Characteristic Classes for Modules over Cyclic Groups

by

Iris Cox Hayslip

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IRIS HAYSILP

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

Chih-Han Sah
Committee Chairman

Leonard Charlap
Thesis Advisor

Michio Kuga

John Smith, Institute for Theoretical Physics (outside member)

The dissertation is accepted by the Graduate School

Charles W. Kim
Dean of the Graduate School
Abstract of the Dissertation

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In this paper we study the Charlap-Vasquez characteristic classes \( v^1(M) \in H^2(\hat{\mathbb{F}}; H^{i-1}(M, H_1(M, k))) \) for the case where \( M \) is an \( n \)-dimensional vector space over the field \( \mathbb{F}_p \), \( p \) prime, and \( \hat{\mathbb{F}} \subset GL(n, p) \). We employ a special resolution of \( Z \) for \( M \), particular choices of subgroups \( \hat{\mathbb{F}} \), and the chain homotopy techniques of Charlap and Vasquez to obtain results on the classes \( v^1(M) \in H^2(\hat{\mathbb{F}}; H^{i-1}(M, H_1(M, k))) \) for \( (n, p) = (2, 3) \) and \( (3, 2) \). These together with the Naturality Theorem of Charlap and Vasquez provide some results on the universal classes \( \overline{v}^1(M) \in H^2(GL(n, p); H^{i-1}(M, H_1(M, k))) \).
In memory of Innes Warner Cox
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INTRODUCTION

Suppose $\mathfrak{g}$ is a group, $M$ a module over $\mathbb{Z}[\mathfrak{g}]$ the integral group ring of $\mathfrak{g}$ and let $D_\ast$ be any resolution of $\mathbb{Z}$ for $M$. One cannot in general define an action of $\mathfrak{g}$ on the modules $D_i$ in a natural way, but the notion of a $\mathfrak{g}$-system approximates an action of $\mathfrak{g}$ on $D_i$. One part of a $\mathfrak{g}$-system consists for each $i$ and each $\sigma \in \mathfrak{g}$ of a $\mathbb{Z}$-homomorphism $A_i(\sigma) : D_i \rightarrow D_i$ which is a chain map satisfying an appropriate semi-linearity condition. Now $A_i(\sigma) A_i(\tau) \neq A_i(\sigma \tau)$ in general, but they are chain homotopic via $U_i(\sigma, \tau) : D_i \rightarrow D_{i+1}$ and these maps $U_i$ are a measure of the obstruction to the existence of an action of $\mathfrak{g}$ on $D_i$. For each $\sigma, \tau \in \mathfrak{g}$, $U_i(\sigma, \tau) \in \text{Hom}(D_i, D_{i+1})$ defines an element in $\text{Hom}(H_i(M, k), H_{i+1}(M, k))$ which is $H^i(M; H_{i+1}(M, k))$ if the coefficient group $k$ is suitably chosen. This defines a 2-cochain $w^{i+1}$ for $\mathfrak{g}$ with coefficients in $H^i(M; H_{i+1}(M, k))$ which is a cocycle, if the resolution $D_\ast$ is suitably chosen (i.e. small), and the cohomology class corresponding to $w^{i+1}$ is

$$v^{i+1}(M) \in H^2(\mathfrak{g}; H^i(M; H_{i+1}(M, k)))$$

and called the characteristic class of $M$. The characteristic classes $v^i(M)$ depend only on $\mathfrak{g}$, $M$ and the action of $\mathfrak{g}$ on $M$.

Now suppose we have the extension $0 \rightarrow M \rightarrow \pi \rightarrow \mathfrak{g} \rightarrow 1$

where $\pi$ is the semidirect product of $\mathfrak{g}$ and $M$. In [2]
Charlap and Vasquez showed that the map $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ in the Hochschild-Serre spectral sequence for this extension can be obtained by forming the cup product with the class $v^r(M)$. When $M$ is free Abelian and finitely generated so we can consider $\hat{\Phi} \in \text{GL}(n,\mathbb{Z})$, Charlap and Vasquez [3] have computed the classes $v^r(M)$ and proved various theorems about characteristic classes in this case including:

(*) Theorem: If $M$ is $\mathbb{Z}$-free, then $v^2(M) = 0$ iff $v^n(M) = 0$ for all $n$.

The problem considered in this paper is that of determining the classes $v^i(M)$ where $M$ is a vector space of dimension $n$ over the finite field $\mathbb{Z}_p$, $p$ prime, and where $\hat{\Phi} \in \text{GL}(n,p)$. Some results on this problem were obtained by Sah in [8] and [9] by methods other than the use of the chain homotopies mentioned above; in particular $v^2(M)$ has another group theoretical interpretation which yields results including the fact that $v^2(M) \neq 0$ for $\hat{\Phi} = \text{GL}(n,2)$ where $n > 3$. However, no such interpretation is known for $v^i(M)$ with $i > 2$.

In this paper we develop some computational techniques for $v^i(M)$ when $\hat{\Phi} \in \text{GL}(n,p)$, for small $n$ and $p$, using a special choice of resolution $D_\Phi$ and direct construction of $\hat{\Phi}$-systems in low dimensions. These techniques are carried out to compute:
1) \( v^2(M) \) and \( v^3(M) \) where \( \tilde{\phi} \subseteq \text{GL}(2,3) \) and

ii) \( v^2(M) \) and \( v^3(M) \) where \( \tilde{\phi} \subseteq \text{GL}(3,2) \).

We hope the procedures used in i) could be extended (with use of a computer) to find some \( v^i(M) \) for \( i > 2 \), \( \tilde{\phi} \subseteq \text{GL}(2,p) \) and \( p > 3 \). Moreover, it may be possible to extend the procedures used in ii) to find \( v^i(M) \) for \( \tilde{\phi} \subseteq \text{GL}(n,2) \) with \( n > 3 \), \( i > 2 \). The main result obtained from ii) is the fact that no theorem analogous to (*) exists in the case where \( M \) is finite.

The calculations rely on particular choices of subgroups \( \tilde{\phi} \subseteq \text{GL}(n,p) \). Results on the classes \( \overline{v}^1(M) \in H^2(\text{GL}(n,p); H^i(M; H_{i+1}(M,k))) \) for the whole group \( \text{GL}(n,p) \) are obtained by using the naturality theorem in [3], which says that if \( v^i(M) \in H^2(\tilde{\phi}; H^{i-1}(M,H_i(M,k))) \), \( \tilde{\phi} \subseteq \text{GL}(n,p) \), \( v^i(M) \neq 0 \) then \( \overline{v}^i(M) \) is also non-zero. Moreover, if \( \tilde{\phi} \) is chosen to be a p-Sylow subgroup of \( \text{GL}(n,p) \) then \( v^1(M) = 0 \) will imply the universal class \( \overline{v}^1(M) = 0 \).
Let \( \Phi \) be a group, and \( M \) a module over \( \mathbb{Z}[\Phi] \), the integral group ring of \( \Phi \); i.e. there is a homomorphism \( \phi : \Phi \to \text{Aut}(M) \). \( M \) is called a \( \Phi \)-module and the action of \( \Phi \) on \( M \) is written \( \sigma \cdot m = \phi(\sigma)(m) \). Since \( M \) is itself a group, it also can act on another group to form an \( M \)-module. Let \( \overline{D} = (D_n, d_n) \) be a projective resolution of the trivial \( M \)-module \( \mathbb{Z} \), i.e. an exact sequence of projective \( M \)-modules

\[
\overline{D} : \cdots \to D_n \xrightarrow{d_n} D_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} D_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0.
\]

**Definition 1**: (\( \Phi \)-system). A \( \Phi \)-system for \( \overline{D} \) consists of two sequences of functions:

\[
A_n : \Phi \to \text{Hom}_{\mathbb{Z}}(D_n, D_n)
\]

and

\[
U_n : \Phi \times \Phi \to \text{Hom}_{\mathbb{Z}}(D_n, D_{n+1})
\]

with the following properties:

1) \( \epsilon A_0(\sigma) = \epsilon \)

11) \( \det A_n(\sigma) = A_{n-1}(\sigma) \det d_n, \ n \geq 1 \)

1ii) \( A_n(\sigma) \) is \( \sigma \)-linear, i.e.

\[
A_n(\sigma)(r \cdot x) = \sigma(r) \cdot A_n(\sigma)(x) \quad \text{for } x \in D_n, \ r \in \mathbb{Z} [M]
\]

1i') \( \det U_0(\sigma, \tau) = A_0(\sigma \tau) - A_0(\sigma) \cdot A_0(\tau) \)

1i') \( \det U_n(\sigma, \tau) + U_{n-1}(\sigma, \tau) \det d_n = A_n(\sigma \tau) - A_n(\sigma) \cdot A_n(\tau), \ n \geq 1 \)

1iii') \( U_n(\sigma, \tau) \) is \( \sigma \tau \)-linear, i.e.

\[
U_n(\sigma, \tau)(r \cdot x) = \sigma \tau(r) \cdot U_n(\sigma, \tau)(x) \quad \text{for } x \in D_n, \ r \in \mathbb{Z} [M].
\]
If $\Gamma$ is both an $M$-module and a $\phi$-module such that 
$\sigma(m \cdot y) = \sigma(m) \cdot \sigma(y)$ for all $\sigma \in \Phi$, $m \in M$, $y \in \Gamma$, then $A_{n}$ can be extended from $D_{n}$ to $\text{Hom}_{\mathbb{Z}}[M](D_{n}, \Gamma)$ by 

$$(A_{n}(\sigma) \cdot f)(x_{n}) = \sigma(f(A_{n}(\sigma^{-1})x_{n})).$$

This map induces an action of $\phi$ on $H_{n}(M, \Gamma)$ which is independent of the $\phi$-system and the choice of resolution $\mathcal{D}$. Analogously, there is an action of $\phi$ on $H_{n}(M, \Gamma)$ induced from 

$$\sigma(x_{n} \otimes y) = [A_{n}(\sigma)(x_{n})] \otimes y \text{ for } x_{n} \in D_{n}, y \in \Gamma.$$ 

This action of $\phi$ on $x \in H_{n}(M, \Gamma)$ will be denoted by $\sigma_{*}(x)$.

Now let $k$ be a principal ideal domain on which $M$ acts trivially. The universal coefficient theorem yields 

$$H_{n}(M, A) \cong \text{Hom}_{k}(H_{n}(M, k), A)$$

where $A$ is a $k$-module on which $M$ acts trivially. The group $H_{n}(M, k)$ can be considered as a trivial $k$-module and trivial $M$-module so is a possible choice for coefficient group $A$.

Let $f^{n} \in \text{Hom}_{\mathbb{Z}}[M](D_{n}, H_{n}(M, k))$ be a cocycle representing the cohomology class in $H_{n}(M, H_{n}(M, k))$ corresponding to the identity map in $\text{Hom}_{k}(H_{n}(M, k), H_{n}(M, k))$. Then $A_{n}(\sigma) \cdot f^{n}$ represents the same cohomology class so for each $\sigma \in \Phi$ there exists $F^{n-1}_{\sigma} \in \text{Hom}_{\mathbb{Z}}[M](D_{n-1}, H_{n}(M, k))$ such that 

$$A_{n}(\sigma) \cdot f^{n} - f^{n} = F^{n-1}_{\sigma} d_{n}.$$ 

Define $u^{n}(\sigma, \tau) \in \text{Hom}_{\mathbb{Z}}[M](D_{n-1}, H_{n}(M, k))$ by
Definition 2:

\[ u^n(\sigma, \tau) = A_{n-1}(\sigma) \cdot F_\tau^{n-1} - F_\tau^{n-1} + F_\sigma^{n-1} + (\sigma \tau)_* [f^n U_{n-1}(\tau^{-1}, \sigma^{-1})]. \]

Theorem I: \( u^n(\sigma, \tau) \) is a cocycle representing an element \( w^n(\sigma, \tau) \in H^{n-1}(M, H_n(M, k)) \). And \( w^n : \hat{\Phi} \times \hat{\Phi} \to H^{n-1}(M, H_n(M, k)) \) is a 2-cocycle for \( \hat{\Phi} \) and represents an element \( v^n \in H^2(\hat{\Phi}; H^{n-1}(M, H_n(M, k))) \). Furthermore, the cohomology class of \( w^n \) depends only on the action of \( \hat{\Phi} \) on \( M \). (It is independent of choices of \( f^n, \ P_\sigma^{n-1} \), the \( \hat{\Phi} \)-system and resolution \( \mathcal{B} \).

