Families of Abelian Varieties of Non-Satake Type Arising from Quaternion Algebras

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Abstract of the Dissertation

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Let G be a semisimple Lie group, and X = G/(max. compact) the corresponding symmetric domain; let $\mathfrak{H}^{(m)}$ be the Siegel upper half space, Sp(2m)/(max. compact). Satake considered the problem of classifying all holomorphic maps $\tau : X \to \mathfrak{H}^{(m)}$, where τ is compatible with a representation $\rho : G \to Sp(2m)$. Such a situation leads to the construction of "group-theoretic" families of abelian varieties $V \to U$, with base space $U = \Gamma\backslash X$, where Γ is a discontinuous subgroup of G.

Satake classified all such τ and ρ with the following additional condition (*): Let $G \simeq G_{nc} \times G_c$, where G_{nc} (resp. G_c)

is the product of the noncompact (resp. compact) simple factors of G. Let proj_{nc} and proj_c be the projections of G onto G_{nc} and G_c. For representations $\rho_{nc} : G_{nc} \to GL(N_1,\mathbb{C})$ and $\rho_c : G_c \to GL(N_2,\mathbb{C})$,

(*) $\rho \sim \rho_{nc}\circ\mathrm{proj}_{nc} \oplus \rho_c\circ\mathrm{proj}_c$.

We construct and roughly classify a class of group-theoretic families of abelian varieties in which (*) does not hold.

Let k be a totally real number field and $S = \{\varphi_1,\ldots,\varphi_m\}$ the set of infinite places of k. Let B be a quaternion algebra with center k, $B \neq M_2(k)$, and S_0 the set of infinite places of k at which B is unramified. The group of units of B having norm one is an algebraic group G; $G_{\mathbb{R}}$ is a semisimple Lie group, and the corresponding symmetric space is a product of upper half planes.

For each $\varphi_i \in S$ we construct a symplectic representation ρ_i of G; consider direct sums of tensor products of the ρ_i. Such a representation ρ is called <u>admissible</u> if 1) it is invariant under the action of $\mathrm{Gal}(K:\mathbb{Q})$, where K is the normal closure of k, and 2) each tensor product in ρ contains at most one ρ_i where $\varphi_i \in S_0$.

Theorem. If ρ is admissible, then some multiple of ρ defines a group-theoretic family of abelian varieties.

If a representation ρ of G defines a group-theoretic family of abelian varieties, then ρ is equivalent to an admissible representation.

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INTRODUCTION

A family of abelian varieties is a C^{∞}-fiber bundle whose fibers are abelian varieties; the total space, base space, and fibers are nonsingular algebraic varieties, and the projection map is a rational map. A group-theoretic family of abelian varieties (abbreviated GTFAV) is a family of abelian varieties that is defined using a symplectic representation of an algebraic group. (This will be defined precisely in §1.)

Background and Applications. The general theory of families of abelian varieties was developed by Kuga and Shimura. Satake classified a large class of algebraic groups and symplectic representations which admit group-theoretic families of abelian varieties. ([S-1], [S-2].)

Families of abelian varieties are closely linked with many areas of algebraic number theory; they also provide examples in algebraic geometry which are amenable to calculation.

If $V \to U$ is a GTFAV, then the base space U is a quotient of a symmetric domain X by an arithmetic subgroup Γ of the group $G_{\mathbb{R}}$ of isometries of X. $G_{\mathbb{R}}$ is a real form of a semisimple algebraic group G defined over $\mathbb{Q}$.

Special cases of GTFAV which arise as moduli spaces for abelian varieties with additional structures, such as endomorphism rings and points of finite order, the fields of moduli and fields of definition of V, of U and of "singular" fibers have been studied by Shimura. In most cases, the field of moduli was obtained as a class field.

Associated to a family of abelian varieties is a Hasse-Weil zeta function, which can be determined in certain cases where the base U is one dimensional. The zeta function coincides with a product of Dirichlet series of GL_2. The case of higher dimensional base space has been investigated by Langlands.

The cohomology of the total space of a GTFAV is given by a spectral sequence in terms of the cohomology of the base and the fiber. If the base space is a product of upper half planes, a theorem of Matsushima and Shimura enables one to compute the dimensions of some of the cohomology spaces. (See [M-Sh], [K-2].) For a group-theoretic family of abelian varieties, the vector-valued cohomology spaces are isomorphic to spaces of automorphic forms on X with respect to Γ. Hence GTFAVs provide a means of realizing spaces of automorphic forms.

Another algebro-geometric problem is to consider algebraic cycles. One can in certain cases calculate the dimension of the space of Hodge cycles in a GTFAV and describe them explicitly ([K-2], [K-3], [T]). If one could prove that such a cycle is not algebraic, this would be a counterexample to the Hodge conjecture. André Weil has suggested that a counterexample to the Hodge conjecture might be found in a generic fiber of a GTFAV.

Summary of Thesis. Let G be a semisimple Lie group, K its maximal compact subgroup, and $X = G/K$ the corresponding symmetric domain; let $\mathfrak{H}^{(m)}$ be the Siegel upper half space, $Sp(2m,\mathbb{R})/(\text{maximal compact})$. I. Satake considered the problem of classifying all holomorphic maps $\tau : X \to \mathfrak{H}^{(m)}$, where τ is compatible with a representation $\rho : G \to Sp(2m)$. Such a situation leads to the construction of "group-theoretic" families of abelian varieties $V \to U$, with base space $U = \Gamma/X$, where Γ is a discontinuous subgroup of G.

Satake classified all such τ and ρ with the following additional condition (*) : Let $G_{\mathbb{R}} \simeq G_{nc} \times G_c$, where G_{nc} (resp. G_c) is the product of the noncompact (resp. compact) simple factors of $G_{\mathbb{R}}$. Let $proj_{nc}$ and $proj_c$ be the projections of $G_{\mathbb{R}}$ onto G_{nc} and G_c. For representations $\rho_{nc} : G_{nc} \to GL(N_1,\mathbb{C})$ and $\rho_c : G_c \to GL(N_2,\mathbb{C})$,

$$(*) \quad \rho \sim \rho_{nc} \circ proj_{nc} \oplus \rho_c \circ proj_c .$$

All previous examples of group-theoretic families of abelian varieties have been of the Satake type.

In this thesis we construct a large class of group-theoretic families of abelian varieties in which (*) does not hold.

Let k be a totally real number field, $|k:\mathbb{Q}| = m$, $S = \{\varphi_1, \ldots, \varphi_m\}$ the set of embeddings of k into $\mathbb{R}$ (the infinite places of k), K the normal closure of k, and $\mathcal{G} = Gal(K:\mathbb{Q})$. Let B be a quaternion algebra with center k, and S_0 the set of infinite places of k at which B is unramified. The group of units of B having norm one is an algebraic group G, defined over $\mathbb{Q}$; $G_{\mathbb{R}}$ is a semisimple Lie group, and the corresponding symmetric space is $\mathfrak{H}^r$, the product of upper half planes.

For each $\varphi_i \in S$, we construct a symplectic representation ρ_i of G (an "atom"); the tensor product of such representations is a "molecule," and the direct sum of tensor products is a "polymer." A polymer is admissible if it is $\mathbb{Q}$-invariant and each molecule contains at most one atom from S_0.

If Γ is an arithmetic subgroup of G such that there is a group-theoretic family of abelian varieties defined over Q with base space $_\Gamma\backslash^{G}\mathbb{R}/(\text{max. compact})$, then the representation associated with the family is an admissible polymer.

Conversely, if ρ is an admissible polymer, then there is an arithmetic subgroup Γ of G, such that there exists a group theoretic family of abelian varieties associated with $\mu\,\rho$ for some multiplicity μ, with base space $_\Gamma\backslash^{G}\mathbb{R}/(\text{max. compact})$.

This essentially classifies such families, up to multiplicity of the representation and the choice of a Γ-invariant lattice in the representation space.

In Chapter I we discuss the ideas and objects involved in the main theorem; Chapter II consists entirely of the proof of the theorem.

In §1, we precisely define GTFAVs and summarize Kuga's method for constructing these families. We also discuss Satake's classification of symplectic representations admitting a GTFAV. In

§2 to §6 we construct the number field, algebras, algebraic group, and representations necessary for the statement of the main theoren in §7.

In §8 we show that any representation of G admitting a GTFAV is equivalent to one of the representations defined in §6. In §9 - §14 we construct a GTFAV from an "admissible" representation of the type defined in §6; the construction follows Kuga's program of §1.

An appendix is included on the theory of central simple algebras and quaternion algebras, which is used heavily in this paper.

Notation and conventions. We assume that every ring has a unit element. All algebras we use will be algebras over fields. By an isomorphism we mean an injective and surjective map.

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ denote the integers, the rational numbers, the real numbers, the complex numbers, and the Hamiltonian quaternions, respectively. By $M_n(k)$ we mean the algebra of $n \times n$ matrices over a field k. By $GL_n(k)$ and $SL_n(k)$ we mean the general linear and special linear groups, as subsets of $M_n(k)$. We let SO(n) and SU(n) denote the (real) special orthogonal and special unitary groups. For a k-vector space W and a nondegenerate alternating bilinear form β on $W \times W$, we let $Sp(W, \beta)$ denote the symplectic group of linear transformations on W.

CHAPTER I

§1. Group-Theoretic Families of Abelian Varieties.

A group-theoretic family of abelian varieties (abbreviated GTFAV) is a family of abelian varieties that arises from a symplectic representation of an algebraic group, in a way we will describe in this section. The main reference here is [K-1].

1.1. General theory. First we discuss Siegel spaces and their various realizations. A reference for this topic is [S-3], Chapter II, §7. Then we describe the construction of a GTFAV, following [K-1].

Let W be a real vector space of dimension 2n, and β a non-degenerate alternating form on W. Then the symplectic group with respect to β is

$$Sp(W,\beta) = \{T \in GL(W) : \beta(Tv,Tw) = \beta(v,w) \text{ for all } v,w \in W\} .$$

The symplectic group is an algebraic group. If F is a $\mathbb{Q}$-vector space, β is a $\mathbb{Q}$-bilinear form, and $W = F \otimes_{\mathbb{Q}} \mathbb{R}$, then $Sp(W,\beta)$ is an algebraic group defined over $\mathbb{Q}$.

The space

$$\mathfrak{H}^{(n)} = \mathfrak{H}(W,\beta) = Sp(W,\beta)/_{(\text{max. compact})}$$

is called the Siegel upper half space. This is a hermitian symplectic space; that is, $\mathfrak{H}^{(n)}$ has a $Sp(W,\beta)$-invariant complex structure.

Let $\mathfrak{S}(W,\beta)$ be the set of automorphisms J of W such that $J^2 = -1_W$ and $\beta(u,Jv)$ is symmetric and positive definite. Extend β and J to $W \otimes \mathbb{C}$ by $\mathbb{C}$-linearity, and let $W_\pm(J)$ be the eigenspaces of $W \otimes \mathbb{C}$ corresponding to the eigenvalues $\pm i$ of J, respectively. Define a hermitian form h_β on $W \otimes \mathbb{C}$ by

$$h_\beta(v,w) = i\beta(\overline{v},w).$$

It can be shown that, for any $J \in \mathfrak{S}(W,\beta)$, $\beta|W_-(J) \times W_-(J) = 0$ and $h_\beta|W_-(J) \times W_-(J)$ is negative definite.

Let $Gr_n(W\otimes\mathbb{C})$ be the Grassmannian manifold of all n-dimensional complex subspaces of $W \otimes \mathbb{C}$. The space $Gr_n(W\otimes\mathbb{C})$ is known to be a complex manifold. Let $\mathfrak{S}'(W,\beta)$ be the set of points V (subspaces of $W \otimes \mathbb{C}$) in $Gr_n(W\otimes\mathbb{C})$ such that $\beta|V \times V = 0$ and $h_\beta|V \times V$ is negative definite. Then $\mathfrak{S}'(W,\beta)$ is an open subset of an algebraic submanifold of Gr_n. The map $J \mapsto W_-(J)$ gives a one-to-one correspondence between $\mathfrak{S}(W,\beta)$ and $\mathfrak{S}'(W,\beta)$, and this identification gives $\mathfrak{S}(W,\beta)$ the structure of a complex manifold.

The group $Sp(W,\beta)$ acts on $\mathfrak{S}(W,\beta)$ by $g\cdot J = gJg^{-1}$ for $g \in Sp(W,\beta)$, $J \in \mathfrak{S}(W,\beta)$. By the identification of $\mathfrak{S}(W,\beta)$ with $\mathfrak{S}'(W,\beta)$, we have an action of $Sp(W,\beta)$ on $\mathfrak{S}'(W,\beta)$, which agrees with the natural action:

$$g\cdot W_-(J) = W_-(gJg^{-1}) = g(W_-(J)).$$

By means of this action, $\mathfrak{S}(W,\beta)$ can be holomorphically identified with $\mathfrak{h}^{(n)}(W,\beta)$. We will usually work with the realizations $\mathfrak{S}$ and $\mathfrak{S}'$ of the Siegel space.

We can form the product space $E = \mathfrak{S}(W,\beta) \times W$. The space E has a unique complex structure that makes it into a holomorphic vector bundle, $E \to \mathfrak{S}$, such that the fiber W_J above the point $J \in \mathfrak{S}$ is a complex vector space with complex structure J.

Suppose that $Sp(W,\beta)$ is defined over $\mathbb{Q}$. Let G be a semisimple algebraic group defined over $\mathbb{Q}$. If ρ is a representation of G into $GL(W\otimes\mathbb{C})$ that is also a rational map of algebraic groups, and the map ρ is defined over $\mathbb{Q}$, then we say that ρ is an algebraic group representation defined over Q. Consider ρ restricted to $G_{\mathbb{R}}$. If $\beta(\rho(g)u, \rho(g)v) = \beta(u,v)$ for all $g \in G_{\mathbb{R}}$, $u,v \in W$, then ρ is a symplectic representation.

Since G is semisimple, $G_{\mathbb{R}}$ is a semisimple Lie group, and $X = G_{\mathbb{R}}/(\text{max. compact})$ is a symmetric space. Suppose X is hermitian, so that X has a $G_{\mathbb{R}}$-invariant complex structure. An <u>Eichler map</u> τ associated to a symplectic representation ρ is a map $\tau : X \to \mathfrak{S}(W,\beta)$ such that $\tau(gx) = \rho(g)\tau(x)$. If a holomorphic Eichler map exists, then we can pull back the holomorphic vector bundle $E \to \mathfrak{S}$ to a holomorphic vector bundle over X:

$$\begin{array}{ccc} \tau^*(E) & \dashrightarrow & \mathfrak{S}(W,\beta) \times W = E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ \tau\ } & \mathfrak{S}(W,\beta) \end{array}$$

Suppose that Γ is a torsion-free discrete subgroup of G that acts properly discontinuously on X. Since the action of G on X is holomorphic, $_{\Gamma}\backslash^{X}$ is a complex manifold. We assume that Γ has been chosen so that $_{\Gamma}\backslash^{X}$ is compact. If there exists a lattice L in W such that $\rho(\Gamma)L = L$, then Γ is an arithmetic subgroup of G. We can take the semidirect product $\Gamma \rtimes L$ of Γ and L, which acts on $\tau^*(E) = X \times W$ by

$$(\gamma, v)(x, w) = (\gamma x, \rho(\gamma)v + w).$$

The group $\Gamma \rtimes L$ acts on $X \times W$ discontinuously, and the action is holomorphic. Hence the quotient of $X \times W$ by the action of $\Gamma \rtimes L$ is a holomorphic fiber bundle $V \xrightarrow{\pi} U$, where $U = {_{\Gamma}\backslash^{X}}$, and the fiber is the complex torus W/L. If β takes integral values on $L \times L$, then β provides W/L with a polarization, so that W/L is an abelian variety. By applying Kodaira's Embedding Theorem, it is shown that both V and U are compact complex submanifolds of a projective space, and therefore, by Chow's Theorem, are algebraic varieties. In fact, the projection $\pi : V \to U$ is an everywhere-defined rational map of varieties.

Such a fiber bundle $V \xrightarrow{\pi} U$, constructed using W, β, G, ρ, τ, Γ, and L, is called a <u>group theoretic family of abelian varieties</u>.

2.2. Satake's classification. In [S-1], Satake classified all the irreducible representations of simple Lie groups of non-compact type for which there exist holomorphic Eichler maps. Here we summarize part of his argument and give his results for the special case used in this paper, namely, $G_{\mathbb{R}} = SL_2(\mathbb{R})^{m_0} \times SU(2)^{m_1}$.

Let $G_{\mathbb{R}} = SL_2(\mathbb{R})^{m_0} \times SU(2)^{m_1}$, and let ρ be a representation of $G_{\mathbb{R}}$ for which there exists a holomorphic Eichler map. Set

$$G_{nc} = SL_2(\mathbb{R})^{m_0} \quad \text{and} \quad G_c = SU(2)^{m_1}.$$

Then G_{nc} is noncompact, G_c is compact, and $G_{\mathbb{R}} \cong G_{nc} \times G_c$.

Since G is semisimple, ρ is the direct sum of irreducible subrepresentations. For an irreducible subrepresentation ρ_i of ρ, set

$$\rho^{[i]} = \bigoplus_{\rho_j \sim \rho_i} \rho_j.$$

Such a representation is called <u>primary</u>. We can write $\rho = \bigoplus_i \rho^{[i]}$. Then, as Satake showed, each $\rho^{[i]}$ also has a holomorphic Eichler map, which is determined by the irreducible representation ρ_i. ([S-3], Chapter IV, Lemma 4.2 and Proposition 4.3.) Thus the problem of classifying representations is reduced to that of classifying irreducible representations.

Satake shows that, if $G_j \in G_{nc}$, then $\rho_i | G_j$ is nontrivial for at most one G_j. ([S-3], Chapter IV, §5.) So if $G = G_{nc}$, then ρ must be of the form

$$\rho = \bigoplus_j \bigoplus_i \mu_{i,j} \rho_{ij} \circ \mathrm{proj}_j$$

where μ_{ij} is a multiplicity ρ_{ij} is an irreducible representation on G_j, and proj_j is projection: $G \to G_j$.

For $G = SL_2(\mathbb{R})$, if ρ is irreducible and nontrivial, then ρ must be the identity representation into $Sp(2,\mathbb{R}) = SL_2(\mathbb{R})$. Hence, if $G = \Pi G_j$, $G_j = SL_2(\mathbb{R})$, then a representation ρ defining a GTFAV must be of the form

$$\rho = \bigoplus_j \mu_j \operatorname{proj}_j .$$

§2. Algebras

We first prove some results necessary for the construction of the algebraic group G and the representation ρ. The propositions in this section concern constructing new algebras from a fixed algebra by changing the scalar multiplication.

Lemma 2.1. If k and K are fields, φ an embedding of k into K, and B a K-algebra, then φ provides B with the structure of a k-algebra.

Proof. Define a scalar multiplication of B by k, $\mu : k \times B \to B$, by $\mu(\alpha, v) = \varphi(\alpha)v$. For convenience, we denote $\mu(\alpha, v)$ by $\alpha \cdot v$. Then, for α, β in k, and v, v' in B, we have

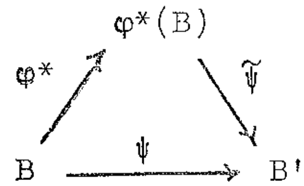
$$(\alpha\beta)\cdot v = \varphi(\alpha\beta)v = \varphi(\alpha)\varphi(\beta)v = \alpha\cdot(\beta\cdot v),$$

$$(\alpha+\beta)\cdot v = \varphi(\alpha+\beta)v = \varphi(\alpha)v + \varphi(\beta)v = \alpha\cdot v + \beta\cdot v,$$

$$\alpha\cdot(v+v') = \varphi(\alpha)(v+v') = \alpha\cdot v + \alpha\cdot v' \qquad \blacksquare$$

We denote the k-algebra obtained from B via φ by $R_\varphi(B)$, or by $\varphi^*(B)$, and the identity map from B to $R_\varphi(B)$ by φ^*.

More precisely, $R_\varphi(B)$ is the pair $(\varphi^*(B), \varphi^*)$ of the k-algebra $\varphi^*(B)$ and the ring isomorphism $\varphi^* : B \to \varphi^*(B)$ with $\alpha\cdot\varphi^*(b) = \varphi^*(\varphi(d)b)$ for $\alpha \in k$, $b \in B$. The algebra $R_\varphi(B)$ is characterized by the following universality: If B' is a k-algebra, and $\psi : B \to B'$ is a ring homomorphism such that $\alpha\psi(b) = \psi(\varphi(\alpha)b)$, then there exists a unique k-algebra homomorphism $\tilde{\psi} : \varphi^*(B) \to B'$ such that the following diagram commutes:



<u>Proposition 2.2</u>. Let k and K be fields, φ an embedding of k into K, and B a k-algebra. Then there exists a K-algebra $\tilde{B}$ and an injective ring homomorphism $\tilde{\varphi} : B \to \tilde{B}$ such that $\tilde{\varphi}|k = \varphi$. If φ is surjective, then $\tilde{\varphi}$ is surjective.

We will use the symbol $B \underset{\varphi}{\otimes} K$ to denote both the pair $(\tilde{B}, \tilde{\varphi})$ and the algebra $\tilde{B}$.

<u>Proof</u>. By Lemma 2.1, $K = \varphi^*(K)$ is a k-algebra via the embedding φ. We can then form the tensor product $\tilde{B} = B \underset{k}{\otimes} \varphi^*(K)$ of k-algebras. We will denote this as $\tilde{B} = B \underset{\varphi}{\otimes} K$. We can also consider $\tilde{B}$ as a K-algebra in the usual way. Let $\tilde{\varphi} : B \to \tilde{B}$ be defined by $\tilde{\varphi}(v) = v \otimes 1$. This is clearly a ring homomorphism.

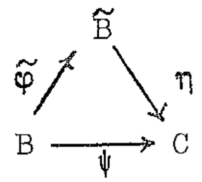
We identify k with the subspace $k1_B$ in B, and K with the subspace $1_B \otimes K$ in $\tilde{B}$. For $\alpha \in k$,

$$\tilde{\varphi}(\alpha 1_B) = \alpha 1_B \otimes 1 = 1_B \otimes \alpha \cdot 1 = 1_B \otimes \varphi(\alpha).$$

Hence $\tilde{\varphi}|k = \varphi$.

The kernel of $\tilde{\varphi}$ is $\{0\}$, i.e. $\tilde{\varphi}$ is injective. Suppose that φ is a surjective. We can define $\tilde{\varphi}^{-1}$ by $\tilde{\varphi}^{-1}(v \otimes \beta) = \varphi^{-1}(\beta)v$. Thus $\tilde{\varphi}$ is surjective. ∎

Proposition 2.3. Let k and K be fields, and φ an embedding of k into K. Let B be a k-algebra, C a K-algebra, and ψ a ring homomorphism from B to C that $\psi|k = \varphi$. Then there exists a unique K-algebra homomorphism η from $\widetilde{B} = B \underset{\varphi}{\otimes} K$ to C such that the following diagram commutes:



If ψ is a ring isomorphism, then η is also.

Proof. Define a map $f : B \times K \to C$ by $f(v,\alpha) = \alpha\psi(v)$. Claim: f is k-bilinear.

It is clear that f is bi-additive. If $\gamma \in k$, then

$$f(\gamma v,\alpha) = \alpha\psi(\gamma v) = \alpha\psi(\gamma)\psi(v) = \alpha\varphi(\gamma)\psi(v)$$

$$= \gamma\cdot\alpha\psi(v) = \gamma\cdot f(v,\alpha) \qquad \text{and}$$

$$f(v,\gamma\cdot\alpha) = (\gamma\cdot\alpha)\psi(v) = \gamma\cdot f(v,\alpha).$$

So f is k-bilinear. Also f is multiplicative in the following sense. For (v,α) and (v',α') in $B \times K$, we have

$$f(vv',\alpha\alpha') = \alpha\alpha'\psi(vv')$$

$$= \alpha\psi(v)\alpha'\psi(v') = f(v,\alpha)f(v',\alpha').$$

Now, by the universality of the tensor product, there exists a unique k-linear map $\eta : \widetilde{B} = B \underset{\varphi}{\otimes} K \to C$ so that the following diagram commutes:

$$\begin{array}{ccc} & B \underset{\varphi}{\otimes} K & \\ {\scriptstyle \pi} \nearrow & & \searrow {\scriptstyle \eta} \\ B \times K & \xrightarrow[f]{} & C \end{array},$$

where π sends (v,α) to $v \otimes \alpha$. Since f is multiplicative, η is a ring homomorphism. Actually η is determined in the generators of $B \underset{\varphi}{\otimes} K$ by $\eta(v\otimes\alpha) = \alpha\psi(v)$.

To check that this η is well defined, let γ be in k.

Then $$\eta(v\otimes\gamma\cdot\alpha) = \gamma\cdot\alpha\psi(v) = \varphi(\gamma)\alpha\psi(v)$$

$$= \psi(\gamma)\alpha\psi(v) = \alpha\psi(\gamma v) = \eta(\gamma v\otimes\alpha).$$

This map η is a k-linear ring homomorphism and makes the diagram commute. Hence it is the unique η of the previous paragraph.

Let β be an element of K. Then

$$\eta(\gamma(v\otimes\alpha)) = \eta(v\otimes\beta\alpha) = \beta\alpha\psi(v) = \beta\eta(v\otimes\alpha),$$

so η is K-linear; that is, η is a K-algebra homomorphism.

We can identify B with B x 1 in B x K. Then $f|B = \psi$, and $\pi|B = \tilde{\varphi}$. Hence the diagram

$$\begin{array}{ccc} & \tilde{B} & \\ {\scriptstyle \tilde{\varphi}} \nearrow & & \searrow {\scriptstyle \eta} \\ B & \xrightarrow[\psi]{} & C \end{array}$$ commutes.

Now suppose that $\psi : B \to C$ is a ring isomorphism. Then

$\psi|k = \varphi$ is an isomorphism of k onto K, the center of C. So, by Proposition 2.2, $\tilde{\varphi}$ is a ring isomorphism. Then $\eta = \psi \circ \tilde{\varphi}^{-1}$ is a ring isomorphism ∎

By Proposition 2.3, $B \otimes_{\varphi} K = (\tilde{B}, \tilde{\varphi})$ is characterized by the following universality: Suppose $(\tilde{B}', \tilde{\varphi}')$ is another pair such that

(a) $\tilde{B}'$ is a K-algebra

(b) $\tilde{\varphi}' : B \to \tilde{B}'$ is a ring homomorphism such that $\tilde{\varphi}'|k = \varphi$

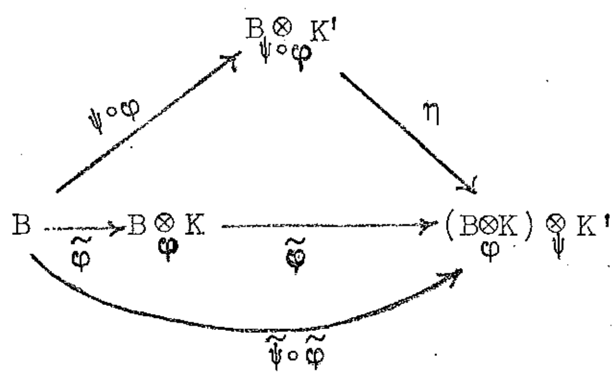
(c) for any K-algebra C and ring homomorphism $\psi : B \to C$ with $\psi|k = \varphi$, there exists a K-algebra homomorphism $\eta' : \tilde{B}' \to C$ such that $\eta' \circ \tilde{\varphi}' = \psi$.

