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RIEMANNIAN FIBRATIONS OF EUCLIDEAN SPACES

A Dissertation presented

by

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to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August 1981

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

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Acknowledgments

I sincerely thank my advisor, Professor Detlef Gromoll, for his guidance and his assistance in the preparation of this dissertation. Also I would like to thank the Mathematics Department of State University of New York for funding my stay at Stony Brook.

To my beloved parents and to my dear D^2.

Introduction

In this paper we mainly work with riemannian fibrations. Some of the problems we consider are also interesting for riemannian foliations, and many of our arguments are valid in that case, in particular in the case of 1-dimensional leaves. We study local properties and the global structure of riemannian fibrations of R^n over some complete riemannian manifold. In general, riemannian fibrations do not exist, even locally, for a general metric. However, there are a lot in case of constant sectional curvature. The corresponding problem for euclidean spheres, has been solved by D. Gromoll and K. Grove [GG]. In our case, it turns out such fibrations are globally (not locally) fairly rigid. The techniques we apply are quite different from the compact case. Because of the non-compactness of the fibers, we cannot use the volume of the fiber as in the compact case. We have to apply the basic construction of riemannian manifolds of non-negative sectional curvature. Essentially we show that non-trivial riemannian fibrations of flat euclidean space exist, and they are all obtained by free isometric actions of lower dimensional euclidean space . We should point out that for the hyperbolic spaces, there is no rigidity. So the rigidity of riemannian fibration seems to be a phenomenon of complete riemannian manifolds of non-negative sectional curvature.

Let N and M be C^{∞} riemannian manifolds. By a riemannian submersion, we mean a C^{∞}-mapping $\pi : N \longrightarrow M$ from N onto M such that (i) π is of maximal rank, and (ii) π_* preserves the lengths of horizontal vectors, i.e. vectors orthogonal to the fiber $\pi^{-1}(x)$ for $x \in M$. Our main theorem is that every riemannian fibration from flat euclidean space R^n onto some riemannian manifold M, which is necessarily complete, has a totally geodesic fiber in R^n. By the existence of the totally geodesic fiber, we prove that each riemannian fibration $\pi : R^n \longrightarrow M^{n-k}$ corresponds to a Lie group representation of R^k in $O(n-k)$. Hence we can classify the riemannian fibration of euclidean space with (complete) connected fibers by group representations of R^k in $O(n-k)$. In particular, the riemannian foliations of R^n with one dimensional leaves are fibrations and are classified by the (non-vanishing) Killing vector fields of R^n.

In this paper, we first give some known results and facts about riemannian submersions and riemannian manifolds of non-negative sectional curvature. Next, we give an informal discussion of the local properties of riemannian fibrations. Then we study the global structure for the general k-dimensional fiber case. Applying the basic construction introduced by Cheeger and Gromoll, [CG], we prove the classification theorem for riemannian fibrations of R^n. Finally we look closely at the riemannian foliation with one dimensional leaves. By using elementary methods, we prove that any one dimensional riemannian foliation of R^n is a riemannian fibration, and it is determined by a non-zero Killing field of R^n.

1. Preliminaries

In this section, we outline some known results and facts about riemannian submersions and riemannian manifolds of nonnegative sectional curvature. For all the basic facts we refer to the papers by O'Neill, Cheeger and Gromoll, and the book by Cheeger and Ebin.

Let $\pi : N^n \longrightarrow M^{n-k}$. A submersion π is a differentiable map such that at each point π_* has maximal rank n-k. It follows from the implicit function theorem that $\pi^{-1}(p)$ is a closed, smooth k-dimensional submanifold regularly imbedded in N for every $p \in M$. The submanifolds $\pi^{-1}(p)$ are called fibers for all $p \in M$. Let V denote the tangent space to $\pi^{-1}(p)$ at $q \in \pi^{-1}(p)$. Assume that N and M have riemannian metrics and set $H = V^{\perp}$. We call H and V the horizontal and vertical subspaces, respectively. Hence a tangent vector to N at $q \in \pi^{-1}(p)$ is horizontal if orthogonal to the fiber $\pi^{-1}(p)$, vertical if tangent to the fiber.