Definition 3 (Characteristic Class): The cohomology class of \( w^n \in H^2(\hat{\Phi}; H^{n-1}(M, H_n(M, k))) \) is called the \( n \)th characteristic class of \( M \) and is denoted \( v^n(M) \) or \( v^n \).

Theorem II (Naturality): Let \( h : \hat{\Phi} \to \hat{\Phi}' \) be a group homomorphism. Then \( \hat{\Phi}' \) acts on \( M \) via \( h \) and the action of \( \hat{\Phi} \), i.e. \( \sigma'(m) = (h(\sigma'))m \). Denote by \( M' \) the \( \hat{\Phi}' \)-module \( M \) under this action. Then \( h \) induces a homomorphism

\[ h^* : H^2(\hat{\Phi}, H^{n-1}(M, H_n(M, k)) \to H^2(\hat{\Phi}', H^{n-1}(M', H_n(M', k))) \]

and \( h^*(v^n(M)) = v^n(M') \).

The proofs of theorems I and II can be found in [3].

Suppose \( M \) is taken to be free Abelian of rank \( n \) and \( \hat{\Phi} \) a subgroup of \( \text{GL}(n, \mathbb{Z}) \). Then Theorem II implies that a non-zero characteristic class \( v^1 \in H^2(\hat{\Phi}; H^{1-1}(M, \mathbb{Z}))(M, k)) \) will lift
to a non-zero class $v^1 e H^2(\text{GL}(n, \mathbb{Z}); H^{i-1}(M, H^i(M, k)))$. The classes $v^1 e H^2(\text{GL}(n, \mathbb{Z}); H^{i-1}(M, H^i(M, k)))$ will be called universal. Also, $M$ may be taken to be a module of rank $n$ over a finite field, $\mathbb{F}_p$, of prime characteristic and $\hat{\varphi}$ a subgroup of $\text{GL}(n, \mathbb{F}_p)$.

The principal importance of the characteristic classes arises from the interpretation of the second differential in the Hochschild-Serre spectral sequence for a split group extension. Let $G$ be the semidirect product of $M$ and $\hat{\varphi}$. Given the extension

$$0 \rightarrow M \rightarrow G \rightarrow \hat{\varphi} \rightarrow 1$$

there is a spectral sequence (Hochschild-Serre) $E^2_{p,q}(B) = H^p(G; B)$ where $E^2_{2,q}(B) = H^p(\hat{\varphi}; H^q(M, B))$ for a suitable coefficient module $B$. In [2] and [3] Charlap and Vasquez showed that the second differential $d_{2,q} : E^2_{2,q}(B) \rightarrow E^2_{2+2,q-1}(B)$ in this spectral sequence may be interpreted as a cup product with the characteristic class $v^n$.

Another interpretation of the characteristic classes is as follows. Let $M$ a $\hat{\varphi}$-module and $\overline{D} : \cdots \rightarrow D_n \xrightarrow{d_n} D_{n-1} \rightarrow \cdots \rightarrow D_0 \rightarrow \mathbb{Z} \rightarrow 0$ any resolution of $\mathbb{Z}$ for $M$. Then $\hat{\varphi}$ doesn't act on the component modules $D_i$ in any natural way. But a $\hat{\varphi}$-system for $\overline{D}$ is an approximation to an action of $\hat{\varphi}$ on $D_i$, i.e. if there are no $\mathbb{Z}$-homomorphisms $A_i(\sigma) : D_i \rightarrow D_1$ which satisfy $A_i(\sigma) \cdot A_1(\tau) = A_i(\sigma \tau)$
then the homotopy part of the $\tilde{g}$-system $U_1(\sigma,\tau)$ is non-zero and measures the obstruction to an action of $\tilde{g}$ on $D_1$.

Also in [3], Charlap and Vasquez computed $\nu^1(M)$ for $M$ a $\mathbb{Z}$-free module and as a result proved the following.

**Theorem III:** If $M$ is $\mathbb{Z}$-free, $2\nu^1(M) = 0$.

**Theorem IV:** If $M$ is $\mathbb{Z}$-free, then $\nu^2(M) = 0$ iff $\nu^1(M) = 0$ for all $i$.

(An alternate proof of Theorem III which doesn't use $\tilde{g}$-systems or specific calculation of $\nu^1(M)$ appears in [8].)
Finite Case

We consider in the following, some cases where $M$ is a vector space of dimension $n$ over a field with $p$ elements, $p$ a prime, and $\mathfrak{H}$ is a subgroup of $\text{GL}(n,p)$. We wish to investigate the characteristic classes $\nu^i(M)$ and to determine if a theorem analogous to Theorem IV exists in this case.

No formula for computing $v^n$ in this case is known, however, some of the groups $H^2(\text{GL}(n,p); H^{n-1}(M; H(M,k)))$ which contain the universal classes have been computed in [8] by other methods, (see Appendix). We compute $v^2$ and $v^3$ in some examples for finite $M$ and particular subgroups of $\text{GL}(2,3)$ and $\text{GL}(3,2)$. The procedure constructing the $\mathfrak{H}$-system in these cases in theory works for $\text{GL}(n,p)$ with arbitrary $p$ and $n$ if $\mathfrak{H}$ is chosen to be particularly simple but the technique is very unwieldy. The question arises: which subgroups of $\text{GL}(n,p)$ may be chosen to insure that a non-zero universal class $\nu^i \times H^2(\text{GL}(n,p); H^{i-1}(M; H(M,k)))$ will restrict to the class $v^i \times H^2(\mathfrak{H}; H^{i-1}(M; H(M,k)))$?

Let $\mathfrak{H}^*$ be the subgroup of $\text{GL}(n,p)$ consisting of upper triangular matrices with 1's on the diagonal. Then $\mathfrak{H}^*$ is a $p$-Sylow subgroup of $\text{GL}(n,p)$ and a non-zero universal class must be detected on this subgroup. However, construction of the $\mathfrak{H}$-system even for $\mathfrak{H} = \mathfrak{H}^*$, $n \geq 2$ is too cumbersome. For $n = 2$, $\mathfrak{H}^* \cong \mathbb{Z}_p$, and the same procedure
as in Example 1 below should work. In Example 2, the case for \( GL(3,2) \), \( \hat{\mathfrak{s}} \ast = D_2 \) (dihedral group) and the construction of a \( \hat{\mathfrak{s}} \ast \)-system is too difficult. Therefore, we choose a subgroup \( \hat{\mathfrak{s}} = \) the group of all matrices in \( GL(3,2) \) generated by

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

so \( \hat{\mathfrak{s}} \cong \mathbb{Z}_2 \). It turns out that this choice of \( \hat{\mathfrak{s}} \) is sufficient to detect the universal class \( \nu^3 \), i.e. show \( \nu^3 \neq 0 \).

The calculation of \( v^n \) for the free Abelian case relied upon the choice of a particularly simple resolution for \( M \) which facilitated the construction of the chain map \( A_n \) and homotopy \( U_n \) in the \( \hat{\mathfrak{s}} \)-system for arbitrarily large \( n \). We are unable to do this in the finite case. Perhaps there is a more efficient resolution than the one employed in these examples.

**Example 1:** \( (GL(2,3)) \)

Let \( M = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \) and let \( \hat{\mathfrak{s}} \) be the subgroup of \( GL(2,3) \) consisting of upper triangular matrices with 1's on the diagonal,

\[
\hat{\mathfrak{s}} = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} : * \in \mathbb{Z}_3 \right\}
\]
so \( \hat{\phi} \cong \mathbb{Z}_3 \). Define action of \( \hat{\phi} \) on \( M \) in the natural way but write it multiplicatively:

\[
\sigma(s^i, t^j) = (s^{i+j}, t^j)
\]

\[
\sigma^2(s^i, t^j) = (s^{i+2j}, t^j)
\]

for \( \sigma \) the generator of \( \hat{\phi} \), \((s^i, t^j) \in M\), \( s \) and \( t \) generators of \( \mathbb{Z}_3 \).

Now let \( \overline{X} = (X_*, \partial_*) \) be the usual free resolution of \( \mathbb{Z} \) for \( \mathbb{Z}_3 \), i.e.

\[
\overline{X} : \cdots \xrightarrow{\partial_n} X_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} X_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0
\]

\( X_i = \mathbb{Z} [\mathbb{Z}_3] \)

for all \( i \).

\[
\partial_i = \begin{cases} 
\text{multiplication by } 1 - t & \text{when } i \text{ is odd} \\
\text{multiplication by } 1 + t + t^2 & \text{when } i \text{ is even.}
\end{cases}
\]

Now construct a resolution \( (D_*, d_*) = \overline{D} \) for \( M \) by the tensor product of resolutions \( \overline{D} = \overline{X} \otimes \overline{X} \), specifically

1) \( D_n = \bigoplus_{i+j=n} X_i \otimes X_j \)

\[
d_n = \otimes (\partial_i \otimes \text{id}_j) \otimes (-1)^i (\text{id}_i \otimes \partial_j)
\]

where \( \text{id}_j \) denotes the identity map on \( X_j \). The augmentation \( \bar{\epsilon} : D_0 \to \mathbb{Z} \) is obtained from the composition
where \( f(n \otimes m) = nm \) and \( \varepsilon \) is the usual augmentation in \( \overline{X} \) namely if \( X_0 = \sum \sigma n_\sigma \) then \( \varepsilon(X_0) = \sum \sigma n_\sigma \).

Each \( \overline{D}_n \) is an \( M \)-module via the action \( (m,n)(a \otimes b) = (ma) \otimes (nb) \) for \( (m,n) \in M; a,b \in \mathbb{Z}[[\mathbb{Z}_3]] \). We fix notation for elements in the components \( \overline{D}_1 \) of \( \overline{D} \) as follows. Since \( X_i = \mathbb{Z}[[\mathbb{Z}_3]] \) for all \( i \) and \( D_i = \bigoplus_{p+q=i} X_p \otimes X_q \), denote by \( s_p \) the generator of \( \mathbb{Z}_3 \) inside \( X_p \) and by \( t_q \) the generator of \( \mathbb{Z}_3 \) inside \( X_q \). So \( s_p \otimes t_q \) denotes a generating element of the summand of \( D_1 \) corresponding to \( X_p \otimes X_q \).

