some $w \in \mathcal{G}_{R,q} \cap \mathcal{N}_q$ by a great circular arc of length $< \eta$ in $\mathcal{G}_{R,q}$. Then, by Corollary 5.8, $d(x, \exp w) < \varepsilon$. But $w \in \mathcal{G}_{R,q} \cap \mathcal{N}_q$ implies that $\exp w \in S(R;q)$.

q.e.d.

Proposition 5.10 (Mapping multiplicity lemma): Let $\varkappa$ and R be as in the preceding proposition. Consider the exponential map on $\mathcal{D}_{R,q}$. Let $W \subset M$ be the set of those points that have more than one preimage under $\exp|\mathcal{D}_{R,q}$. Then, if

$$(1 - \upsilon_2)\mathrm{vol}_{S^n} < \mathrm{vol}_M$$

for some $0 < \upsilon_2$ small enough so that

$$m_R(\mathcal{G}_{R,q} \sim \mathcal{N}_q) < \delta$$

as in Lemma 5.2, then

$$\mathrm{vol}_W < \frac{\delta}{R^{n-1}} \int_0^R \left(\frac{e^{\nu r}}{r}\right)^{n-1} dr \quad .$$

Proof: Each point in W has at least one preimage in $\mathcal{D}_{R,q} \sim \mathcal{N}_q$. Thus, by Kronecker's formula,

$$\mathrm{vol}_W \leq \int_{\mathcal{D}_{R,q} \sim \mathcal{N}_q} Gr_q^{\frac{1}{2}} \, d\tilde{m}$$

Using Fubini's theorem, by Corollary 5.7,

$$\int_{\mathcal{D}_{R,q} \sim \mathcal{N}_q} \mathrm{Gr}_q^{\frac{1}{2}}\, d\tilde{m} = \int_0^R \int_{\mathcal{S}_{r,q} \sim \mathcal{N}_q} \mathrm{Gr}_q^{\frac{1}{2}}(v)\, dv\, dr$$

$$\leq \int_0^R \left(\frac{e^{\nu r}}{r^2}\right)^{n-1} m_r(\mathcal{S}_{r,q} \sim \mathcal{N}_q)\, dr \quad .$$

But, as in the proof of Lemma 5.2, the fact that $\mathcal{N}_q$ is star-shaped implies that

$$\mathcal{S}_{r,q} \sim \mathcal{N}_q \subset \eta_{r/R}(\mathcal{S}_{R,q} \sim \mathcal{N}_q)$$

for $r < R$. Here, $\eta$ is the homothecy operator again. Therefore, the last integral is

$$\leq \int_0^R \left(\frac{e^{\nu r}}{r^2}\right)^{n-1} m_R(\mathcal{S}_{R,q} \sim \mathcal{N}_q)\, dr$$

$$< \int_0^R \left(\frac{e^{\nu r}}{rR}\right)^{n-1} \delta\, dr$$

as required.

q.e.d.

## 6. Main theorem - part two: A topological construction

Lemma 6.1: If $\frac{1}{2}\mathrm{vol}_{S^n} < \mathrm{vol}_M$, then M is simply connected.

Proof: We prove the contrapositive. Assume that M is not simply connected. Then, there exists a non-trivial covering space $\pi: \tilde{M} \to M$, say of order $k \geqq 2$, and

$$k\ \mathrm{vol}_M = \mathrm{vol}_{\tilde{M}} \quad .$$

But since $n - 1 \leqq \mathrm{Ric}$ in $\tilde{M}$ as well and Myers' theorem (Theorem 10) states that $d_{\tilde{M}} \leqq \pi$, by Corollary 2.6, we see that

$$\mathrm{vol}_{\tilde{M}} \leqq \mathrm{vol}_{S^n} \quad .$$

Putting these informations together,

$$\mathrm{vol}_M \leqq 1/k\ \mathrm{vol}_{S^n} \leqq \tfrac{1}{2}\mathrm{vol}_{S^n} \quad .$$

q.e.d.