Then $(\tilde{B}, \tilde{\varphi})$ is canonically isomorphic to $(\tilde{B}', \tilde{\varphi}')$. That is, there is a unique isomorphism $\epsilon : \tilde{B} \to \tilde{B}'$ such that $\epsilon \circ \tilde{\varphi} = \tilde{\varphi}'$. This justifies the use of the notation $B \otimes_{\varphi} K$.

<u>Proposition 2.4</u>. Let k, K, and K' be fields, $\varphi : k \to K$ and $\psi : K \to K'$ be embeddings, and B a k-algebra. Then $(B \otimes_{\varphi} K) \otimes_{\psi} K' \cong B \otimes_{\psi \circ \varphi} K'$, canonically, as K'-algebras. That is, there exists a unique K'-algebra isomorphism η from $(B \otimes_{\varphi} K) \otimes K'$ to $B \otimes_{\psi \circ \varphi} K'$ such that $\eta \circ \tilde{\psi} \circ \tilde{\varphi} = \widetilde{\psi \circ \varphi}$.

Proof. Since φ is an embedding of k into K, by Proposition 2.2, we can construct the K-algebra $\widetilde{B} = B \otimes_{\varphi} K$ and a ring homomorphism $\widetilde{\varphi} : B \to \widetilde{B}$ so that $\widetilde{\varphi}|k = \varphi$. Similarly, we can construct the K'-algebra $B \otimes_{\varphi} K = (B\otimes_{\varphi} K) \otimes_{\psi} K'$ and a ring isomorphism $\widetilde{\psi} : B \otimes_{\varphi} K \to \widetilde{(B\otimes_{\varphi} K)} \otimes_{\psi} K'$ such that $\widetilde{\psi}|K = \psi$.

Now we use Proposition 2.3. The map $\widetilde{\psi}\circ\widetilde{\varphi}$ is a ring isomorphism from B to $(B\otimes_{\varphi} K)\otimes_{\psi} K'$ with $\widetilde{\psi}\circ\widetilde{\varphi}|k = \psi\circ\varphi$, and $\psi\circ\varphi$ is an embedding of k into K'. So there exists a unique K'-algebra homomorphism η from $B \otimes_{\psi\circ\varphi} K'$ to $(B\otimes_{\varphi} K)\otimes_{\psi} K'$ such that the following diagram commutes:



From the proof of Proposition 2.3, η is given by

$$\eta(v\otimes\gamma) = \gamma\widetilde{\psi}\circ\widetilde{\varphi}(v) = \gamma((v\otimes 1)\otimes 1) = (v\otimes 1)\otimes\gamma .$$

Since

$$(v\otimes\alpha)\otimes\beta = (v\otimes 1)\otimes\psi(\alpha)\beta,$$

we can define $\eta^{-1}((v\otimes\alpha)\otimes\beta) = v\otimes\psi(\alpha)\beta$. It is obvious that $\eta\circ\eta^{-1}$ and $\eta^{-1}\circ\eta$ are the identity maps, so η is an isomorphism. ∎

So we can identify $(B \underset{\varphi}{\otimes} K) \underset{\psi}{\otimes} K'$ with $B \underset{\psi\circ\varphi}{\otimes} K'$, and $\tilde{\psi}\circ\tilde{\varphi}$ with $\widetilde{\psi\circ\varphi}$.

In particular, when $K' = K$, then $\psi = \sigma$ is an automorphism of K. Then we have

$$B \underset{\varphi}{\otimes} K \xrightarrow{\tilde{\sigma}} (B \underset{\varphi}{\otimes} K) \underset{\sigma}{\otimes} K \xrightarrow[\cong]{\eta} B \underset{\sigma\circ\varphi}{\otimes} K .$$

Since η is canonical, we identify $\eta\circ\tilde{\sigma}$ with $\tilde{\sigma}$. If τ and σ are automorphisms of K, then we have $\tilde{\tau}\circ\tilde{\sigma} = \widetilde{\tau\circ\sigma} : B \underset{\varphi}{\otimes} K \to B \underset{\tau\circ\sigma\circ\varphi}{\otimes} K$.

§3. Scalar Restriction

In order to construct an algebraic group and a representation defined over $\mathbb{Q}$, we will use the scalar restriction functor. Here we summarize some facts about the scalar restriction functor for algebras and the corresponding functor for algebraic varieties.

3.1. If B is an algebra over a field L, K is a subfield of L, and $i : K \to L$ the inclusion map, then R_i is the functor described in §1. Namely, R_i is the pair $(i^*(B), i^*)$, where $i^*(B)$ is B considered as a K-algebra, and i^* is the identity map from the L-algebra B to the K-algebra $i^*(B)$. When K is a subfield of L, we write $R_{L/K}$ instead of R_i and $R_{L/K}(B)$ instead of $i^*(B)$.

Proposition 3.1. Let k be an extension of $\mathbb{Q}$ of degree m, and $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ the set of embeddings of k into $\mathbb{C}$. Let L be a field containing the compositum $\varphi_1(k)\cdots\varphi_m(k)$. Then, for a k-algebra B,

$$R_{k/\mathbb{Q}}(B) \underset{\mathbb{Q}}{\otimes} L \simeq \bigoplus_{\varphi_i \in \Phi} (B \underset{\varphi_i}{\otimes} L)$$

canonically as an L-algebra.

In other words, there exists a unique isomorphism j of $R_{K/\mathbb{Q}}(B) \otimes L$ into $\oplus B \underset{\varphi_i}{\otimes} L$ that makes the following diagram commute.

$$\begin{array}{ccccc} B & \xrightarrow{i^*} & R_{k/\mathbb{Q}}(B) & \xrightarrow{\otimes 1} & R_{k/\mathbb{Q}}(B) \underset{\mathbb{Q}}{\otimes} L \\ & \searrow^{\oplus \tilde{\varphi}_i} & & & \downarrow \\ & & & & \bigoplus_{\varphi_i \in \Phi} B \underset{\varphi_i}{\oplus} L \end{array}$$

Here the map $\oplus \tilde{\varphi}_i$ means $(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m)$.

The proof of Proposition II.2.1 will be delayed until Section 3.3.

3.2. Now we discuss the functor which restricts the field of definition of an algebraic variety. Some details and proofs can be found in [W-1], §1.3. Unless otherwise stated, we consider algebraic varieties and algebraic groups to be subsets of an affine space $A^N(\Omega)$ over a universal domain Ω, which is algebraically closed. For simplicity, we assume the characteristic of Ω is zero.

Let k and k_0 be fields such that $k_0 \subseteq k \subseteq \Omega$ and $|k:k_0| = m$. Let $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ be the set of embeddings of k into Ω fixing k_0. Suppose that V and W are varieties defined over k and k_0 respectively, and p is a morphism from W to V defined over k. If I is the ideal defining V, let I^φ be the ideal of all polynomials in I with φ applied to their coefficients. Let $V^\varphi = V(I^\varphi)$. We can define a map $p^\varphi : W \to V^\varphi$ by applying this procedure to the graph of P. If the map $(p^{\varphi_1}, \ldots, p^{\varphi_m})$: $W \to V^{\varphi_1} \times \ldots \times V^{\varphi_m}$ is biregular, then we say that W is the variety obtained from V by scalar restriction from k to k_0. We write $\{W,p\} = \mathcal{R}_{k/k_0}(V)$, or simply $W = \mathcal{R}_{k/k_0}(V)$. The existence of such a variety is proved in [W-1].

As an example, let $V = A^n(\Omega)$, the affine (actually linear) space over Ω, V is defined over $\mathbb{Q}$. So $V^{\varphi_i} = A^n(\Omega)$, hence $\prod_i V^{\varphi_i} = A^{mn}(\Omega)$. Also, $\mathcal{R}_{k/k_0}(A^n(\Omega))$ is known to be $A^{mn}(\Omega)$, and the map

$$p : \mathcal{R}_{k/k_0}(A^n(\Omega)) = A^{mn}(\Omega) \to A^n(\Omega)$$

is a surjective linear map defined over k. The map

$$\prod_i p^{\varphi_i} : \mathcal{R}_{k/k_0}(A^n(\Omega)) \to \prod_i A^n(\Omega)^{\varphi_i} = A^{mn}(\Omega)$$

is a bijective linear map defined over $K = \varphi_1(k)\cdots\varphi_m(k)$.

Actually, $\mathcal{R}_{k/k_0}$ is functor from the category of varieties defined over k to the category of triples (W,V,p), where W and V are defined over k_0 and k respectively, and p is defined over k.

Since an algebraic group G defined over k is a variety together with operations which are k-rational maps of varieties, G can be considered as a collection of commutative diagrams in the category of varieties over k. Because $\mathcal{R}_{k/k_0}$ is a functor, it takes these commutative diagrams to commutative diagrams defined over k_0, so that $\mathcal{R}_{k/k_0}(G)$ can be considered as an algebraic group defined over k_0. The map $p : \mathcal{R}_{k/k_0}(G) \to G$ is a surjective group homomorphism defined over k, and $\prod_i p^{\varphi_i} : \mathcal{R}_{k/k_0}(G) \to \prod_i G^{\varphi_i}$ is a group isomorphism defined over K.

The same reasoning applies to algebra-varieties. (An algebra-variety is a k-algebra considered as a k-variety, whose operations are k-rational maps.) Algebraic groups and algebra-varieties will be discussed further in §3.3.

Let K be any extension of k_0, and let Ω be a field containing $\overline{K}$. For k and $\Phi = \{\varphi_i\}$ as described above, we say that $\varphi_i \sim \varphi_j$ if there exists $\omega \in \mathrm{Aut}_K(\Omega)$ such that $\varphi_i = \omega \circ \varphi_j$. Let Φ' be a set of representatives for these equivalence classes. We have

$$k \underset{k_0}{\otimes} K = \underset{\Phi'}{\oplus} \varphi_i(k)K.$$

Proposition 3.2. ([W-1], Theorem 1.3.1):

$$\mathcal{R}_{k/k_0}(V) \approx \prod_{\Phi'} \mathcal{R}_{\varphi_i(k)K/K}(V^{\varphi_i})$$

canonically, and the isomorphism is defined over K.

Proposition 3.3. ([W-1], Theorem 1.3.2):

$$(\mathcal{R}_{k/k_0}(V))_K \approx \prod_{\Phi'} (V^{\varphi_i})_{\varphi_i(k)K}$$

canonically.

3.3. In this section, we discuss the relation between the functor R_{k/k_0} for algebras and the functor $\mathcal{R}_{k/k_0}$ for varieties, algebraic groups, and algebra-varieties.

Let B be an n-dimensional k-algebra. We will construct an algebra-variety $\mathfrak{V}(B)$ defined over k. Choose a basis $\{e_1,\ldots,e_n\}$

for B over k. Let the variety $\mathfrak{V}(B)$ be $B \otimes_k \Omega$ considered as the affine space $A^n(\Omega)$. Hence $\mathfrak{V}(B)$ has an additive group structure over $\mathbb{Q}$. If multiplication in B is defined on the basis elements by $e_i e_j = \sum_h c_{ij}^h e_h$, with $c_{ij}^h \in k$, consider multiplication in $\mathfrak{V}(B)$ to be the bilinear map $\mu(e_i, e_j) = \sum_h c_{ij}^h e_h$. Since this is defined over k, $\mathfrak{V}(B)$ is an algebra-variety defined over k.

We sometimes write $\mathfrak{V}(B)$ as $\mathfrak{V}(B/k)$ to emphasize that B is a k-algebra, and also to indicate that $\mathfrak{V}(B)$ is defined over k.

The algebra-variety $\mathfrak{V}(B)$ has the following properties.

(i) $\mathfrak{V}(B/k)_k = B$, where $(\quad)_k$ means the set of k-rational points.

(ii) For any field L, with $k \subseteq L \subseteq \Omega$, $\mathfrak{V}(B/k)_L = B \otimes_k L$

(iii) $\mathfrak{V}(B \otimes_k L/L) = \mathfrak{V}(B/k)$.

(iv) $\mathfrak{V}(B_1 \oplus B_2/k) = \mathfrak{V}(B_1/k) \oplus \mathfrak{V}(B_2/k)$.

(v) If φ is an embedding of k into Ω,

$$\mathfrak{V}(B/k)^{\varphi} = \mathfrak{V}(B \otimes_{\varphi} \varphi(k)/\varphi(k)).$$

If $\varphi(k) \subseteq K$,

$$\mathfrak{V}(B/k)^{\varphi} = \mathfrak{V}(B \otimes_{\varphi} K/K).$$

For any variety W defined over k, we have $(W_k)^{\varphi} = (W^{\varphi})_{\varphi(k)}$. By taking the $\varphi(k)$-rational points of both sides of the first formula in (v), we have

$$(\mathfrak{V}(B/k)_k)^{\varphi} = (\mathfrak{V}(B/k)^{\varphi})_{\varphi(k)}$$

$$= \mathfrak{V}(B \otimes_{\varphi} \varphi(k)/\varphi(k))_{\varphi(k)} = B \otimes_{\varphi} \varphi(k).$$

Now $\mathfrak{V}(B/k)_k = B$. By applying the field homomorphism φ to their coordinates, we have the set of $\varphi(k)$-rational points of $\mathfrak{V}(B\otimes_\varphi\varphi(k)/\varphi(k))$, which is $B\otimes_\varphi\varphi(k)$. The map from B to $B\otimes_\varphi\varphi(k)$ resulting from applying φ to the coordinates is essentially the same as the "lift" $\tilde{\varphi}$ of φ constructed in §2. That is, $\tilde{\varphi}$ is the map just described followed by the inclusion of $B\otimes_\varphi\varphi(k)$ into $B\otimes_\varphi K$.

(vi) For a subfield $k_0 \subseteq k$ of finite index, $\Phi = \{\varphi_i\}$ the set of embeddings of k into Ω over k_0, and $K = \varphi_1(k)\cdots\varphi_m(k)$,

$$\bigoplus_\Phi \mathfrak{V}(B/k)^{\varphi_i} = \bigoplus_\Phi \mathfrak{V}(B\otimes_{\varphi_i}\varphi_i(k)/\varphi_i(k))$$

$$= \bigoplus_\Phi \mathfrak{V}(B\otimes_{\varphi_i}K/K) = \mathfrak{V}(\bigoplus_\Phi(B\otimes_{\varphi_i}K)/K).$$

Similarly to the algebraic group situation, $\mathfrak{R}_{k/k_0}$ of an algebra-variety $\mathfrak{V}$ is an algebra-variety. The map $p : \mathfrak{R}_{k/k_0}(\mathfrak{V}) \to \mathfrak{V}$ is a surjective algebra homomorphism defined over k, and

$$\Sigma\, p^{\varphi_i} : \mathfrak{R}_{k/k_0}(\mathfrak{V}) \to \oplus\, \mathfrak{V}^{\varphi_i}$$

is an algebra isomorphism defined over K. Here, since $\mathfrak{V}^{\varphi_i}$ is an Ω-algebra, we use the direct sum instead of the cartesian product.

For a k-algebra B, we have

(vii) $\mathfrak{R}_{k/k_0}(\mathfrak{V}(B/k)) = \mathfrak{V}(R_{k/k_0}(B)/k_0)$.

(viii) The set $\mathfrak{V}(B/k)^{\times}$ of invertible elements of $\mathfrak{V}(B/k)$ is an algebraic group defined over k, and $\mathfrak{V}(B/k)^{\times}_{k} = B^{\times}$.

(ix) Suppose that B has a norm map ν; that is $\nu : B \to k$ is a polynomial map defined over k such that $\nu(xy) = \nu(x)\nu(y)$. Extend ν to $B \otimes \Omega$. Then ν maps $B^{\times}$ to $k^{\times}$ and $(B\otimes\Omega)^{\times} = \mathfrak{V}(B)^{\times}$ to $\Omega^{\times}$. Denote the kernels of these maps by B^{1} and $\mathfrak{V}(B)^{1}$ respectively. Then $\mathfrak{V}(B)^{1}$ is an algebraic group defined over k, and $\mathfrak{V}(B)^{1}_{k} = B^{1}$.

(x) For any field L, $k \subseteq L \subseteq \Omega$,

$$\mathfrak{V}(B/k)^{1}_{L} = (\mathfrak{V}(B/k)_{L})^{1} = (B \underset{k}{\otimes} L)^{1} .$$

Finally, as an application of these formulae, we prove Proposition 3.1.

Apply the map $\Sigma\, p^{\varphi_i}$ of the paragraph before (vii) to $\mathfrak{V} = \mathfrak{V}(B/k)$. We have

$$\Sigma\, p^{\varphi_i} : \mathcal{R}_{K/q}(\mathfrak{V}) \xrightarrow{\approx} \oplus\, \mathfrak{V}^{\varphi_i},$$

so we can take the L-rational points of both sides, since $\Sigma\, p^{\varphi_i}$ is defined over $K \subseteq L$. By (v), we have $\mathfrak{V}(B/k)^{\varphi_i} = \mathfrak{V}(B \underset{\varphi_i}{\otimes} K/K)$. Since $\varphi_i(k) \subseteq K \subseteq L$, we can take the L-rational points

$$(\mathfrak{V}(B/k)^{\varphi_i})_{L} = \mathfrak{V}(B \underset{\varphi_i}{\otimes} K/K)_{L} = (B \underset{\varphi_i}{\otimes} K) \otimes L = B \underset{\varphi_i}{\otimes} L$$

(by (ii).) On the other hand, by (vii),

$$\mathcal{R}_{k/\mathbb{Q}}(\mathfrak{V}(B/k))_L = \mathfrak{V}(\mathcal{R}_{k/\mathbb{Q}}(B)/\mathbb{Q})_L$$

$$= R_{k/\mathbb{Q}}(B) \underset{\mathbb{Q}}{\otimes} L, \qquad \text{by (ii).}$$

So we have

$$R_{k/\mathbb{Q}}(B) \underset{\mathbb{Q}}{\otimes} L = \mathcal{R}_{k/\mathbb{Q}}(\mathfrak{V}(B/k))_L$$

$$= \oplus(\mathfrak{V}(B/k)_L^{\varphi_i} = \oplus\, B \underset{\varphi_i}{\otimes} L\,,$$

and we have proved the proposition. ∎

§4. Construction of Fields and Algebras

Let k be a totally real algebraic number field, with degree m over $\mathbb{Q}$. Let $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ be the distinct embeddings of k into $\mathbb{R}$. (Equivalently, Φ is the set of infinite places or completions of k.) For convenience, we put $S = \{1, \ldots, m\}$, the set of indices of Φ.

Let K be the compositium of $\varphi_1(k), \ldots, \varphi_m(k)$, so K is a subfield of $\mathbb{R}$. If $\{\psi_1, \ldots, \psi_d\}$ is the set of embeddings of K into $\mathbb{C}$, then $\psi_h \circ \varphi_i$ is an embedding of k into $\mathbb{C}$, hence equal to some φ_j. Since the elements of the fields $\varphi_j(k)$ generate K, $\psi_h(K) = K$ for all h; thus K is Galois over $\mathbb{Q}$ and totally real.

Proposition 4.1. For a subset $S_0 \subseteq S$, there exists a quaternion algebra B with center k such that

$$B \underset{\varphi_i}{\otimes} \mathbb{R} \cong \begin{cases} M_2(\mathbb{R}) & \text{if } i \in S_0 \\ \mathbb{H} & \text{if } i \notin S_0 \end{cases}.$$

Proof. Let $S_1 = S - S_0$. Pick any finite set T of finite places of k such that $|S_1| + |T|$ is even. Then, by Theorem A.2, there exists a (unique) quaternion algebra over k ramified at exactly the places in $S_1 \cup T$. In particular, at $i \in S_1$ $B \underset{\varphi_i}{\otimes} \mathbb{R}$ is a division ring with center $\mathbb{R}$, hence isomorphic to $\mathbb{H}$. At the other places of S, B is unramified, hence $B \underset{\varphi_i}{\otimes} \mathbb{R} \cong M_2(\mathbb{R})$. ∎

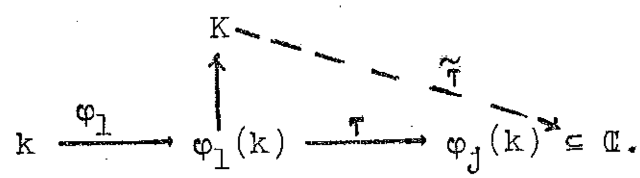
We form K-algebras $B \underset{\varphi_i}{\otimes} K$, and denote $B \underset{\varphi_i}{\otimes} K$ by B_i.

Let $\mathcal{G} = \mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_i : i = 1,\ldots,d\}$. Since $\sigma\circ\varphi_i$ maps k into $K \subseteq \mathbb{R}$, $\sigma\circ\varphi_i$ is one of the φ_j. Hence $\mathcal{G}$ acts on Φ, and on S.

<u>Proposition 4.2</u>. If $\sigma \in \mathcal{G}$, then $\sigma : \Phi \to \Phi$ is one-to-one and onto the action of $\mathcal{G}$ on Φ is transitive.

<u>Proof</u>. The action of $\sigma \in \mathcal{G}$ on Φ is one-to-one, for if $\sigma\circ\varphi_i = \sigma\circ\varphi_j$, then $\sigma^{-1}\circ\sigma\circ\varphi_i = \sigma^{-1}\circ\sigma\circ\varphi_j$, so $\varphi_i = \varphi_j$. Any φ_j is the image of $\sigma^{-1}(\varphi_j)$, so σ is surjective.

Now, we wish to show that for any $\varphi_j \in \Phi$, there exists $\tau \in \mathcal{G}$ such that $\tau\circ\varphi_1 = \varphi_j$. Since $\varphi_1(k)$ and $\varphi_j(k)$ are both isomorphic to k, we can find an isomorphism $\tau : \varphi_1(k) \to \varphi_j(k)$. Since $\varphi_j(k)$ is a subfield of $\mathbb{R} \subseteq \mathbb{C}$, τ can be considered as embedding of $\varphi_1(k)$ into $\mathbb{C}$. We can extend τ to the algebraic extension K of $\varphi_1(k)$, and call the extension $\tilde{\tau}$. We have the following diagram:



But K is Galois, so the image of K under $\tilde{\tau}$ is K. Hence $\tilde{\tau}$ may be considered as an element of $\mathcal{G}$. Restricting $\tilde{\tau}$ to $\varphi_1(k)$, we have $\tilde{\tau}\circ\varphi_1 = \varphi_j$. ∎

If we let ι denote inclusion of K into $\mathbb{R}$, by the remark after Proposition 2.4, we can identify

$$(B\underset{\varphi_i}{\otimes} K)\underset{\iota}{\otimes}\mathbb{R} = B\underset{\varphi_i}{\otimes}\mathbb{R} \cong \begin{cases} M_2(\mathbb{R}) & \text{if } i \in S_0 \\ \mathbb{H} & \text{if } i \notin S_0 \end{cases}.$$

We denote $B\underset{\varphi_i}{\otimes}\mathbb{R}$ by $B_{i,\mathbb{R}}$.

We can also take tensor products of the algebras B_i. For a subset A of S, $A = \{a_1,\ldots,a_r\}$, we let $B_A = \underset{i}{\otimes} B_{a_i} = B_{a_1} \underset{K}{\otimes} B_{a_2} \underset{K}{\otimes} \cdots \underset{K}{\otimes} B_{a_r}$. For a finite collection $\mathbb{A}$ of such sets, $\mathbb{A} = \{A_1,\ldots,A_d\}$, we denote $B_{\mathbb{A}} = \underset{j}{\oplus} B_{A_j} = B_{A_1} \oplus\cdots\oplus B_{A_d}$. We write $B_{A,\mathbb{R}}$ for $B_A \otimes \mathbb{R}$ and $B_{\mathbb{A},\mathbb{R}}$ for $B_{\mathbb{A}} \underset{K}{\otimes} \mathbb{R}$.

Terminology. We will use Kuga's terminology for describing these algebras. An index $i \in S$ of an algebra B_i will be called an atom. A set of indices $A = \{a_1,\ldots,a_r\}$ corresponding to a tensor product will be called a molecule. A set of molecules $\mathbb{A} = \{A_1,\ldots,A_d\}$ corresponding to a direct sum will be called a polymer. Sometimes we write $\mathbb{A} = A_1+\cdots+A_d$. We will also refer to the corresponding algebras as atoms, molecules, and polymers.

We say a polymer is homogeneous if each of its molecules contains the same number of atoms.

Since $\mathcal{G}$ acts on atoms, it acts on molecules by $\sigma\{a_1,\ldots,a_r\} = \{\sigma(a_1),\ldots,\sigma(a_r)\}$, and on polymers by $\sigma\{A_1,\ldots,A_d\} = \{\sigma(A_1),\ldots,\sigma(A_d)\}$.

§5. Construction of Algebraic Group

From the algebra B of §4 we will construct an algebraic group defined over $\mathbb{Q}$. The real points of this group will be a semi-simple Lie group.

From now on k and K will be the totally real number fields of §4, and the universal domain $\Omega = \mathbb{C}$. We identify S with the set of infinite places of k, S_0 the subset of S for which B is unramified, and $S_1 = S - S_0$. Let ν be the reduced norm of the algebra B, or its extension to $B \otimes \mathbb{C}$.

Let $\mathfrak{V}(B/k)$ be the algebra-variety defined from B, and set $G' = \mathfrak{V}(B/k)^1$ = the kernel of the norm map $\nu : \mathfrak{V}(B/k)^x \to \mathbb{C}^x$. By §3.3(ix), G' is an algebraic group defined over k. Set $G = \mathcal{R}_{k/\mathbb{Q}}(G')$, and $\mathfrak{V} = \mathcal{R}_{k/\mathbb{Q}}(\mathfrak{V}(B/k))$. So G is an algebraic group defined over $\mathbb{Q}$, and $\mathfrak{V}$ is an algebra-variety defined over $\mathbb{Q}$. Note that $R_{k/\mathbb{Q}}(B)$ is a $\mathbb{Q}$-algebra, and $\mathfrak{V} = \mathfrak{V}(R_{k/\mathbb{Q}}(B)/\mathbb{Q})$, by §3.3(vii). We have

$$G'_k = G_{\mathbb{Q}} \subseteq \mathfrak{V}^x_{\mathbb{Q}} = R_{k/\mathbb{Q}}(B)^x \cong B^x.$$

Proposition 5.1.

(a) $R_{k/\mathbb{Q}}(B) \underset{\mathbb{Q}}{\otimes} \mathbb{R} \cong \underset{S}{\oplus}(B \underset{\varphi_i}{\otimes} \mathbb{R})$ canonically,

so that $R_{k/\mathbb{Q}}(B) \underset{\mathbb{Q}}{\otimes} \mathbb{R} \cong \underset{S_0}{\oplus} M_2(\mathbb{R}) \oplus \underset{S_1}{\oplus} \mathbb{H} = m_0 M_2(\mathbb{R}) \oplus m_1 \mathbb{H}$.