**Lemma 1:** The following \( \mathbb{Z} \)-homomorphisms \( A_i : D_1 \to D_1 \) satisfy (i), (ii) and (iii) of Definition 1:

\[
A_0(1) = A_1(1) = A_2(1) = \text{identity map}
\]
\[
A_0(\sigma)(s_0^i \otimes t_0^j) = s_0^{i+1} \otimes t_0^j
\]
\[
A_0(\sigma^2)(s_0^i \otimes t_0^j) = s_0^{i+2} \otimes t_0^j
\]
\[
A_1(\sigma)(s_1^i \otimes t_0^j) = s_1^{i+1} \otimes t_0^j
\]
\[
A_1(\sigma^2)(s_1^i \otimes t_0^j) = s_1^{i+2} \otimes t_0^j
\]
\[
A_1(\sigma)(s_0^i \otimes t_1^j) = s_0^{i+1} \otimes t_1^j + s_1^{i+1} \otimes t_0^{j+1}
\]
\[
A_1(\sigma^2)(s_0^i \otimes t_1^j) = s_0^{i+2} \otimes t_1^j + s_1^{i+1} \otimes t_0^{j+1} + s_1^{i+2} \otimes t_0^{j+1}
\]
\[
A_2(\sigma)(s_1^i \otimes t_1^j) = s_1^{i+1} \otimes t_1^j
\]
\[ A_2(\sigma)(s_2^j \otimes t_0^j) = s_2^{i+j} \otimes t_0^j \]
\[ A_2(\sigma^2)(s_1^j \otimes t_1^j) = s_1^{i+2j} \otimes t_1^j \]
\[ A_2(\sigma^2)(s_2^j \otimes t_0^j) = s_2^{i+2j} \otimes t_0^j \]
\[ A_2(\sigma)(s_0^j \otimes t_2^j) = s_0^{i+j} \otimes t_2^j - s_1^{i+j} \otimes t_1^j + s_1^{i+j} \otimes t_1^{j+1} - s_1^{i+j} \otimes t_1^{j+2} \]
\[ + s_1^{i+j} \otimes t_1^{j+1} + s_2^{i+j} \otimes t_0^j \]
\[ A_2(\sigma^2)(s_0^j \otimes t_2^j) = s_0^{i+2j} \otimes t_2^j + s_1^{i+2j} \otimes t_1^j + s_1^{i+2j} \otimes t_1^{j+1} \]
\[ + s_1^{i+2j} \otimes t_1^{j+1} + s_2^{i+2j} \otimes t_0^j + s_2^{i+2j} \otimes t_0^{j+1} \]
\[ A_3(\sigma)(s_3^j \otimes t_0^j) = s_3^{i+j} \otimes t_0^j \]
\[ A_3(\sigma^2)(s_3^j \otimes t_0^j) = s_3^{i+2j} \otimes t_0^j \]
\[ A_3(\sigma)(s_2^j \otimes t_1^j) = s_2^{i+j} \otimes t_1^j + s_3^{i+j} \otimes t_0^j \]
\[ A_3(\sigma^2)(s_2^j \otimes t_1^j) = s_2^{i+2j} \otimes t_1^j + s_3^{i+2j} \otimes t_0^j + s_3^{i+2j} \otimes t_0^j \]
\[ A_3(\sigma)(s_1^j \otimes t_2^j) = s_1^{i+j} \otimes t_2^j + s_3^{i+j} \otimes t_0^j \]
\[ A_3(\sigma^2)(s_1^j \otimes t_2^j) = s_1^{i+2j} \otimes t_2^j + s_3^{i+2j} \otimes t_0^j + s_3^{i+2j} \otimes t_0^j \]
\[ A_3(\sigma)(s_0^j \otimes t_3^j) = s_0^{i+j} \otimes t_3^j + s_1^{i+j} \otimes t_2^j + s_2^{i+j} \otimes t_1^j + s_3^{i+j} \otimes t_0^j \]
\[ A_3(\sigma^2)(s_0^j \otimes t_3^j) = s_0^{i+2j} \otimes t_3^j + s_1^{i+2j} \otimes t_2^j + s_2^{i+2j} \otimes t_1^j + s_3^{i+2j} \otimes t_0^j \]
\[ + s_2^{i+2j} \otimes t_1^j + s_3^{i+2j} \otimes t_0^j + s_3^{i+2j} \otimes t_0^j \]

**Proof:** Define \( A_0(\sigma) : D_0 \to D_0 \) by \( A_0(\sigma)(s_0^j \otimes t_0^j) = s_0^{i+j} \otimes t_0^j \). This clearly satisfies (1) in Definition 1. The other \( A_1 \) are defined by starting in dimension 0 and solving equations.
(ii) for $A_1$ subject to condition (iii). Then define $A_i$ by solving (ii) using $A_{i-1}$ subject to (iii). For example we compute $A_3(\sigma)(s_1^i \otimes t_2^j)$. $A_3$ must satisfy (ii)

$$d_3A_3(s_1^i \otimes t_2^j) = A_2d_3(s_1^i \otimes t_2^j)$$

$$= A_2(\sigma)[s_0^i \otimes t_2^j - s_0^{i+1} \otimes t_2^j - s_1^i \otimes t_1^j - s_1^{i+1} \otimes t_1^{j+1} - s_1^i \otimes t_1^{j+2}]$$

$$= s_0^{i+j} \otimes t_2^j - s_1^{i+j} \otimes t_1^j - s_1^{i+j+1} \otimes t_1^j - s_1^i \otimes t_1^{j+2}$$

$$+ s_2^i \otimes t_0^j - s_0^{i+1+j} \otimes t_1^j + s_1^{i+j+1} \otimes t_1^j$$

$$+ s_1^{i+j+1} \otimes t_1^{j+2} + s_1^{i+j+2} \otimes t_1^{j+2} - s_2^{i+j+1} \otimes t_0^j$$

$$- s_1^{i+j} \otimes t_1^j - s_1^{i+j+1} \otimes t_1^j - s_1^{i+j+2} \otimes t_1^j$$

$$= s_0^{i+j} \otimes t_2^j - s_0^{i+j+1} \otimes t_2^j - s_1^{i+j} \otimes t_1^j - s_1^{i+j+1} \otimes t_1^j$$

$$- s_1^{i+j+2} + s_2^{i+j} \otimes t_0^j - s_2^{i+j+1} \otimes t_0^j$$

$$= d_3(s_1^i \otimes t_2^j + s_3^{i+j} \otimes t_0^j).$$

The following is a typical check for $\sigma$-linearity. Let $r = (s^k, t^l) \in M$, then $\sigma(r) = (s^{k+l}, t^l)$, and let

$$d = s_1^{i} \otimes t_2^j \in D_3.$$ Then

$$A_3(\sigma)(rd) = A_3(\sigma)(s_1^{i+k} \otimes t_2^j + t^l)$$

$$= s_1^{i+k+l} \otimes t_2^j + s_3^{i+k+l} \otimes t_0^j$$

$$= (s^{k+l}, t^l)(s_1^{i+j} \otimes t_2^j + s_3^{i+j} \otimes t_0^j)$$

$$= \sigma(r) \cdot A_3(\sigma)(d).$$

Other $\sigma$-linearity checks are done in a similar fashion. \qed
It is possible that computation of $A_1$ could be done by computer. The difficult case in all examples is for $A_1(\tau)(s_0 \otimes t_1^m)$.

Given $A_1$ as above, $A_1(\sigma \tau) \neq A_1(\sigma) \cdot A_1(\tau)$ in general and the next step is to find a chain homotopy $U$.

**Lemma 2:** The following maps $U_i(\sigma, \tau) : D_i \to D_{i+1}$ satisfy (i'), (ii') and (iii') in Definition 1:

- $U_0(\sigma, \tau) = 0$ for all $\sigma, \tau \in \mathbb{F}$
- $U_1(1, \tau) = U_1(\tau, 1) = 0$ for all $\tau \in \mathbb{F}$
- $U_1(\sigma, \tau)(s_1 \otimes t_0^j) = 0 \quad \forall \sigma, \tau \in \mathbb{F}$
- $U_1(\sigma, \sigma)(s_0 \otimes t_1^j) = 0$
- $U_1(\sigma, \sigma^2)(s_0 \otimes t_1^j) = -s_2^1 \otimes t_0^{j+1}
- U_1(\sigma^2, \sigma)(s_0 \otimes t_1^j) = -s_2^1 \otimes t_0^{j+1}
- U_1(\sigma^2, \sigma^2)(s_0 \otimes t_1^j) = -s_2^{1+j} \otimes t_0^{j+1}
- U_2(\sigma, \tau)(s_2 \otimes t_0^j) = 0 \quad \forall \sigma, \tau \in \mathbb{F}
- U_2(1, \tau)(x) = U_2(\tau, 1)(x) = 0 \quad \forall \tau \in \mathbb{F}$, $x \in D_2$
- $U_2(\sigma, \sigma)(s_1 \otimes t_1^j) = 0$
- $U_2(\sigma, \sigma^2)(s_1 \otimes t_1^j) = s_3^1 \otimes t_0^{j+1}
- U_2(\sigma^2, \sigma)(s_1 \otimes t_1^j) = s_3^1 \otimes t_0^{j+1}
- U_2(\sigma^2, \sigma^2)(s_1 \otimes t_1^j) = s_3^{1+j} \otimes t_0^{j+1}
- U_2(\sigma, \sigma)(s_0 \otimes t_2^j) = s_2^{1+2j} \otimes t_1^{j+1} + 2(s_2^{1+2j} \otimes t_1^{j+2})
Now choose \( k = \mathbb{Z}_3 \) a trivial \( \tilde{\mathbb{F}} \)-module. Consider \( H_n(M,k) \) as a trivial \( M \)-module and trivial \( k \)-module. We need \( f^n \circ \text{Hom}_{\mathbb{Z}[M]}(D_n, H_n(M,k)) \) corresponding to the identity map in \( \text{Hom}_K(H_n(M,k), H_n(M,k)) \). Then, by the Künneth theorem \( H_n(M,k) \) is isomorphic to a direct sum of copies of \( k \), one copy for each pair \((p,q)\) such that \( p+q = n \), i.e.

\[
H_n(M,k) = \bigoplus_{p+q=n} \langle g_{pq} \rangle
\]

where \( \langle g_{pq} \rangle = k = \mathbb{Z}_3 \) and \( g_{pq} \) is a generator. Now define:

\[
f^n(s_i \otimes t_j^m) = g_{ij} \quad \text{for } s_i \otimes t_j^m \in D_n.
\]

Note that \( f^n(s_i \otimes t_j^m) = f^n(l_i \otimes l_j) \) since \( H_n(M,k) \) is a trivial \( M \)-module and \( f^n \) is an \( M \)-homomorphism \( [s_i \otimes t_j^m = (s_i^m, t_j^m) \cdot (l_i \otimes l_j)] \) where \((s_i^m, t_j^m) \in M\) and \( \cdot \) denotes the \( M \)-action on \( D_n \).

The action of \( \tilde{\mathbb{F}} \) on \( H_n(M,k) \) is given by the following:

\[
\begin{bmatrix}
\sigma_\#(g_{01}) \\
\sigma_\#(g_{10})
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\
0 & 1 \end{bmatrix} \begin{bmatrix} g_{01} \\
g_{10} \end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_\#^2(g_{01}) \\
\sigma_\#^2(g_{10})
\end{bmatrix} = \begin{bmatrix} 1 & 2 \\
0 & 1 \end{bmatrix} \begin{bmatrix} g_{01} \\
g_{10} \end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_\#^2(g_{01}) \\
\sigma_\#^2(g_{10})
\end{bmatrix} = \begin{bmatrix} 1 & 2 \\
0 & 1 \end{bmatrix} \begin{bmatrix} g_{01} \\
g_{10} \end{bmatrix}
\]
\[ = r^3(s_2^4 \otimes t_1^4) . \]

Consequently \( \mathbb{F}_n \tau = 0 \) for \( n \leq 2, \tau  \neq \phi \), so for this example
\[ u^n(\sigma, \tau) = (\sigma \tau)_* [r^n u_{n-1}(\tau^{-1}, \sigma^{-1})]. \]

**Lemma 3:** For the resolution (1); \( \sigma, \tau  \neq \phi, u^n(\sigma, \tau) \in \text{Hom}_{\mathbb{Z}}[M](D_n, H_{n+1}(M, k)) \) for \( n = 2, 3 \) are given by the following:

\[
\begin{align*}
    u^2(\rho, \tau)(s_0^1 \otimes t_1^j) & = 0 & \text{for all } \rho, \tau  \neq \phi \\
    u^2(\tau, 1) & = u^2(1, \sigma) = 0 & \text{for all } \tau  \neq \phi \\
    u^2(\sigma, \sigma)(s_0^1 \otimes t_1^j) & = -g_{20} \\
    u^2(\sigma, \sigma^2)(s_0^1 \otimes t_1^j) & = -g_{20} \\
    u^2(\sigma^2, \sigma)(s_0^1 \otimes t_1^j) & = -g_{20} \\
    u^2(\sigma^2, \sigma^2)(s_0^1 \otimes t_1^j) & = 0 \\
    u^3(\rho, \tau)(s_2^4 \otimes t_0^j) & = 0 & \text{for all } \rho, \tau  \neq \phi \\
    u^3(\tau, 1) & = u^3(1, \tau) = 0 & \text{for all } \tau  \neq \phi \\
    u^3(\sigma, \sigma)(s_1^1 \otimes t_1^j) & = g_{30} \\
    u^3(\sigma, \sigma^2)(s_1^1 \otimes t_1^j) & = g_{30} \\
    u^3(\sigma^2, \sigma)(s_1^1 \otimes t_1^j) & = g_{30} \\
    u^3(\sigma^2, \sigma^2)(s_1^1 \otimes t_1^j) & = 0 \\
    u^3(\tau, \rho)(s_0^1 \otimes t_2^j) & = 0 & \text{for all } \tau, \rho  \neq \phi.
\end{align*}
\]

**Proof:** The following is a typical calculation:
\[ u^2(\sigma, \tau)(s_0 \otimes t_1^j) = l_* f^2(U_1(\sigma^{-1}, (\sigma^2)^{-1})(s_0 \otimes t_1^j)) \]
\[ = l_* f^2(U_1(\sigma^2, \sigma)(s_0 \otimes t_1^j)) \]
\[ = l_* f^2[-s_2 \otimes t_0^{j+1}] \]
\[ = l_*(-g_{20}) = -g_{20}. \]

Other calculations are done similarly.