π is called a riemannian submersion if $\pi_*|_H$ is an isometry, i.e. π_* preserves the length of horizontal vectors. Let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of the tangent spaces of N onto the subspaces H and V, respectively. Define the tensor T for arbitrary vector fields E and F by :

$$T_E F = \mathcal{H} \nabla_{\mathcal{V}E} \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H}F \ .$$

T has the following properties :

1) T_E is a skew adjoint operator on tangent spaces of N reversing horizontal and vertical subspaces.

2) $T_E = T_{\mathcal{V}E}$.

3) For vertical vector fields V and W, T is symmetric i.e. $T_V W = T_W V$.

Along a fiber, T is the second fundamental tensor of the fiber provided we restrict ourselves to vertical vector fields.

Define the integrability tensor A associated with the submersion : $A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$ with the following properties :

1) At each point A_E is skew adjoint operator on TN reversing the horizontal and vertical subspaces.

2) $A_E = A_{\mathcal{H}E}$.

3) For X, Y horizontal, A is alternating, i.e. $A_X Y = - A_Y X$.

For X,Y horizontal vector fields, V, W be vertical vector fields, $A_X Y = \frac{1}{2}[X, Y]$ and $\nabla_V W = T_V W + \hat{\nabla}_V W$, where $\hat{\nabla}$ denotes the riemannian connection along a fiber with respect to the induced metric.

Definition : A basic vector field is a horizontal vector field which is π-related to a vector field X_* on M, i.e.

$$\pi_* X_u = X_{*\,\pi(u)} \,,\forall\, u \in N \,.$$

For X, Y basic and π-related to X_*, Y_* respectively,

a) $\langle X, Y \rangle_N = \langle X_* , Y_* \rangle_M$.

b) $\mathcal{H}[X,Y]$ is basic and π-related to $[X_*, Y_*]$.

c) $\mathcal{H}\nabla_X Y$ is basic and π-related to $\nabla^*_{X_*} Y_*$.

Proposition 1.1

$\pi : N \longrightarrow M$ be a riemannian submersion. Let K, K_* and $\hat{K}$ be the sectional curvature of N, M and the fibers. If X,Y are horizontal vectors at a point of N, and V,W are vertical, then:

1) $K(P_{VW}) = \hat{K}(P_{VW}) - \dfrac{\langle T_V V, T_W W\rangle - \| T_V W\|^2}{\| V\wedge W\|^2}$.

2) $K(P_{XV})\|X\|^2\|V\|^2 = \langle(\nabla_X T)_V V, X\rangle + \|A_X V\|^2 - \|T_V X\|^2$.

3) $K(P_{XY}) = K_*(P_{X_* Y_*}) - \dfrac{3\|A_X Y\|^2}{\|X\wedge Y\|^2}$.

Proof : cf. O'Neill's paper $[O_1]$.

Proposition 1.2

Let $\pi : N \longrightarrow M$ be a riemannian submersion. Assume M is complete. If X,Y are basic vector fields, then $A_X Y$ restricted to any totally geodesic fiber is a Killing vector field of that fiber.

Proof : It is straight forward.

A curve in N is said to be horizontal if its tangent vector field is horizontal. The projection of a horizontal geodesic of N is a geodesic of M.

If $\sigma : [0,1] \longrightarrow M$ is a curve, $x_0 \in \pi^{-1}(\sigma(0))$, there is at most one horizontal lift $\sigma_1 : [0,1] \longrightarrow N$ of σ beginning at x_0, and the horizontal lift exists locally.

Proposition 1.3

Let $\pi : N \longrightarrow M$ be a riemannian submersion. If N is complete, so is M. The horizontal lift of paths of M exist globally, and π is a fiber space.

Proof : See $[H_1]$, Proposition 3.2 .

Proposition 1.4

If $\sigma : [0,1] \rightarrow M$ is a path, there is a diffeomorphism $h_\sigma : \pi^{-1}(\sigma(0)) \longrightarrow \pi^{-1}(\sigma(1))$ obtained by mapping each x_o in $\pi^{-1}(\sigma(0))$ into the end point of the horizontal lift of σ starting at x_o . Hence any two fibers are diffepmorphic.