(b) $G_{\mathbb{R}} \cong \underset{S}{\Pi}(B \underset{\varphi_i}{\otimes} \mathbb{R})^1$ canonically, so that

$$G_{\mathbb{R}} \cong \prod_{S_0} SL_2(\mathbb{R}) \times \prod_{S_1} \mathbb{H}^1 = SL_2(\mathbb{R})^{m_0} \times SU(2)^{m_1}.$$

Proof. Part (a) is exactly Proposition 3.1, with $L = \mathbb{R}$ together with the fact that B was defined so that $B \underset{\varphi_i}{\otimes} \mathbb{R} \cong M_2(\mathbb{R})$ if $i \in S_0$, and $B \underset{\varphi_i}{\otimes} \mathbb{R} \cong \mathbb{H}$ if $i \in S_1$.

Part (b). First recall the equivalence relation on Φ in §3.2. If $i \neq j$, there is no ω in $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ such that $\omega \circ \varphi_i = \varphi_j$. Thus the set Φ' of representatives of equivalence classes is just Φ.

We have

$$
\begin{aligned}
G_{\mathbb{R}} &= (\mathcal{R}_{k/\mathbb{Q}}(G'))_{\mathbb{R}} = \mathcal{R}_{k/\mathbb{Q}}(\mathfrak{V}(B/k)^1)_{\mathbb{R}} \\
&= \prod_S ((\mathfrak{V}(B/k)^1)_{\mathbb{R}}^{\varphi_i} && \text{by Prop.3.3} \\
&= \prod_S (\mathfrak{V} \underset{\varphi_i}{\otimes} K/K)^1)_{\mathbb{R}} \\
&= \prod_S (\mathfrak{V}(B \underset{\varphi_i}{\otimes} K/K)_{\mathbb{R}})^1 && \text{by §3.3(x)} \\
&= \prod_S ((B \underset{\varphi_i}{\otimes} K) \otimes \mathbb{R})^1 && \text{by §3.3(x)} \\
&= \prod_S (B \underset{\varphi_i}{\otimes} \mathbb{R})^1 .
\end{aligned}
$$

Now the reduced norm of an element of B is defined to be the determinant of the image of the element in $M_n(\mathbb{C})$ under the reduced representation. The reduced representation sends b to $b \otimes 1$ in $B \otimes \mathbb{C} \cong M_n(\mathbb{C})$. Since, for $i \in S_0$, $B \underset{\varphi_i}{\otimes} \mathbb{R} \cong M_2(\mathbb{R}) \subseteq B \underset{\varphi_i}{\otimes} \mathbb{C} \cong M_2(\mathbb{C})$, we have

$$(B \underset{\varphi_1}{\otimes} \mathbb{R})^1 \cong \{M \in M_2(\mathbb{R}) : \det M = 1\} = SL_2(\mathbb{R}).$$

Consider the representation $\rho : \mathbb{H} \to M_2(\mathbb{C})$ defined by

$$1 \mapsto I,\ i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

For $x = x_1 = ix_2 + jx_3 + kx_4$,

$$\rho(x) = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} .$$

So the image of $\mathbb{H}$ is the set of matrices in $M_2(\mathbb{C})$ of the form $\begin{pmatrix} a & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$. We have $\nu(x) = \det \rho(x)$, so $\mathbb{H}^1$ is isomorphic to the set of elements of the form $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ having determinant one. Hence, for $i \in S_1$, $(B \otimes_{\varphi_i} \mathbb{R})^1 \cong \mathbb{H}^1 \cong SU(2)$.

Finally, we have

$$G_{\mathbb{R}} = \prod_S (B \otimes_{\varphi_i} \mathbb{R})^1 \cong \prod_{S_0} SL_2(\mathbb{R}) \times \prod_{S_1} SU(2) .$$ ∎

We have shown that $G_{\mathbb{R}}$ is a semisimple Lie group. Notice that, since $\mathbb{H}^1 = \{(x_1,x_2,x_3,x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, $\mathbb{H}^1 \cong SU(2)$ is topologically a 3-sphere, hence compact. The maximal compact subgroup $\subseteq$ of $G_{\mathbb{R}}$ is $\prod_{S_0} SO(2) \times \prod_{S_1} SU(2)$, so the symmetric space $G_{\mathbb{R}}/\subseteq$ is

$$\Big(\prod_{S_0} SL_2(\mathbb{R}) \times \prod_{S_1} SU(2)\Big)\Big/\Big(\prod_{S_0} SO(2) \times \prod_{S_1} SU(2)\Big)$$

$$= \prod_{S_0} (SL_2(\mathbb{R})/SO(2)) \times \prod_{S_1} (SU(2)/SU(2))$$

$$= \prod_{S_0} \mathfrak{h} \times \prod_{S_1} p = \mathfrak{h}^{m_0} .$$

Here p denotes a single point. Since $\mathfrak{h}$ is known to be an hermitian symmetric domain, $G_{\mathbb{R}/\underline{c}}$ is also.

§6. Representation

For any polymer $\mathbb{A}$ we will define a representation $\rho_{\mathbb{A}}$ of the algebraic group G.

Following the proof of Proposition 5.1(b), we see that G, considered as the set of its complex points, is

$$G = G_{\mathbb{C}} = \prod_{S} (B \otimes_{\varphi_i} \mathbb{C})^1 \cong SL_2(\mathbb{C})^m.$$

Let proj_i be the projection map of G onto its simple factor $G_i = (B \otimes_{\varphi_i} \mathbb{C})^1 \cong SL_2(\mathbb{C})$. For an atom a, let ρ_a be the map proj_a considered as a representation of G on the vector space $\mathbb{C}^2$. For a molecule $A = \{a_1, \ldots, a_r\}$, let $\rho_A = \rho_{a_1} \otimes \ldots \otimes \rho_{a_r}$. For a polymer $\mathbb{A} = \{A_1, \ldots, A_d\}$, let $\rho_{\mathbb{A}} = \rho_{A_1} \oplus \ldots \oplus \rho_{A_d}$.

§7. Statement of Main Theorem

Recall that, in order to construct a group-theoretic family of abelian varieties, it is necessary to have: a semisimple algebraic group G defined over $\mathbb{Q}$ such that $X = G_{\mathbb{R}}/(\text{max. compact})$ is a hermitian symmetric space; a $\mathbb{Q}$-vector space F and a non-degenerate alternating $\mathbb{Q}$-bilinear form β on F; an algebraic group representation $\rho : G \to Sp(F\otimes\mathbb{C}, \beta)$ defined over $\mathbb{Q}$; a holomorphic Eichler map $\tau : X \to \mathfrak{h}^{(N)}$ that is compatible with $\rho|G_{\mathbb{R}}$; a cocompact torsion-free arithmetic subgroup Γ of G; and a lattice $L \subseteq F \otimes \mathbb{R}$ such that β takes integral values on $L \times L$ and $\rho(\Gamma)L = L$. When we say that a representation ρ defines a GTFAV, we mean that there exist F, β, τ, Γ, and L that satisfy all these conditions.

We will say that a polymer is admissible if it is G-invariant and every molecule of the polymer contains at most one atom form S_0.

We assume that the quaternion algebra B is not $M_2(k)$.

<u>Main Theorem</u>. Let G be the algebraic group defined in §5, and, for a polymer $\mathbb{A}$, let $\rho_{\mathbb{A}}$ be the associated representation of G defined in §6.

(1) Suppose that ρ is a symplectic algebraic group representation of G, defined over $\mathbb{Q}$, that defines a group-theoretic family of abelian varieties. Then there exists an admissible polymer $\mathbb{A}$ such that $\rho \sim \rho_{\mathbb{A}}$ over $\mathbb{C}$.

(2) Let $\mathbb{A}$ be an admissible polymer. Then some multiple $\mu\rho_{\mathbb{A}}$ of $\rho_{\mathbb{A}}$ is $\mathbb{C}$-equivalent to an algebraic group representation P of G defined over $\mathbb{Q}$, and P defines a group-theoretic family of abelian varieties.

CHAPTER II - Proof of the Theorem

A. §8. Proof of the First Part

We are given an algebraic group representation ρ of G into $Sp(F\underset{\mathbb{Q}}{\otimes}\mathbb{C},\beta)$ defined over $\mathbb{Q}$, and an arithmetic subgroup Γ of G. The restriction of ρ to $G_{\mathbb{R}}$ is a representation of $G_{\mathbb{R}}$ into $Sp(F\otimes\mathbb{R},\beta) = Sp(W,\beta)$ that defines the GTFAV.

Write $G_{i,\mathbb{R}}$ for the simple component $(B\underset{\varphi_i}{\otimes}\mathbb{R})^1 = B^1_{i,\mathbb{R}}$ of $G_{\mathbb{R}}$, so that $G_{\mathbb{R}} = \prod_S G_{i,\mathbb{R}}$. Let $G_{nc} = \prod_{S_0} G_{i,\mathbb{R}} \times \prod_{S_1} \{1\}$, which is called the non-compact part of $G_{\mathbb{R}}$, and let $G_c = \prod_{S_0}\{1\} \times \prod_{S_1} G_{i,\mathbb{R}}$, the compact part of $G_{\mathbb{R}}$. We have $G_{nc} \cong SL_2(\mathbb{R})^{m_0}$ and $G_c \cong SU(2)^{m_1}$.

Let $\rho_{\mathbb{C}}$ denote ρ considered as a representation of $G_{\mathbb{R}}$ into $GL(W_{\mathbb{C}})$. Then $\rho_{\mathbb{C}}$ is $\mathbb{C}$-equivalent to a direct sum of irreducible representations:

$$\rho_{\mathbb{C}} \sim \rho^{(1)}\oplus\ldots\oplus\rho^{(M)}.$$

Each irreducible subrepresentation $p^{(j)}$ is of the form

$$\rho^{(j)} = \rho_1^{(j)}\otimes\ldots\otimes\rho_m^{(j)},$$

and $\rho_i^{(j)} = r_i^{(j)}\circ proj_i$, where $proj_i$ is projection of $G_{\mathbb{R}}$ onto $G_{i,\mathbb{R}}$, and $r_i^{(j)}$ is an irreducible representation of $G_{i,\mathbb{R}}$.

Let $G_{\mathbb{C}}$ be the complex points of the algebraic group G, and let $G_{i,\mathbb{C}} = (B \underset{\varphi_i}{\otimes} \mathbb{C})^1 = B^1_{i,\mathbb{C}} \cong SL_2(\mathbb{C})$. Then $G_{i,\mathbb{C}}$ is the complexification of the Lie group $G_{i,\mathbb{R}}$, and $G_{\mathbb{C}} = \prod_S G_{i,\mathbb{C}} \cong SL_2(\mathbb{C})^m$. The representations $\rho_{\mathbb{C}}$, $\rho^{(j)}$ are holomorphic representations of $G = G_{\mathbb{C}}$, and $\rho_{\mathbb{C}} = \rho$. So $r_i^{(j)}$ is extended to a holomorphic representation of $G_{i,\mathbb{C}} = SL_2(\mathbb{C})$.

It is well known that any irreducible holomorphic representation of $SL_2(\mathbb{C})$ is $\mathbb{C}$-equivalent to one of the symmetric tensor representations, s_n, of dimension $n + 1$. For $n = 0$ or 1, s_0 is the trivial representation and s_1 is the identity representation. Hence we can put $r_i^{(j)} = s_{n(j,i)}$. So we have

$$\rho^{(j)} = \bigotimes_{i \in S} s_{n(j,i)} \circ \mathrm{proj}_i .$$

Since the representation $\rho = \rho_{\mathbb{C}}$ of $G_{\mathbb{R}}$ defines a GTFAV, ρ must be compatible with a holomorphic Eichler map τ:

$$\begin{array}{ccc} G_{\mathbb{R}} & \xrightarrow{\ \rho\ } & Sp(W_{\mathbb{R}}, \beta) \\ \downarrow & & \downarrow \\ X = \mathfrak{h}^{m_0} & \xrightarrow{\ \tau\ } & \mathfrak{h}^{(N)} \end{array}$$

where $\mathfrak{h}^{(N)}$ is the Siegel upper half space of degree N. Now, the action of G_c on X is trivial; hence $\rho|G_{nc}$ must be compatible with the Eichler map:

$$\begin{array}{ccc} G_{nc} = SL_2(\mathbb{R})^{m_0} & \xrightarrow{\ \rho|G_{nc}\ } & Sp(W_{\mathbb{R}}, \beta) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ \tau\ } & \mathfrak{h}^{(N)} \end{array} .$$

Satake has completely classified the representations of $SL_2(\mathbb{R})^M$ compatible with a holomorphic Eichler map. (See §1.) By his results, $\rho|G_{nc}$ must be of the form

$$(*) \qquad \rho|G_{nc} \sim \bigoplus_{i \in S_0} \mu_i(s_1 \circ proj_i)$$

where the multiplicities μ_i are nonnegative integers.

We also have

$$\rho \sim \bigoplus_j \Big(\bigotimes_{i \in S} (s_{n(j,i)} \circ proj_i) \Big),$$

so that

$$(**) \qquad \rho|G_{nc} \sim \bigoplus_j \Big[\Big(\bigotimes_{i \in S_0} s_{n(j,i)} \circ proj_i \Big) \otimes \Big(\bigotimes_{h \in S_1} s_{n(j,h)}(1) \Big) \Big]$$

$$= \bigoplus_j \Big[\nu_j \bigotimes_{i \in S_0} (s_{n(j,i)} \circ proj_i) \Big],$$

where the multiplicity $\nu_j = \prod_{h \in S_1} (n(j,h)+1)$,

since $s_{n(j,h)}(1)$ is the identity matrix of size $n(j,h) + 1$.

Comparing $(*)$ and $(**)$, we see that, for all j and i, $n(j,i) \leq 1$, and that for each j, there is at most one i such that $n(j,i) = 1$. Thus any irreducible subrepresentation $\rho^{(j)}$ of $\rho_{\mathbb{C}}$ has the form

$$(***) \qquad \rho^{(j)} \sim \begin{cases} (s_1 \circ proj_i) \otimes \Big(\bigotimes_{h \in S_1} s_{n(h)} \circ proj_h \Big) \\ \text{or} \\ \bigotimes_{h \in S_1} s_{n(h)} \circ proj_h \end{cases},$$

where $i \in S_0$. Note that $s_1 \circ \mathrm{proj}_i = \mathrm{proj}_i$.

Now we will show that, for $h \in S_1$, $n(j,h)$ is also zero or one. We must investigate the representation proj_i more closely.

Recall that $G' = \mathfrak{V}(B/k)^1$ is an algebraic group defined over k, $G = \mathcal{R}_{k/\mathbb{Q}}(G')$ is an algebraic group defined over $\mathbb{Q}$, and that $G \subseteq \mathcal{R}_{k/\mathbb{Q}}(\mathfrak{V}(B/k))^{\times} = \mathfrak{V}(R_{k/\mathbb{Q}}(B)/\mathbb{Q})^{\times}$. Since $\mathfrak{V}(R_{k/\mathbb{Q}}(B)/\mathbb{Q})$ $= \oplus \mathfrak{V}(B \otimes_{\varphi_i} K/K)$, we have

$$\mathfrak{V}(R_{k/\mathbb{Q}}(B)/\mathbb{Q})^{\times} = \Pi \mathfrak{V}(B \otimes_{\varphi_i} K/K)^{\times}.$$

Denote by proj_i the projection of $\mathfrak{V}(R_{k/\mathbb{Q}}(B)/\mathbb{Q})$ to the i^{th} component $\mathfrak{V}(B \otimes_{\varphi_i} K/K)$ of the direct sum. This is a homomorphism of algebra-varieties defined over K. We also denote the restriction of proj_i to a subgroup of $\mathfrak{V}(R_{k/\mathbb{Q}}(B)/\mathbb{Q})^{\times}$ by proj_i.

Then the projection of $\mathfrak{V}(R_{k/\mathbb{Q}}(B)/\mathbb{Q})$ to $\mathfrak{V}(B \otimes_{\varphi_i} K/K)$ is an algebraic group homomorphism defined over K. So the restriction of this homomorphism to G is also defined over K. Since $K \subseteq \mathbb{R}$, we can take the real or complex points of these algebraic groups. We have

$$G_{\mathbb{R}} \xrightarrow{(\mathrm{proj}_i)_{\mathbb{R}}} \mathfrak{V}(B \otimes_{\varphi_i} K/K)^{\times}_{\mathbb{R}} = (B \otimes_{\varphi_i} \mathbb{R})^{\times}$$

$$= \left\{ \begin{matrix} GL_2(\mathbb{R}) \\ \mathbb{H}^{\times} \end{matrix} \right\} \subseteq GL_2(\mathbb{C})$$

and

$$G_{\mathbb{C}} \xrightarrow{(\mathrm{proj}_i)_{\mathbb{C}}} \mathfrak{V}(B\otimes_{\varphi_i} K/K)^{\times}_{\mathbb{C}} = (B\otimes_{\varphi_i} \mathbb{C})^{\times} \cong Gl_2(\mathbb{C}).$$

The maps $(\mathrm{proj}_i)_{\mathbb{R}}$ and $(\mathrm{proj}_i)_{\mathbb{C}}$ are the same as the representations described above. Hence the representations proj_i in (*), (**), and (***) are rational representations of the algebraic group G restricted to $G_{\mathbb{R}}$ or $G_{\mathbb{C}}$.

Let $\mathrm{Inj}(\mathbb{C})$ be the set of injective homomorphisms of $\mathbb{C}$ into $\mathbb{C}$. Any $\sigma \in \mathrm{Inj}(\mathbb{C})$ induces an automorphism of $\overline{\mathbb{Q}}$, and, conversely, any automorphism σ of $\overline{\mathbb{Q}}$ can be extended to an element σ of $\mathrm{Inj}(\mathbb{C})$. We also denote by σ the restriction of $\sigma \in \mathrm{Inj}(\mathbb{C})$ to a Galois extension K of $\mathbb{Q}$.

Since proj_i is a rational representation of the algebraic group G defined over $\mathbb{Q}$, $(\mathrm{proj}_i)^{\sigma}$ is also a rational representation of G for any σ. Since proj_i is defined over K, proj_i^{σ} depends only on $\sigma|K$, that is, on σ considered as an element of $\mathrm{Gal}(K:\mathbb{Q})$.

<u>Proposition 8.1</u>. $(\mathrm{proj}_i)^{\sigma} = \mathrm{proj}_{\sigma(i)}$.

<u>Proof</u>. Apply (v) of § 3.3. For $\sigma : K \to K \subseteq \Omega = \mathbb{C}$, we have

$$\mathfrak{V}(B\otimes_{\varphi_i} K)^{\sigma} = \mathfrak{V}((B\otimes_{\varphi_i} K)\otimes_{\sigma} K/K)$$

$$= \mathfrak{V}(B_{\varphi_{\sigma(i)}}\otimes K/K).$$

Apping σ to $\mathfrak{V}(R_{K/\mathbb{Q}}(B)/\mathbb{Q}) \xrightarrow{\mathrm{proj}_i} \mathfrak{V}(B\otimes_{\varphi_i} K/K)$, we have

$$\mathfrak{V}(R_{K/\mathbb{Q}}(B)/\mathbb{Q}) \xrightarrow{(\text{proj}_i)^\sigma} \mathfrak{V}(B \underset{\varphi_i}{\otimes} K/K)^\sigma = \mathfrak{V}(B_{\varphi_{\sigma(i)}} \otimes K/K).$$

Since proj_i is surjective, proj_i^σ is also surjective. We have $\text{proj}_i \circ \text{proj}_i = \text{proj}_i$, so $\text{proj}_i^\sigma \circ \text{proj}_i^\sigma = \text{proj}_i^\sigma$; i.e., proj_i^σ is also a projection. The kernel of proj_i is $\bigoplus_{j \neq i} \mathfrak{V}(B \underset{\varphi_j}{\otimes} K/K)$, so the kernel of proj_i^σ is $\bigoplus_{j\neq i} \mathfrak{V}(B \underset{\varphi_j}{\otimes} K/K)^\sigma = \bigoplus_{j \neq \sigma(i)} \mathfrak{V}(B \underset{\varphi_j}{\otimes} K/K)$. The projection $\text{proj}_{\sigma(i)}$ is also surjective, and has the same kernel. So both proj_i^σ and $\text{proj}_{\sigma(i)}$, restricted to the i^{th} component, are automorphisms of the i^{th} component. Since $\mathfrak{V}(B_{\varphi_{\sigma(i)}} \otimes K/K) \cong M_2(\mathbb{C})$ is a $\mathbb{C}$-simple algebra, every automorphism is inner. Hence there is a matrix $A \in GL_2(\mathbb{C})$ such that $\text{proj}_i^\sigma(x) = AxA^{-1}$ for all x in $\mathfrak{V}(B_{\varphi_{\sigma(i)}} \otimes K/K)$. Since $\text{proj}_{\sigma(i)}$ is the identity map, we have $\text{proj}_i^\sigma(x) = A\,\text{proj}_{\sigma(i)}(x)A^{-1}$; that is, $\text{proj}_i^\sigma \sim \text{proj}_{\sigma(i)}$ over $\mathbb{C}$ as algebra homomorphisms. Restricting to G, we have $\text{proj}_i^\sigma \sim \text{proj}_{\sigma(i)}$ as group representations, over $\mathbb{C}$. ∎

<u>Proposition 8.2</u>. For $\rho \sim \bigoplus_j(\bigotimes_i(s_{n(j,i)} \circ \text{proj}_i))$, $\rho = \rho^\sigma$ for all $\sigma \in \mathcal{G}$.

<u>Proof</u>. First, ρ is a rational representation. Because ρ defines a GTFAV, $\rho(\Gamma)$ fixes a lattice L, where $L \subseteq R_{K/\mathbb{Q}}(B_A) = \mathfrak{V}(R_{K/\mathbb{Q}}(B_A)/\mathbb{Q})_{\mathbb{Q}}$. Also, $\Gamma \subseteq G_{\mathbb{Q}} \subseteq \mathfrak{V}(R_{K/\mathbb{Q}}(B)/\mathbb{Q})_{\mathbb{Q}}^{x}$. Hence both Γ and L subsets of $\mathbb{Q}$-points of varieties defined over $\mathbb{Q}$. For $\gamma \in \Gamma$, $\lambda \in L$, we have $\rho(\gamma)(\lambda) = \lambda'$ for some $\lambda' \in L$. Applying σ, we have $\rho^\sigma(\gamma^\sigma)(\lambda^\sigma) = (\lambda')^\sigma$,

or $\rho^{\sigma}(\gamma)(\lambda) = \lambda'$. Since L spans the linear space $R_{K/\mathbb{Q}}(B_A)$, $\rho(\gamma) = \rho^{\sigma}(\gamma)$ for all $\gamma \in \Gamma$. Now Γ is Zariski dense in G, so $\rho = \rho^{\sigma}$ on G. ∎

We have $\rho = \rho^{\sigma} \sim (\rho^{(1)})^{\sigma}\oplus\ldots\oplus(\rho^{(M)})^{\sigma}$, so each $(\rho^{(h)})^{\sigma}$ is equivalent to one of the original irreducible $\rho^{(j)}$. Also,

$$(\rho^{(h)})^{\sigma} = (\bigotimes_i \rho_i^{(h)})^{\sigma} = \bigotimes_i (\rho_i^{(h)})^{\sigma}.$$

For each i, $(\rho_i^{(h)})^{\sigma} = (s_{n(h,i)}\circ \mathrm{proj}_i)^{\sigma} = s_{n(h,i)}{}^{\sigma}\circ \mathrm{proj}_i^{\sigma} = s_{n(h,i)}\circ \mathrm{proj}_{\sigma(i)}$. Since $\mathcal{G}$ acts transitively on S, we can choose σ so that $\sigma(i) \in S_0$ for $i \notin S_0$. Hence $n(h,i)$ is zero or one. Thus, for $i \notin S_0$, $\rho_i^{(j)}$ is $s_1\circ \mathrm{proj}_i$ or $s_0\circ \mathrm{proj}_i$.

We have shown that any representation $\rho_{\mathbb{Q}}$ of $G_{\mathbb{Q}}$ such that ρ defines a GTFAV is equivalent, over $\mathbb{C}$, to a representation of $G_{\mathbb{R}}$ of the form

$$\rho \sim \bigoplus_j \mu_j \left(\bigotimes_{i\in A_j} \mathrm{proj}_i \right)$$

where A_j is a subset of S, and $A_j \cap S_0$ contains at most one element. We have also shown that ρ is invariant under the action of $\mathcal{G}$.

For an atom a, once an isomorphism is fixed between $B_{a,\mathbb{R}}$ and $SL_2(\mathbb{R})$ or $SU(2)$, the representation ρ_a of $G_{\mathbb{R}}$ is equivalent to proj_a. Hence we have shown that ρ is equivalent over $\mathbb{C}$ to the polymer representation $\rho_{\mathbb{A}}$, where $\mathbb{A} = \Sigma\, \mu_j A_j$. ∎

§9. Proof of the Second Part

B. Multiplicity and Rationality of the Representation

There are three reasons that the multiplicity of an admissible representation may have to be increased.

9.1. The construction of a GTFAV in this paper uses a polymer of the form $\mathbb{A} = \sum_{\sigma \in \mathcal{G}} \sigma(A)$, where A is a molecule. While $\mathbb{A}$ is clearly $\mathcal{G}$-invariant, it may not be the smallest $\mathcal{G}$-invariant polymer containing A. We call a polymer prime if it contains no proper $\mathcal{G}$-invariant subpolymers.

Proposition 9.1. Every $\mathcal{G}$-invariant polymer is the sum of prime polymers. If $\mathbb{A} = \sum_{\sigma \in \mathcal{G}} \sigma(A)$, then $\mathbb{A} = \mu \mathbb{A}'$, where $\mathbb{A}'$ is prime. Every prime polymer is homogeneous.

Proof. Suppose that a polymer $\mathbb{A}$ is $\mathcal{G}$-invariant, but not prime. Then there is a subpolymer $\mathbb{A}' \neq \mathbb{A}$ that is $\mathcal{G}$-invariant. Since the action of $\mathcal{G}$ on the set of atoms is one-to-one and onto, the action of $\mathcal{G}$ on molecules is also one-to-one. Because $\mathbb{A}$ is $\mathcal{G}$-invariant, σ is an automorphism on the set of molecules in $\mathbb{A}$ for $\sigma \in \mathcal{G}$. This is also true of $\mathbb{A}'$, so σ must map $\mathbb{A} - \mathbb{A}'$ to itself. Hence $\mathbb{A}$ is the sum of two proper $\mathcal{G}$-invariant subpolymers. Because $\mathbb{A}$ has finitely many molecules, we will have reduced $\mathbb{A}$ to the sum of prime polymers in a finite number of steps.

Let $\mathcal{G}_A$ be subgroup of all $\sigma \in \mathcal{G}$ such that $\sigma(A) = A$. Clearly $\sigma(A) = \sigma'(A)$ if and only if σ and σ' are in the same coset of $\mathcal{G}_A$.