**Proposition 1:** For \( \phi \in \text{GL}(2,3) \), \( M = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \), \( v^2(M) \) and \( v^3(M) \) are both zero.

**Proof:** We have \( u^2(\sigma, \tau) \in \text{Hom}_M(D_1, H_2(M, k)) \) representing the class \( w^2(\sigma, \tau) \in H^1(M; H_2(M, k)) \). And \( w^2 : \phi \times \phi \to H^1(M; H_2(M, k)) \) is a \( \phi \)-cocycle which represents \( v^2 \in H^2(\phi; H^1(M; H_2(M, k))) \). To determine whether \( v^2 = 0 \), we need to know if \( w^2 \) is a coboundary, i.e., if there is a map \( \psi : \phi \to H^1(M, H_2(M, k)) \) such that

3) \[ u^2(\sigma, \tau) = \sigma \psi(\tau) - \psi(\sigma \tau) + \psi(\sigma) \]

where \( \sigma \cdot \psi(\tau) \) denotes the action of \( \phi \) on \( H^1(M; H_2(M, k)) \). For any \( \sigma \in \phi \), \( \psi(\sigma) \in H^1(M; H_2(M, k)) \). Let \( \psi_\sigma \) represent this class; \( \psi_\sigma \in \text{Hom}_M[\mathbb{Z}[M]; D_1, H_2(M, k)] \). Now, if \( u^2(\sigma, \tau) \) represents a coboundary then

4) \[ u^2(\sigma, \tau)(x_1) = (\sigma \psi_\tau - \psi_\sigma \tau + \psi_\sigma)(x_1) \] for any \( x_1 \in D_1 \).

\[ H_2(M, k) = \langle g_{02} \rangle + \langle g_{20} \rangle + \langle g_{11} \rangle \] (where \( \langle g_{ij} \rangle \equiv k = \mathbb{Z}_3 \)) so for any \( \sigma \in \phi \), there exist \( n_{ij}, m_i \in \mathbb{Z}_3 \) such that
3) \[ \psi_\sigma(1_0 \otimes 1_1) = n_1^2 g_{02} + n_1^2 g_{20} + n_0^2 g_{11} \]
\[ \psi_\sigma(1_1 \otimes 1_0) = m_1^2 g_{02} + m_1^2 g_{20} + m_0^2 g_{11} \]

and since \( H^2(M, k) \) is a trivial \( M \)-module, we have

\[ \psi_\sigma(s_0^1 \otimes t_1^1) = \psi_\sigma(1_0 \otimes 1_1) \]
\[ \psi_\sigma(s_1^1 \otimes t_0^1) = \psi_\sigma(1_1 \otimes 1_0) \quad \text{for} \, \sigma \in \Sigma. \]

Now \( v^2 = 0 \) because \( \psi \) can be defined by setting (in equations (5)):

\[ m_1^i = n_1^i = 0 \quad \text{for} \, i = 0, 1, 2 \]
\[ m_0^1 = l, \, n_0^1 = l \]
\[ m_0^1 = 2, \, n_0^2 = 1 \]
\[ m_0^2 = 0, \, n_0^2 = 0 \]
\[ m_0^2 = 1, \, n_0^2 = 1. \]

We have \( u^3(\sigma, \tau) \in \text{Hom}_{\mathbb{Z}}[M] (D_2, H_3(M, k)) \) which represents \( w^3(\sigma, \tau) \in H^2(M, H_3(M, k)) \). The cocycle \( w^3 : \hat{\tau} \times \hat{\tau} \rightarrow H^2(M, H_3(M, k)) \)
represents the characteristic class \( v^3 \in \text{H}^2(\hat{\tau}; H^2(M, H_3(M, k))) \).
The class \( v^3 = 0 \) if there exists a map \( \psi : \hat{\tau} \rightarrow H^2(M, H_3(M, k)) \)
whose boundary is \( v^3 \), i.e. \( w^3(\sigma, \tau) = \sigma \psi(\tau) - \psi(\sigma \tau) + \psi(\sigma) \).
Denote by \( \psi_\sigma : D_2 \rightarrow H_3(M, k) \) a representative cocycle of \( \psi(\sigma) \).
Now $H_3(M,k) = \langle g_{03} \rangle + \langle g_{30} \rangle + \langle g_{12} \rangle + \langle g_{21} \rangle$ where \( \langle g_{14} \rangle \cong \mathbb{Z}_3 \). We will define $\psi_\tau$ on generators of $D_2$ by determining elements of $\mathbb{Z}_3$: $\ell^1_\tau$, $m^1_\tau$, $n^1_\tau$ for any $\tau \in \hat{S}$, $i = 1,2,3,4$ such that

$$
\psi_\tau(l_0 \oplus l_2) = \ell^1_\tau g_{03} + \ell^2_\tau g_{30} + \ell^3_\tau g_{12} + \ell^4_\tau g_{21}
$$

$$
\psi_\tau(l_1 \oplus l_1) = m^1_\tau g_{03} + m^2_\tau g_{30} + m^3_\tau g_{12} + m^4_\tau g_{21}
$$

$$
\psi_\tau(l_2 \oplus l_0) = n^1_\tau g_{03} + n^2_\tau g_{30} + n^3_\tau g_{12} + n^4_\tau g_{21}
$$

and satisfying

6) $u^3(\tau, \rho)(x_2) = (\tau \psi_\rho - \psi_\tau \rho + \psi_\tau)(x_2)$

for any $\tau, \rho \in \hat{S}$, $x_2 \in D_2$. In this case the map $\psi$ can be defined by taking:

7) $n^1_\tau = \ell^1_\tau = 0$ for all $\tau \in \hat{S}$ and for all $i$

$$
\begin{align*}
1 & = \ell^1_\sigma = 2 \\
0 & = \ell^2_\sigma = 2 \\
1 & = \ell^3_\sigma = 2 \\
0 & = \ell^4_\sigma = 2
\end{align*}
$$

$$
\begin{align*}
1 & = m^1_\sigma = 2 \\
0 & = m^2_\sigma = 2 \\
1 & = m^3_\sigma = 2 \\
0 & = m^4_\sigma = 2
\end{align*}
$$

where $\sigma$ generates $\hat{S}$

$$
\begin{align*}
0 & = m^1_1 \\
0 & = m^1_i \\
0 & = m^1_i
\end{align*}
$$

for all $i$.

The $n^1_\tau$, $m^1_\tau$, $\ell^i_\tau$ are determined by solving a system of equations
determined by 6). For instance, if we write out 6) for \( x_2 = (1_1 \otimes 1_1) \) and \( \rho = \tau = \sigma \), the generator of \( \Phi \) we obtain:

8) \( u^3(\sigma, \sigma)(1_1 \otimes 1_1) = (\sigma \psi_\sigma - \psi^2 \sigma)(1_1 \otimes 1_1) \).

The left hand side is equal to \( g_{30} \) by (2) and the right hand side can be expanded to obtain:

\[
g_{30} = (\sigma \psi_\sigma - \psi^2 \sigma)(1_1 \otimes 1_1) \\
= \sigma \psi_\sigma(1_1 \otimes 1_1) - \psi^2 \sigma(1_1 \otimes 1_1) + \psi_\sigma(1_1 \otimes 1_1)
\]

\[
g_{30} = \sigma_*(\psi(\sigma_2^2(1_1 \otimes 1_1))) - \psi^2 \sigma(1_1 \otimes 1_1) + \psi_\sigma(1_1 \otimes 1_1)
\]

\[
g_{30} = \sigma_*(\psi(1_1 \otimes 1_1)) - \psi^2 \sigma(1_1 \otimes 1_1) + \psi_\sigma(1_1 \otimes 1_1)
\]

\[
g_{30} = \sigma_*(m_{\sigma}^1 g_{03}^1 + m_{\sigma}^2 g_{30}^1 + m_{\sigma}^3 g_{12}^3 + m_{\sigma}^4 g_{21}^4)
\]

\[
- (m_{\sigma}^1 g_{03}^1 + m_{\sigma}^2 g_{30}^1 + m_{\sigma}^3 g_{11}^3 + m_{\sigma}^4 g_{21}^4)
\]

\[
+ (m_{\sigma}^1 g_{03}^1 + m_{\sigma}^2 g_{30}^1 + m_{\sigma}^3 g_{12}^3 + m_{\sigma}^4 g_{21}^4)
\]

\[
g_{30} = m_{\sigma}^1 (g_{03}^1 + g_{12}^3 + g_{21}^4 + g_{30}^1) + m_{\sigma}^2 g_{30}^1 + m_{\sigma}^3 (g_{12}^3 + g_{30}^1)
\]

\[
+ m_{\sigma}^4 (g_{21}^4 + g_{30}^1) - (m_{\sigma}^1 g_{03}^1 + m_{\sigma}^2 g_{30}^1 + m_{\sigma}^3 g_{11}^3 + m_{\sigma}^4 g_{21}^4)
\]

\[
+ m_{\sigma}^1 g_{03}^1 + m_{\sigma}^2 g_{30}^1 + m_{\sigma}^3 g_{11}^3 + m_{\sigma}^4 g_{21}^4
\]

Equating coefficients of \( g_{ij} \) on both sides yields:

\[
m_{\sigma}^1 - m_{\sigma}^2 + m_{\sigma}^3 = 0
\]

\[
m_{\sigma}^1 + 2m_{\sigma}^2 + m_{\sigma}^3 + m_{\sigma}^4 - m_{\sigma}^2 = 1
\]
\[
\begin{align*}
7^1_m + 2m_3^1 - m_2^3 &= 0 \\
7^1_m + 2m_4^1 - m_2^4 &= 0.
\end{align*}
\]

Writing out 6) for all \(T, P \in \mathcal{D}_2\) and other \(x_2 \in \mathcal{D}_2\) yields a system of equations in \(t^1_T, m^1_T, n^1_T\), for which 7) is a solution.

In this case \(v^2\) is known to be zero since the group \(H^2(GL(2,3); H^2(M_2(H_2(M,k)))\) containing the universal class \(v^2\) is zero, (see Appendix). We have shown that \(v^3 = 0\) for \(\mathfrak{p}\) the p-Sylow subgroup above. Since \(\overline{v}^3\) must be detected on a Sylow subgroup \(v^3 = 0\) implies \(\overline{v}^3 = 0\). In other words no non-zero element of \(H^2(SL(2,3); H^2(M_2(H_2(M,k)))\) arises from a characteristic class.