On the other hand, it is a classical theorem of J. H. C. Whitehead (1948), that a simply connected compact cell complex is homotopically equivalent to $S^n$ as soon as all homology groups, or equivalently all cohomology groups, vanish except in the top and bottom dimensions. The following is well-known:

Lemma 6.2 (Topological lemma): Let V be an orientable topological (not necessarily differentiable) manifold of dimension n, and suppose that there is a continuous mapping

$$\varphi: \quad S^n \to V$$

of Brouwer degree 1. Then

$$H^j(V;\mathbb{Z}) = 0$$

for $j = 1, \ldots, n-1$.

Proof: Suppose that $\xi \in H^j(V;\mathbb{Z})$. Then,

$$H^j(V;\mathbb{Z}) \ni H^*\varphi(\xi) = 0 \quad .$$

Thus,

$$[S^n] \cap H^*\varphi(\xi) = 0 \quad ,$$

where [ ] denotes the homology class of the fundamental cyle and $\cap$: $H_{j+k}(\cdot;\mathbb{Z}) \times H^j(\cdot;\mathbb{Z}) \to H_k(\cdot;\mathbb{Z})$ is the cap product. From the functoriality,

$$H_{n-j}(V;\mathbb{Z}) \ni H_*\varphi[S^n] \cap \xi = 0 \quad .$$

By our assumption on the Brouwer degree, $H_*\varphi[S^n] = [V]$. Therefore,

$$[V] \cap \xi = 0 \quad .$$

But, as V was an orientable manifold, the mapping $[V] \cap$: $H^j(V;\mathbb{Z}) \to H_{n-j}(V;\mathbb{Z})$ is just the Poincaré duality isomorphism. Hence, $\xi$, which was arbitrary, must = 0.

q.e.d.

By virtue of the above two lemmas and Whitehead's theorem, in order to complete our proof, it suffices to show that given an arbitrary upper bound $\varkappa$ for K, it is possible to find a $0 < \nu$ so that if

$$(1 - \nu)\mathrm{vol}_{S^n} \leq \mathrm{vol}_M \quad ,$$

then we can construct a degree 1 continuous mapping $\varphi: \; S^n \to M$.

<u>Complete proof of the Main theorem</u>: For the given $\varkappa$, let $\varepsilon := \pi/(4\sqrt{\varkappa})$. Then, by Proposition 3.6 (Cheeger-Heintze-Karcher), we can find an $\nu_3$ so that for any M with

$$(1 - \nu_3)\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad ,$$

we have

$$4\varepsilon < i_M \quad .$$

By our Covering lemma (Corollary 2.9), if we set $\delta := 2\varepsilon$ and $\rho := \varepsilon$, we can also find a $0 < \nu_4$ for which if

$$(1 - \nu_4)\mathrm{vol}_{S^n} < \mathrm{vol}_M$$

is satisfied and if p, q $\in$ M realize $d(p,q) = d_M$, then

$$D(3\varepsilon;p) \cup D(b(\nu_4) - \varepsilon;q) = M$$

(the function $b(\nu)$ was defined in Section 2).

Now, set $R := b(\nu_4) - \varepsilon$ and let $0 < \nu_1$ be the number in the Geodesic endpoint lemma (Proposition 5.9), so that whenever

$$(1 - \upsilon_1)\mathrm{vol}_{S^n} < \mathrm{vol}_M$$

and $x \in \exp(\mathcal{G}_{R,q})$, then

$$d(x, S(b(\upsilon_4) - \varepsilon; q)) < \varepsilon \quad .$$

Next, using the Mapping multiplicity lemma (Proposition 5.10), we can define a $0 < \upsilon_2$ such that whenever

$$(1 - \upsilon_2)\mathrm{vol}_{S^n} < \mathrm{vol}_M$$

is satisfied and if $W$ is the set of points with more than one preimage under $\exp|\mathcal{D}_{b(\upsilon_4)-\varepsilon, q}$, then

$$\mathrm{vol}_W < (1 - \upsilon_1)\mathrm{vol}_{S^n} - \mathrm{vol}_{\hat{B}(4\varepsilon)} \quad .$$

Finally, we define

$$\upsilon := \min\{\tfrac{1}{2}, \upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\} \quad .$$

Then, $0 < \upsilon < 1$ and it depends only on $n$ and $\kappa$. Suppose that $M$ has

$$(1 - \upsilon)\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad .$$

We now construct an explicit mapping

$$\varphi: \quad S^n \longrightarrow M$$

which has Brouwer degree 1.