Hence $\mathbb{A} = \sum_{\sigma \in \mathcal{G}} \sigma(A) = |\mathcal{G}_A| \sum_{\sigma \in \mathcal{G}/\mathcal{G}_A} \sigma(A) = |\mathcal{G}_A|\mathbb{A}'$. The set of coset representatives for $\mathcal{G}/\mathcal{G}_A$ acts transitively on the molecules of $\mathbb{A}'$, so $\mathbb{A}'$ has no $\mathcal{G}$-invariant subpolymer. Hence $\mathbb{A}'$ is prime.

On the other hand, any prime polymer containing A must also contain all the distinct $\sigma(A)$ where $\sigma \in \mathcal{G}$. So every prime polymer is of the form $\sum_{\sigma \in \mathcal{G}/\mathcal{G}_A} \sigma(A)$, and, in particular, is homogeneous. ∎

In order to construct a GTFAV, we need a polymer (or a sum of polymers) of the form $\sum_{\sigma \in \mathcal{G}} \sigma(A)$. Suppose the given polymer $\mathbb{A}$ is prime. Then $|\mathcal{G}_A|\mathbb{A} = \sum_{\sigma \in \mathcal{G}} \sigma(A)$, where A is any molecule of $\mathbb{A}$. If $\mathbb{A} = \mu\mathbb{A}'$, where $\mathbb{A}'$ is prime, then $\mu|\mathcal{G}_A|\mathbb{A}'$ is of the desired form. If $\mathbb{A} = \Sigma\mu_i\mathbb{A}_i$, where $\mathbb{A}_i$ is prime, then $\Sigma\mu_i|\mathcal{G}_{A_i}|\mathbb{A}_i$ is a sum of polymers of the desired form. If we let μ be the least common multiple of the integers $|\mathcal{G}_{A_i}|$, then $\mu\mathbb{A}$ is the sum of polymers of the desired form. Henceforth we will assume that the given $\mathcal{G}$-invariant polymer $\mathbb{A}$ is $\sum_{\sigma \in \mathcal{G}} \sigma(A)$.

9.2. While the polymer reprentation $\rho_{\mathbb{A}}$ is convenient to describe, it is not always defined over $\mathbb{Q}$. Because $\rho_{\mathbb{A}}$ is the direct sum of tensor products of the representations proj_i, our proof of Proposition 8.1 shows that $\rho_{\mathbb{A}}$ is an algebraic group homomorphism (but not necessarily an algebraic group representation over $\mathbb{Q}$.) In this section, we construct an algebraic group representation

$P_{\mathbb{A}}$ of G, defined over $\mathbb{Q}$, and $\mathbb{C}$-equivalent to $\mu\rho_{\mathbb{A}}$ for some multiplicity μ. (Actually, μ is one or two.)

In order to descrive $P_{\mathbb{A}}$, we will use the scalar restriction functor $R_{K/\mathbb{Q}}$ for algebras.

If B is a k-algebra, and $k \subseteq K$ we have the natural embedding $i : B \to B \otimes_k K$ by $b \mapsto b \otimes 1$. Since the scalar restriction functor gives an isomorphic ring, we have the induced inclusion

$$R_{K/\mathbb{Q}}(B) \to R_{K/\mathbb{Q}}(B \otimes_k K).$$

We denote this inclusion by $r_{K/\mathbb{Q}}(i)$. The new symbol is needed because i is not a morphism of K-algebras.

If φ is a k-algebra homomorphism from a k-algebra B to a K-algebra C, then φ induces a K-algebra homomorphism $\varphi : B \otimes_k K \to C$. So we have

$$R_{K/\mathbb{Q}}(\varphi) : R_{K/\mathbb{Q}}(B \otimes_k K) \to R_{K/\mathbb{Q}}(C).$$

Combining this with the result of the previous paragraph, we have

$$R_{K/\mathbb{Q}}(\varphi) \circ r_{K/\mathbb{Q}}(i) = r_{K/\mathbb{Q}}(\varphi) : R_{k/\mathbb{Q}}(B) \to R_{K/\mathbb{Q}}(C).$$

We can also define $R_{K/\mathbb{Q}}(\varphi)$ and $r_{K/\mathbb{Q}}(\varphi)$ for multilinear maps and for multiplicative maps. The resulting maps are again multilinear and multiplicative, respectively.

For an atom a, let $\tilde{\varphi}_a : B \to B_a$ be the map defined in §4. For a molecule $A = \{a_1, \ldots, a_r\}$, define

$$\tilde{\varphi}_A : B \to B_A = \bigotimes_{a_i \in A} B_{a_i}$$

by

$$\tilde{\varphi}_A(b) = \tilde{\varphi}_{a_1}(b)\otimes\ldots\otimes\tilde{\varphi}_{a_r}(b).$$

Then $\tilde{\varphi}_A$ is multiplicative (but not linear).

Let F_A be a minimal left ideal of B_A; denote the action, by left multiplication, of B_A on F_A by ℓ_A. In fact, $\ell_A : B_A \to \mathrm{End}_K(F_A)$ is a K-algebra homomorphism. We have the following maps:

$$\begin{array}{ccccc} B & \xrightarrow{\tilde{\varphi}_A} & B_A & \xrightarrow{\ell_A} & \mathrm{End}_K(F_A) \\ \cup & & \cup & & \cup \\ B^x & \xrightarrow{\tilde{\varphi}_A} & B_A{}^x & \xrightarrow{\ell_A} & \mathrm{Aut}_K(F_A) \end{array} .$$

Since $\tilde{\varphi}_A$ is multiplicative and ℓ_A is a K-algebra homomorphism, $\ell_A \circ \tilde{\varphi}_A$ is multiplicative, and so $\ell_A \circ \tilde{\varphi}_A$ induces a group homomorphism of B^x into $\mathrm{Aut}_K(F_A)$. Hence we have a representation of B^x or any subgroup of B^x into $\mathrm{Aut}_K(F_A)$.

Now since $R_{K/\mathbb{Q}}(B) \cong B$ and $R_{K/\mathbb{Q}}(B_A) \cong B_A$ canonically as rings, we have the map

$$r_{K/\mathbb{Q}}(\tilde{\varphi}_A) : R_{K/\mathbb{Q}}(B) \to R_{K/\mathbb{Q}}(B_A).$$

Since $\tilde{\varphi}_A$ is multiplicative, so is $r_{K/\mathbb{Q}}(\tilde{\varphi}_A)$.

Because $R_{K/\mathbb{Q}}(B_A) \cong B_A$ as rings, their ideals are in one-to-one correspondence. Hence $R_{K/\mathbb{Q}}(F_A)$ is a minimal left ideal of

$R_{K/\mathbb{Q}}(F_A)$. The K-algebra homomorphism ℓ_A induces the $\mathbb{Q}$-algebra homomorphism

$$R_{K/\mathbb{Q}}(\ell_A) : R_{K/\mathbb{Q}}(B_A) \to \mathrm{End}_{\mathbb{Q}}(R_{K/\mathbb{Q}}(F_A)),$$

and the action of $R_{K/\mathbb{Q}}(\ell_A)$ on $R_{K/\mathbb{Q}}(F_A)$ is left multiplication.

For brevity, we set $\tilde{\Phi}_A = r_{K/\mathbb{Q}}(\tilde{\varphi}_A)$ and $L_A = R_{K/\mathbb{Q}}(\ell_A)$. Combining the two maps, we have

$$\begin{array}{ccccc} R_{k/\mathbb{Q}}(B) & \xrightarrow{\tilde{\Phi}_A} & R_{K/\mathbb{Q}}(B_A) & \xrightarrow{L_A} & \mathrm{End}_{\mathbb{Q}}(R_{K/\mathbb{Q}}(F_A)) \\ \cup & & \cup & & \cup \\ R_{k/\mathbb{Q}}(B)^{\times} & \xrightarrow{\tilde{\Phi}_A} & R_{K/\mathbb{Q}}(B_A)^{\times} & \xrightarrow{L_A} & \mathrm{Aut}_{\mathbb{Q}}(R_{K/\mathbb{Q}}(F_A)). \end{array}$$

Because $\tilde{\Phi}_A$ is multiplicative and L_A is an algebra homomorphism, $L_A \circ \tilde{\Phi}_A$ is multiplicative, and induces a group homomorphism of $R_{k/\mathbb{Q}}(B)^{\times}$ into $\mathrm{Aut}_{\mathbb{Q}}(R_{K/\mathbb{Q}}(F_A))$. In particular, since $G_{\mathbb{Q}} \subseteq R_{k/\mathbb{Q}}(B)^{\times}$, $L_A \circ \tilde{\Phi}_A$ restricted to $G_{\mathbb{Q}}$ gives a representation of $G_{\mathbb{Q}}$ into $\mathrm{Aut}_{\mathbb{Q}}(R_{K/\mathbb{Q}}(F_A))$. Let $P_A = L_A \circ \tilde{\Phi}_A | G_{\mathbb{Q}}$.

Although our objective is to construct a representation of G into $\mathrm{Aut}_{\mathbb{Q}}(R_{K/\mathbb{Q}}(F_A))$, and later to construct a bilinear form and a complex structure on F_A, we will generally do this for the algebra $R_{K/\mathbb{Q}}(B_A)$, then restrict to the ideal $R_{K/\mathbb{Q}}(F_A)$. We use the following notations:

$\hat{\ell}_A$ is left multiplication of B_A on B_A, giving an algebra homomorphism of B_A into $\mathrm{End}_K(B_A)$

$\hat{L}_A$ is left multiplication of $R_{K/\mathbb{Q}}(B_A)$ on $R_{K/\mathbb{Q}}(B_A)$

$\hat{P}_{\mathbb{A}} = \hat{L}_A \circ \tilde{\Phi}_A | G_{\mathbb{Q}}$.

Sometimes we will use $\hat{F}_A$ to denote B_A.

Let $\tilde{\Phi}_{A,\mathbb{R}}$ and $L_{A,\mathbb{R}}$ be the scalar extensions of $\tilde{\Phi}_A$ and L_A from $\mathbb{Q}$ to $\mathbb{R}$. Then

$$\tilde{\Phi}_{A,\mathbb{R}} : G_{\mathbb{R}} \to (R_{K/\mathbb{Q}}(B_A) \otimes_{\mathbb{Q}} \mathbb{R})^{x} = \prod_{\sigma \in G} (B_{\sigma(A),\mathbb{R}})^{x} = B^{x}_{\mathbb{A},\mathbb{R}} .$$

Similarly, $\tilde{\Phi}_{A,\mathbb{C}}$ maps $G_{\mathbb{C}}$ into $B^{x}_{\mathbb{A},\mathbb{C}}$. We also have

$$L_{A,\mathbb{R}} : B^{x}_{\mathbb{A},\mathbb{R}} \to \mathrm{Aut}_{\mathbb{R}}(R_{K/\mathbb{Q}}(F_A) \otimes \mathbb{R})$$

and

$$L_{A,\mathbb{C}} : B^{x}_{\mathbb{A},\mathbb{C}} \to \mathrm{Aut}_{\mathbb{C}}(R_{K/\mathbb{Q}}(F_A) \otimes \mathbb{C}).$$

Combining these maps and restricting to $G_{\mathbb{R}}$ and $G_{\mathbb{C}}$, we have representations

$$P_{\mathbb{A},\mathbb{R}} : G_{\mathbb{R}} \to \mathrm{Aut}_{\mathbb{R}}(R_{K/\mathbb{Q}}(F_A) \otimes \mathbb{R})$$

$$P_{\mathbb{A},\mathbb{C}} : G_{\mathbb{C}} \to \mathrm{Aut}_{\mathbb{C}}(R_{K/\mathbb{Q}}(F_A) \otimes \mathbb{C}).$$

Similarly, we have

$$\hat{P}_{\mathbb{A},\mathbb{R}} : G_{\mathbb{R}} \to \mathrm{Aut}_{\mathbb{R}}(R_{K/\mathbb{Q}}(B_A) \otimes \mathbb{R})$$

$$\hat{P}_{\mathbb{A},\mathbb{C}} : G_{\mathbb{C}} \to \mathrm{Aut}_{\mathbb{C}}(R_{K/\mathbb{Q}}(B_A) \otimes \mathbb{C}).$$

We wish to compare the complex equivalence class of $P_{\mathbb{A}}$ with that of $\rho_{\mathbb{A}}$.

First we need to know more about the ideal $F_{\mathbb{A}}$.

Proposition 9.2. B_A is isomorphic either to $M_N(K)$ for some N, or to $M_N(B(A))$ for some N, where B(A) is a division quaternion algebra over K.

Proof. If we prove the result for $|A| = 2$, the rest will follow by induction. So let $A = (i,j)$. Let $h(B) = (h_1(B), h_2(B), \ldots, h_r(B) \ldots)$ $\in \mathbb{Z}/2\mathbb{Z} \times \ldots \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \times \ldots$ be the Hasse invariant of a simple algebra B in the Brauer group of K. The first m places are understood to be the infinite places of K. The algebra B is trivial (that is, isomorphic to $M_N(K)$) if and only if $h(B) = 0$. By the remarks in the appendix on central simple algebras and Theorem A.2, B is equivalent to a quaternion algebra B' (that is, isomorphic to $M_N(B')$) if and only if

(i) $|\mathrm{Ram}(B)|$ is finite and even and

(ii) for every place $i \in \mathrm{Ram}(B)$, $h_i(B) = \frac{1}{2}$.

So we must show that $B_i \otimes B_j$ satisfies these conditions, or that $h(B_i \otimes B_i) = 0$.

We will abbreviate Ram (B_i) by R_i, $\mathrm{Ram}(B_j)$ by R_j, and $\mathrm{Ram}(B_i \otimes B_j)$ by R. Let p(S) be the number of elements in the set S, mod 2. First suppose that R_i and R_j are disjoint. Then $R = R_i \cup R_j$ clearly satisfies (i) and (ii). Now, if R_i and R_j are not disjoint, set $T = R_i \cap R_j$. For $\ell \in T$, $h_\ell(B_1 \otimes B_2)$ $= h_\ell(B_1) + h_\ell(B_2) = \frac{1}{2} + \frac{1}{2} = 0$. Since $R_i - T$ and $R_j - T$ are disjoint, $R = (R_i - T) \cup (R_j - T)$, so that R is finite. Now $p(R) = p(R_i) - p(T) + p(R_j) - p(T) \equiv p(R_i) + p(R_j) \equiv 0 \mod 2$. Condition (ii) is clearly satisfied. We have shown that $B_i \otimes B_j \cong M_2(B')$ for a quaternion algebra B'. If B' is not a division algebra, then $B' \cong M_2(K)$, so $B_i \otimes B_j \cong M_4(K)$. ▫

We have shown that B_A is isomorphic to $M_n(K)$ or $M_N(B(A))$. In fact, if $B_A \cong M_N(K)$, $N = 2^r$, where $r = |A|$, and, if $B_A \cong M_{N'}(B(A))$, $N' = 2^{r-1}$.

Case (i). If $B_A \cong M_N(K)$, then $F_A \cong K^N$, the direct sum of N copies of K. The representation $P_{\mathbb{A}}$ maps $G_{\mathbb{Q}}$ into $\mathrm{Aut}_{\mathbb{Q}}(R_{K/\mathbb{Q}}(F_A))$. Since $F_A \cong K^N$, $R_{K/\mathbb{Q}}(F_A) \cong \mathbb{Q}^{Nd}$, where $d = |K : \mathbb{Q}|$. Hence $G_{\mathbb{Q}}$ is mapped into $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}^{Nd}) \subseteq \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}^{Nd}) \cong M_{Nd}(\mathbb{Q})$. Also, $P_{\mathbb{A},\mathbb{C}}$ maps $G_{\mathbb{C}}$ into $\mathrm{Aut}_{\mathbb{C}}(\mathbb{Q}^{Nd} \otimes \mathbb{C}) \subseteq \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{Nd}) \cong M_{Nd}(\mathbb{C})$. Since $G_{\mathbb{Q}} \subseteq G_{\mathbb{C}}$, we can consider $P_{\mathbb{A},\mathbb{C}}$ as mapping $G_{\mathbb{Q}}$ into $M_{Nd}(\mathbb{C})$.

Case (ii). If $B_A \cong M_{N'}(B(A))$, then $F_A \cong B(A)^{N'}$. Then the representation $P_{\mathbb{A}}$ maps $G_{\mathbb{Q}}$ into $\mathrm{Aut}_{\mathbb{Q}}(R_{K/\mathbb{Q}}(B(A)^{N'}))$. Also, $P_{\mathbb{A},\mathbb{C}}$ maps $G_{\mathbb{C}}$ into $\mathrm{Aut}_{\mathbb{C}}(R_{K/\mathbb{Q}}(B(A)^{N'}) \otimes_{\mathbb{Q}} \mathbb{C})$. We have

$$R_{K/\mathbb{Q}}(B(A)^{N'}) \otimes \mathbb{C} \cong [R_{K/\mathbb{Q}}(B(A)) \otimes \mathbb{C}]^{N'}$$

$$\cong [\bigoplus_{\sigma \in \mathcal{G}} (B(A) \underset{\sigma}{\otimes} \mathbb{C})]^{N'} \cong [M_2(\mathbb{C})]^{N'd}.$$

Hence $P_{\mathbb{A},\mathbb{C}}$ maps $G_{\mathbb{C}}$, and $G_{\mathbb{Q}}$, into $\mathrm{Aut}_{\mathbb{C}}([M_2(\mathbb{C})]^{N'd}) \subseteq \mathrm{End}_{\mathbb{C}}([M_2(\mathbb{C})]^{N'd}$ $\subseteq \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{4N'd}) \cong M_{4N'd}(\mathbb{C})$.

Now we return to the representation $\rho_{\mathbb{A}}$. Recall that we have assumed that $\mathbb{A}$ is of the form $\mathbb{A} = \sum_{\sigma \in \mathcal{G}} \sigma(A)$. The representation $\rho_{\mathbb{A}}$ sends $G_{\mathbb{Q}} \subseteq B$ into $\bigoplus_{\sigma \in \mathcal{G}} (\bigotimes_{a_i \in A} (B_{\sigma(a_i)} \otimes_K \mathbb{C}))$ via the map $\sum_{\sigma \in \mathcal{G}} \tilde{\varphi}_{\sigma(A)}$. We have

$$\bigoplus_{\sigma \in \mathcal{G}} (\bigotimes_{a_i \in A} (B_{\sigma(a_i)} \otimes_K \mathbb{C})) = \bigoplus_{\mathcal{G}}((\bigotimes_{A} B_{\sigma(a_i)}) \otimes_K \mathbb{C})$$

$$= \bigoplus_{\mathcal{G}}(B_{\sigma(A)} \otimes_K \mathbb{C}) = \bigoplus_{\mathcal{G}}(B_A \otimes_{\sigma} \mathbb{C})$$

$$\cong R_{K/\mathbb{Q}}(B_A) \otimes_{\mathbb{Q}} \mathbb{C}.$$

In fact, $\Sigma \tilde{\varphi}_{\sigma(A)} = R_{K/\mathbb{Q}}(\tilde{\varphi}_A)$. On the other hand,

$$\bigoplus_{\mathcal{G}}(\bigotimes_{A}(B_{\sigma(a_i)} \otimes \mathbb{C})) \cong \bigoplus_{\mathcal{G}}(\bigotimes_{\mathbb{A}} M_2(\mathbb{C})) = M_N(\mathbb{C})^d \subseteq M_{Nd}(\mathbb{C})$$

where $N = 2^{|A|}$.

Thus, as set maps of G into $R_{K/\mathbb{Q}}(B_A) \otimes_{\mathbb{Q}} \mathbb{C}$, $\rho_{\mathbb{A}}$ and $P_{\mathbb{A},\mathbb{C}}$ are identical. However, the representation spaces of $\rho_{\mathbb{A}}$ and $P_{\mathbb{A},\mathbb{C}}$ are sometimes different. The representation space of $\rho_{\mathbb{A}}$ is a minimal left ideal of $M_{Nd}(\mathbb{C})$, hence isomorphic to $\mathbb{C}^{Nd}$. In Case (i), the representation space of $P_{\mathbb{A},\mathbb{C}}$ is also $\mathbb{C}^{N}d$. Since $\rho_{\mathbb{A}}$ and $P_{\mathbb{A},\mathbb{C}}$ are the same as set maps, $\rho_{\mathbb{A}} \sim P_{A,\mathbb{C}}$ over $\mathbb{C}$.

In Case (ii), the representation space of $P_{\mathbb{A},\mathbb{C}}$ is isomorphic to $\mathbb{C}^{4N'd} = \mathbb{C}^{2Nd}$, since $N' = 2^{|A|-1}$. The representation space $R_{K/\mathbb{Q}}(F_A) \otimes \mathbb{C}$ is a left ideal of $R_{K/\mathbb{Q}}(B_A) \otimes \mathbb{C}$, hence the direct sum of minimal left ideals. We have just seen that a that a minimal left ideal of $R_{K/\mathbb{Q}}(B_A) \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}^{Nd}$, and the representation space of $P_{\mathbb{A},\mathbb{C}}$ is $\mathbb{C}^{2Nd}$. Hence, in Case (ii), $2\rho_{\mathbb{A}} \sim P_{\mathbb{A},\mathbb{C}}$ over $\mathbb{C}$.

In the construction of the GTFAV, we use the representation $P_{\mathbb{A},\mathbb{R}}$. As we have just seen, $P_{\mathbb{A},\mathbb{R}}$ is equivalent, over $\mathbb{C}$, to $\rho_{\mathbb{A}}$ in Case (i), and to $2\rho_{\mathbb{A}}$ in Case (ii). The representation space of $P_{\mathbb{A},\mathbb{R}}$ is $R_{K/\mathbb{Q}}(F_A) \underset{\mathbb{Q}}{\otimes} \mathbb{R}$. We will call this vector space W. We sometimes write $W_{\mathbb{Q}}$ for $R_{K/\mathbb{Q}}(F_A)$. If we set $\mathbb{A} = (A_1,\ldots,A_d)$, then W is the direct sum of d vector spaces, each corresponding to a molecule. We write this decomposition as $W = W_{A_1} \oplus \ldots \oplus W_{A_d}$, or sometimes $W = W_1 \oplus \ldots \oplus W_d$.

In Case (i), for each i, $W_i \cong 2^r \cdot \mathbb{R}$, where $|A_i| = r$. In Case (ii), $W_i \cong 2 \cdot 2^r$ or $W_i \cong 2^{r-1} \cdot \mathbb{H}$. In the second situation, we consider $\mathbb{H}$ as an $\mathbb{R}$-vector space by letting $\mathbb{H}$ act on itself by the regular representation.

When considering the representation $\hat{P}_{\mathbb{A},\mathbb{R}}$, we denote the representation space $R_{K/\mathbb{Q}}(B_A) \otimes \mathbb{R}$ by $\hat{W}$. We set $B_{A_i} \otimes \mathbb{R} = \hat{W}_{A_i}$, so that $\hat{W} = \hat{W}_{A_1} \oplus \ldots \oplus \hat{W}_{A_d}$.

We set $P_{A_i} = P_{\mathbb{A},\mathbb{R}} | W_{A_i}$, so that $P_{\mathbb{A},\mathbb{R}} = \oplus P_{A_i}$, and

$\hat{P}_{A_i} = \hat{P}_{\mathbb{A},\mathbb{R}} | \hat{W}_{A_i}$, so that $\hat{P}_{\mathbb{A},\mathbb{R}} = \oplus \hat{P}_{A_i}$.

The third reason that a multiple of $\mathbb{A}$ is needed will appear later when we construct a bilinear form and complex structure.

§10. Construction of the Fiber Bundle $V \xrightarrow{\pi} U$

In this section, we find an arithmetic subgroup Γ of G and a lattice L in W.

Theorem 10.1. ([V], IV, Theorem 1.1): Let $G_{nc} = \prod_{i \in S_0} B^1_{i,\mathbb{R}} = \prod_{i \in S_0} SL(2,\mathbb{R})$. Let $\mathfrak{O}$ be an order in B, and $\mathfrak{O}^1$ the group of units of $\mathfrak{O}$ with norm one. Denote by $\tilde{\varphi} = \Sigma\, \tilde{\varphi}_i$ the standard injection of B into $\prod_{i \in S_0} B_{i,\mathbb{R}}$. Then the group $\tilde{\varphi}(\mathfrak{O}^1)$ is isomorphic to $\mathfrak{O}^1$. It is a discrete subgroup of G_{nc}. $G_{nc}/\tilde{\varphi}(\mathfrak{O}^1)$ has finite volume, and is compact if B is a division ring.

In fact, if B is not a division ring, then $B \cong M_2(k)$, and $G_{\mathbb{R}} = G_{nc}$. The construction gives a Satake type GTFAV, with $G_{nc}/\tilde{\varphi}(\mathfrak{O}^1)$ non-compact. This is the reason we have required that $B \neq M_2(k)$.

Proposition 10.2. ([V], IV, Proposition 1.6): Let $\mathfrak{O}^1$ be as above. Then $\mathfrak{O}^1$ contains a torsion-free subgroup of finite index.

Let $\mathfrak{O}$ be any order of B, and $\mathfrak{O}^1$ its group of norm-one units. Let Γ be any torsion-free subgroup of $\mathfrak{O}^1$. If we consider Γ as a subset of $R_{k/\mathbb{Q}}(B)$, then $\Gamma \subseteq G_{\mathbb{Q}}$. So Γ is also a subgroup of $G_{\mathbb{R}}$, via the map $\sum_S \tilde{\varphi}_i$ of B into $\prod_S B_{i,\mathbb{R}}$.

Proposition 10.3. Let C be the maximal compact subgroup of $G_{\mathbb{R}}$.

Then $U = {}_{\Gamma}\backslash G_{\mathbb{R}}/C$ is a compact complex manifold.

Proof. Since $G_{\mathbb{R}} \cong G_{nc} \times G_c$ and G_c is compact, we can identify $G_{\mathbb{R}}/C$ with $G_{nc}/C\cap G_{nc}$, and ${}_{\Gamma}\backslash G_{\mathbb{R}}/C$ with ${}_{(\Gamma\cap G_{nc})}\backslash G_{nc}/{}_{(C\cap G_{nc})}$. By Theorem 10.1, ${}_{(\Gamma\cap G_{nc})}\backslash G_{nc}/{}_{(C\cap G_{nc})}$ is compact, so ${}_{\Gamma}\backslash G_{\mathbb{R}}/C$ is also compact. Since Γ has been chosen to be torsion-free, ${}_{\Gamma}\backslash G_{\mathbb{R}}/C$ is a manifold. Now $G_{\mathbb{R}}/C = \mathfrak{h}^{m_0}$. It is known that $\mathfrak{h}^{m_0}$ is a complex manifold, and $G_{\mathbb{R}}$ acts bihomomorphically on $\mathfrak{h}^{m_0}$. (See, e.g., [S-3], II, §7.) Thus ${}_{\Gamma}\backslash G_{\mathbb{R}}/C$ inherits a $G_{\mathbb{R}}$-invariant complex structure. ∎

In fact, by the general theory in §1, $U = {}_{\Gamma}\backslash G_{\mathbb{R}}/C$ is algebraic.