The procedure of Example I could be used to explicitly compute \(v^2, v^3\) for \(p > 3, \mathfrak{p} \leq GL(2,p)\) but calculations would be more difficult. It is known, (see Appendix) that \(\overline{v}^2(M) \neq 0\) for \(p \geq 5\).

**Example 2: \((GL(3,2))\)**

Let \(M = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2\) and let \(\mathfrak{p}\) be the subgroup of \(GL(3,2)\) generated by

\[
\sigma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.
\]
Then $\sigma^2 = 1$ so $\hat{\phi} = \mathbb{Z}_2$. Define action of $\hat{\phi}$ on $M$ by

$$\sigma(r^s, s^n, t^m) = (r^{s+m}, s^{n+m}, t^m) \quad r, s, t \text{ generators of } \mathbb{Z}_2.$$ 

Now suppose $\overline{X} : \cdots \to X_n \xrightarrow{\partial_n} X_{n-1} \to \cdots \to X_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$ is the usual resolution of $\mathbb{Z}$ for $\mathbb{Z}_2$, i.e. $X_1 = \mathbb{Z}[\mathbb{Z}_2]$ with $\partial_1 = \text{multiplication by } (1+t)$ for $i$ even and $\partial_1 = \text{multiplication by } (1-t)$ for $i$ odd. Now let $\overline{D}$ be defined as the three-fold tensor product of the resolution $\overline{X}$:

9) \[ \overline{D} = \overline{X} \otimes \overline{X} \otimes \overline{X} \]

$$D_n = \bigoplus_{p+q_1+q_2=n} X_p \otimes X_{q_1} \otimes X_{q_2}$$

$$d_n = \bigoplus \partial_p \otimes \text{id}_{q_1} \otimes (-1)^p \text{id}_p \otimes [\partial_{q_1} \otimes \text{id}_{q_2} + (-1)^q \text{id}_{q_1} \otimes \partial_{q_2}]$$

where $q = q_1 + q_2$ and $\text{id}_j$ denotes the identity map on $X_j$.

We can now construct a $\hat{\phi}$-system for this resolution. Here $r_i^p \otimes s_j^q \otimes t_k^q$ will denote an element of $D_i$, $i = p+q_1+q_2$, corresponding to the summand $X_p \otimes X_{q_1} \otimes X_{q_2}$ and extended to be $M$-linear.

**Lemma 2.1:** The following $\mathbb{Z}$-homomorphisms $A_n(\tau) : D_n \to D_n$ satisfy the conditions of Definition 1.

$$A_i(1) = \text{identify map for all } i$$

$$A_0(\sigma)(r_0^i \otimes s_0^j \otimes t_0^k) = r_0^{i+k} \otimes s_0^{j+k} \otimes t_0^{k}$$
\[\begin{align*}
A_1(\sigma)(r_1^i \otimes s_0^j \otimes t_0^k) &= r_1^{i+k} \otimes s_0^{j+k} \otimes t_0^k \\
A_1(\sigma)(r_0^i \otimes s_1^j \otimes t_0^k) &= r_0^{i+k} \otimes s_1^{j+k} \otimes t_0^k \\
A_1(\sigma)(r_0^i \otimes s_0^j \otimes t_1^k) &= r_1^{i+k} \otimes s_0^{j+k} \otimes t_1^k + r_0^{i+k} \otimes s_0^{j+k+l} \otimes t_0^{k+l} \\
&+ r_0^{i+k} \otimes s_1^{j+k} \otimes t_0^k \\
A_2(\sigma)(r_2^i \otimes s_0^j \otimes t_0^k) &= r_2^{i+k} \otimes s_0^{j+k} \otimes t_0^k \\
A_2(\sigma)(r_0^i \otimes s_2^j \otimes t_0^k) &= r_0^{i+k} \otimes s_2^{j+k} \otimes t_0^k \\
A_2(\sigma)(r_1^i \otimes s_1^j \otimes t_0^k) &= r_1^{i+k} \otimes s_1^{j+k} \otimes t_0^k \\
A_2(\sigma)(r_0^i \otimes s_1^j \otimes t_1^k) &= r_1^{i+k} \otimes s_1^{j+k} \otimes t_1^k + r_1^{i+k} \otimes s_1^{j+k+l} \otimes t_0^{k+l} \\
&+ r_2^{i+k} \otimes s_0^{j+k+l} \otimes t_0^k - r_2^{i+k} \otimes s_0^{j+k} \otimes t_0^k \\
A_2(\sigma)(r_1^i \otimes s_0^j \otimes t_1^k) &= r_1^{i+k} \otimes s_0^{j+k} \otimes t_1^k + r_1^{i+k} \otimes s_0^{j+k+l} \otimes t_0^{k+l} \\
A_2(\sigma)(r_0^i \otimes s_2^j \otimes t_2^k) &= r_0^{i+k} \otimes s_2^{j+k} \otimes t_2^k + r_1^{i+k} \otimes s_1^{j+k+l} \otimes t_1^k \\
&+ r_1^{i+k} \otimes s_1^{j+k+l} \otimes t_0^{k+l} + r_1^{i+k} \otimes s_1^{j+k} \otimes t_0^k \\
A_3(\sigma)(r_3^i \otimes s_0^j \otimes t_0^k) &= r_3^{i+k} \otimes s_0^{j+k} \otimes t_0^k \\
A_3(\sigma)(r_0^i \otimes s_3^j \otimes t_0^k) &= r_0^{i+k} \otimes s_3^{j+k} \otimes t_0^k \\
A_3(\sigma)(r_2^i \otimes s_1^j \otimes t_0^k) &= r_2^{i+k} \otimes s_1^{j+k} \otimes t_0^k \\
A_3(\sigma)(r_1^i \otimes s_2^j \otimes t_0^k) &= r_1^{i+k} \otimes s_2^{j+k} \otimes t_0^k \\
A_3(\sigma)(r_0^i \otimes s_1^j \otimes t_1^k) &= r_2^{i+k} \otimes s_1^{j+k} \otimes t_1^k + r_3^{i+k} \otimes s_0^{j+k} \otimes t_0^k \\
&+ r_2^{i+k} \otimes s_1^{j+k} \otimes t_0^k + r_0^{i+k} \otimes s_2^{j+k} \otimes t_0^k
\[ A_3(\sigma)(r_1^i \otimes s_1^j \otimes t_1^k) = r_1^i s_1^j s_1^1 t_1^1 + r_2^i s_1^j t_1^1 t_0^1 + r_3^i s_1^j t_0^1 t_0^1 + r_1^k s_1^j t_1^1 t_0^1 + r_2^k s_1^j t_0^1 t_0^1 + r_3^k s_0^j t_0^1 t_0^1 \]
With the above definitions, \( A_n(1) - A_n(\sigma) \cdot A_n(\sigma) \neq 0 \) in general and a homotopy \( U_n(\rho, \tau) : D_n \to D_{n+1} \) may be defined as follows:

**Lemma 2.2:** The following maps \( U_1 \) satisfy the conditions of Definition 1.

\[
U_1(\sigma, l) = U_1(l, \sigma) = 0 \quad \text{for all } l
\]

\[
U_0(\sigma, \sigma) = 0
\]

\[
U_1(\sigma, \sigma)(r_1^i \otimes s_1^j \otimes t_0) = 0
\]

\[
U_1(\sigma, \sigma)(r_0^i \otimes s_1^j \otimes t_0) = 0
\]

\[
U_1(\sigma, \sigma)(r_0^i \otimes s_0^j \otimes t_1) = -r_1^i \otimes s_1^j \otimes t_0 + r_0^i + k \otimes s_2^j \otimes t_0 - r_2^i \otimes s_0^j \otimes t_0
\]

\[
U_2(\sigma, \sigma)(r_0^i \otimes s_1^j \otimes t_1) = -r_2^i \otimes s_1^j \otimes t_1 + r_0^i \otimes s_3^j \otimes t_0
\]

\[
U_2(\sigma, \sigma)(r_2^i \otimes s_0^j \otimes t_0) = 0
\]

\[
U_2(\sigma, \sigma)(r_0^i \otimes s_2^j \otimes t_0) = 0
\]

\[
U_2(\sigma, \sigma)(r_1^i \otimes s_1^j \otimes t_0) = 0
\]

\[
U_2(\sigma, \sigma)(r_1^i \otimes s_0^j \otimes t_0) = 0
\]

\[
U_2(\sigma, \sigma)(r_1^i \otimes s_0^j \otimes t_1) = r_1^i + 1 \otimes s_2^j \otimes t_0 + r_3^i \otimes s_0^j \otimes t_0
\]

\[
U_2(\sigma, \sigma)(r_0^i \otimes s_1^j \otimes t_2) = -r_2^i \otimes s_1^j \otimes t_0 + r_1^i \otimes s_2^j \otimes t_0 - r_2^i \otimes s_1^j \otimes t_0
\]

\[
- r_2^i \otimes s_1^j \otimes t_0 + r_1^i \otimes s_1^j \otimes t_1
\]

(continued)
\[- r_2^1 \otimes s_0^j \otimes t_1^{k+1} - r_0^1 \otimes s_2^j \otimes t_1^{k+1} \]
\[+ r_0^{i+1} \otimes s_3^j \otimes t_0^k + r_3^{i+1} \otimes s_0^j \otimes t_0^k \]
\[+ r_3^{i+1} \otimes s_0^j \otimes t_0^{k+1} \]

The proofs of Lemmas 2.1 and 2.2 are omitted since the computations involved are analogous to those appearing in the proofs of Lemmas 1.1 and 1.2.

We now let the coefficient group \( k = \mathbb{Z}_2 \). Define \( f^n(r_a \otimes s_b \otimes t_c) = g_{abc} \) where \( r_a \otimes s_b \otimes t_c \) and \( s_a \otimes X_b \otimes X_c \) are the corresponding generator of \( \mathit{H}_n(M; k) = \bigoplus \mathbb{Z}_2 \langle g_{abc} \rangle \) and each \( \mathbb{Z}_2 \) is isomorphic to six copies of \( \mathbb{Z}_2 \) and written:

\[ \mathit{H}_2(M, k) = \langle g_{002} \rangle + \langle g_{020} \rangle + \langle g_{200} \rangle \]
\[+ \langle g_{011} \rangle + \langle g_{101} \rangle + \langle g_{110} \rangle. \]

Now by the same argument as that in Example 1 we may take \( f^n = 0 \) and \( u^n(\sigma, \tau) \in \hom_{\mathbb{Z}}[M](D_{n-1}, \mathit{H}_n(M, k)) \) are defined by the following lemma. The proof is omitted since the procedure imitates that used in proof of Lemma 1.3.

**Lemma 2.3:** The cocycles \( u^n(\sigma, \tau) \in \hom_{\mathbb{Z}}[M](D_{n-1}, \mathit{H}_n(M, k)) \) for \( n = 1, 2, 3 \) can be defined as follows:

\[ u^n(\sigma, \tau) = (\sigma \tau)_*[f^n U_{n-1}(\tau^{-1}, \sigma^{-1})] \]
\[ u^n(\sigma, 1) = u^n(1, \sigma) = 0 \quad \forall n \]

\[ u^1(\sigma, \sigma) = 0 \]

\[ u^2(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = 0 \]

\[ u^2(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = 0 \]

\[ u^2(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = -g_{110} - g_{020} - g_{200} \]

\[ u^3(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = -g_{210} + g_{030} \]

\[ u^3(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = g_{120} + g_{300} \]

\[ u^3(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = -g_{210} - g_{120} - g_{210} + g_{111} \]

\[ - g_{201} - g_{201} - g_{021} + g_{030} + g_{300} \]

\[ u^3(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = 0 \]

\[ u^3(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = 0 \]

\[ u^3(\sigma, \sigma)(r^1_0 \otimes s^1_0 \otimes t^k_0) = 0 \]

**Proposition 2:** For \( \phi \in \text{GL}(3, 2) \) generated by \( \sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \),

and \( M = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \), \( v^2(M) = 0 \).