Proof : $[H_1]$.

If $p_1, p_2 \in M$ are two distinct points, $q_i, q_i' \in \pi^{-1}(p_i) =: F_i$, then

$$\begin{aligned} \mathrm{dist}(F_1,F_2) &= \mathrm{dist}(q_1,F_2) = \mathrm{dist}(q_1',F_2) \\ &= \mathrm{dist}(F_1,q_2) = \mathrm{dist}(F_1,q_2') \\ &= \mathrm{dist}(p_1,p_2) \quad , \end{aligned}$$

which shows that every two fibers are everywhere equidistant. Hence we have the stability property of the fibers : If one fiber comes within a certain distance of another fiber at one point, it remains within that distance. Thus, if one of the fiber is bounded (unbounded) in N, then every other fiber is bounded (unbounded).

Proposition 1.5

Let $\pi : N \longrightarrow M$ be a riemannian submersion. If γ is a geodesic in N, that is horizontal at some one point, then γ is always horizontal, and hence $\pi \circ \gamma$ is a geodesic of M.

Proof : cf. [0_2] .

Proposition 1.6

Let $\pi : N \longrightarrow M$ be a riemannian submersion. $\gamma : [a,b] \longrightarrow N$ a horizontal geodesic segment. Then the following integers are equal :

1) The order of $\gamma(b)$ as a focal point of the fiber F_a along γ.

2) The order of $\gamma(a)$ as a focal point of the fiber F_b along γ.

3) The order of conjugacy of F_a and F_b along γ.

4) The order of conjugacy of the end points of $\pi \circ \gamma$ along $\pi \circ \gamma$.

Proof : cf. [0_2] .

In general, conjugate points occur sooner in M : if $\gamma(a)$ has a conjugate point $\gamma(t)$, then there exists a conjugate point $\pi \circ \gamma(t')$ of $\pi \circ \gamma(a)$ with $t' \leq t$.

Proposition 1.7

X is a lift along the fiber F iff

$$\nabla_V X = A_X V + T_V X$$

for each vertical field V tangent to F.

Proof : cf. [0_1] .

It is an immediate consequence that any fibration is determined uniquely by one fiber and a frame of basic fields along that fiber.

We recall the basic construction of [CG] :

Let M be a complete riemannian of non-negative sectional curvature. A non-empty subset C of M will be called totally convex if for any p and q in C, and <u>any</u> geodesic $c : [0,1] \longrightarrow M$ from p to q, c lies in C.

<u>Theorem 1.8</u>

Let C be a closed totally convex subset of an arbitrary riemannian manifold M. Then C has the structure of an imbedded k-dimensional submanifold of M with smooth totally geodesic interior and (possibly non-smooth) boundary ∂C.

For a closed totally convex set C, if $\partial C \neq \phi$,
set $C^a =: \{ p \in C : d(p, \partial C) \geq a \}$ for $a \geq 0$.

<u>Theorem 1.9</u>

Let M have non-negative curvature, and let C be a closed totally convex subset of M with non-empty boundary. If

$$a_{max} = \max \{ a \in R : C^a \neq \phi \} = \max_{p \in C} d(p, \partial C) ,$$

then $\dim C^{a_{max}} < \dim C$.

A ray $\gamma : [0,1] \longrightarrow M$ is a geodesic parametrized by arc length, each finite segment of which realises the distance between its end points. In any non-compact manifold M, there is at least one ray starting at each point of M.

Let $B_r(p)$ as usual denote the open metric ball of radius r, centered at p. Given γ, define B_γ to be $\bigcup_{t>0} B_t(\gamma(t))$

and C_γ to be the complement $(B_\gamma)'$ of M. Since the balls $B_t(\gamma(t))$ are open, $\gamma(0) \in (B_\gamma)' = C_\gamma$. Hence $C_\gamma \neq \emptyset$.

Theorem 1.10

Let M be non-compact manifold of non-negative sectional curvature. Then for any ray γ, C_γ is totally convex.