We recall some number-theoretic definitions and facts. (See, e.g., [W-2], especially V, §1, §2.) For k a number field, a k-lattice in a finite-dimensional k-algebra B is a finitely generated $\mathcal{O}_k$ module that contains a basis for B over k. ($\mathcal{O}_k$ denotes the ring of integers of k.) An $\mathbb{R}$-lattice in an $\mathbb{R}$-algebra is a finitely generated discrete subgroup that contains a basis. Any $\mathbb{R}$-lattice in an n-dimensional $\mathbb{R}$-algebra is isomorphic to $\mathbb{Z}^n$.

An order in a k-algebra is a subring that is also a k-lattice. Let k' be a finite extension of k, $\mathcal{O}_{k'}$ and $\mathcal{O}_k$ their rings of integers, and A' a k'-algebra. If an $\mathcal{O}_{k'}$-module L is a k'-lattice in A', then it is a k-lattice in $R_{k'/k}(A')$. The intersection of two k-lattices is a k-lattice. For any two k-lattices L and L', there exists an integer n such that $n L \subseteq L'$.

Let $\mathcal{O}_B$ be the maximal order of B that contains the order $\mathcal{O}$ used in the selection of Γ. The image of $\mathcal{O}_B$ under $\tilde{\varphi}_i$ is contained in $\mathcal{O}_B \underset{\tilde{\varphi}_i(\mathcal{O}_k)}{\otimes} \mathcal{O}_K = \mathcal{O}_i$ which is an order in B_i. Hence the image of $\mathcal{O}_B$ under $\tilde{\varphi}_A$ is contained in $\mathcal{O}_{a_1} \underset{\mathcal{O}_K}{\otimes} \cdots \underset{\mathcal{O}_K}{\otimes} \mathcal{O}_{a_r}$, which is an order in B_A. Let $\hat{L}$ denote $\mathcal{O}_A$ considered as a subset of $R_{K/\mathbb{Q}}(B_A) = R_{K/\mathbb{Q}}(\hat{F}_A)$. Set $L = \hat{L} \cap W_{\mathbb{Q}}$, where $W_{\mathbb{Q}} = R_{K/\mathbb{Q}}(F_A)$.

<u>Proposition 10.4</u>. L is a $\mathbb{Q}$-lattice in $W_{\mathbb{Q}}$, $\rho(\Gamma)L = L$, and W/L is a real torus.

<u>Proof</u>. Since $\mathcal{O}_A$ is an order in B_A, it is a K-lattice. Hence $\hat{L}$ is a $\mathbb{Q}$-lattice in $R_{K/\mathbb{Q}}(B_A)$. Since F_A is an ideal of B_A, L is a finitely generated $\mathbb{Z}$-module in $W_{\mathbb{Q}}$. We must show that L contains a $\mathbb{Q}$-basis of $W_{\mathbb{Q}}$. Take a basis $\{e_1, \ldots, e_s\}$ for $W_{\mathbb{Q}}$ and extend it to a basis $\{e_1, \ldots, e_t\}$ for $R_{K/\mathbb{Q}}(B_A)$. Let L' be the $\mathbb{Q}$-lattice $\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_t$. Now, there is some integer n so that $nL' \subseteq \hat{L}$. Hence $nL' \cap W_{\mathbb{Q}} \subseteq \hat{L} \cap W_{\mathbb{Q}} = L$. Since $\{ne_1, \ldots, ne_s\}$ is in $nL' \cap W_{\mathbb{Q}}$, and the set is a basis, L is a $\mathbb{Q}$-lattice in $W_{\mathbb{Q}}$.

Now we will show that $\rho(\Gamma)$ maps L onto itself. Since $\rho(\Gamma) = \tilde{\varphi}_A(\Gamma) \subseteq \tilde{\varphi}_A(\mathcal{O}_B) \subseteq \mathcal{O}_A$, and $\mathcal{O}_A$ is a ring, $\rho(\Gamma)\mathcal{O}_A \subseteq \mathcal{O}_A$. By definition of ρ, $\rho(\Gamma)W_{\mathbb{Q}} \subseteq W_{\mathbb{Q}}$. Hence $\rho(\Gamma)L \subseteq L$; that is, for any $\gamma \in \Gamma$, $\rho(\gamma)L \subseteq L$. So also $\rho(\gamma^{-1})L \subseteq L$, and hence

$L = \rho(\gamma)\rho(\gamma^{-1})L \subseteq \rho(\gamma)L$. We have

$$L \subseteq \rho(\gamma)L \subseteq L \text{ for all } \gamma \in \Gamma, \text{ so } \rho(\gamma)L = L.$$

We have shown that L is a finitely generated $\mathbb{Z}$-module in $W_{\mathbb{Q}}$ that contains a basis for $W_{\mathbb{Q}}$. So L is a finitely generated discrete subgroup of $W = W_{\mathbb{Q}} \otimes \mathbb{R}$ which contains a basis; namely, L is an $\mathbb{R}$-lattice in W. Hence $L \cong \mathbb{Z}^{\dim W}$, and W/L is a real torus. ∎

As described in §1, we can form the semidirect product $\Gamma \ltimes L$. We then have a fiber bundle $V = {}_{\Gamma \ltimes L}\backslash^{X \times W} \xrightarrow{\pi} {}_{\Gamma}\backslash^{X} = U$, where the base space U is the compact complex manifold ${}_{\Gamma}\backslash G_{\mathbb{R}}/C$ and, for any $x \in U$, the fiber $\pi^{-1}(x)$ is a torus isomorphic to W/L.

§11. Alternating Bilinear Form

In order to show that ρ defines a GTFAV, we must produce a real-valued bilinear form β on the representation space W such that

(i) β is alternating and nondegenerate,

(ii) β is invariant under the action of $G_{\mathbb{R}}$,

(iii) β takes integral values on $L \times L$, and

(iv) if J_x is the complex structure on the fiber $W_x \cong W$ above $x \in X$, then $\beta(u, J_x v)$ is symmetric and positive definite.

Properties (i) and (ii) show that ρ is a symplectic representation. Property (iv) will be used to explicitly construct the Eichler map. Properties (i), (iii), and (iv) provide each fiber W_x/L of the fiber bundle $V \to U$ with the structure of polarized abelian variety.

In this section we construct β and show that it is alternating. The nondegeneracy of β will follow from the fact that $\beta(u, J_x v)$ is positive definite.

We will define $\hat{\beta}$ on $R_{K/\mathbb{Q}}(B_A)$, then restrict to β on the ideal $R_{K/\mathbb{Q}}(F_A)$.

11.1. The construction of β differs according to the type of the molecule. For an admissible molecule A, we say A is of type I if $|A \cap S_0| = 1$, and of type II if $A \cap S_0 = \emptyset$. If every molecule in a polymer is of type I, we say the polymer is <u>rigid</u>. If not, we say it is of <u>mixed type</u>. Note that, if S_0 is non-empty, then there is at least one molecule in $\mathbb{A}$ of type I: if $a \in A \in \mathbb{A}$ is an atom, there exists $\sigma \in \mathcal{G}$ so that $\sigma(a) \in S_0$, and $\sigma(a) \in \sigma(A) \in \sigma(\mathbb{A}) = \mathbb{A}$. So if all molecules of A are of type II, then $S = S_1$. Hence $G_{\mathbb{R}}$ is compact, X is a single point, and the family of abelian varieties is a single abelian variety.

First we prove a few lemmas necessary for the construction of β.

<u>Lemma 11.1</u>. There exists a nonzero element η in $\mathcal{O}_B$ such that $\eta' = -\eta$, and for every $i \in S$, $\varphi_i(\nu(\eta))$ is positive.

<u>Proof</u>. By Theorem A.3, B contains a field L which is a totally imaginary quadratic extension of k. Hence $L = k(\sqrt{d})$, where $d \in k$ is totally negative; that is, $\varphi_i(d) < 0$ for all $i \in S$. Let $\eta = \sqrt{d}$. The involution on B induces the nontrivial Galois automorphism σ on L, where $\sigma(\eta) = -\eta$. So, if we consider η as an element of B, then $\eta' = -\eta$. Now $\nu(\eta) = \eta\eta'$, so $\varphi_i(\nu(\eta)) = \varphi_i(\eta\eta') = \varphi_i(-d) > 0$ for all $i \in S$. If η is not in $\mathcal{O}_B$, we can multiply it by a suitably large rational integer N; the same properties hold for $N\eta$. ∎

We note for future reference that, if $\alpha \in \mathcal{O}_k$, then $\alpha' = \alpha$. If α is a nonzero element of $\mathcal{O}_k$, then $\alpha\eta \in \mathcal{O}_B$, $(\alpha\eta)' = -\alpha\eta$, and $\varphi_i(\nu(\alpha\eta)) = \varphi_i(\alpha^2\nu(\eta)) = \varphi_i(\alpha)^2\nu(\eta) > 0$.

For any simple algebra B over a field K, let τ_B denote the reduced trace in B.

Lemma 11.2. $\tau_{B_A}(b) = \prod_{a_i \in A} \tau_{B_{a_i}}(b_{a_i})$ for $b = \bigotimes_{a_i \in A} b_{a_i} \in B_A = \bigotimes_{a_i \in A} B_{a_i}$.

Proof. The reduced trace is the trace of the reduced representation, where $\rho_{red}(b) = b \otimes 1 \in B \otimes_K \mathbb{C} \cong M_N(\mathbb{C})$. Under the isomorphism

$$(B_{a_1} \otimes_K \mathbb{C}) \otimes \ldots \otimes (B_{a_r} \otimes_K \mathbb{C}) \xrightarrow{\cong} B_A \otimes_K \mathbb{C},$$

the element $(b_{a_1} \otimes 1) \otimes \ldots \otimes (b_{a_r} \otimes 1)$ goes to $(b_{a_1} \otimes \ldots \otimes b_{a_r}) \otimes 1$. Hence the reduced representation on B_A is the tensor product of the reduced representations on B_{a_i}. The trace of a tensor product of matrices is the product of the traces of the factors so $\tau_{B_A}(b) = \prod_{a_i \in A} \tau_{B_{a_i}}(b_{a_i})$ ∎.

Lemma 11.3. The canonical involution in $B_i = B \otimes_{\varphi_i} K$ is induced from that on B by $(b \otimes_{\varphi_i} \alpha)' = b' \otimes_{\varphi_i} \alpha$. There is an involution on $B_A = B_{a_1} \otimes \ldots \otimes B_{a_r}$ defined by $(b_{a_1} \otimes \ldots \otimes b_{a_r})' = b'_{a_1} \otimes \ldots \otimes b'_{a_r}$.

Proof. The map $(\ )'$ on B_i is well defined because

$$b_1' \otimes \alpha + b_2' \otimes \alpha = (b_1' + b_2') \otimes \alpha = ((b_1 + b_2) \otimes \alpha)'$$

$$= (b_1 \otimes \alpha + b_2 \otimes \alpha)' = b_1' \otimes \alpha + b_2' \otimes \alpha,$$

$$b'\otimes\alpha_1 + b'\otimes\alpha_2 = b'\otimes(\alpha_1+\alpha_2) = (b\otimes(\alpha_1+\alpha_2))'$$

$$= (b\otimes\alpha_1)' + (b\otimes\alpha_2)' = b'\otimes\alpha_1 + b'\otimes\alpha_2,$$

and, for $\lambda \in k$,

$$(\lambda b\otimes\alpha)' = (\lambda b)'\otimes\alpha = \lambda b'\otimes\alpha = b'\otimes\varphi_i(\lambda)\alpha = b'\otimes\lambda\cdot\alpha$$

$$= (b\otimes\lambda\cdot\alpha)'.$$

The map clearly fixes K, so the map is K-linear. Moreover, the elements of K are the only elements fixed by $(\ \ )'$. Fix a basis $\{e_1,\ldots,e_n\}$ for K over $\varphi(k)$. Then $x \in B_i$ is uniquely expressed as $x = \Sigma\, b_i \underset{\varphi}{\otimes} e_i$, with $b_i \in B$. So $x' = \Sigma\, b_i'\otimes e_i$. If $x = x'$, then $b_i' = b_i$, by the uniqueness of the expression. But the only elements of B fixed by the involution are in k, so $b_i \in k$; hence $x \in k\underset{\varphi}{\otimes}K \cong K$.

The map is anti-multiplicative, since

$$((b_1\otimes\alpha_1)(b_2\otimes\alpha_2))' = (b_1b_2\otimes\alpha_1\alpha_2)' = (b_1b_2)'\otimes\alpha_1\alpha_2$$

$$= b_2'b_1'\otimes\alpha_2\alpha_1 = (b_2\otimes\alpha_2)'(b_1\otimes\alpha_1)'.$$

Also $(b\otimes\alpha)'' = (b'\otimes\alpha)' = b''\alpha = b\otimes\alpha$, so this map must be the canonical involution on B_i.

Now we consider the map $(\ \ )'$ on B_A. By the definition of the tensor product, the map is K linear. The map is anti-multiplicative:

$$((b_{a_1}\otimes\ldots\otimes b_{a_r})(c_{a_1}\otimes\ldots\otimes c_{a_r}))' = (b_{a_1}c_{a_1}\otimes\ldots\otimes b_{a_r}c_{a_r})'$$

$$= c'_{a_1}b'_{a_1}\otimes\ldots\otimes c'_{a_r}b'_{a_r} = (c_{a_1}\otimes\ldots\otimes c_{a_r})'(b_{a_1}\otimes\ldots\otimes b_{a_r})'.$$

The map composed with itself is the identity, since this is true of each component involution. Hence the map is an involution on B_A. ∎

Lemma 11.4. For $u \in B_A$, $\tau_{B_A}(u') = \tau_{B_A}(u)$.

Proof. The element u is a linear combination of elements of the form $u_1\otimes\ldots\otimes u_r$, where $u_r \in B_{a_r}$. If $\tau_{B_{a_i}}(u_i') = \tau_{B_{a_i}}(u_i)$, then clearly the lemma is proved, by 11.2. Under the reduced representation, the involution in B_{a_i} extends to the unique involution in $M_2(\mathbb{C})$. For $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in M_2(\mathbb{C})$, $\begin{pmatrix} a & b \\ c & d\end{pmatrix}' = \begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$. If we set $\rho_{red}(u_i) = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, then $\tau(u_i') = tr(\rho_{red}(u_i'))$ $= tr(\rho_{red}(u_i)') = tr\begin{pmatrix} d & -b \\ -c & a\end{pmatrix} = a+d = tr(\rho_{red}(u_i)) = \tau(u_i)$. ∎

Lemma 11.5. Let $f_1,\ldots,f_n$ be bilinear forms on vector spaces $V_1,\ldots,V_n$ respectively. Then there is a unique bilinear form $f = \otimes f_i$ on $\otimes V_i$ defined by

$$f(\otimes v_i, \otimes w_i) = \prod f_i(v_i, w_i).$$

If all the f_i are nondegenerate, then so is f. If an odd number of the f_i are alternating and the rest are symmetric, then f is alternating. If an even number of the f_i are alternating, and the rest are symmetric, then f is symmetric.

Proof. The first two statements are proved in [B], Chapter 9, §1, No.9. Suppose that all the f_i are either alternating or symmetric, and that the number of alternating form suffices to check vectors of the form $v = \otimes v_i$, $w = \otimes w_i$. We have

$$f(w,v) = \Pi f_i(w_i,v_i)$$

$$= \prod_{f_i \text{alt.}} (-f_i(v_i,w_i)) \prod_{f_j \text{sym.}} (v_j,w_j) = (-1)^m f(v,w).$$

So f is alternating if m is odd, and symmetric if m is even. ∎

11.2. Type I. Here we construct a bilinear form β^I. We will see in §13 that, for the complex structure J_x to be defined in §12, $\beta^I(u,J_x v)$ is a symmetric form that is positive definite on subspaces W_i corresponding to molecules of type I.

By Lemma 11.1, we can choose $\eta \in \mathcal{O}_B^x$ such that $\eta' = -\eta$ and $\varphi_i(\nu(\eta))$ is positive for all $i \in S$.

For use in §13, we have the following lemma.

Lemma 11.6. There exist matrices M_i in $M_2(\mathbb{R})$ and real numbers λ_i such that

$$\tilde{\varphi}_i(\eta) = M\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}M^{-1} \in M_2(\mathbb{R}) \cong B_i \underset{K}{\otimes} \mathbb{R}$$

where $i \in S_0$.

Proof. Since $B_i \otimes \mathbb{R}$ is a matrix algebra, the trace of $\tilde{\varphi}_i(\eta)$ as a matrix is $\tau(\tilde{\varphi}_i(\eta))$, and the determinant of $\tilde{\varphi}_i(\eta)$ is

$\nu(\tilde{\varphi}_i(\eta))$. Since $\eta \in B$, we have, from Lemma 11.3,

$$\mathrm{tr}(\tilde{\varphi}_i(\eta)) = \tau(\tilde{\varphi}_i(\eta)) = \tilde{\varphi}_i(\eta) + \tilde{\varphi}_i(\eta)'$$
$$= \tilde{\varphi}_i(\eta) + \tilde{\varphi}_i(\eta') = \tilde{\varphi}_i(\eta) + \tilde{\varphi}_i(-\eta) = 0.$$

Also,

$$\det(\tilde{\varphi}_i(\eta)) = \nu(\tilde{\varphi}_i(\eta)) = \tilde{\varphi}_i(\eta)\tilde{\varphi}_i(\eta)'$$
$$= \tilde{\varphi}_i(\eta\eta') = \tilde{\varphi}_i(\nu(\eta)) > 0,$$

by the choice of η. Hence there is some $\lambda_i \in \mathbb{R}$ such that $\det(\tilde{\varphi}_i(\eta)) = \lambda_i^2$. We may assume that $\lambda_i > 0$. So the characteristic polynomial of $\tilde{\varphi}_i(\eta)$ is $X^2 + \lambda_i^2 = 0$, and its eigenvalues are $\pm\sqrt{-1}\,\lambda_i$.

The matrix $\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$ also has trace zero and determinant λ_i^2. Since $\tilde{\varphi}_i(\eta)$ and $\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$ are both matrices with distinct eigenvalues and the same characteristic polynomial, they are similar. Since they are real matrices, they are similar via real matrices; that is, there exists $M \in M_2(\mathbb{R})$ such that

$$\tilde{\varphi}_i(\eta) = M\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}M^{-1}. \qquad \blacksquare$$

From now on, we will fix isomorphisms between $B_i \otimes \mathbb{R}$ and $M_2(\mathbb{R})$ for $i \in S_0$ so that

$$\tilde{\varphi}_i(\eta) \to \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}.$$

If $1 \notin S_0$, we fix an isomorphism arbitrarily. (Actually, in this section and in §13, we will adjust η by a multiplicative constant $\alpha \in \mathfrak{O}_k$. However, here we will assume that the proper choice of η has already been made.)

Define maps e and f from $B_a \times B_a$ to B_a by

$$e(u,v) = uv'$$

$$f(u,v) = u\tilde{\varphi}_a(\eta)v'.$$

Let τ be the reduced trace on B_a. Then $\tau \circ e$ and $\tau \circ f$ map $B_a \times B_a$ to K.

Lemma 11.7. $\tau \circ e$ and $\tau \circ f$ are K-bilinear forms $\tau \circ e$ is symmetric, and $\tau \circ f$ is alternating.

Proof. Since the involution is K-linear, the maps e and f are K-bilinear. Since τ is K-linear, $\tau \circ e$ and $\tau \circ f$ are K-bilinear.

Recall that $\tau(v') = \tau(v)$. Then

$$\tau \circ e(u,v) = \tau(uv') = \tau((uv')') = \tau(vu') = \tau \circ e(v,u),$$

so $\tau \circ e$ is symmetric. Also,

$$\begin{aligned}\tau \circ f(u,v) &= \tau(u\tilde{\varphi}_a(\eta)v') = \tau((u\tilde{\varphi}_a(\eta)v')') \\ &= \tau(v\tilde{\varphi}_a(\eta)'u') = \tau(v\tilde{\varphi}_a(\eta')u') \\ &= \tau(v\tilde{\varphi}_a(-\eta)u') = -\tau \circ f(v,u),\end{aligned}$$

so $\tau \circ f$ is alternating.

By Lemma 11.5, we can define a bilinear from E_{a_j} on $B_A \times B_A$ by

$$E_{a_j}\left(\underset{a_i \in A}{\otimes} v_{a_i}, \underset{a_i \in A}{\otimes} w_{a_i}\right) = \tau \circ f(v_{a_j}, w_{a_j}) \prod_{a_i \neq a_j} \tau \circ e(v_{a_i}, w_{a_i}).$$

We sometimes write

$$E_{a_j} = \tau \circ f_{a_j} \otimes \left(\underset{a_i \neq a_j}{\otimes} \tau \circ e_{a_i}\right).$$

By Lemma 11.2, we can also write

$$E_{a_j} = \tau \circ \left[f_{a_j} \otimes \left(\underset{a_i \neq a_j}{\otimes} e_{a_i}\right)\right],$$

where τ is the reduced trace in B_A. Again by Lemma 11.5, E_{a_j} is alternating because $\tau \circ f$ is alternating and $\tau \circ e$ is symmetric.

The map $\sum_{a_j \in A} E_{a_j}$ is again an alternating K-bilinear form on $B_A \times B_A$, since all the summands are. Since this function has values in K, we can apply the trace $tr_{K/\mathbb{Q}}$ of number fields, which has values in $\mathbb{Q}$. Recall that i^* is the identity map from B_A to $R_{K/\mathbb{Q}}(B_A)$. Then we have

$$R_{K/\mathbb{Q}}(B_A) \times R_{K/\mathbb{Q}}(B_A) \xrightarrow{(i^*)^{-1} \times (i^*)^{-1}} B_A \times B_A \xrightarrow{\Sigma E_{a_i}} K \xrightarrow{tr_{K/\mathbb{Q}}} \mathbb{Q}.$$

Let

$$\hat{\beta}^I = tr_{K/\mathbb{Q}} \circ \left(\sum_{a_j \in A} E_{a_j}\right) \circ (i^*)^{-1} \times (i^*)^{-1}.$$

Except when we wish to make a distinction, we will use i^* to identify B_A and $R_{K/\mathbb{Q}}(B_A)$. So we abbreviate $\hat{\beta}^I$ by $tr_{K/\mathbb{Q}}\circ\Sigma E_{a_j}$. We have

$$(11.8) \qquad \hat{\beta}^I = tr_{K/\mathbb{Q}}\circ \sum_{a_j\in A} E_{a_j} = tr_{K/\mathbb{Q}}\circ \sum_{a_j\in A} \tau[f_{a_j}\otimes(\bigotimes_{i\neq j} e_{a_i})]$$

$$= tr_{K/\mathbb{Q}}\circ\tau\circ \sum_{a_j\in A} f_{a_j}\otimes(\bigotimes_{i\neq j} e_{a_i})$$

since τ is linear.

Set $\beta^I = \hat{\beta}^I | R_{K/\mathbb{Q}}(F_A) \times R_{K/\mathbb{Q}}(F_A)$.

Proposition 11.9. The map β^I

(a) is an alternating $\mathbb{Q}$-bilinear form on $R_{K/\mathbb{Q}}(F_A) \times R_{K/\mathbb{Q}}(F_A)$,

(b) takes integral values on $L \times L$, and

(c) is invariant under the action of $G_{\mathbb{Q}}$; that is, $\beta^I(P_{\mathbb{A}}(g)(u), P_{\mathbb{A}}(g)(v)) = \beta^I(u,v)$ for all $g \in G_{\mathbb{Q}}$ and $u,v \in R_{K/\mathbb{Q}}(F_A) = W_{\mathbb{Q}}$.

Proof. We will prove (a), (b), and (c) for $\hat{\beta}^I$ on $R_{K/\mathbb{Q}}(B_A)$; the same facts are then clearly true when $\hat{\beta}^I$ is restricted to the ideal $R_{K/\mathbb{Q}}(F_A)$.

(a) Since $tr_{K/\mathbb{Q}}$ and 1^* are $\mathbb{Q}$-linear, and ΣE_{a_j} is K-linear and alternating, $\hat{\beta}^I$ is $\mathbb{Q}$-linear and alternating.

(b) By (11.8), $\hat{\beta}^I = tr_{K/\mathbb{Q}}\circ\tau\circ\Sigma f_{a_j}\otimes(\bigotimes_{i\neq j} e_{a_i})$.

By Lemma 11.3, the bilinear map $\Sigma f_{a_j}\otimes(\bigotimes_{i\neq j} e_{a_i})$ can be written as $u\eta_A v'$, where $u,v \in B_A$, and $\eta_A = \sum_{a_j\in A} 1\otimes\ldots\otimes\tilde{\varphi}_{a_j}(\eta)\otimes\ldots\otimes 1$. Since

$\tilde{\varphi}_{a_j}(\eta) \in \mathcal{O}_{a_j}$, and $1 \in \mathcal{O}_{a_i}$ for all a_i, $\eta_A \in \mathcal{O}_A$.

Since $\mathcal{O}_A$ is an order in the K-algebra B_A, there is a finite set $\{e_1, \ldots, e_s\}$ of generators for $\mathcal{O}_A$ over $\mathcal{O}_K$. So any $v \in \mathcal{O}_A$ can be written $v = \Sigma c_i e_i$, where each $c_i \in \mathcal{O}_K$. Then $v' = \Sigma c_i e_i'$, and the elements e_i', are in B_A, but not necessarily in $\mathcal{O}_A$. However, for each e_i', there is a rational integer d_i such that $d_i e_i' \in \mathcal{O}_A$. Let D be the least common multiple of the d_i. Then $Dv' \in \mathcal{O}_A$ for any $v \in \mathcal{O}_A$.

If we replace η with $D\eta$, then η_A is replaced by $D\eta_A$, and for any u and v in $\mathcal{O}_A$, $uD\eta_A v' \in \mathcal{O}_A$. Note that $D \in \mathbb{Z} \subseteq \mathcal{O}_K$, so $D\eta$ has the properties stated in Lemma 11.1.

For this new η, $u\eta_A v'$ maps $\hat{L} \times \hat{L}$ to $\mathcal{O}_A$. Since τ maps $\mathcal{O}_A$ to $\mathcal{O}_K$, and $tr_{K/\mathbb{Q}}$ maps $\mathcal{O}_K$ to $\mathbb{Z}$, we have shown that $\hat{\beta}^I$ takes integral values on $\hat{L} \times \hat{L}$.

(c) Recall that $\hat{P}_{\mathbb{A}} = \hat{L}_A \circ \tilde{\Phi}_A$ is a map from $G_{\mathbb{Q}}$ to $Aut_{\mathbb{Q}}(R_{K/\mathbb{Q}}(B_A))$. If $g \in G_{\mathbb{Q}}$, $v \in R_{K/\mathbb{Q}}(B_A)$, then $\hat{P}_{\mathbb{A}}(g)(v) = \tilde{\Phi}_A(g)v$. If we use i^* to identify B_A with $R_{K/\mathbb{Q}}(B_A)$, then we have $\hat{P}_{\mathbb{A}}(g)(v) = \tilde{\varphi}_A(g)v$.