**Proof:** We have \( u^2(\rho, \tau) \in \text{Hom}_{\mathbb{Z}}[M](D_1, H_2(M, k)) \) defining \( w^2(\rho, \tau) \in H^1(M; H_2(M, k)) \) and \( w^2 : \phi \times \phi \to H^1(M; H_2(M, k)) \).
is a cocycle which represents $v^2 \in H^2(\tilde{\sigma}; H^1(M; H_2(M, k)))$. The class $v^2$ is a coboundary if there exists a map

$$\psi : \tilde{\sigma} \to H^1(M; H_2(M, k))$$

such that

$$v^2(\rho, \tau) = \rho \psi(\tau) - \psi(\rho \tau) + \psi(\rho).$$

Denote by $\psi_\rho$ an element of $\text{Hom}_{\mathbb{Z}[M]}(D_1, H_2(M, k))$ which represents $\psi(\rho) \in H^1(M; H_2(M, k))$. If $w^2(\sigma, \tau)$ represents a coboundary then

11) $u^2(\rho, \tau)(x_1) = (\rho \psi_\tau - \psi_{\rho \tau} + \psi_\rho)(x_1)$ for all $x_1 \in D_1$ and all $\rho, \tau \in \tilde{\sigma}$.

To determine if such a $\psi$ exists we follow an analogous procedure as in Example 1. We determine how $\psi_\tau$ must be defined on $l_p \otimes l_q_1 \otimes l_q_2 \in D_1$ and use the fact that $H_2(M, k)$ is a trivial $M$-module so

$$\psi_\tau(l_p \otimes l_q_1 \otimes l_q_2) = \psi_\tau(r_p \otimes s_{q_1} \otimes t_{q_2}).$$

If such a $\psi$ exists there are elements $t^1$, $m^1$, $n^1$ in $\mathbb{Z}_2$ such that:

$$\psi(1_0 \otimes 1_0 \otimes 1_0) = t^1_1 g_{002} + t^2_1 g_{020} + t^3_1 g_{200} + n^1_1 g_{001} + m^1_1 g_{010} + t^5_1 g_{101} + t^6_1 g_{110}$$

$$\psi(1_0 \otimes 1_1 \otimes 1_0) = m^1_1 g_{002} + m^2_1 g_{020} + m^3_1 g_{200} + n^1_1 g_{001} + m^1_1 g_{010} + t^5_1 g_{101} + t^6_1 g_{110}$$
\[
\psi_T(l^i_1 \otimes 1^0_0 \otimes 1^0_0) = n^1_T g^{002}_T + n^2_T g^{020}_T + n^3_T g^{200}_T + n^4_T g^{011}_T + n^5_T g^{101}_T + n^6_T g^{110}_T.
\]

Writing out equation (11) for various \(x_1 \in D_1\) and \(\rho, \tau \in \Psi\) and substituting \(u^2\) values from (10) yields a set of equations in \(l^i_T, m^i_T, n^i_T\). The action of \(\Phi\) on \(H_2(M, k)\) is used in computing \(\rho \psi_T(x_1)\) on the right hand side of (11):

\[
\rho \psi_T(x_1) = \rho_*(\psi_T(A_1(\rho^{-1})(x_1))).
\]

The action of \(\Phi\) on \(H_2(M, k)\) is given by:

\[
l_* = 1d
\]

\[
\begin{bmatrix}
\sigma_*(g^{002}_T) \\
\sigma(g^{020}_T) \\
\sigma_*(g^{200}_T) \\
\sigma_*(g^{011}_T) \\
\sigma_*(g^{101}_T) \\
\sigma_*(g^{110}_T)
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
g^{002}_T \\
g^{020}_T \\
g^{200}_T \\
g^{011}_T \\
g^{101}_T \\
g^{110}_T
\end{bmatrix}
\]

The system of equations in \(l^i_T, m^i_T, n^i_T\) has a solution in this case:

\[
m^i_1 = n^i_1 = \psi^i_1 = 0 \quad \text{for all } i
\]
\[ m_1^1 = 0 \quad 1 \neq 2,3 \]
\[ m_1^2 = 1 \]
\[ n_1^1 = 0 \quad 1 \neq 5 \]
\[ n_1^5 = 1 \]
\[ u_1^1 = 0 \quad \text{for all } 1. \]

Therefore \( v^2 = 0 \) since this explicitly exhibits an \( M \)-homomorphism \( \psi \) whose coboundary is \( v^2 \).

We now compute \( v^3 \in H^2(\mathbb{Z}, H^2(M; H_3(M, k))) \). Again \( v^3 \) is represented by \( w^3 : \tilde{\mathbb{Z}} \times \tilde{\mathbb{Z}} \to H^2(M; H_3(M, k)) \) where \( w^3(\sigma, \tau) \in H^2(M; H_3(M, k)) \) is represented by
\[ u^3(\sigma, \tau) \in \text{Hom}_{\mathbb{Z}}[\tilde{M}] (D_2, H_3(M, k)) \] and \( u^3(\sigma, \tau) \) are tabulated in \( 10 \). We must determine whether \( v^3 = 0 \). We follow a procedure similar to the previous cases but write out a few more details.

\( H_3(M, k) \) is a direct sum of ten copies of \( \mathbb{Z}_2 \) written with generators:

\[ H_3(M, k) = \langle g_{003} \rangle + \langle g_{030} \rangle + \langle g_{300} \rangle \]
\[ + \langle g_{012} \rangle + \langle g_{102} \rangle + \langle g_{201} \rangle \]
\[ + \langle g_{111} \rangle + \langle g_{021} \rangle + \langle g_{120} \rangle \]
\[ + \langle g_{210} \rangle. \]
The action of $\sigma$ on $H_3(M, k)$ is given by:

$$
\sigma_*(g_{003}) = g_{003} + g_{030} + g_{300} + g_{012} + g_{102} \\
+ g_{201} + g_{111} + g_{021} + g_{120} + g_{210}
$$

$$
\sigma_*(g_{030}) = g_{030}
$$

$$
\sigma_*(g_{300}) = g_{300}
$$

$$
\sigma_*(g_{012}) = g_{012} + g_{111} - g_{210} + g_{030}
$$

$$
\sigma_*(g_{102}) = g_{102} - g_{111} + g_{120} + g_{300}
$$

$$
\sigma_*(g_{201}) = g_{201} + g_{210} + g_{300}
$$

$$
\sigma_*(g_{111}) = g_{111}
$$

$$
\sigma_*(g_{021}) = g_{021} + g_{030} + g_{120}
$$

$$
\sigma_*(g_{120}) = g_{120}
$$

$$
\sigma_*(g_{210}) = g_{210}
$$

We recall how this action was determined: an element of $H_3(M, k)$ is represented by an element $x \otimes \gamma \in D_3 \otimes \mathbb{Z} [M]^k$ and $\sigma \in \Phi$ acts on $x \otimes \gamma$ diagonally,

$$
\sigma_*(x \otimes \gamma) = (A_3(\sigma)(x)) \otimes \gamma
$$

Now $v^3 = 0$ if there exists a homomorphism

$$
\psi : \Phi \rightarrow H^2(M; H_3(M, k))
$$

such that
12) \( u^3(\rho, \tau)(x_2) = [\rho \psi(\tau) - \psi(\rho \tau) + \psi(\rho)](x_2) \)

\( \forall x_2 \in D_2; \forall \rho, \tau \in \Phi. \)

Represent \( \psi(\tau) \in H^2(M; H^3(M, k)) \) by \( \psi : D_2 \to H^3(M, k) \) so \( \psi \)
is an \( M \)-homomorphism, \( H^3(M, k) \) is a trivial \( M \)-module. Here again \( \rho \psi \)
is computed by

\[ \rho \psi(x) = \rho_*(\psi(A_2(\rho^{-1}))(x)). \]

Again for all \( \tau \in \Phi \), \( \psi(x) \in H^2(M, k) \) so there exist elements
\( a^i_\tau, b^i_\tau, c^i_\tau, d^i_\tau, e^i_\tau, f^i_\tau \) in \( Z_2 \) such that:

\( \psi_\tau(1_0 \otimes 1_0 \otimes 1_2) = a^1_\tau(g_{003}) + a^2_\tau(g_{030}) + a^3_\tau(g_{300}) + a^4_\tau(g_{012}) + a^5_\tau(g_{102}) + a^6_\tau(g_{201}) + a^7_\tau(g_{111}) + a^8_\tau(g_{021}) + a^9_\tau(g_{120}) + a^{10}_\tau(g_{210}) \)

\( \psi_\tau(1_0 \otimes 1_2 \otimes 1_0) = b^1_\tau(g_{003}) + b^2_\tau(g_{030}) + b^3_\tau(g_{300}) + b^4_\tau(g_{012}) + b^5_\tau(g_{102}) + b^6_\tau(g_{201}) + b^7_\tau(g_{111}) + b^8_\tau(g_{021}) + b^9_\tau(g_{120}) + b^{10}_\tau(g_{210}) \)

\( \psi_\tau(1_2 \otimes 1_0 \otimes 1_0) = c^1_\tau g_{003} + c^2_\tau g_{030} + c^3_\tau g_{300} + c^4_\tau g_{012} + c^5_\tau g_{102} + c^6_\tau g_{201} + c^7_\tau g_{111} + c^8_\tau g_{021} + c^9_\tau g_{120} + c^{10}_\tau g_{210} \)
\[ \psi_\tau(l_0 \otimes l_1 \otimes l_1) = d^1_\tau g_{003} + d^2_\tau g_{030} + d^3_\tau g_{300} + d^4_\tau g_{012} \\
+ d^5_\tau g_{102} + d^6_\tau g_{201} + d^7_\tau g_{111} + d^8_\tau g_{021} \\
+ d^9_\tau g_{120} + d^{10}_\tau g_{210} \]

\[ \psi_\tau(l_1 \otimes l_0 \otimes l_1) = e^1_\tau g_{003} + e^2_\tau g_{030} + e^3_\tau g_{300} + e^4_\tau g_{012} \\
+ e^5_\tau g_{102} + e^6_\tau g_{201} + e^7_\tau g_{111} + e^8_\tau g_{021} \\
+ e^9_\tau g_{120} + e^{10}_\tau g_{210} \]

\[ \psi_\tau(l_1 \otimes l_1 \otimes l_0) = f^1_\tau g_{003} + f^2_\tau g_{030} + f^3_\tau g_{300} \\
+ f^4_\tau g_{012} + f^5_\tau g_{102} + f^6_\tau g_{201} + f^7_\tau g_{111} \\
+ f^8_\tau g_{021} + f^9_\tau g_{120} + f^{10}_\tau g_{210} \]

Again \( \psi_\tau(r^i_p \otimes q^j_2 \otimes t^k) = \psi_\tau(l_1 \otimes l_1 \otimes l_1) \) since \( M \) acts trivially on \( H_3(M;k) \).

We write 12) using generators:

13) \( u^3(\rho,\tau)(x_2) = (\rho \psi_\tau - \psi_\rho \tau + \psi_\rho)(x_2) \) for \( x_2 \in D_2 \).

Proposition 3: For \( \psi \) the given subset of \( GL(3,2) \) and using resolution 9) we obtain a non-zero characteristic class \( v^3(M) \in H^2(\psi;H^2(M;H_3(M;k))) \).