Theorem 1.11

With M as above and $p \in M$, there exists a family of compact totally convex sets C_t, $t \geq 0$, such that

1) $t_2 \geq t_1$, implies $C_{t_2} \supset C_{t_1}$, and

$$C_{t_1} = \left\{ q \in C_{t_2} : d(q, \partial C_{t_2}) \geq t_2 - t_1 \right\}, \text{ in particular}$$

$$\partial C_{t_1} = \left\{ q \in C_{t_1} : d(q, \partial C_{t_2}) = t_2 - t_1 \right\} .$$

2) $\bigcup_{t \geq 0} C_t = M$.

3) $p \in C_0$.

Theorem 1.12

M contains a compact totally geodesic submanifold S without boundary, which is totally convex, $0 \leq \dim S \leq \dim M$. In particular, S has non-negative sectional curvature. S is called the soul of M. The manifold M is diffeomorphic to the normal bundle of S in M. In particular, M is diffeomorphic to R^n iff any soul S is a point.

Theorem 1.13

Let M have non-negative sectional curvature, C be closed and convex (totally convex, respectively), $\partial C \neq \emptyset$, and let $\psi : C \longrightarrow R$ be defined by $\psi(x) = d(x, \partial C)$. Then for any normal geodesic segment s contained in C, the function $\psi \circ c(t)$ is (weakly) convex, i.e.

$$\psi \circ c(\alpha t_1 + \beta t_2) \geq \alpha \psi \circ c(t_1) + \beta \psi \circ c(t_2),$$

where $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

For the proofs see Cheeger and Ebin [CE], or Cheeger and Gromoll [CG].

2. Some Local Aspects.

In this section we give an informal discussion about local properties of riemannian fibrations. It will not be essential to our main theorem, but it has its own interest.

We first consider the case that the fiber dimension is one. Even for the one dimensional case, it is rather complicated to analyse the local property of riemannian fibrations. Given a regular C^3-curve c in R^n , we will describe the necessary and sufficient condition for c to be (locally) a fiber of some riemannian fibration in R^n.

We first derive a necessary condition. So suppose c is a fiber of some local riemannian fibration in R^n, which we may assume to be parametrized by arc length. If X is a basic field defined along c, then it satisfies the equation

$$\dot{X} = A_X T - \mathcal{X}_x T \quad , \qquad (*)$$

where T is the unit tangent field. Conversely, it is true if X is a (horizontal) vector field along the fiber satisfying (*) then it is basic. Using O'Neill's A-tensor, we can define an operator Ω on the horizontal space by $\Omega(X) = - A_X T$. Then Ω is a skew symmetric operator because of the skew symmetry of the A-tensor. If we choose an orthonormal frame at a point on the curve, say X_i , then $\Omega = (\omega_{ij})$ is a skew symmetric matrix with $\omega_{ij} = \langle A_{x_i} x_j , T \rangle$.

Lemma 2.1

Ω is parallel, i.e. $\Omega' = 0$ relative to the induced connection of the normal bundle.

Proof : Let $\omega(X,Y) = \langle A_X Y, T \rangle$. Then

$$\omega'(X,Y) = \omega(X,Y)' - \omega(X',Y) - \omega(X,Y') .$$

Since $\omega(X,Y)' = 0$ as $\omega(X,Y)$ is constant along the fiber, and

$$\omega(X',Y) = \langle A_{X'}Y , T \rangle = -\langle A_Y X', T \rangle = \langle A_X T, A_Y T \rangle$$
$$= \langle A_X T, Y' \rangle = -\langle A_X Y', T \rangle = -\omega(X,Y') .$$

Thus $\omega' = 0$ and hence $\Omega' = 0$.