So

$$(*) \quad \hat{\beta}^I(\hat{P}_{\mathbb{A}}(g)(u), \hat{P}_{\mathbb{A}}(g)(v)) = \hat{\beta}^I(\tilde{\varphi}_A(g)u, \tilde{\varphi}_A(g)v)$$

$$= tr_{K/\mathbb{Q}}\tau(\tilde{\varphi}_A(g)u\eta_A(\tilde{\varphi}_A(g)v)')$$

$$= tr_{K/\mathbb{Q}}\tau(\tilde{\varphi}_A(g)u\eta_A v'\tilde{\varphi}_A(g)').$$

Also,

$$\tilde{\varphi}_A(g)' = \bigotimes_{a_i \in A} (\tilde{\varphi}_{a_i}(g))', \text{ considering } g \in G_{\mathbb{Q}} \subseteq R_{K/\mathbb{Q}}(B)^{\times} \cong B^{\times}.$$

By the same argument used in the proof of Proposition 5.1(b), $G_K = \prod_{i \in S} B_i^1$. The inclusion map $G_{\mathbb{Q}} \to G_K$ is given by $g \mapsto (\tilde{\varphi}_1(g), \ldots, \tilde{\varphi}_m(g))$, considering $G_{\mathbb{Q}}$ as a subset of $B^{\times}$. So if $g \in G_{\mathbb{Q}}$, $\nu(\tilde{\varphi}_i(g)) = 1$, where ν is the reduced norm in B_i. Since B_i is a quaternion algebra, $1 = \nu(\tilde{\varphi}_i(g)) = \tilde{\varphi}_i(g)(\tilde{\varphi}_i(g))'$. Hence $\tilde{\varphi}_i(g)' = \tilde{\varphi}_i(g)^{-1}$ for all $i \in S$. So $\tilde{\varphi}_A(g)' = \otimes(\tilde{\varphi}_{a_i}(g))^{-1}$ $= \tilde{\varphi}_A(g)^{-1}$. Then the last line of (*) becomes

$$tr_{K/\mathbb{Q}}\tau(\tilde{\varphi}_A(g)u\eta_A v'\tilde{\varphi}_A(g)^{-1}) = tr_{K/\mathbb{Q}}\tau(u\eta_A v') = \hat{\beta}^I(u,v),$$

since τ is invariant under inner automorphisms. Thus $\hat{\beta}^I$ is invariant under the action of $G_{\mathbb{Q}}$. ∎

We have constructed $\mathbb{Q}$-bilinear forms $\hat{\beta}^I$ and β^I on $\hat{W}_{\mathbb{Q}} \times \hat{W}_{\mathbb{Q}}$ and $W_{\mathbb{Q}} \times W_{\mathbb{Q}}$ with values in $\mathbb{Q}$. The forms can be extended, by $\mathbb{R}$-linearity, to bilinear forms $\hat{\beta}^I : \hat{W}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \times W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} = \hat{W} \times \hat{W}$ $\to \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}$ and $\beta^I : W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \times W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} = W \times W \to \mathbb{R}$. Then these forms are clearly alternating and $\mathbb{R}$-bilinear. Since $L \subseteq W_{\mathbb{Q}} \subseteq W_{\mathbb{R}}$, property (b) of the proposition still holds. By the same argument as in the proof of property (c), we see that $\hat{\beta}^I$ and β^I on $\hat{W} \times \hat{W}$ and $W \times W$ are $G_{\mathbb{R}}$-invariant.

If $\mathbb{A}$ is a rigid polymer, we set $\beta = \beta^I$.

11.3. Type II. If $\mathbb{A}$ contains molecules of type II, it is necessary to construct another bilinear form β^{II}. To do this, we must use the polymer $\mathbb{A} + \mathbb{A} = 2\mathbb{A}$. Then $2\mathbb{A} = \sum_{\sigma \in g} 2\sigma(A)$.

Denote $B_A \oplus B_A$ by B_{2A}, and the ideal $F_A \oplus F_A$ by F_{2A}. In fact, $B_A \oplus B_A \cong B_A \underset{K}{\otimes} K^2$, and

$$R_{K/\mathbb{Q}}(B_A \oplus B_A) \cong R_{K/\mathbb{Q}}(B_A) \oplus R_{K/\mathbb{Q}}(B_A) \cong R_{K/\mathbb{Q}}(B_A) \underset{\mathbb{Q}}{\otimes} \mathbb{Q}^2.$$

Similarly, $F_{2A} \cong F_A \underset{K}{\otimes} K^2$, and

$$R_{K/\mathbb{Q}}(F_{2A}) = R_{K/\mathbb{Q}}(F_A) \oplus R_{K/\mathbb{Q}}(F_A) \cong R_{K/\mathbb{Q}}(F_A) \underset{\mathbb{Q}}{\otimes} \mathbb{Q}^2.$$

Let g be the alternating bilinear form on $K^2 \times K^2$ defined by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let e be the function from $B_a \times B_a$ to B_a defined in §11.2. As before, $\tau \circ e$ is a symmetric K-bilinear form on $B_a \times B_a$. By Lemma 11.5, we can define a K-bilinear form C on $B_A \otimes K^2 \times B_A \otimes K^2$ by

$$C(u\otimes a, v\otimes b) = \bigotimes_{a_i \in A} \tau \circ e_{a_i} \otimes g(u\otimes a, v\otimes b)$$

$$= \prod_{a_i \in A} \tau \circ e(u_{a_i}, v_{a_i}) \cdot g(a,b),$$

where $u = \otimes u_{a_i}$ and $v = \otimes v_{a_i}$ are in B_A, and a and b are in K^2, Since $\tau \circ e$ is symmetric and g is alternating, C is alternating.

By Lemma 11.2, C can be written as $C = \tau(\bigotimes_{a_i \in A} e_{a_i}) \otimes g$.

As before, we have a sequence of maps

$$R_{K/\mathbb{Q}}(B_{2A}) \times R_{K/\mathbb{Q}}(B_{2A}) \xrightarrow{(i^*)^{-1} \times (i^*)^{-1}} B_{2A} \times B_{2A} \xrightarrow{C} K \xrightarrow{tr_{K/\mathbb{Q}}} \mathbb{Q}.$$

Let D be tne positive integer in the proof of Propolition 11.9 (b), and let δ be a positive integer to be fixed in §13. Set $\hat{\beta}^{II} = tr_{K/\mathbb{Q}} \circ \delta DC \circ ((i^*)^{-1} \times (i^*)^{-1})$, or, briefly, $\hat{\beta}^{II} = tr_{K/\mathbb{Q}} \circ \delta DC$.

Let $\beta^{II} = \hat{\beta}^{II}|R_{K/\mathbb{Q}}(F_{2A}) \times R_{K/\mathbb{Q}}(F_{2A})$.

<u>Proposition 11.10</u>. The map β^{II}

(a) is an alternating $\mathbb{Q}$-bilinear form on $W_{\mathbb{Q}} \times W_{\mathbb{Q}}$,

(b) takes integral values on $L \times L$, and

(c) is invariant under the action of $G_{\mathbb{Q}}$.

<u>Proof</u>. As before, we prove (a), (b), and (c) for $\hat{\beta}^{II}$; the same things are then true for β^{II}.

(a) Since i^* and $tr_{K/\mathbb{Q}}$ and $\mathbb{Q}$-linear, and C is k-bilinear and alternating, $\hat{\beta}^{II}$ is K-bilinear and alternating.

(b) The lattice $\hat{L}$ we get using the polymer $2\mathbb{A}$ is

$i^*(\mathcal{O}_A) \oplus i^*(\mathcal{O}_A) = i^*(\mathcal{O}_A \otimes_{\mathbb{Z}} \mathbb{Z}^2)$ in B_{2A}.

Since $e(u_a, v_a) = u_a v_a'$, we can write

$$\left(\bigotimes_{a_i \in A} e\right)(u,v) = uv'$$

for $u,v \in B_A$. Then, for $u,v \in \mathcal{O}_A$, $a,b \in \mathbb{Z}^2$,

$$\beta^{II}(u\otimes a, v\otimes b) = tr_{K/\mathbb{Q}}(\delta D\cdot\tau(uv')\cdot g(a,b))$$

$$= tr_{K/\mathbb{Q}}(\delta\tau(uDv')\cdot g(a,b)).$$

By the choice of D, $Dv' \in \mathcal{O}_A$, so $uDv' \in \mathcal{O}_A$, and $\tau(uDv') \in \mathcal{O}_K$. Clearly $g(a,b) \in \mathbb{Z}$, and $\delta \in \mathbb{Z}$, so $\delta\tau(uDv')\cdot g(a,b) \in \mathcal{O}_K$. Since $tr_{K/\mathbb{Q}}$ takes $\mathcal{O}_K$ to $\mathbb{Z}$, we have shown that $\hat{\beta}^{II}|\hat{L} \times \hat{L}$ is integral-valued.

(c) We must show that, for $\gamma \in G_{\mathbb{Q}}$, and $u,v \in B_{2A}$,

$$\hat{\beta}^{II}(\hat{P}_{2\mathbb{A}}(\gamma)(u),\ \hat{P}_{2\mathbb{A}}(\gamma)(v)) = \hat{\beta}^{II}(u,v).$$

Note that, since $B_{2A} = B_A \otimes K^2$, $\hat{P}_{2\mathbb{A}}$ decomposes as a tensor product $\hat{P}_{2\mathbb{A}} = \hat{P}_{\mathbb{A}} \otimes 1_{K^2}$. Then

$$\hat{\beta}^{II}(\hat{P}_{2\mathbb{A}}(\gamma)(u\otimes a),\ \hat{P}_{2\mathbb{A}}(\gamma)(v\otimes b))$$

$$= \hat{\beta}^{II}(\hat{P}_{\mathbb{A}}(\gamma)(u)\otimes a, \hat{P}_{\mathbb{A}}(\gamma)(v)\otimes b)$$

$$= tr_{K/\mathbb{Q}}\{\delta D\tau[\hat{P}_{\mathbb{A}}(\gamma)(u)\hat{P}_{\mathbb{A}}(\gamma)(v))']g(a,b)\}$$

As in the proof of Proposition 11.9(c),

$$\hat{P}_{\mathbb{A}}(\gamma)(u)[\hat{P}_{\mathbb{A}}(\gamma)(v)]' = \tilde{\varphi}_A(\gamma)u[\tilde{\varphi}_A(\gamma)v]'$$

$$= \tilde{\varphi}_A(\gamma)uv'\tilde{\varphi}_A(\gamma)' = \tilde{\varphi}_A(\gamma)uv'\tilde{\varphi}_A(\gamma)^{-1}.$$

Since the reduced trace is invariant under inner automorphisms, we have

$$\hat{\beta}^{II}(\hat{P}_{2\mathbb{A}}(\gamma)(u\otimes a),\ \hat{P}_{2\mathbb{A}}(\gamma)(v\otimes b)) = \hat{\beta}^{II}(u\otimes a, v\otimes b).$$ ∎

For a mixed type polymer $\mathbb{A}$, we use $\mathbb{A}$ to construct β^I and $2\mathbb{A}$ to construct β^{II}. Set $\beta = \beta^I \oplus \beta^I + \beta^{II}$, which is a bilinear form on $R_{K/\mathbb{Q}}(F_{2A}) \times R_{K/\mathbb{Q}}(F_{2A}) = W_{\mathbb{Q}} \times W_{\mathbb{Q}}$. The map clearly satisfies properties (a), (b), and (c) of Propositions 11.9 and 11.10.

As before, we can extend β by $\mathbb{R}$-linearity to an alternating $\mathbb{R}$-bilinear form on $W\times W$, which is integer valued on $L\times L$ and $G_{\mathbb{R}}$-invariant. Similarly, we can extend $\hat{\beta}$ to an $\mathbb{R}$-bilinear form on $\hat{W}\times\hat{W}$ which has the same properties.

§12. Complex Structure

In this section, we define a complex structure on each fiber $W_x = \pi^{-1}(x)$ of the fiber bundle $X \times W \xrightarrow{\pi} X$. This is so we can explicitly construct an Eichler map in §14. For the polymer $\mathbb{A} = \sum_{\sigma \in \mathcal{G}} \sigma(A) = \sum A_i$, let $W_{A_i} = W_i = F_{A_i} \otimes \mathbb{R}$ be the subspace of W corresponding to A_i. We will construct a complex structure J_{A_i} for each subspace W_{A_i}; the construction depends on the type of A_i.

First, let A be a type I molecule. Let j be the element of $G_{\mathbb{R}} = \prod_{i \in S_0} G_i \times \prod_{h \in S_1} G_h$:

$$j = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1, \ldots, 1\right),$$

where each 1 is in G_h for $h \in S_1$. We can consider j as an element of $G_{\mathbb{R}}$ because we have fixed isomorphisms in §11.

For the molecule A, set

$$\hat{J}_A^I = \hat{P}_{A,\mathbb{R}}(j)$$

$$J_A^I = P_{A,\mathbb{R}}(j).$$

Since $j^2 = -1$, $\hat{J}_A^I$ and J_A^I are complex structures on $\hat{W}_A$ and W_A respectively.

If A is a type II molecule, we must use the molecule $2A$. Then, as in §11, $B_{2A} \otimes \mathbb{R} = B_{A,\mathbb{R}} \oplus B_{A,\mathbb{R}} = B_A \otimes_K \mathbb{R}^2 = B_{A,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}^2$. Since we have fixed isomorphisms, $B_{A,\mathbb{R}} \cong \bigotimes_{a_i \in A} \mathbb{H}$. Let ι be the

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$$J_A^I = P_{A,\mathbb{R}}(j).$$

Since $j^2 = -1$, $\hat{J}_A^I$ and J_A^I are complex structures on $\hat{W}_A$ and W_A respectively.

If A is a type II molecule, we must use the molecule 2A. Then, as in §11, $B_{2A} \otimes \mathbb{R} = B_{A,\mathbb{R}} \oplus B_{A,\mathbb{R}} = B_A \otimes_K \mathbb{R}^2 = B_{A,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}^2$. Since we have fixed isomorphisms, $B_{A,\mathbb{R}} \cong \bigotimes_{a_i \in A} \mathbb{H}$. Let ι be the authomorphism of $\mathbb{R}^2$ defined by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let 1 be the identity transformation on B_A. Set $\hat{J}_A^{II} = 1 \otimes \iota$. Then $\hat{J}_A^{II}$ is a complex structure on $\hat{W}_{2A} = \hat{W}_A \otimes \mathbb{R}^2$, and restricts to a complex structure J_A^{II} on $W_{2A} = W_A \otimes \mathbb{R}^2$.

If $\mathbb{A}$ is a rigid polymer, we set

$$\hat{J} = \bigoplus_{A_i \in \mathbb{A}} \hat{J}^I_{A_i} = (\hat{J}^I_{A_1}, \ldots, \hat{J}^I_{A_d})$$

on $\hat{W} = \oplus \hat{W}_{A_i}$, restricting to $J = \oplus J^I_{A_i}$ on $W = \oplus W_{A_i}$.

If $\mathbb{A}$ is a mixed type polymer, set

$$\hat{J} = \bigoplus_{\substack{A_i \in \mathbb{A} \\ \text{type I}}} (\hat{J}^I_{A_i} \oplus \hat{J}^I_{A_i}) \oplus \bigoplus_{\substack{A_j \in \mathbb{A} \\ \text{type II}}} \hat{J}^{II}_{A_j}$$

$$= (\hat{J}^I_{A_1}, \hat{J}^I_{A_1}, \ldots, \hat{J}^I_{A_c}, \hat{J}^I_{A_c}, \hat{J}^{II}_{A_{c+1}}, \ldots, \hat{J}^{II}_{A_d})$$

on $\hat{W} = \oplus \hat{W}_{2A_i}$. This restricts to

$$J = \bigoplus_{A_i \text{ type I}} (J^I_{A_i} \oplus J^I_{A_i}) \oplus \bigoplus_{A_j \text{ type II}} J_{A_j}$$

on $W = \oplus W_{2A_i}$.

Now we fix a base point $x_0 = (i, \ldots, i)$ in $\mathfrak{h} \times \ldots \times \mathfrak{h} = \mathfrak{h}^{m_0} = X$. Identify x_0 with the coset C of $G_{\mathbb{R}}/C$. Let the complex structure $J_{x_0} = J_0$ on the fiber W_{x_0} above x_0 be J as defined above. For an arbitrary point $x \in X$, take $g \in G_{\mathbb{R}}$ such that $gx_0 = x$. Let $J_x = P(g) J_0 P(g)^{-1}$ be the complex structure on W_x, where $P = P_{\mathbb{A},\mathbb{R}}$

or $P_{2\mathbb{A},\mathbb{R}}$, depending on whether $\mathbb{A}$ is rigid or of mixed type. Also let $\hat{J}_x = \hat{P}(g)\hat{J}_0\hat{P}(g)$.

<u>Proposition 12.1</u>. J_x is well defined.

<u>Proof</u>. It suffices to prove that $\hat{J}_x$ is well defined. We must show that, if $x = g_1x_0 = g_2x_0$, then $\hat{P}(g_1)\hat{J}_0\hat{P}(g_1)^{-1} = \hat{P}(g_2)\hat{J}_0\hat{P}(g_2)^{-1}$. If $g_1x_0 = g_2x_0$, then $g_1^{-1}g_2x_0 = x_0$, so $g_1^{-1}g_2 \in C$, where $C = SO(2)^{m_0} \times SU(2)^{m_1}$. So the problem reduces to showing that $\hat{P}(g)\hat{J}_0\hat{P}(g)^{-1} = \hat{J}_0$ if $g \in C$.

If $\mathbb{A}$ is rigid, $\hat{P} = \hat{P}_{\mathbb{A},\mathbb{R}} = \bigoplus_{A_i \in \mathbb{A}} \hat{P}_{A_i,\mathbb{R}}$; if $\mathbb{A}$ is of mixed type, then $\hat{P} = \hat{P}_{2\mathbb{A},\mathbb{R}} = \hat{P}_{\mathbb{A},\mathbb{R}} \otimes 1_{\mathbb{R}^2}$. If $g = (g_1,\ldots,g_m)$ is the expression for g in terms of the simple components of $G_{\mathbb{R}}$, then $\hat{P}_{A_i,\mathbb{R}}$ sends $g = (g_1,\ldots,g_m)$ to $\bigotimes_{a_h \in A_i} g_{a_h} \in B_{A_i,\mathbb{R}}$. If A is a type I molecule, we have

$$\hat{P}_{A,\mathbb{R}}(g)\hat{P}_{A,\mathbb{R}}(j)\hat{P}_{A,\mathbb{R}}(g)^{-1} = \hat{P}_{A,\mathbb{R}}(gjg^{-1})$$

$$= \hat{P}_{A,\mathbb{R}}[(\bigotimes_{a_h \in S_0} g_{a_h}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}g_{a_h}^{-1}) \otimes (\bigotimes_{a_h \in S_1} g_{a_h}g_{a_h}^{-1})]$$

$$= \hat{P}_{A,\mathbb{R}}[(\bigotimes_{a_h \in S_0} g_{a_h}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}g_{a_h}^{-1}) \otimes (\bigotimes_{a_h \in S_1} 1)].$$

A straightforward calculation shows that $g_{a_h}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}g_{a_h}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ if and only if $g_{a_h} \in SO(2)$. Hence we have shown for type I molecules that $\hat{P}(g)\hat{J}_0\hat{P}(g) = \hat{J}_0$ if $g \in C$.

If A is a type II molecule, we have

$$\hat{P}_{2A,\mathbb{R}}(g)\hat{J}_0^{II}\hat{P}_{2A,\mathbb{R}}(g)^{-1}$$

$$= (\hat{P}_{A,\mathbb{R}}(g)\otimes 1_{\mathbb{R}^2})(1_{B_A}\otimes\iota)(\hat{P}_{A,\mathbb{R}}(g)\otimes 1_{\mathbb{R}^2})^{-1}$$

$$= \hat{P}_{A,\mathbb{R}}(g)\hat{P}_{A,\mathbb{R}}(g)^{-1}\otimes\iota = 1_{B_A}\otimes\iota = \hat{J}_0^{II}.$$

So $\hat{J}_0^{II}$ is well defined, and in fact, $\hat{J}_x^{II} = \hat{J}_0^{II}$ for all $x \in X$. ∎

§13. Positive Definite Symmetric Bilinear Form

In this section, we show that the bilinear form $s_x(u,v) = \beta(u,J_x v)$ is symmetric and positive definite on $W \times W$. In doing this, we fix the constants η and δ used in the construction of β.

13.1. Preliminary lemmas.

Proposition 13.1. Let $\Phi = \{\varphi_i\}$ be the set of embeddings of a totally real number field k into $\mathbb{R}$, S the set of indices of Φ, and T a proper subset of S. Let $m = |S| > 1$. Let $\mathcal{O}_k$ be the ring of integers of k. Fix a real number $N > 1$ and set $\epsilon = N^{\frac{1}{1-m}}$, so that $0 < \epsilon < 1 < N$. Then there exists $\alpha \in \mathcal{O}_k$ such that

$$\varphi_i(\alpha) > N \qquad \text{for } i \in T$$

$$0 < \varphi_i(\alpha) < \epsilon \qquad \text{for } i \in S - T.$$

We need a lemma, which uses several standard theorems of number theory.

Theorem 13.2. ([W-2], Chapter IV, Theorem 5). For every place v of k, let $|\ |_v$ be the valuation at v. If $z \in k^{\times}$, then $\prod_v |z|_v = 1$.

Theorem 13.3. ([W-2], Chapter V, Theorem 1). For every finite place v of k, let r_v be the set of elements x of k_v such that $|x|_v \le 1$. Then $\mathcal{O}_k = \bigcap_{\text{finite } v} (k \cap r_v)$.

<u>Theorem 13.4.</u> (Minkowski's Theorem). Let L be a lattice in $\mathbb{R}^n$, and let V be the volume of L. If D is a convex subset of $\mathbb{R}^n$, symmetric about the origin, with $\mathrm{vol}(D) \geq 2^n V$, then D contains a non-zero lattice point.

<u>Lemma 13.5.</u> Given $N > 1$ and one infinite place j of k, let $\epsilon = N^{\frac{1}{1-m}}$, where $m = |S|$. Then there exists $\alpha \in \mathfrak{O}_k$ such that

$$\varphi_j(\alpha) > N$$

$$0 < \varphi_i(\alpha) < \epsilon \qquad \text{for } i \in S,\ i \neq j.$$

<u>Proof.</u> Let φ_∞ be the injection $\varphi_\infty = (\varphi_1, \ldots, \varphi_m)$ of k into $\bigoplus_S k \underset{\varphi_i}{\otimes} \mathbb{R} \cong \mathbb{R}^m$. By the proof of Theorem 13.3 in [W-2], $\varphi_\infty(\mathfrak{O}_k)$ is a lattice in $\mathbb{R}^m$. Let V be the volume of this lattice. Choose $a \in \mathbb{R}$ such that $a > \max\{N, VN\}$. Define a rectangular parallelopiped D in $\mathbb{R}^m$ by

$$D = (-\epsilon, \epsilon) x \ldots x(-\epsilon, \epsilon) x(-a, a) x(-\epsilon, \epsilon) x \ldots x(-\epsilon\ \epsilon),$$

where $(-a,a)$ is in the j-th place. The set D is convex, symmetric about the origin, and

$$\mathrm{vol}(D) = (2\epsilon)^{m-1} 2a = 2^m \epsilon^{m-1} a > 2^m \epsilon^{m-1} V \epsilon^{1-m} = 2^m V.$$

So, by Minkowski's Theorem, D contains a nonzero point of $\varphi_\infty(\mathfrak{O}_k)$. That is, there is $\alpha \in \mathfrak{O}_k$ such that $|\varphi_i(\alpha)| < \epsilon$ for $i \neq j$ in S, and $|\varphi_j(\alpha)| < a$.

It remains to be shown that $|\varphi_j(\alpha)|$ is actually large, and that α can be chosen so that all $\varphi_i(\alpha)$ are positive. Since $\alpha \neq 0$, by Theorem 13.2, $\Pi|\alpha|_v = 1$, where the product is taken over all places of k. Then

$$(*) \quad 1 = \prod_v |\alpha|_v = \prod_{i \in S} |\varphi_i(\alpha)| \prod_{\text{finite } v} |\alpha|_v < \epsilon^{m-1} |\varphi_j(\alpha)| \prod_{\text{finite } v} |\alpha|_v .$$

Theorem 13.3 says that $\alpha \in r_v$ for all finite places v, so $\prod_{\text{finite } v} |\alpha|_v \leq 1$. Combining this with (*), we have

$$1 < \epsilon^{m-1} |\varphi_j(\alpha)|, \text{ so } |\varphi_j(\alpha)| > \epsilon^{1-m} = N.$$

If not all the $\varphi_i(\alpha)$ are positive, replace α with α^2. Then $\varphi_i(\alpha^2) = (\varphi_i(\alpha))^2 = |\varphi_i(\alpha)|^2$, so that

$$\varphi_i(\alpha^2) < \epsilon^2 < \epsilon \qquad \text{for } i \neq j$$

and

$$\varphi_j(\alpha^2) > N^2 > N. \qquad \blacksquare$$

<u>Proof of Proposition 13.1</u>. Choose ϵ_0 so that

$$\epsilon_0 < \min\{(1+|T|)^{\frac{1}{1-m}}\epsilon,\ |T|^{-1}\epsilon\}.$$

Set $N_0 = \epsilon_0^{1-m}$. Since $\epsilon < 1$, ϵ_0 is also less than one. By the previous lemma, for every place $t \in T$, we can find $\alpha^{(t)} \in \mathfrak{O}_k$ and M_0 depending only on $\mathfrak{O}_k$ and N_0 such that

$$M_0 > \varphi_t(\alpha^{(t)}) > N_0$$

$$0 < \varphi_i(\alpha^{(t)}) < \epsilon_0 \qquad \text{for } i \in S,\ i \neq t.$$

Let $\alpha = \sum_{t \in T} \alpha^{(t)}$. for $i \in S - T$,

$$\varphi_i(\alpha) = \sum_T \varphi_i(\alpha^{(t)}) < |T| \epsilon_0 < \epsilon.$$

For $j \in T$,

$$\varphi_j(\alpha) = \varphi_j(\alpha^{(j)}) + \sum_{\substack{t \in T \\ t \neq j}} \varphi_j(\alpha^{(t)})$$

$$\geq \varphi_j(\alpha^{(j)}) - \sum_{t \neq j} \varphi_j(\alpha^{(t)}) \geq N_0 - (|T|-1) \epsilon_0$$

$$> (1+|T|)N - |T| + 1 \geq N + 1 \geq N. \qquad \blacksquare$$

Lemma 13.6. Let $\{\beta_i\}$ be positive definite symmetric bilinear forms on real vector spaces $\{V_i\}$. Then $\otimes \beta_i$ is positive definite on $\otimes V_i$.