Proof: We write out 13) for various \( \rho, \tau \in \psi \) and \( x_2 \in D_2 \) (note that \( \psi_\rho = 0 \) since \( \psi \) is a homomorphism) to get a system of equations in \( a^i_\tau, b^i_\tau, c^i_\tau, d^i_\tau, e^i_\tau, f^i_\tau, \tau \in \psi \). Since in this
example $\mathbb{Z}_2$ generated by $\sigma$, we drop the subscripts for simplicity, i.e. $d^1 = d^1_\sigma$. The values for $u^3(\rho, \tau)$ are substituted into the left hand side of 13) using 10). Recall that

$$p_\tau(x) = p_\tau(A_n(\rho^{-1})(x))$$

where $\sigma_*$ denotes the action of $\mathbb{Z}_2$ on $H_*(M,k)$.

14) $0 = u^3(\mathbb{Z}, \sigma)(1_0 \otimes 1_2 \otimes 1_0) = (\mathbb{Z}_2 \otimes 1_1 \otimes \mathbb{Z}_2)(1_0 \otimes 1_2 \otimes 1_0)$

\[= \sigma_*(\mathbb{Z}_2(\sigma^{-1})(1_0 \otimes 1_2 \otimes 1_0)) + \mathbb{Z}_2(1_0 \otimes 1_2 \otimes 1_0)\]

\[= b^1(g_{003} + g_{030} + g_{300} + g_{012} + g_{102} + g_{201} + g_{111} + g_{021} + g_{120} + g_{210})\]

\[+ b^2(g_{030}) + b^3(g_{300})\]

\[+ b^4(g_{012} + g_{111} - g_{210} + g_{030})\]

\[+ b^5(g_{102} - g_{111} + g_{120} + g_{300})\]

\[+ b^6(g_{201} + g_{210} + g_{300}) + b^7(g_{111})\]

\[+ b^8(g_{021} + g_{030} + g_{120}) + b^9(g_{120})\]

\[+ b^{10}(g_{210}) + b^1(g_{003}) + b^2(g_{030})\]

\[+ b^3(g_{300}) + b^4(g_{012}) + b^5(g_{102}) + b^6(g_{201})\]

\[+ b^7(g_{111}) + b^8(g_{021}) + b^9(g_{120}) + b^{10}(g_{210}).\]
Equating coefficients of \( g_{102} \) on left and right hand sides of (14) yields:

\[
0 = 2b^5 + b^1
\]

or \( b^1 = 0 \) since all calculations are mod 2. Equating coefficients of \( g_{300} \) in (14) yields:

\[
2b^3 + b^1 + b^5 + b^6 = 0
\]

which implies

15)

\[
b^5 + b^6 = 0
\]

Writing out equation 13) for \( u^3(\sigma,\sigma)(l_2 \otimes l_0 \otimes l_0) \) results in the same equation as (14) except "\( b \)" is replaced by "\( f \)". And writing out equation 13) for \( u^3(\sigma,\sigma)(l_1 \otimes l_0 \otimes l_0) \) yields the same equation as (14) except "\( b \)" is replaced by "\( c \)".

This is true since \( A_2 \) is defined on all of \( l_0 \otimes l_2 \otimes l_0, \)
\( l_2 \otimes l_0 \otimes l_0, \) and \( l_1 \otimes l_1 \otimes l_0 \) in the same way and \( u^3 = 0 \) in each case. Therefore, equating coefficients for \( g_{102} \) (and \( g_{300} \)) as above yields:

16) \( f^1 = 0 \)

\[
f^5 + f^6 = 0
\]

17) and \( c^1 = 0 \)

\[
c^5 + c^6 = 0.
\]
Now

\[(18) \quad g_{210} + g_{030} = u^3(\sigma, \sigma)(l_0 \otimes l_1 \otimes l_1) = (\psi_\sigma - \psi_1 + \psi_\sigma)(l_0 \otimes l_1 \otimes l_1)\]

\[= \sigma_\ast \psi_\sigma(A_2(\sigma)(l_0 \otimes l_1 \otimes l_1)) + \psi_\sigma(l_0 \otimes l_1 \otimes l_1)\]

\[= \sigma_\ast \psi_\sigma(l_0 \otimes l_1 \otimes l_1 + r_1 \otimes s_1 \otimes t_0)\]

\[+ r_2 \otimes s_2 \otimes t_0 - l_2 \otimes l_0 \otimes t_0) + \psi_\sigma(l_0 \otimes l_1 \otimes l_1)\]

\[= \sigma_\ast(\psi_\sigma(l_0 \otimes l_1 \otimes l_1 + r_1 \otimes s_1 \otimes t_0))\]

\[+ \psi_\sigma(l_0 \otimes l_1 \otimes l_1)\]

\[= \sigma_\ast(d^1 g_{003} + d^2 g_{030} + d^3 g_{030} + d^4 g_{012} + d^5 g_{102}\]

\[+ d^6 g_{201} + d^7 g_{111} + d^8 g_{021} + d^9 g_{120}\]

\[+ d^{10} g_{210} + f^1 g_{003} + f^2 g_{030} + f^3 g_{300}\]

\[+ f^4 g_{012} + f^5 g_{102} + f^6 g_{201} + f^7 g_{111}\]

\[+ f^8 g_{021} + f^9 g_{120} + f^{10} g_{210}) \quad \psi_\sigma(l_0 \otimes l_1 \otimes l_1)\]

\[= (2d^1 + f^1) g_{003}\]

\[+ (2d^2 + f^2 + d^1 + f^1 + d^4 + f^4 + d^8 + f^8) g_{030}\]

\[+ (2d^3 + f^3 + d^1 + f^1 + d^5 + f^5 + d^6 + f^6) g_{300}\]

\[+ (2d^4 + f^4 + d^1 + f^1) g_{012}\]

\[+ (2d^5 + f^5 + d^1 + f^1) g_{102}\]
\[ + (2d^6 + f^6 + d^1 + f^1)g_{201} \]
\[ + (2d^7 + f^7 + d^1 + f^1 + d^4 + f^4 - d^5 - f^5)g_{111} \]
\[ + (2d^8 + f^8 + d^1 + f^1)g_{021} \]
\[ + (2d^9 + f^9 + d^1 + d^5 + f^5 + d^8 + f^8)g_{120} \]
\[ + (2d^{10} + f^{10} + d^1 + f^1 - d^4 - f^4 + d^6 + f^6)g_{210} \]

Equating coefficients of \( g_{300} \) yields:

19) \( 2d^3 + f^3 + d^1 + f^1 + d^5 + f^5 + d^6 + f^6 = 0 \).

Writing out 13) for \( u^3(\sigma, \sigma)(l_1 \otimes l_0 \otimes l_1) \) yields the same right hand side as 18) but replacing "d" with "e" and the left hand side will equal \( g_{120} + g_{300} \). So equating coefficients of \( g_{300} \) in this equation yields:

20) \( 2e^3 + f^3 + e^1 + f^1 + e^5 + f^5 + e^6 + f^6 = 1 \).

Writing out 13) for \( u^3(\sigma, \sigma)(l_0 \otimes l_0 \otimes l_2) \) and equating coefficients of \( g_{102} \) and \( g_{201} \) yield:

\[ b^5 + c^5 + d^5 + e^5 + f^5 + a^1 + b^1 + c^1 + d^1 + e^1 + f^1 = 0 \]

and \( b^6 + c^6 + d^6 + e^6 + f^6 + a^1 + b^1 + c^1 + d^1 + e^1 + f^1 = 0 \)

which implies (adding)

\[ b^5 + b^6 + c^5 + c^6 + d^5 + d^6 + e^5 + e^6 + f^5 + f^6 = 0 \]
which with 15), 16) and 17) above yields

\[ 21) \quad d^5 + d^6 + e^5 + e^6 = 0. \]

Adding equations 19) and 20) yields

\[ d^1 + e^1 + d^5 + e^5 + d^6 + e^6 = 1 \]

and this with 21) implies

\[ 22) \quad d^1 + e^1 = 1. \]

Now equating coefficients of \( g_{012} \) in 18) yields

\[ 23) \quad f^4 + d^1 = 0 \]

and equating coefficients of \( g_{012} \) in 13) for \( u^3(v, v)(l_1 \otimes l_0 \otimes l_1) \) yields

\[ 24) \quad f^4 + e^1 = 0. \]

Adding 23) and 24) yields \( d^1 + e^1 = 0 \) which contradicts 22).

We conclude that \( v^3 \neq 0 \) since it is impossible to find \( a^1, b^1, c^1, d^1, e^1, f^1 \) which could define a map \( \psi \) such that

\[ u^3(\rho, \tau)(x_2) = [\rho \psi(\tau) - \psi(\rho \tau) + \psi(\rho)](x_2) \]

for all \( x_2 \in D_2 \); and for all \( \rho, \tau \in \mathbb{F} \).

Proposition 3 implies that the universal class \( \bar{v}^3(M) \in H^2(GL(3, \mathbb{C}) H^2(M; H_3(M, k))) \) is also non-zero by the
naturality theorem (Theorem II). But since \( \mathfrak{g} \) in this example was not chosen a Sylow subgroup, Proposition 2 does not imply that the universal class 
\[ v^2 \mathbb{H}^2(\text{GL}(3,2); \mathbb{H}^1(M, \mathbb{H}_2(M,k))) \] 
is zero. However, Proposition 2 and 3 prove that no analog to Theorem IV exists for finite \( M \). Actually, we have shown that no analog to Theorem IV exists where \( M \) is a sum of copies of \( \mathbb{Z}_2 \). It is still possible that such a theorem holds for \( M \) a sum of copies of \( \mathbb{Z}_p \) with \( p > 2 \) since \( p = 2 \) is often an anomalous case. However, only the case of \( p = 3 \) would matter since \( v^2(M) \) is known to be non-zero when \( M \) is a sum of copies of \( \mathbb{Z}_p \), \( p > 5 \) (see Appendix).

The same procedure as that in Example 2 above might be useful for computing \( v^2(M) \) and \( v^3(M) \) where \( M = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 \) since one could choose \( \mathfrak{g} \subseteq \text{GL}(4,2) \) to be the subgroup generated by 
\[ \sigma = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
so \( \sigma^2 = \text{id} \) and \( \mathfrak{g} \) would again be isomorphic to \( \mathbb{Z}_2 \) making construction of a \( \mathfrak{g} \)-system manageable. It is not clear whether this choice would detect non-zero universal classes. The class \( v^2(M) \) is not zero for \( \text{GL}(4,2) \) (see Appendix) but \( v^3(M) \) is not known.

The choice of resolution in calculating \( v^1(M) \) in these cases was made such that the \( F^1_0 \) in Definition 2 are zero.
(The resolution chosen for calculating $v^1(M)$ for free Abelian $M$ also had this property see [3].) The definition of the cocycle $u^R(\sigma, \tau)$ which leads to $v^R$ can be thought of as splitting into two parts: one containing the $F_\sigma^i$'s and the other containing $U_1$'s. If the resolution chosen is "small" the $F_\sigma^i$ are zero and only the $U_1$'s contribute to the characteristic class. If the resolution chosen for $M$ is "large" (i.e. the standard resolution) then the $U_1$ are zero ($\mathfrak{g}$ acts on $D_1$) and only the $F_\sigma^i$ contribute to the characteristic classes. Computation of $v^1(M)$ is easiest when the resolution is chosen in order to force $F_\sigma^1 = 0$. 
Appendix

The following is a compilation of some other known results concerning characteristic classes in the finite case, almost all of which are due to Sah [8, 9]. The methods, some of which are outlined below, do not entail the use of the chain homotopy arguments. Described are two approaches which can be used for gaining information regarding the characteristic classes \( v^1(M) \) (especially \( v^2 \)) where \( \mathfrak{g} = \text{GL}(n, p) \), and \( M \) is \( n \)-dimensional vector space over \( \mathbb{F}_p \). First, the groups \( H^2(\mathfrak{g}; H^{n-1}(M, H_n(M, k))) \) which contain the \( v^n(M) \) can sometimes be computed directly or at least vanishing or nonvanishing results obtained. Secondly, the class \( v^2(M) \) has an alternate interpretation in terms of group extensions and automorphisms. These extensions can be studied and the \( v^2(M) \) can be realized as obstructions to the lifting of group actions. The following contains outlines of some of these methods and lists consequential results.