Since Ω is parallel, if we are given Ω at a point, we know Ω along the fiber. It then follows from (*) that any basic field X along c is determined by its values at that point. Now let X,Y be basic fields along the curve c, i.e. satisfying the equation (*). Consider the variation of curves

$$c_\epsilon(t) = c(t) + X(t) + \epsilon Y(t) .$$

For fixed t, $c_\epsilon(t)$ is a line segment in R^n, and

$$\frac{dc_\epsilon}{d\epsilon}\Big|_{t=o} = Y(t),$$

$$\frac{dc_\epsilon}{dt} = \dot{c}(t) + \dot{X}(t) + \epsilon \dot{Y}(t) ,$$

$$|\dot{c}(t)|^2 = |A_X T|^2 + (1-\mathcal{K}_x)^2 \, |T|^2 ,$$

and $\langle Y(t), \dot{c}_o(t) \rangle = \langle \dot{X}(t), Y(t) \rangle = -\langle A_X Y, T \rangle$ where $T = \dot{c}(t)$. Hence Y makes constant angle with $\dot{c}_o(t)$ along $c_o(t)$. So, the horizontal projection of Y has constant length along the curve c_o iff $\mathcal{K}_x$ = constant along c.

When c is a fiber of a riemannian fibration with projection π, $c_o(t)$ is also a fiber induced by the basic field X along c. Y is a lift (not necessarily horizontal) along c_o. Then $\pi_*(Y)$ is of constant length along π_* and this forces $\mathcal{K}_x$ = constant along c, provided $A_X T \neq 0$.

On the other hand, it follows from this discussion: if $\mathcal{K}_x$ = constant along the curve c for each basic field X, then locally we have a riemannian fibration, by exponentiating the basic fields along c. Note that $\mathcal{K}_x$ = constant along the curve c iff T' is basic, since $\mathcal{K}_x = \langle \dot{X}, T \rangle = -\langle X, T' \rangle$. Also, a vector field X is basic iff $X' = -\Omega(X)$. Hence the necessary and sufficient condition for a regular C^3-curve to be a fiber of a local riemannian fibration is that T' is basic, i.e.

$$\Omega(T') + T'' = 0$$

on the horizontal space, provided Ω is non-degenerate, i.e. $\det \Omega \neq 0$. (Since Ω is parallel, $\det\Omega$ is constant along c.)

Let us briefly analyse the simplest nontrivial case, when c is a curve in R^3. Ω then is a skew symmetric operator on the two dimensional normal space of c. So either $\Omega = 0$, or Ω is non-degenerate. In the latter case, its action is just a dilatation through 90° in the normal space. Using Frenet's formulas : $\frac{dT}{dt} = \varkappa N$, $\frac{dN}{dt} = -\varkappa T + \tau B$, $\frac{dB}{dt} = -\tau N$.

$\Omega(T') = \Omega(\varkappa N) = \mu\varkappa B$, $\mu \neq 0$ is the constant of dilatation,

$$T'' = (\varkappa N)' = \left[\frac{d}{dt}(\varkappa N)\right]^h = \varkappa' N + \tau\varkappa B$$

Since $\Omega(T') + T'' = 0$, we must have $\varkappa' = 0$ and $\varkappa(\tau + \mu) = 0$. So $\varkappa =$ constant along the curve and $\tau =$ constant at the same time. In this case, c must be a helix in R^3, and even the local fibers are "rigid". Notice however, that in the case $\Omega \equiv 0$, i.e. $A \equiv 0$, by the above, any regular C^2-curve c in R^n is the fiber of a local (flat) riemannian fibration, so in that case the local situation is highly non-rigid. If dim ker $\Omega = 1$, then there exists $X \neq 0$ such that $\Omega(X) = 0$. If X is basic and $\Omega(X) = 0$ at one point, then $\Omega(X) \equiv 0$ along the curve c_o . Let us consider the neighbouring fibers induced by X, say $c(t) = c_o(t) + X(t)$. Then $\dot{c}(t) = \dot{c}_o(t) + \dot{X}(t) = T_o + A_X T_o + \varkappa_x T_o$

$= (1 + \varkappa_x)T_o$. So in this case, $\Omega(X) = 0$;

all the nearby fibers induced by sX are parallel to c_o , and hence there is no restriction on the curvature of the curve c_o in X-direction.