Proof. By Lemma 11.5, $\otimes \beta_i$ is well defined and symmetric. Choose bases for the spaces V_i so that the matrices for β_i are diagonal, with diagonal entries $\lambda_j^i \geq 0$ for all i,j. Then the matrix for $\otimes \beta_i$ is also diagonal, and its diagonal entries are of the form $\prod_i \lambda_j^i$. Since every λ_j^i is positive, all products are positive, and $\otimes \beta_i$ is positive definite. $\blacksquare$

Lemma 13.7. ([B], §7, No.3, Proposition 6). Let β_1 and β_2 be symmetric bilinear forms on a real vector space V, with β_1 positive definite. Then there is a basis for V such that the matrix of β_1 is the identity matrix, and the matrix of β_2 is diagonal.

<u>Corollary 13.8.</u> If β_1 and β_2 are as above, then there exists a real number N such that $N\beta_1 + \beta_2$ is positive definite.

<u>Proof.</u> Choose a basis for V as in Lemma 13.6, and let $\mathrm{diag}(d_1,\ldots,d_n)$ be the matrix of β_2. Choose $N > \max\{|d_i|\}$. ∎

13.2. In this section, we show that the bilinear forms $s_x(u,v) = \beta(u,J_x v)$ are symmetric and positive definite.

<u>Lemma 13.9.</u> If $s_{x_0}(u,v)$ is symmetric and positive definite, then so is $s_x(u,v)$ for any $x \in X$.

<u>Proof.</u> Since β is $G_{\mathbb{R}}$-invariant, we have, for $g(x_0) = x$,

$$s_x(u,v) = \beta(u,J_x v) = \beta(u,P(g)JP(g)^{-1}v)$$

$$= \beta(P(g)^{-1}u,\ P(g)^{-1}P(g)JP(g)^{-1}v)$$

$$= s_{x_0}(P(g)^{-1}u,P(g)^{-1}v).$$

So s_x is symmetric and positive definite if s_{x_0} is. ∎

Throughout this section, we consider $\hat{\beta}$ as an $\mathbb{R}$-bilinear form on $\hat{W}\times\hat{W}$. Recall that

$$\hat{W} = R_{K/\mathbb{Q}}(B_A) \underset{\mathbb{Q}}{\otimes} \mathbb{R} = \bigoplus_{\sigma\in G} B_{\sigma(A)} \underset{K}{\otimes} \mathbb{R} = \oplus\, \hat{W}_{\sigma(A)}.$$

Since $\hat{\beta}$ was originally defined on $R_{K/\mathbb{Q}}(B_A)$, $\hat{\beta}$ as a form on $\hat{W}$ decomposes as $\hat{\beta} = \oplus\, \hat{\beta}_{\sigma(A)}$, where $\hat{\beta}_{\sigma(A)} = \hat{\beta}|\hat{W}_{\sigma(A)}$. The same remarks hold for β on W.

Proposition 13.10. If $\mathbb{A}$ is a rigid polymer, then $s(u,v) = \beta^{I}(u,Jv)$ is a symmetric form on $W \times W$.

Proof. It will suffice to prove this for $\hat{\beta}_A$ on $\hat{W}_A$ for any molecule A. For each A, $\hat{J}_A = \hat{J}^{I}$. For simplicity, we will drop the subscript A. First note that $\hat{J}' = \hat{J}^{-1}$, and recall that τ is invariant under inner automorphisms and the involution ( )' on B_A. For $u,v \in \hat{W}_A$, we have

$$\sum_{a_i \in A} E_{a_i}(u,\hat{J}v) = \tau(u\eta_A(\hat{J}v)') = \tau((u\eta_A(\hat{J}v)')')$$

$$= \tau(\hat{J}v\eta_A' u') = \tau(-\hat{J}v\eta_A u')$$

$$= \tau(-\hat{J}^2 v\eta_A u'\hat{J}^{-1}) = \tau(v\eta_A(\hat{J}u)')$$

$$= \Sigma E_{a_i}(v,\hat{J}u).$$

Hence $s(u,v) = s(v,u)$.

Proposition 13.11. If $\mathbb{A}$ is a mixed type polymer, then $s(u,v) = \beta(u,Jv)$ is a symmetric form on $W \times W$.

Proof. Since $\mathbb{A}$ is of mixed type, $\hat{\beta} = \hat{\beta}^{I} \oplus \hat{\beta}^{I} + \hat{\beta}^{II} = \hat{\beta}^{I} \otimes 1_{\mathbb{R}^2} \oplus \hat{\beta}^{II}$ and, for each molecule A of $\mathbb{A}$, $\hat{J}_A = \hat{J}^{I} \otimes 1_{\mathbb{R}^2}$ or $\hat{J} = \hat{J}^{II} = 1_{B_{A,\mathbb{R}}} \otimes \iota$ on $\hat{W}_{2A}$. It is sufficient to check $\hat{s}(u,v) = \hat{\beta}(u,\hat{J}v)$ on $\hat{W}_{2A}$. We have four cases.

(a) $\hat{\beta}^{I} \otimes 1(u \otimes a, \hat{J}^{I} \otimes 1(v \otimes b))$

$$= \hat{\beta}^{I}\hat{J}^{I} \otimes 1(u \otimes a, v \otimes b) = \hat{\beta}^{I}(u,\hat{J}^{I}v)1(a,b).$$

This is the tensor product of two bilinear forms. Since 1 is symmetric, and $\hat{\beta}^{I}(u,\hat{J}^{I}v)$ is symmetric, by Proposition 13.10, their product is symmetric, by Lemma 11.5.

(b) $\hat{\beta}^{II}(u\otimes a, \hat{J}^{I}\otimes 1(v\otimes b))$

$$= \hat{\beta}^{II}(u\otimes a, \hat{J}^{I}v\otimes b)$$

$$= \delta D\tau(\bigotimes_{a_i\in A} e_{a_i}\otimes g)(u\otimes a, \hat{J}^{I}v\otimes b).$$

It is enough to check symmetry on elements of the form $u = \bigotimes_{a_i\in A} u_{a_i}$, $v = \bigotimes_{a_i\in A} v_{a_i}$. If a_h is the atom of A which is in S_0, then

$$\hat{J}^{I} = \hat{P}_A(j) = \iota \otimes (\bigotimes_{a_i\neq a_h} 1),$$

where $\iota = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So the calculation above becomes $\delta D\cdot\tau\circ e(u_{a_h},\iota v_{a_h})\cdot \prod_{i\neq h} \tau\circ e(u_{a_i},v_{a_i})\cdot g(a,b)$. We have

$$\tau\circ e(u_{a_h},\iota v_{a_h}) = \tau(u_{a_h}(\iota v_{a_h})')$$

$$= \tau((u_{a_h}(\iota v_{a_h})')') = \tau(\iota v_{a_h}u'_{a_h})$$

$$= \tau(\iota^2 v_{a_h}u'_{a_h}\iota^{-1}) = -\tau(v_{a_h}u'_{a_h}\iota')$$

$$= -\tau(v_{a_h},\iota u_{a_h}),$$

so this form is alternating. The form g is alternating, and $\tau\circ e$ is symmetric, by Lemma 11.7. Since $\hat{\beta}^{II}(u\otimes a,\hat{J}^{I}\otimes 1(v\otimes b))$ is a constant times the tensor product of all these forms, it is symmetric, by Lemma 11.5.

(c) $\hat{\beta}^{I} \otimes 1(u\otimes a, \hat{J}^{II}(v\otimes b))$

$$= \hat{\beta}^{I} \otimes 1(u\otimes a, v\otimes \iota b)$$

$$= \hat{\beta}^{I}(u,v)1(a,\iota b) = \hat{\beta}^{I}(u,v)\iota(a,b).$$

Here we have considered the linear transformation ι as a bilinear form. Since β^{I} and ι are both alternating, their tensor product is symmetric, by Lemma 11.5.

(d) $\hat{\beta}^{II}(u\otimes a, \hat{J}^{II}(v\otimes b))$

$$= \hat{\beta}^{II}(u\otimes a, v\otimes \iota b)$$

$$= \delta D \cdot (\bigotimes_{a_i \in A} \tau \circ e_{a_i})(u,v)\cdot g(a,\iota b)$$

$$= \delta D \cdot (\otimes \tau \circ e_{a_i})(u,v)\cdot g\iota(a,b)$$

$$= \delta D \cdot (\otimes \tau \circ e_{a_i})(u,v)\cdot 1(a,b).$$

Since this is a constant times the tensor product of symmetric forms, it is symmetric.

∎

In order to prove that s(u,v) is positive definite, we examine more closely the behavior of linear functions under scalar extensions.

First, note that the isomorphism

$$R_{K/\mathbb{Q}}(B_A) \otimes_{\mathbb{Q}} \mathbb{R} \cong \bigoplus_{G} B_A \otimes_{\sigma} \mathbb{R} = \bigoplus_{G} F_{\sigma(A)} \otimes_{K} \mathbb{R}$$

is given by

$$u \otimes_{\mathbb{Q}} r \mapsto \sum_{G} \tilde{\sigma}(u) \otimes_{K} r.$$

On the left side, u is considered as an element of $R_{K/\mathbb{Q}}(B_A)$; on the right, it is considered as an element of B_A, by the identification i^*.

If f is a $\mathbb{Q}$-linear map from $R_{K/\mathbb{Q}}(B_A)$ to a $\mathbb{Q}$-vector space V, then its $\mathbb{R}$-linear extension

$$f_{\mathbb{R}} : R_{K/\mathbb{Q}}(B_A) \underset{\mathbb{Q}}{\otimes} \mathbb{R} \to V \underset{\mathbb{Q}}{\otimes} \mathbb{R}$$

is given by $f_{\mathbb{R}}(u \otimes r) = f(u) \otimes r$. For brevity, we will write $f(r \cdot u) = r \cdot f(u)$. Using the isomorphism above, we have, for $u \in R_{K/\mathbb{Q}}(B_A)$, $r \in \mathbb{R}$,

$$f_{\mathbb{R}}(u \underset{\mathbb{Q}}{\otimes} r) = f_{\mathbb{R}}(\underset{G}{\Sigma} \tilde{\sigma}(u) \underset{K}{\otimes} r) = \Sigma\, f_{\mathbb{R}}(\tilde{\sigma}(u) \underset{K}{\otimes} r).$$

Identifying B_A with $R_{K/\mathbb{Q}}(B_A)$, we have

$$f_{\mathbb{R}}(u \underset{\mathbb{Q}}{\otimes} r) = \Sigma\, f(\tilde{\sigma}(u)) \underset{K}{\otimes} r.$$

If $\alpha \in K$, then

$$\begin{aligned} f_{\mathbb{R}}(\alpha u \underset{\mathbb{Q}}{\otimes} r) &= \Sigma\, f_{\mathbb{R}}(\tilde{\sigma}(\alpha u) \underset{K}{\otimes} r) \\ &= \Sigma\, f_{\mathbb{R}}(\sigma(\alpha)\tilde{\sigma}(u) \underset{K}{\otimes} r) \\ &= \Sigma\, f_{\mathbb{R}}(\tilde{\sigma}(u) \underset{K}{\otimes} \sigma(\alpha) r) \\ &= \Sigma\, f(\tilde{\sigma}(u)) \underset{K}{\otimes} \sigma(\alpha) r, \end{aligned}$$

or briefly,

$$f(r \cdot (\alpha u)) = \underset{G}{\Sigma}\, \sigma(\alpha) r \cdot f(\tilde{\sigma}(u)).$$

The following lemmas will be useful in simplifying the proof.

Lemma 13.12. If $a \in S_0$, then the bilinear form $\tau \circ f_a(u, \hat{P}_a(j)v)$ is positive definite on $B_{a,\mathbb{R}}$.

Proof. Since $a \in S_0$, $\hat{P}_a(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \iota$. Recall that $f(u,v) = u\tilde{\varphi}_a(\eta)v'$, where $u,v \in B_{a,\mathbb{R}} \cong M_2(\mathbb{R})$. Since we have fixed an isomorphism in §11, $\tilde{\varphi}_a(\eta) = \begin{pmatrix} 0 & \lambda_a \\ -\lambda_a & 0 \end{pmatrix} = \lambda_a \iota$, with λ_a a positive real number. Set $u = v \neq 0$. Then

$$\tau \circ f_a(v, \hat{P}_a(j)v) = \tau(v\lambda_a \iota(\iota v)')$$

$$= \lambda_a \tau(v\iota v'\iota').$$

A calculation shows that, for any $v \in M_2(\mathbb{R})$, $\iota v'\iota' = {}^t v$. So the previous line becomes $\lambda_a \tau(v\,{}^t v)$. For $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$v\,{}^t v = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix},$$

so that

$$\lambda_a \tau(v\,{}^t v) = \lambda_a(a^2+b^2+c^2+d^2) > 0$$ ∎

Lemma 13.13. If $a \in S_1$, then the bilinear form $\tau \circ e_a(u, \hat{P}_a(j)v)$ is positive definite on $B_{a,\mathbb{R}}$.

Proof. Since $a \in S_1$, $B_{a,\mathbb{R}} \cong \mathbb{H}$ and $\hat{P}_a(j) = 1$. Take $u = v = a + bi + cj + dk \neq 0$ in $B_{a,\mathbb{R}}$. Then

$$\tau \circ e_a(v, \hat{P}_a(j)v) = \tau(vv') = 2\nu(v)$$

$$= 2(a^2+b^2+c^2+d^2) \geq 0. \qquad \blacksquare$$

Proposition 13.14. If $\mathbb{A}$ is rigid, then $s(u,v) = \beta(u,Jv)$ is a positive definite form on $W \times W$.

Proof. It suffices to check $\hat{\beta}(u,\hat{J}v)$ on $\hat{W}\times\hat{W}$. We will first investigate the form on a molecule $A \in \mathbb{A}$.

Suppose that the atom a_g of A is in S_0. Replace the element $\eta \in B$ used in the construction of β^I with $\alpha\eta$, where α is a nonzero element of Θ_k to be fixed at the end of the proof.

Then, for $u,v \in B_{A,\mathbb{R}}$ of the form $u = \underset{a_i \in A}{\otimes} u_{a_i}$, $v = \underset{a_i \in A}{\otimes} v_{a_i}$,

$$\hat{\beta}_A(u,\hat{J}_A v) = \sum_{a_i \in A} E_{a_i}(u,\hat{J}_A v)$$

$$= \sum_{a_i \in A} [\tau \circ f_{a_i} \otimes (\underset{h \neq i}{\otimes} \tau \circ e_{a_h})](u,\hat{J}_A v)$$

$$= \sum_{a_i \in A} \tau \circ f_{a_i}(u_{a_i}, \hat{P}_{a_i}(j)v_{a_i}) \circ \prod_{h \neq i} \tau \circ e_{a_h}(u_{a_h}, \hat{P}_{a_h}(j)v_{a_h})$$

$$= \sum_{a_i \in A} \tau(u_{a_i}\tilde{\varphi}_{a_i}(\alpha\eta)(\iota v_{a_i})') \prod_{h \neq i} \tau \circ e_{a_h}(u_{a_h}, v_{a_h}).$$

Since $\tilde{\varphi}_{a_i}(\alpha\eta) = \varphi_{a_i}(\alpha)\tilde{\varphi}_{a_i}(\eta)$, and $\varphi_{a_i}(\alpha) \in K \subseteq \mathbb{R}$, we can factor $\varphi_{a_i}(\alpha)$ out of each term of the sum. Since we showed in Proposition 13.10 that the forms $E_{a_j}(u,\hat{J}_A v)$ are symmetric, we can consider $E_{a_j}(v,\hat{J}_A v)$ as a quadratic form. So, we rewrite setting $u = v$, the last line of the calculation above as

$$\sum_{a_i \in A} \varphi_{a_i}(\alpha) Q_{a_i}(v),$$

where $Q_{a_i}(v) = \tau \circ f_{a_i} \otimes (\bigotimes_{h \neq i} \tau \circ e_{a_h})(v, \hat{J}_A v)$. If $a_i = a_g$, then $a_i \in S_0$ and $a_h \in S_1$ for $h \neq i$. We have shown in Lemmas 13.12 and 13.13 that each factor of this tensor product is positive definite. So, by Lemma 13.7, $Q_{a_g}(v)$ is positive definite if $a_g \in S_0$. If $a_i \notin S_0$, then Q_{a_i} is not positive definite.

By our remarks on scalar extensions, for $v = \oplus v_{\sigma(A)} \in B_{\mathbb{A},\mathbb{R}}$,

$$\hat{\beta}(v, \hat{J}v) = \sum_{\sigma \in \mathcal{G}} \sum_{a_i \in A} \sigma \circ \varphi_{a_i}(\alpha) Q_{\sigma(a_i)}(v_{\sigma(A)})$$

$$= \sum_{\sigma \in \mathcal{G}} \sum_{a_i \in A} \varphi_{\sigma(a_i)}(\alpha) Q_{\sigma(a_i)}(v_{\sigma(A)}).$$

The form $Q_{\sigma(a_i)}$ is positive definite when $\sigma(a_i) \in S_0$. By Corollary 13.8, for each molecule $\sigma(A)$, there exists a real number $N_{\sigma(A)}$ such that

$$N_{\sigma(A)} Q_{\sigma(a_i)}(v_{\sigma(A)}) + \sum_{h \neq i} Q_{\sigma(a_h)}(v_{\sigma(A)})$$

is positive definite, where $\sigma(a_i) \in S_0$. Take $N = \max\{N_{\sigma(A)}\}$. By Proposotion 13.1, there is an element $\alpha \in \mathfrak{O}_k$ such that

$$\varphi_{\sigma(a_i)}(\alpha) > N \qquad \text{for } \sigma(a_i) \in S_0$$

$$0 < \varphi_{\sigma(a_h)}(\alpha) < 1 \qquad \text{for } \sigma(a_h) \in S_1.$$

Fix this α. Then

$$\hat{s}(v,v) = \hat{\beta}(v,\hat{J}v)$$

$$= \sum_{\sigma \in G} [\varphi_{\sigma(a_1)}(\alpha) Q_{\sigma(a_1)}(v_{\sigma(A)}) + \sum_{\sigma(a_h) \in S_1} \varphi_{\sigma(a_h)}(\alpha) Q_{\sigma(a_h)}(v_{\sigma(A)})]$$

is positive definite. ∎

Proposition 13.15. If $\mathbb{A}$ is of mixed type, then $s(u,v) = \beta(u,Jv)$ is a positive definite form on $W \times W$.

Proof. Again, we need only check $\hat{\beta}(u,\hat{J}v)$ on $\hat{W} \times \hat{W}$. First we investigate the molecules separately. As in Proposition 13.11, we have four cases. Let A be a molecule of $\mathbb{A}$.

(a) From the proof of Proposition 13.11, Case (a),

$$\hat{\beta}^I_A \otimes 1(u \otimes a, \hat{J}^I_A \otimes 1(v \otimes b))$$

$$= \hat{\beta}_A(u,\hat{J}^I_A v) 1(a,b).$$

So this form is the tensor product of the forms $\hat{\beta}^I_A(u,\hat{J}^I_A v)$, discussed above, and $1(a,b)$, which is positive definite. We will use a different constant α, which we will fix at the end of this proof. Now we consider $\hat{\beta}^I_A(v,\hat{J}^I_A v)$ and $1(a,a)$ as quadratic forms.

Using the notation of the proof of Proporition 13.14, we have

$$\hat{\beta}^I_A(v,\hat{J}^I_A v) 1(a,a)$$

$$= [\varphi_{a_1}(\alpha) Q_{a_1}(v) + \sum_{a_h \in S_1} \varphi_{a_h}(\alpha) Q_{a_h}(v)] 1(a,a),$$

where a_i is the atom of A that is in S_0. For clarity, we put $Q_{a_h}^{I,I}$ for Q_{a_h}. To simplify the notation, we put

$$Q_A^{I,I,S_0} = \varphi_{a_i}(\alpha)Q_{a_i}^{I,I} \otimes 1_{\mathbb{R}^2}$$

and

$$Q_A^{I,I,S_1} = \sum_{a_h \in S_1} \varphi_{a_h}(\alpha)Q_{a_h}^{I,I} \otimes 1_{\mathbb{R}^2},$$

so that

$$\hat{\beta}_A^I(v,\hat{J}_A^I v)1(a,a) = (Q_A^{I,I,S_0} + Q_A^{I,I,S_1})(v\otimes a).$$

The proof of Proposition 13.14 showed that $Q_{a_i}^{I,I}$ is positive definite if $a_i \in S_0$. Since $1_{\mathbb{R}^2}$ is positive definite, Q_A^{I,I,S_0} is positive definite.

(b) Since $\hat{\beta}_A^{II}(u\otimes a,\hat{J}_A^I \otimes 1(v\otimes b))$ is a symmetric bilinear form, we can consider it as a quadratic form. We set

$$Q_A^{II,I}(v) = D\tau(\bigotimes_{a_i \in A} e_{a_i} \otimes g)(v,(\hat{J}_A^I \otimes 1)v),$$

so that

$$\hat{\beta}_A^{II}(v,(\hat{J}_A^I\otimes 1)v) = \delta Q_A^{II,I}(v).$$

Combining this with the form of Case (a), we obtain an expression for $\hat{\beta}_A(v,\hat{J}_A v) = \hat{\beta}(v,Jv)|\hat{W}_{2A}$, where A is type I:

$$\begin{aligned}\hat{\beta}_A(v,\hat{J}v) &= (\hat{\beta}_A^I\otimes 1+\hat{\beta}_A^{II})(v,(\hat{J}_A^I\otimes 1)v)\\ &= (Q_A^{I,I,S_0}+Q_A^{I,I,S_1}+\delta Q_A^{II,I})(v).\end{aligned}$$

Here Q_A^{I,I,S_0} is positive definite; Q_A^{I,I,S_1} and $Q_A^{II,I}$ are not.

(c) If A is a type II molecule, then

$$\hat{\beta}_A^I \otimes 1(u\otimes a, \hat{J}_A^{II}(v\otimes b)) = \hat{\beta}_A^I(u,v)\iota(a,b)$$

is a symmetric form. In fact, this can be written as the sum of symmetric forms:

$$\hat{\beta}_A^I \otimes \iota = \sum_{a_i \in A} E_{a_i} \otimes \iota = \sum \tau[f_{a_i} \otimes (\underset{h\neq i}{\otimes} e_{a_h})] \otimes \iota .$$

Replacing η by $\alpha\eta$, we have

$$\hat{\beta}_A^I \otimes \iota(v\otimes a, v\otimes a) = \sum_{a_i \in A} \tau[v_{a_i}\tilde{\varphi}_{a_i}(\alpha\eta)v'_{a_i} \otimes (\underset{h\neq i}{\otimes} e_{a_h}(v_{a_h}, v_{a_h}))] \otimes \iota(a,a)$$

$$= \sum_{a_i \in A} \varphi_{a_i}(\alpha)\tau[f_{a_i}(v_{a_i}, v_{a_i}) \otimes (\underset{h\neq i}{\otimes} e_{a_h}(v_{a_h}, v_{a_h}))] \otimes \iota(a,a).$$

Set

$$Q_A^{I,II} = \hat{\beta}_A^I \otimes \iota$$

and

$$Q_{a_i}^{I,II}(v\otimes a) = E_{a_i} \otimes \iota(v\otimes a, v\otimes a),$$

so that

$$Q_A^{I,II} = \sum_{a_i \in A} \varphi_{a_i}(\alpha) Q_{a_i}^{I,II}.$$

(d) When A is type II, we have

$$\hat{\beta}_A^{II}(u\otimes a, \hat{J}_A^{II}(v\otimes b)) = \delta D(\underset{a_i \in A}{\otimes} \tau \circ e_{a_i})(u,v)\cdot 1(a,b).$$

We showed in Lemma 13.13 that $\tau \circ e_a$ is positive definite if $a \in S_1$, which is true for every $a_i \in A$. Since $1(a,b)$ is also positive definite, their tensor product is positive definite, by Lemma 13.6. Set

$$Q_A^{II,II}(v) = D\cdot[(\bigotimes_{a_i \in A} \tau\circ e_{a_i})\otimes 1](v,v)$$

so that

$$\hat{\beta}_A^{II}(v,\hat{J}_A^{II}v) = \delta Q_A^{II,II}(v).$$

Combining this with the result of Case (c), we obtain an expression for $\hat{\beta}_A(v,\hat{J}_A^{II}v) = \hat{\beta}(v,Jv)|\hat{W}_{2A}$, where A is of type II:

$$\hat{\beta}_A(v,\hat{J}_A^{II}v) = (\hat{\beta}_A^{I}\otimes 1+\hat{\beta}_A^{II})(v,\hat{J}_A^{II}v) = (Q_A^{I,II}+\delta Q_A^{II,II})(v).$$

Having found expressions for $^\wedge(v,v)$ on each subspace $\hat{W}_{2A}$ of $\hat{W}$, we are now ready to choose the constants α and δ. Note that the quadratic forms $Q_a^{I,I}$, $Q_a^{I,II}$, $Q_A^{II,I}$, and $Q_A^{II,II}$ are independent of α and δ.

The constant α will be chosen so that

$$\varphi_a(\alpha) > N \qquad \text{for } a \in S_0$$

$$0 < \varphi_a(\alpha) < \epsilon \qquad \text{for } a \in S_1$$

for some $N > 1$ where $\epsilon = N^{\frac{1}{1-m}}$, so that $\epsilon < 1$. Here $m = |S|$. If A is a type II molecule, then every $a_i \in A$ is in S_1. Hence $\varphi_{a_i}(\alpha) < 1$ for every $a_i \in A$; this fact does not depend on the choice of α, as long as α is chosen as above. Thus, for any $v \in \hat{W}_{2A}$,

$$|\varphi_{a_i}(\alpha)Q_{a_i}^{I,II}(v)| \leq |Q_{a_i}^{I,II}(v)|.$$

Recall that $Q_A^{I,II} = \sum_{a_i \in A} Q_{a_i}^{I,II}$. If F is a positive definite quadratic form such that $F + \sum_{a_i \in A} Q_{a_i}^{I,II}$ is positive definite, then because of the inequality above,

$$F + \Sigma\, \varphi_{a_i}(\alpha)Q_{a_i}^{I,II} = F + Q_A^{I,II}$$

is positive definite.

Now, since $Q_A^{II,II}$ is positive definite, we can find $N_{A,II}$ for each type II molecule A such that

$$N_{A,II}Q_A^{II,II} + \sum_{a_i \in A} Q_{a_i}^{I,II}$$

is positive definite. By the preceding remarks, $N_{A,II}Q_A^{II,II} + Q_A^{I,II}$ is then positive definite. Let $N_{II} = \max\{N_{A,II}\}$. Choose $\delta \in \mathbb{Z}$ such that $\delta > N_{II}$, so $\delta > N_{II,A}$ for every type II molecule A. Then

$$\hat{\beta}_A(v, \hat{J}_A^{II} v) = (\delta Q_A^{II,II} + Q_A^{I,I})(v)$$

is positive definite for each type II molecule A.