Some computations of the groups \( H^2(\mathfrak{g}; H^{n-1}(M, H_n(M, k))) \) rely on the description of \( H^*(M, k) \) as a tensor product of polynomial rings on \( M^* \) and exterior algebras on \( M^* \) (recall \( M \) is a vector space of dimension \( n \) over the finite field \( \mathbb{F}_p \) and \( k = \mathbb{F}_p \)). \( M^* \) denotes the vector space dual of \( M, \text{Hom}(M, k) \). The formula for \( p > 2 \) is given by:

\[
1) \quad H^1(M, k) = \oplus_{0 \leq i < j \leq n/2} [\wedge^{j-i-1}(M_1^*) \otimes S^i(M_2^*)]
\]
where $M_1 \cong M_2 \cong M$ but $M_1$ lives only in degree 1. Here
\( \wedge^1 \) denotes the 1th exterior power and \( S^1 \) denotes homogeneous polynomials of degree 1 in \( n \)-variables.

**Example:**

\[
\begin{align*}
H^0(M,k) &= \wedge^0(M^*) \\
H^1(M,k) &= \wedge^1 \otimes S^0 = \wedge^1 \\
H^2(M,k) &= \wedge^2 \otimes S^0 + \wedge^0 \otimes S^1 = \wedge^2 + M^* \\
H^3(M,k) &= \wedge^3 \otimes S^0 + \wedge^1 \otimes S^1 \\
&= \wedge^3(M^* \otimes M^*)
\end{align*}
\]

using \( \wedge^0 = S^0 = \mathbb{F}_p \), \( \wedge^1 = S^1 = M^* \). All exterior powers and symmetric algebras above are over the dual space \( M^* \), i.e. \( \wedge^1 \) means \( \wedge^1(M^*) \). However, the homology groups \( H_*(M) \) can be written as vector space duals of \( H^*(M) \) so there are formulas dual to those above.

Now \( \nu^m(M) \varepsilon H^2(G,H^{m-1}(M,H_m(M,k))) \) and we rewrite the group before using the above splitting. The coefficients \( k \) may be suppressed since \( k \) is \( \mathbb{F}_p \),

\[
\begin{align*}
H^2(G,H^{m-1}(M,H_m(M,k))) \\
\text{11)} & \cong H^2(G, \text{Hom}(H_{m-1}(M),H_m(M))) \\
\text{111)} & \cong H^2(G, \text{Hom}(H^m(M),H^{m-1}(M)))
\end{align*}
\]

using only Universal Coefficient Theorem or the fact that
$H^*$ is dual of $H_*$.

The next step is to break up the coefficients in ii) or iii) using i). For instance if $H^m$ and $H^{m-1}$ are rewritten in iii) in terms of symmetric and exterior algebras then one can use a weight (eigenvalue) argument to determine possible homomorphisms in $\text{Hom}(H^n(M),H^{m-1}(M))$. Sometimes such a procedure will show that no non-trivial homomorphisms can exist, forcing the group to be zero.

The center of $\mathfrak{g} = \text{GL}(n,p)$ is of order prime to $p$. Since $\mathfrak{g}$ acts trivially on its cohomology and $\sigma \mathfrak{g}$ operates on cochains by

$$(\sigma \mathfrak{g})(c) = \sigma \mathfrak{g}(\sigma^{-1}(c)) \text{ with } \sigma^{-1}(c) = \sigma^{-1}c$$

the coefficient module for $\mathfrak{g}$ can be replaced by its fixed points under the action of the center. In other words, after splitting the coefficients $H^{n-1}(M,H_n(M))$ up according to i) some parts of the resulting sums may be ignored since they can contribute nothing to the cohomology $H^2(\mathfrak{g})$.

**Example:** Suppose $p > 2$.

$$\varphi^2 \epsilon H^2(\text{GL}(n,F_p);H^1(M,H_2(M)))$$

$$= H^2(\mathfrak{g};\text{Hom}(H_1(M),H_2(M)))$$

$$= H^2(\mathfrak{g};\text{Hom}(M,M\otimes^\mathfrak{g} A^2(M)))$$

$$1 \nu) = H^2(\mathfrak{g};\text{Hom}(M,M))$$
since the center removes the part of coefficients corresponding to $\text{Hom}(M, \Lambda^2(M))$.

Now when $p = 3$ and $n = 2$, $\text{Hom}(M, M)$ is the direct sum of $F_3$ and a free module for a 3-Sylow subgroup of $\text{GL}(2,3)$, so $H^2(\text{GL}(2,3), F_3) = 0$ (with trivial action), so the group containing $v^2$ for $\phi = \text{GL}(2,3)$ is zero. This argument (due to Sah) serves as an example of how such calculations can yield vanishing results for characteristic classes. Equation iv) is valid for any prime $p \geq 3$, and the group was computed for $n = 2$ [Prop. 4.5, Sah [9]]:

$$H^2(\text{GL}(2,p), \text{Hom}(M, M)) = \begin{cases} F_p & p > 3 \\ 0 & p = 2, 3 \end{cases}$$

In fact, it was shown in [8, 9] that $v^2 \neq 0$ for $\text{GL}(2,p)$ with $p \geq 5$.

The above procedures don't work for $p = 2$. The problem occurs since

$$0 \to \Lambda^2(M) \to H^2(N, k) \to M \to 0$$

is exact but does not split (it splits for $p \geq 3$). Therefore, the splitting of the coefficients as above doesn't hold. In this case the class $v^2 \in H^2(\text{GL}(n,2); H^1(M, H_2(M)))$ and there is a mapping

$$H^2(\text{GL}(n,p), H^1(M, H_2(M))) \to H^2(\text{GL}(n,2), H^1(M, M))$$
and the image under this map is the characteristic class associated to
\[ 0 \to M \to \left( \mathbb{Z}/4\mathbb{Z} \right)^n \to M \to 0. \]

This was shown \[9\] to be non-zero for \( n \geq 4 \). Therefore, \( v^2 \neq 0 \) for \( \text{GL}(n,2) \), \( n \geq 4 \). When \( n = 2 \) the coefficient modules are free for a 2-Sylow subgroup. Namely, \( H^1(M) \cong S^1(M^*) = \) homogeneous polynomials of degree 1 in two variables. The monomials \( x_1^a x_2^b \), \( a+b = 1 \) form a basis, \( \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \) is a 2-Sylow subgroup. Therefore, \( H^1(M) \) is free when \( i \) is odd and \( H^{m-1}(M, H^m(M)) \) is always free. Therefore, \( v^m = 0 \) for all \( m \) when \( \phi = \text{GL}(2,2) \).

In the correspondence of \( v^2 \) to \( d_2 \) in the Hochschild-Serre spectral sequence for the split extension

\[ v) \quad 0 \to M \to \pi \to \phi \to 1 \]

\( v^2 = d_2^0 \cdot 2(f^2) \) where \( f^2 \in H^0(\text{GL}(n,p), H^2(M, H_2(M,k))) \); and \( [f^2] \in H^2(M, H_2(M,k)) \) corresponds to identity in \( \text{Hom}(H_2(M), H_2(M)) \).

Let \( L \) = any \( \pi \) module which is \( M \) trivial (i.e. \( H_2(M,k) \) will work). Define \( E = \{(l,m) | l \in L, m \in M\} \) and define for each 2-cocycle \( f \) a group structure \( E(f) \) on \( E \):

\[ E(f) : (l_1, x)(l_2, y) = (l_1 + l_2 + f(x,y), xy). \]

This leads to an exact sequence

\[ 0 \to L \to E(f) \to M \to 1 \]
which is a central extension of $L$ by $M$. Each $\sigma \in \Phi = \text{GL}(n,p)$ defines a bijective map $E \to E$ by $\sigma(u, x) = (\sigma(u), \sigma(x))$, which induces a new group structure on $E$:

$$\sigma : E(f) \to E(\sigma f).$$

For a 1-cochain $g$ (in $H^1(M, L)$), define $g : E \to E$ by $g(u, x) = (u + g(x), x)$. Now suppose $[f] \in H^0(\Phi; H^2(M, L))$. For each $\sigma \in \Phi$, there exists a 1-cochain $g_\sigma \in H^1(M, L)$ such that $\sigma f \sigma = \partial_L g_\sigma$, or equivalently $g_\sigma : E(\sigma f) \to E(f)$ is a group isomorphism which is the identity on each of $M$ and $L$. The action of $\Phi$ on the 1-cochain $g$ is $\sigma g = \sigma g_\sigma \sigma^{-1}$ where $\circ$ denotes composition of maps in $E$. Now $g_\sigma \sigma (g_\tau) \sigma g_\tau^{-1} = \sigma(g_\tau) - g_\tau \circ g_\sigma$ so we have an automorphism $E(f)$ which is the identity on both $L$ and $M$. Let $A$ be the group of all automorphisms of $E(f)$ which induce identity on $L$ and $M$. Let $B = \text{ker} \partial_L g$. Then $g_\sigma \circ \sigma \in B$ so there exists an action:

$$h : \Phi \to A/B.$$

We can identify $B$ with $H^1(M, L)$ and in the Hochschild-Serre spectral sequence for $(v)$, $d_2^{0, 2}[f]$ is the cohomology class in $H^2(\Phi; H^1(M, L))$ determined by $\partial_L g$. The map $h$ defines a group extension:

$$\begin{align*}
vi) \quad & 0 \to H^1(M, L) \to X \to \Phi \to 1
\end{align*}$$
corresponding to the class $d_{2}^{0,2}[f]$. There is a homomorphism $\eta : X \rightarrow B$ and an exact sequence

$$1 \rightarrow E(f) \rightarrow E(f) \times X \rightarrow X \rightarrow 1.$$ 

Now $d_{2}^{0,2}[f]$ is 0 if vi) splits, and this is equivalent to finding a split

$$1 \rightarrow E(f) \rightarrow E_{f} \times \hat{\phi} \rightarrow \hat{\phi} \rightarrow 1,$$

considering $\hat{\phi}$ as a subgroup of $X$. In such a case the action $h$ can be lifted to an action $\overline{h} : \hat{\phi} \rightarrow B$. Now, in the above procedure take $L = H_{2}(M,k)$, $k = \mathbb{F}_{p}$. Then $[f] = [f^{2}]$ corresponds to identity in $\text{Hom}(H_{2}(M),H_{2}(M))$. This yields a way of determining $v^{2}(M)$ by checking to see if the $\hat{\phi}$-action $h : \hat{\phi} \rightarrow A_{B}$ lifts to $\overline{h} : \hat{\phi} \rightarrow B$.

For $p > 2$ combine the above sequence

$$0 \rightarrow H_{2}(M,k) \rightarrow E(f) \rightarrow M \rightarrow 1$$

with the splitting $H_{2}(M,k) \cong M \oplus \Lambda^{2}(M)$ to get two exact sequences

vii) $$0 \rightarrow M \rightarrow E(f_{1}) \rightarrow M \rightarrow 0$$

viii) $$0 \rightarrow \Lambda^{2}(M) \rightarrow E(f_{2}) \rightarrow M \rightarrow 0.$$

The part of $v^{2}$ corresponding to viii) is killed by the center so the determination of $v^{2}$ reduces to checking lift-
ings of $\hat{\mathfrak{g}}$-action on both ends of $\mathfrak{v}_{1}$). Actually in this case $E(f_{1}) = (\mathbb{Z}/p^{2}\mathbb{Z})^{n}$, and in [8] Sah shows that $\hat{\mathfrak{g}} = \text{GL}(n,p)$ action can't be lifted for $p > 3$ on $\mathfrak{v}_{1}$) in this case. Therefore, $v_{2}^{2} \neq 0$ for $p > 3$. 
Bibliography