Ω is skew symmetric on the horizontal space H, and $H = \ker\Omega + \mathrm{Im}\Omega$ is an orthogonal splitting. Because Ω is parallel, we also have $\ker\Omega$ and $\mathrm{Im}\Omega$ parallel, thus there exists a parallel frame field E_i along the curve c_o with

$\Omega(E_i) = E_{2i+1}$, $\Omega(E_{2i+1}) = - E_{2i}$, $1 \leq i \leq r$, and $\Omega(E_s) = 0$, $s \geq 2r + 1$.

For the case $\Omega \equiv 0$, i.e. the flat case, there is no restriction whatever for the (regular C^2) curve to be a fiber of a local riemannian fibration; as we have seen we can define a riemannian fibration locally by parallel transport of the tangent vector field in horizontal directions. This will give us a flat quotient manifold.

From the discussion above, the case in between the two extremal cases, $\Omega = 0$ and Ω non-degenerate, is somewhat more complicated. What we conclude is that the more degeneracy of Ω is imposed, the less the rigidity of the curve. Only in the case that Ω is non-degenerate will the fibers of local fibrations in R^n necessarily be fairly rigid already. As soon as Ω is degenerate, riemannian fibrations in R^n are locally " infinite dimensional", and ample. Our main rigidity theorem is therefore a global phenomenon, and must make use of global arguments.

For the general case of a k-dimensional fiber ($n \geq k \geq 2$), the analysis is more difficult. We just give the necessary conditions for a k-dimensional submanifold in R^n to be fiber of a riemannian fibration.

(I) $\quad T_V T_W - T_W T_V + \overline{R}(V,W) = 0$ on the horizontal space, i.e.

$$\langle (T_V T_W - T_W T_V)X, Y \rangle = \langle \overline{R}(W,V)X, Y \rangle = \langle \nabla_V(A_X Y), W \rangle - \langle \nabla_W(A_X Y), V \rangle ,$$

where X,Y horizontal; V,W vertical; and $\bar{R}$ is the curvature tensor of the normal bundle.

(II) $\mathcal{V}(\nabla_V \nabla_W X - \nabla_W \nabla_V X) = 0$, i.e.

$$\langle \nabla_V (T_W X), U \rangle - \langle \nabla_W (T_V X), U \rangle$$

$$= \langle A_X V,\ T_W U \rangle \ - \langle A_X W,\ T_V U \rangle \ .$$

These give the necessary relations between the A-tensor with the second fundamental tensor T of the submanifold.

The case of fiber codimensional one is, of course, rather trivial. Any hypersurface in R^n is locally a fiber of the unique flat fibration given by equidistant hypersurfaces and these are all such fibrations.

3. Global Riemannian Fibrations of Euclidean Spaces

Let $\pi : R^n \longrightarrow M^{n-k}$ be a riemannian fibration of flat euclidean space R^n. From section 1, it follows that M^{n-k} is complete, and M has nonnegative sectional curvature everywhere (as a consequence of O'Neill's formula).

We first observe that topologically everything is rather simple. Our main concern will be geometric rigidity aspects of this fibration. Since π is a fiber map, we have the long exact homotopy sequence :

$$\cdots \longrightarrow \pi_j(F) \longrightarrow \pi_j(R^n) \longrightarrow \pi_j(M^{n-k}) \longrightarrow \pi_{j-1}(F) \longrightarrow \pi_{j-1}(R^n) \rightarrow \cdots ,$$

where F is the fiber. Since R^n is contractible, $\pi_j(R^n) = 0$ for $j \geq 1$, we have $\pi_j(M^{n-k}) \cong \pi_{j-1}(F)$, for $j \geq 1$. In particular, when $j = 1$, $\pi_1(M) = \pi_0(F)$. So, if the fiber is connected, then M^{n-k} must be simply connected, i.e. $\pi_1(M) = 0$. If the fibers are not connected, we have the factorization $R^n \xrightarrow{\tilde{\pi}} \tilde{M} \longrightarrow M$ of the riemannian fibration π, where $\tilde{M}$ is the simply connected covering of M, and $\tilde{\pi}$ has connected fibers. Since the universal covering $\tilde{M} \longrightarrow M$ in general is fairly well understood [CG] , we will from now on restrict our attention to the case when M is simply connected, i.e. the fibers of π are connected.