Now let A be a type I molecule. Since $Q_{a_j}^{I,I}$ is positive definite, there exists $N_{A,I}$ such that

$$N_{A,I}Q_{a_j}^{I,I} + \sum_{a_h \in A \cap S_1} Q_{a_h}^{I,I} + \delta Q_A^{II,I}$$

is positive definite. Let $N_I = \max\{N_{A,I}\}$. By Proposition 13.1, we can choose $\alpha \in \Theta_k$ such that

$$\varphi_a(\alpha) \geq N_I \qquad \text{for } a \in S_0$$

$$0 < \varphi_i(\alpha) < \epsilon_I \qquad \text{for } a \in S_1,$$

where $\epsilon_I = N_I^{\frac{1}{1-m}} < 1$. Then

$$(\varphi_{a_j}(\alpha)Q_{a_j}^{I,I} + \sum_{a_h \neq a_j} \varphi_{a_h}(\alpha)Q_{a_h}^{I,I} + \delta Q_A^{II,I})(v)$$

$$= (Q_A^{I,I,S_0} + Q_A^{I,I,S_1} + \delta Q_A^{II,I})(v)$$

$$= \hat{\beta}_A(v, \hat{J}_A^I v)$$

is positive definite for every type I molecule A.

We have shown that each component $\hat{\beta}_A(v,\hat{J}v)$ of $\hat{\beta}(v,\hat{J}v)$ is positive definite. Hence $\hat{\beta}(v,\hat{J}v)$ is positive definite. ∎

§14. Eichler Map

In this section we show that the representation $\rho = P_{\mathbb{A},\mathbb{R}}$ admits a holomorphic Eichler map. That is, we construct a holomorphic $\tau : X = G_{\mathbb{R}/C} \to \mathfrak{S}'(W,\beta)$, where $\mathfrak{S}'(W,\beta)$ is the Siegel upper half space discussed in §1. The map τ must satisfy

$$\rho(g)\tau(x) = \tau(gx)$$

for every $g \in G_{\mathbb{R}}$, $x \in X$. By the general theory in §1, this shows that $V \xrightarrow{\pi} U$ is a holomorphic fiber bundle.

First we prove a few lemmas. Recall that, for a complex structure J on W, $W_-(J)$ is the $-i$-eigensapce of the $\mathbb{C}$-linear extension of J on $W\otimes\mathbb{C}$.

Lemma 14.1. $W_-(J) = \{v + iJv : v \in W\}$.

Proof. $J(v+iJv) = Jv + iJ^2v = -iv + Jv = -i(v+iJv)$.

Lemma 14.2. Let $W = \mathbb{R}^2$. Set $G = SL_2(\mathbb{R})$, $C = SO(2)$, and identify G/C with $\mathfrak{H}$ by $gC \mapsto g(i)$. Let $J_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $J_x = gJ_ig^{-1}$ for $x = g(i) \in \mathfrak{H}$. Then $W_-(J_x)$ is the subspace $\{\binom{x}{1}z : z \in \mathbb{C}\}$ of $W\otimes\mathbb{C}$.

Proof. First note that, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$,

$$i(ac+bd) = 1 + i(ai+b)(-ci+d) = -1 + i(-ai+b)(ci+d).$$

Take $\binom{1}{0}$, $\binom{0}{1}$ as a basis for W over $\mathbb{R}$. Then

$$\{e_1 = \tbinom{1}{0} + iJ_x\tbinom{1}{0},\ e_2 = \tbinom{0}{1} + iJ_x\tbinom{0}{1}\}$$

is an $\mathbb{R}$-basis for $W_-(J_x)$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then

$$e_1 = \binom{1}{0} + igJ_i g^{-1}\binom{1}{0} = \binom{1}{0} + i\begin{pmatrix} -ac-bd \\ -c^2-d^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - i(ac+bd) \\ -i(c^2+d^2) \end{pmatrix} = -i(c^2+d^2)\begin{pmatrix} \frac{1-i(ac+bd)}{-i(c^2+d^2)} \\ 1 \end{pmatrix}$$

$$= -i(c^2+d^2)\begin{pmatrix} \frac{-i(ai+b)(-ci+d)}{-i(ci+d)(-ci+d)} \\ 1 \end{pmatrix} = -i(c^2+d^2)\binom{x}{1}$$

and

$$e_2 = \binom{0}{1} + igJ_i g^{-1}\binom{0}{1} = \binom{0}{1} + i\begin{pmatrix} a^2+b^2 \\ ac+bd \end{pmatrix}$$

$$= \begin{pmatrix} i(a^2+b^2) \\ 1 + i(ac+bd) \end{pmatrix} = (1 + i(ac+bd))\begin{pmatrix} \frac{i(a^2+b^2)}{1+i(ac+bd)} \\ 1 \end{pmatrix}$$

$$= (1 + i(ac+bd))\begin{pmatrix} \frac{i(-ai+b)(ai+b)}{i(-ai+b)(ci+d)} \\ 1 \end{pmatrix}$$

$$= (1 + i(ac+bd))\binom{x}{1} .$$

Hence

$$W_-(J_x) = \{re_1 + se_2 : r,s \in \mathbb{R}\}$$

$$= \{[r(-i)(c^2+d^2) + s(1+i(ac+bd))]\binom{x}{1} : r,s \in \mathbb{R}\}$$

$$= \{z\binom{x}{1} : z \in \mathbb{C}\}.$$

The last equality is true because if we set $z = u + iv$, $u, v \in \mathbb{R}$, we can solve for r and s:

$$z = u + iv = r(-i)(c^2+d^2) + s(1+i(ac+bd))$$

$$= s + i[-r(c^2+d^2) + s(ac+bd),$$

so $s = u$ and $r = \dfrac{u(ac+bd) - v}{c^2 + d^2}$. ∎

Corollary 14.3. Let G, C, J_i, and J_x be as in Lemma 14.2. Let $\hat{W} = M_2(\mathbb{R})$. Let $\hat{J}_i$ (respectively $\hat{J}_x$) be left multiplication of J_i (respectively J_x) on $\hat{W}$. Then the $-i$ eigenspace $\hat{W}_-(\hat{J}_x)$ is

$$\{\binom{x}{1}z : z \in \mathbb{C}\} \otimes \mathbb{C}^2.$$

Proof. The left multiplication of J_x on $M_2(\mathbb{R})$ is the same as the action of J as an automorphism of the direct sum of the two minimal left ideals of $M_2(\mathbb{R})$. Thus

$$\hat{W}_-(\hat{J}_x) = W_-(J_x) \otimes W_-(J_x) = W_-(J_x) \otimes \mathbb{C}^2$$

$$= \{\binom{x}{1}z : z \in \mathbb{C}\} \otimes \mathbb{C}^2.$$ ∎

We will construct an Eichler map $\tau : X \to \mathfrak{S}'(W, \beta)$ by first constructing $\hat{\tau} : X \to \mathfrak{S}'(\hat{W}, \hat{\beta})$ using the complex structures $\hat{J}_x$, and then restricting to W. To ensure that this works, we have the following lemmas.

Lemma 14.4. Let W be an even dimensional real vector space, and

β a nondegenerate alternating $\mathbb{R}$-bilinear form on $W \times W$. Let W_1 be a subspace of W, and let $\beta_1 = \beta | W_1 \times W_1$. Suppose that $f : X \to \mathfrak{S}(W,\beta)$ is a holomorphic map of a complex manifold X into the Siegel space. Let J_x denote $f(x)$ considered as a complex structure on W; since $J_x \in \mathfrak{S}(W,\beta)$, $\beta(u,J_x v)$ is symmetric and positive definite. If $J_x(W_1) = W_1$ for all $x \in X$, then

(a) β_1 is nondegenerate,

(b) W_1 is even dimensional, and

(c) the map $f_1 : X \to \mathfrak{S}(W_1,\beta_1)$ defined by $x \mapsto J_x | W_1$ is holomorphic.

Proof. Let $\dim W = 2n$. Set $s_x(u,v) = \beta(u,J_x v)$, and let $s_{1,x}$ denote s_x restricted to $W_1 \times W_1$. Since s_x is symmetric and positive definite, so is $s_{1,x}$. Therefore, β_1 is a nondegenerate alternating form on $W_1 \times W_1$, so $\dim W_1$ must be even. Set $\dim W_1 = 2m$.

We use the identification of $\mathfrak{S}(W,\beta)$ with $\mathfrak{S}'(W,\beta)$ described in §1 to consider $\mathfrak{S}(W,\beta)$ as a subvariety of the complex Grassmann variety $Gr_n(W \otimes \mathbb{C})$. Clearly, the eigenspace $W_{1,-}(J_x)$ is $W_-(J_x) \cap (W_1 \otimes \mathbb{C})$. Therefore, $W_{1,-}(J_x)$ is a point in the Schubert variety

$$s_{n,m}(W\otimes\mathbb{C}, W_1\otimes\mathbb{C}) = \left\{ \begin{array}{l} U : U \text{ is an } n\text{-dimensional } \mathbb{C}\text{-subspace of } W \otimes \mathbb{C}, \\ \quad \dim_{\mathbb{C}}(U \cap (W_1 \otimes \mathbb{C})) = m \end{array} \right\} .$$

The set $S_{n,m}(W\otimes\mathbb{C}, W_1\otimes\mathbb{C})$ is an open subset of a subvariety of

$Gr_n(W\otimes\mathbb{C})$. Let g be the map of $S_{n,m}(W\otimes\mathbb{C}, W_1\otimes\mathbb{C})$ to $Gr_m(W_1\otimes\mathbb{C})$ given by $g(U) = U\cap(W_1\otimes\mathbb{C})$. Then g is a holomorphic rational map. We have $W_{1,-}(J_x) = g(W_-(J_x))$, or, in other words, $f_1 = g\circ f$. Since f and g are holomorphic, f_1 is also. ∎

Lemma 14.5. Let W_1 and W_2 be $\mathbb{C}$-vector spaces, J a semisimple endomorphism of W_1, and λ an eigenvalue of J with eigenspace $W_1(\lambda) \subseteq W_1$. Then λ is also an eigenvalue of the endomorphism $J \otimes 1_{W_2}$ of $W_1 \otimes W_2$, and the eigenspace of $J \otimes 1_{W_2}$ is $W_1(\lambda) \otimes W_2 \subseteq W_1 \otimes W_2$.

Proof. Let $\lambda_1,\ldots,\lambda_r$ be the eigenvalues of J and $e_1,\ldots,e_s$ the eigenvectors of J. Since J is semismiple, $e_1,\ldots,e_s$ span W_1. Assume that the first q vectors $e_1,\ldots,e_q$ span $W_1(\lambda)$. Take any basis $f_1,\ldots,f_t$ of W_2. Then $B = \{e_i \otimes f_j : i = 1,\ldots,s;\ j = 1,\ldots,t\}$ is a basis of $W_1 \otimes W_2$, and $B_\lambda = \{e_i \otimes f_j : i = 1,\ldots,q;\ j = 1,\ldots,t\}$ spans $W_1(\lambda) \otimes W_2$. Clearly, $(J\otimes 1)(e_i\otimes f_j) = \lambda(e_i\otimes f_j)$ if $e_i \otimes f_j \in B_\lambda$. Conversely, if λ' is an eigenvalue of $J \otimes 1$ with eigenvector $e_i \otimes f_j$, then λ' is the eigenvalue of J corresponding to e_i; that is, if $\lambda' = \lambda$, then $e_i \in W_1(\lambda)$, $e_i \otimes f_j \in B_\lambda$. ∎

Proposition 14.6. Define $\hat{\tau} : X \to \mathfrak{S}'(\hat{W},\hat{\beta})$ by $\hat{\tau}(x) = \hat{W}_-(\hat{J}_x)$, where $\hat{J}_x$ is the complex structure defined in §12. Then the restricted map $\tau : X \to \mathfrak{S}'(W,\beta)$ described in Lemma 14.3 is well defined, and τ is a holomorphic Eichler map for the representation $\rho_{A,R}$.

Proof. First we prove that $\hat{\tau}$ is a holomorphic Eichler map for $\hat{\rho}_{A,R}$.

As explained in §1, the action of $Sp(\hat{W},\hat{\beta})$ on $\mathfrak{S}'(\hat{W},\hat{\beta})$ is given by

$$\gamma\cdot\hat{W}_-(J) = \hat{W}_-(\gamma J\gamma^{-1}).$$

Then, for $g \in G_{\mathbb{R}}$, $x \in X$,

$$\begin{aligned}\hat{\tau}(gx) &= \hat{W}_-(\hat{J}_{gx}) = \hat{W}_-(\bigoplus_{A\in\mathbb{A}}\hat{J}_{A,gx}) \\ &= \hat{W}_-(\bigoplus_{A\in\mathbb{A}}\hat{P}_{A,\mathbb{R}}(g)\hat{J}_{A,x}\hat{P}_{A,\mathbb{R}}(g)^{-1}) \\ &= \hat{W}_-(\hat{P}_{\mathbb{A},\mathbb{R}}(g)(\bigoplus_{A\in\mathbb{A}}\hat{J}_{A,x})\hat{P}_{\mathbb{A},\mathbb{R}}(g)^{-1}) \\ &= \hat{W}_-(\hat{P}_{\mathbb{A},\mathbb{R}}(g)\hat{J}_x\hat{P}_{\mathbb{A},\mathbb{R}}(g)^{-1}) \\ &= \hat{P}_{\mathbb{A},\mathbb{R}}(g)\cdot\hat{W}_-(J_x) = \hat{P}_{\mathbb{A},\mathbb{R}}(g)\cdot\hat{\tau}(x).\end{aligned}$$

Here $\hat{J}_A$ and $\hat{P}_{A,\mathbb{R}}$ represent $\hat{J}_{2A}$ and $\hat{P}_{2A,\mathbb{R}}$ if A is of type II. Thus we have shown that $\hat{\tau}$ is an Eichler map for $\hat{P}_{\mathbb{A},\mathbb{R}}$.

Now we show that $\hat{\tau}$ is holomorphic. We will also use $\hat{W}_A$ to denote $\hat{W}_{2A}$ if A is of type II. We have

$$\hat{W}_-(\hat{J}_x) = \hat{W}_-(\bigoplus_{A\in\mathbb{A}}\hat{J}_{A,x}) = \bigoplus_{A\in\mathbb{A}}\hat{W}_{A,-}(\hat{J}_{A,x}).$$

Hence it suffices to find the eigenspace of $\hat{J}_{A,x}$ on $\hat{W}_A$ for each A. Write $G_{\mathbb{R}} = G_1 x\ldots xG_m$ in terms of its simple components, and $X = X_1 x\ldots xX_m$ as the corresponding decomposition of X. If $h \in S_0$, $X_h = \mathfrak{h}$. If $h \in S_1$, then X_h is a single point; we identify X_h with the point $\{i\}$ in $\mathfrak{h}$. Let $M_A = \dim_{\mathbb{R}}(\hat{W}_A)$.

If $\mathbb{A}$ is rigid, then $\hat{J}_{A,x} = \hat{J}^I_{A,x}$ for each molecule A. Let $g = (g_1,\ldots,g_m)$ be an element of $G_{\mathbb{R}}$ such that $gx_0 = x$, where

$x = (x_1, \ldots, x_m)$. If $h \in S_0$, then $g_h(i) = x_h$.

Suppose a_s is the atom of A that is in S_0. Recall that $j = ((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), \ldots, (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), 1, \ldots, 1) \in G_{\mathbb{R}}$. Then

$$\hat{J}_{A,x} = \hat{P}_{A,\mathbb{R}}(g)\hat{P}_{A,\mathbb{R}}(j)\hat{P}_{A,\mathbb{R}}(g)^{-1}$$

$$= \hat{P}_{A,\mathbb{R}}(gjg^{-1}) = g_{a_s}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{a_s}^{-1} \otimes \Big(\bigotimes_{a_h \neq a_s} 1\Big)$$

$$= g_{a_s}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{a_s}^{-1} \otimes 1_{B^{a_s}} ,$$

where $B^{a_s} = \bigotimes_{\substack{a_h \in A \\ a_h \neq a_s}} B_{a_h}$. By Corollary 14.3, the $-i$-eigenspace of $g_{a_s}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{a_s}^{-1}$ is the 2-dimensional subspace $\left\{\begin{pmatrix} x_{a_s} \\ 1 \end{pmatrix} z : z \in \mathbb{C}\right\} \otimes \mathbb{C}^2$ of $\mathbb{C}^4$. By Lemma 14.5, the $-i$-eigenspace of $g_{a_s}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{a_s}^{-1} \otimes 1_{B^{a_s}}$ on $(B_{a_s} \otimes B^{a_s}) \otimes \mathbb{C} = B_A \otimes \mathbb{C} = \hat{W}_A \otimes \mathbb{C}$ is

$$\left\{\begin{pmatrix} x_{a_s} \\ 1 \end{pmatrix} z : z \in \mathbb{C}\right\} \otimes \mathbb{C}^2 \otimes \mathbb{C}^{\frac{1}{4}M_A} = \left\{\begin{pmatrix} x_{a_s} I_{\frac{1}{2}M_A} \\ I_{\frac{1}{2}M_A} \end{pmatrix} z : z \in \mathbb{C}^{\frac{1}{2}M_A}\right\} .$$

Here $I_{\frac{1}{2}M_A}$ is the identity matrix of size $\frac{1}{2}M_A$

If $\mathbb{A}$ is of mixed type, and A is of type I, then the argument above holds. Then

$$\hat{W}_{2A,-}(\hat{J}^{I}_{A,x} \otimes 1) = \left\{\begin{pmatrix} x_{a_s} I_{M_A} \\ I_{M_A} \end{pmatrix} z : z \in \mathbb{C}^{M_A}\right\} ,$$

where $a_s \in A$ is the atom in S_0. If A is of type II, then $\hat{J}^{II}_{A,x} = \hat{J}^{II}_{A} = 1_{\hat{W}_A} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The $-i$-eigenspace of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{C}^2$ is

$$\{\begin{pmatrix} i \\ 1 \end{pmatrix} z : z \in \mathbb{C}\},$$

so, by Lemma 14.5,

$$\hat{W}_{2A,-(\hat{J}^{II}_{A,x})} = \{\begin{pmatrix} i \\ 1 \end{pmatrix} z : z \in \mathbb{C}\} \otimes \mathbb{C}^{M_A} = \left\{ \begin{pmatrix} iI_{M_A} \\ I_{M_A} \end{pmatrix} z : z \in \mathbb{C}^{M_A} \right\}.$$

So, if $\mathbb{A}$ is rigid,

$$\hat{W}_-(\hat{J}_x) = \bigoplus_{A \in \mathbb{A}} \hat{W}_{A\ -(\hat{J}^{I}_{A,x})} = \bigoplus_{A \in \mathbb{A}} \left\{ \begin{pmatrix} x_{a_s} I_{\frac{1}{2}M_A} \\ I_{\frac{1}{2}M_A} \end{pmatrix} z : z \in \mathbb{C}^{\frac{1}{2}M_A} \right\}.$$

If $\mathbb{A}$ is of mixed type, then

$$\hat{W}_-(\hat{J}_x) = \left[\bigoplus_{A \text{ type I}} W_{2A,-(\hat{J}^{I}_{A,x} \otimes 1)} \right] \oplus \left[\bigoplus_{A \text{ type II}} \hat{W}_{2A,-(\hat{J}^{II}_{A,x})} \right]$$

$$= \bigoplus_{A \text{ type I}} \left\{ \begin{pmatrix} x_{a_s} I_{M_A} \\ I_{M_A} \end{pmatrix} z : z \in \mathbb{C}^{M_A} \right\} \oplus \bigoplus_{A \text{ type II}} \left\{ \begin{pmatrix} x_{a_h} I_{M_A} \\ I_{M_A} \end{pmatrix} z : z \in \mathbb{C}^{M_A} \right\}.$$

Here, a_h is any atom in A, where A is of type II, and $x_{a_h} = i$, the only point in X_{a_h}. In both cases, the subspace $\hat{W}_-(\hat{J}_x)$ of $\hat{W} \otimes \mathbb{C}$ clearly varies holomorphically with each variable x_a. Since this map $\hat{\tau}$ is holomorphic in each variable, $\hat{\tau}$ is holomorphic.

Since we have showed in previous sections that $\hat{\beta}$ is a nondegenerate alternating form on $\hat{W}$, and that $\hat{J}_x(W) = J_x(W) = W$ for all $x \in X$, $\hat{\tau} : X \to \mathfrak{S}'(\hat{W}, \hat{\beta})$ satisfies the hypotheses of Lemma 14.4. Then $\tau : X \to \mathfrak{S}'(W, \beta)$ given by $x \mapsto W_-(J_x)$ is well defined and holomorphic. ∎

This completes the proof of the theorem.

APPENDIX

Central Simple Algebras and Quaternion Algebras

A.1. Central simple algebras. A central simple algebra over a field k is an algebra whose center is k that has no nontrivial two-sided ideals. The theory of central simple algebras is discussed in detail in [W-2]. We will abbreviate "central simple algebra" by CSA.

Every CSA is isomorphic to a matrix algebra over a division algebra, $M_n(D)$. Two CSAs over k are called equivalent if they are matrix algebras over the same division algebra. The set of equivalence classes of CSAs over k forms a group under the tensor product, and is called the Brauer group of k, B(k). The identity element of B(k) is the class of algebras $M_n(k)$; any algebra in this class is called trivial. The inverse of an algebra A is its opposite algebra, A°, constructed from A by defining a new multiplication $\mu(x,y) = yx$.

If L is a field containing k, and A is a CSA over k, then $A \otimes_k L$ is a CSA over L. If L is algebraically closed, then every CSA over L is isomorphic to $M_n(L)$ for some n. Hence, every CSA over k has a representation into $M_n(\bar{k})$, where $\bar{k}$ is the algebraic closure of k. This representation is called the reduced representation, and is given by $A \otimes_k \bar{k} \xrightarrow{\cong} M_n(\bar{k})$. This isomorphism is unique up to inner automorphisms. As a corollary, the dimension of any CSA is a square.

Proposition A.1. ([W-2], Chapter IX, Proposition 6). Let k be an infinite field, and A a CSA of dimension n^2 over k. Then there is a nonzero k-linear form τ and a k-valued polynomial function ν on A; τ is the trace of the reduced representation and ν is the determinant of the reduced representation. Both τ and ν are invariant under inner automorphisms of A, and the polynomial function ν has degree n and coefficients in k.

The maps τ and ν are called the reduced trace and reduced norm on A. Clearly $\tau(xy) = \tau(yx)$ and $\nu(xy) = \nu(x)\nu(y) = \nu(yx)$.

If k is a number field, we consider the places v of k. For any place v, $A_v = A \otimes_k k_v$ is a CSA over k_v. If $A_v \cong M_n(k_v)$, we say that A splits at v. If not, we say that A is ramified at v.

If $k_v = \mathbb{C}$, then the only simple algebras over k_v are matrix algebras, so $B(\mathbb{C})$ is the trivial group. If $k_v = \mathbb{R}$, the only non-trivial division algebra over k_v is the Hamiltonian quaternion algebra, $\mathbb{H}$. Thus $B(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, which we will represent by $\{0, \frac{1}{2}\}$. If v is a finite place, i.e., k_v is a finite extension of a p-adic field, then $B(k_v)$ is isomorphic to the group of roots of unity in $\mathbb{C}$. ([W-2], Chapter XII, Corollary 1 to Theorem 1.) This group is isomorphic to $\mathbb{Q}/\mathbb{Z}$, which we will represent by the set of positive rational numbers less than one.

Let $h_v(A)$ be the Hasse invariant of a simple algebra A over k_v, described in [W-2]. Then h_v is an isomorphism from $B(k_v)$ to $\mathbb{Q}/\mathbb{Z}$ if v is finite and from $B(k_v)$ to $\{0\}$ or $\mathbb{Z}/_{2\mathbb{Z}}$ if v is infinite. By Theorem 2 of Chapter XI in [W-2], the equivalence class of an algebra A over the number field k is uniquely determined by $\{h_v(A)\}$ for all places v of k. In fact, suppose that, for each place v of k, h_v is an element of $\mathbb{Q}/\mathbb{Z}$ such that $h_v = 0$ for all but a finite number of v, $\Sigma\, h_v = 0$, h_v is 0 or $\frac{1}{2}$ if v is real, and $h_v = 0$ if v is complex. Then there is a CSA A over k such that $h_v(A) = h_v$.

A.2. Quaternion algebras. The main reference for this section is [V]. A quaternion algebra is a CSA of dimension four. Thus a quaternion algebra is either a division algebra or a matrix algebra over its center, $M_2(k)$. A quaternion has a unique involution $(\ )'$. The map $(\ )'$ is a k-linear anti-automorphism whose square is the identity map, and which fixes only the elements of k. By means of this involution, one can show that a quaternion algebra A is isomorphic to its opposite algebra A°. Hence $A \otimes A \cong A \otimes A^\circ \cong M_4(k)$, so that a quaternion algebra has order two in the Brauer group. So if k is a number field, the Hasse invariants $h_v(A)$ of a quaternion algebra are all 0 or $\frac{1}{2}$, and $h_v(A) = 0$ if v is complex. Let Ram(A) be the set of places of k at which A is ramified. Then, since $\Sigma\, h_v(A) = 0$, $|\mathrm{Ram}\ A|$ must be even.

<u>Theorem A.2</u>. ([V], Chapter II, Theorem 3.1). For every finite set S of places of k such that $|S|$ is even, there exists a quaternion algebra A over k, unique up to isomorphism, such that $S = \mathrm{Ram}(A)$.

In a quaternion algebra, $\tau(x) = x + x'$, and $\nu(x) = xx'$.

Every element α of A, $\alpha \notin k$, satisfies a quadratic equation $X^2 - \tau(\alpha)X + \nu(\alpha)$. Hence every quaternion algebra contains a (non-unique) maximal subfield which is quadratic over the center k.

If A is a quaternion algebra over $\mathbb{R}$, then $A \cong M_2(\mathbb{R})$ or $A \cong \mathbb{H}$. The involution in $M_2(\mathbb{R})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence

$$\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} = \mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot 1$$

and

$$\nu\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot 1.$$

The involution in $\mathbb{H}$ is given by

$$(a+bi+cj+dk)' = a - bi - cj - dk,$$

so that

$$\tau(a+bi+cj+dk) = 2a$$

and

$$\nu(a+bi+cj+dk) = a^2 + b^2 + c^2 + d^2.$$

For the rest of this section, k is a number field.

Theorem A.3. A quaternion algebra A over k contains a maximal subfield L which quadratic over k and totally imaginary.

To prove Theorem A.3, we need several other results.

Lemma A.4. ([V], Chapter III, Lemma 3.6). There exists a quadratic extension L of k such that $L_v = L \otimes_k k_v$ is a quadratic field extension of k_v for a given finite set of places v of k.

Theorem A.5. ([V], Chapter III, Theorem 3.8). A quadratic extension L of k is a subfield of A if and only if $L_v = L \otimes_k k_v$ is a quadratic extension of k_v when $v \in \mathrm{Ram}(A)$.

Proof of Theorem A.3. By Lemma A.4, there is a quadratic extension L of k such that $L_v = L \otimes_k k_v$ is a quadratic extension of k_v when $v \in \mathrm{Ram}(A)$ or v is real. If v is complex, then $k_v = L_v = \mathbb{C}$. If v is real, then L_v is a quadratic extension of $k_v = \mathbb{R}$, so $L_v = \mathbb{C}$. Thus L is totally imaginary. Theorem A.5 says that L is a subfield of A. ∎

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