Let $p, q \in M$ have

$$d(p,q) = d_M \quad .$$

Let $M'$ be the topological quotient space obtained by identifying all points in $D(4\varepsilon; p)$. Note that $D(4\varepsilon; p)$ is contractible inside $M$. Therefore, there are continuous mappings

$$\mathscr{f}: \quad M \longrightarrow M'$$

$$\mathscr{k}: \quad M' \longrightarrow M$$

such that $\mathscr{f} \circ \mathscr{k}$ and $\mathscr{k} \circ \mathscr{f}$ are both homotopic to the identities and, additionally, such that $\mathscr{k} \circ \mathscr{f} | \neg D(4\varepsilon;p)$ is the identity ($\mathscr{f}$ is just the natural projection). Consider the mapping

$$\mathscr{f} \circ \exp\colon \mathscr{D}_{b(v_4)-\varepsilon,q} \longrightarrow M' \quad .$$

Now, by the Covering lemma,

$$S(b(v_4) - \varepsilon;q) \subset D(3\varepsilon;p) \quad ,$$

while, by the Geodesic endpoint lemma,

$$d(x,S(b(v_4) - \varepsilon;q)) < \varepsilon$$

for all $x \in \exp(\mathscr{S}_{b(v_4)-\varepsilon,q})$. Consequently

$$\exp(\mathscr{S}_{b(v_4)-\varepsilon,q}) \subset D(4\varepsilon;p) \quad .$$

Since the right side of the above is identified to a single point by $\mathscr{f}$, the mapping $\mathscr{f} \circ \exp|\mathscr{D}_{b(v_4)-\varepsilon,q}$ factors through a continuous mapping

$$\mathbb{D}: \quad S^n \longrightarrow M'$$

where the $S^n$ here is obtained specifically by collapsing $\partial\mathscr{D}_{b(v_4)-\varepsilon,q} = \mathscr{S}_{b(v_4),q}$ to a point.

The desired mapping $\varphi\colon \; S^n \longrightarrow M$ is, then, obtained as

$$\varphi := \mathscr{k} \circ \mathbb{D} \ .$$

To show that $\varphi$ has Brouwer degree 1, it suffices to show that there is some open set in M which is covered only once by $\mathscr{k} \circ \mathscr{f} \circ \exp|\mathscr{D}_{b(\upsilon_4)-\varepsilon,q}$ . But, this mapping coincides with exp on $\mathscr{D}_{b(\upsilon_4)-\varepsilon,q}$ , and we have

$$\mathrm{vol}_W < (1 - \upsilon_1)\mathrm{vol}_{S^n} - \mathrm{vol}_{\hat{B}(4\varepsilon)}$$

$$< \mathrm{vol}_M - \mathrm{vol}_{B(4\varepsilon;p)} \leqq \mathrm{vol}_{\exp(\mathscr{D}_{b(\upsilon_4)-\varepsilon,q})}$$

by the volume comparison and our construction. Thus,

$$\exp(\mathscr{D}_{b(\upsilon_4)-\varepsilon,q}) \cap W \neq \emptyset \ .$$

This completes our proof of the Main theorem.

q.e.d.

By way of conclusion to our paper, we mention that if the given $\kappa$ is fairly small, so that $\frac{1}{2} < \min\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$, then our proof shows that for a larger choice of $\upsilon$, viz. $\upsilon := \min\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4\}$,

$$(1 - \upsilon)\mathrm{vol}_{S^n} < \mathrm{vol}_M$$

still implies that M has the homology type of $S^n$. Also, in the event that $\frac{1}{2}, \upsilon_2 < \min\{\upsilon_1, \upsilon_3, \upsilon_4\} =: \bar{\upsilon}$, if

$$(1 - \bar{\upsilon})\mathrm{vol}_{S^n} < \mathrm{vol}_M \quad ,$$

then we can still conclude that M has the cohomology structure of a truncated polynomial ring (such is the cohomology ring of a CROSS). The actual numerical values of the $\nu$'s should be easily and explicitly calculable with an aid of a moderate speed computer.

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## Index of symbols

| | | | |
|---|---|---|---|
| vol | 23 | $\mathfrak{X}''_c$ | 19 |
| $\mathfrak{X}_c$ | 19 | $\mathfrak{X}_M$ | 17 |