For topological reasons [S] , it follows that both the fiber F and the base M, being finite dimensional CW-complexes, must be contractible. In particular, M cannot be compact. By 1.12, [CG] , [GM] , M is in fact diffeomorphic to R^{n-k}, and it will become clear in the following that the fiber is diffeomorphic to R^k. Of course, a compact base M can arise in case of disconnected fibers, for example, when π is the universal covering of flat torus $M = T^n$.

A crucial step toward our main result is the following :

Theorem 3.1

If $\pi : R^n \longrightarrow M^{n-k}$ is a riemannian fibration of flat euclidean space R^n, and the fibers are connected, i.e. M is simply connected. Then there exists a totally geodesic flat fiber in R^n.

The proof of this theorem will be given in several parts: Choose any point $p \in M$, let $\gamma(t)$ be a ray enaminating from p. Let F_p be the fiber over p and $\tilde{p}$ any point on the fiber F_p. Then through $\tilde{p}$, there exists a unique horizontal geodesic $\tilde{\gamma}(t)$ in R^n that covers $\gamma(t)$. Let B_t be the open metric ball in M with radius t centered at $\gamma(t)$. Let $\tilde{B}_t$ be the open ball in R^n with radius t centered at $\tilde{\gamma}(t)$. Then

Lemma 3.2

π maps $\tilde{B}_t$ onto B_t .

Proof : If $\tilde{x} \in \tilde{B}_t$, then $\tilde{d}(\tilde{x}, \tilde{\gamma}(t)) < t$, but

$d(\pi(\tilde{x}), \pi(\tilde{\gamma}(t)) \leq \tilde{d}(\tilde{x}, \tilde{\gamma}(t)) < t$, since π is a distance non-increasing mapping. Thus, $\pi(\tilde{x}) \in B_t$, so $\pi(\tilde{B}_t) \subset B_t$.

Conversely, if $x \in B_t$, then $d(x, \gamma(t)) < t$. Let σ be a minimal geodesic between $\gamma(t)$ and x with $\sigma(0) = \gamma(t)$,

$\sigma(1) = x$. Through $\tilde{\gamma}(t)$, there exists a unique horizontal lift $\tilde{\sigma}$ of σ. Then

$d(\tilde{\sigma}(1), \tilde{\gamma}(t)) = d(\sigma(1), \gamma(t)) < t$.

And $\pi(\tilde{\sigma}(1)) = \sigma(1) = x$, thus $\pi(\tilde{B}_t) \supset B_t$.

Lemma 3.3

Let $B = \bigcup_{t>o} B_t$ and $\tilde{B} = \bigcup_{t>o} B_t$, then B is an open "half space" in M, $\tilde{B}$ is an open half space in R^n, and π maps $\tilde{B}$ onto B.

Proof : This follows directly from lemma 3.2 .

Now consider $C = M - B$, $\tilde{C}_{\tilde{p}} = R^n - \tilde{B}$.

Lemma 3.4

C is a closed, totally convex subset in M, $\tilde{C}_{\tilde{p}}$ is a closed half space of R^n.

Proof : cf. [CG] .

Lemma 3.5

Let $\tilde{C} = \bigcap \tilde{C}_{\tilde{p}}$, intersection over all $\tilde{p} \in F_p$. Then $\tilde{C}$ is a non-empty closed convex subset in R^n, and π maps $\tilde{C}$ onto C.

Proof : $\tilde{C}$ is non-empty because $F_p = \pi^{-1}(p) \subset \tilde{C}_{\tilde{p}}$ for all $\tilde{p} \in \pi^{-1}(p)$, by construction. Since each $\tilde{C}_{\tilde{p}}$ is a closed half space, this intersection is a convex closed subset of R^n. If $\tilde{x} \in \tilde{C}_{\tilde{p}}$ for all $\tilde{p} \in F_p$, $\pi(\tilde{x})$ cannot belong to B, otherwise $\pi(\tilde{x}) \in B_t$ for some $t > 0$. Then there exists a minimal geodesic σ from $\gamma(t)$ to $\pi(\tilde{x})$ with length $d(\gamma(t), \pi(\tilde{x})) = d(\sigma(0), \sigma(1)) < t$. Through $\tilde{x}$ there is a horizontal lift $\tilde{\sigma}$ with $\tilde{\sigma}(1) = \tilde{x}$, $\pi\tilde{\sigma}(0) = \gamma(t)$, and

$d(\tilde{\sigma}(0), \tilde{\sigma}(1)) = d(\sigma(0), \sigma(1)) < t$. We find a unique point in $\pi^{-1}(p)$ say $\tilde{p}_0$ such that $\tilde{\sigma}(0)$ lies on the horizontal lift of $\gamma(t)$ through $\tilde{p}_0$. Then $\sigma(1) = \tilde{x} \in \tilde{B}_t$ by the construction for $\tilde{p}_0$, contradicting $\tilde{x} \in \tilde{C}_{\tilde{p}}$ for all $\tilde{p}$. If $x \in C$, then $x \notin B$ and hence $\pi^{-1}(x) \cap \tilde{B}_t = \phi$, for all $\tilde{p}$. Thus, $\pi^{-1}(x) \subset \tilde{C}$. So, we have proved that $\pi^{-1}(C) = \tilde{C}$.

For each ray $\gamma(t)$ enaminating from p , we obtain a closed totally convex subset C^{γ} in M and a closed convex set $\tilde{C}^{\gamma}$ in R^n. If we take the intersection of these sets over all possible rays from p , we obtain a compact, totally convex subset $C = \bigcap_{\text{All ray from } p} C^{\gamma}$ in M, and a closed convex subset $\tilde{C} = \bigcap \tilde{C}^{\gamma}$ in R^n. Clearly, $\pi^{-1}(C) = \tilde{C}$. By Theorem 1.8 , C is a topological submanifold of M of dimension $0 \leq m \leq n-k$, with totally geodesic interior and probably non-smooth boundary ∂C. Since M is contractible, ∂C must be non-empty, unless C is a point. For all this, compare [CG] . The structure of convex sets in R^n is elementary and well known. It follows that C is a convex set with non-empty interior in some affine subspace E of R^n of dimension m + k .

<u>Lemma 3.6</u>

$$\pi^{-1}(\operatorname{int} C) \subset \operatorname{int}\tilde{C}, \quad \pi^{-1}(\partial C) = \partial\tilde{C}.$$

Proof : π restricted to $\tilde{C}$ is a fibration over the manifold C with boundary. The fiber is a manifold without boundary. Note that $\pi^{-1}(C) = \widetilde{C}$.

Lemma 3.7

The closed convex (m+k)-dimensional subset $\widetilde{C}$ of E is either E itself or $\partial\widetilde{C} \neq \emptyset$.

Proof : Obvious, since $\widetilde{C}$ has non-empty interior in E.

If C is a point, then $\widetilde{C} = E$ is a flat fiber. So we may assume $\partial C \neq \emptyset$. Let us consider

$C^r = \{x \in C : d(x, \partial C) \geq r\}$, for $0 \leq r \leq \max\{d(x,\partial C) | x \in C\}$,

and $\widetilde{C}^r = \{\tilde{x} \in \widetilde{C} : \tilde{d}(\tilde{x}, \partial\widetilde{C}) \geq r\}$.

Then C^r is closed, compact, totally convex subset in M, $\widetilde{C}^r$ is closed convex set in R^n .

Lemma 3.8

$\pi^{-1}(C^r) = \widetilde{C}^r$.

Proof : If $\tilde{x} \in \widetilde{C}^r$, then $\tilde{d}(\tilde{x}, \partial\widetilde{C}) \geq r$,

$d(\pi(\tilde{x}), \partial C) = \tilde{d}(\tilde{x}, \pi^{-1}(\partial C)) = \tilde{d}(\tilde{x}, \partial\widetilde{C}) \geq r$,

hence $\pi(\widetilde{C}^r) \subset C^r$. If $x \in C^r$, $d(x, \partial C) \geq r$,

then for $\tilde{x} \in \pi^{-1}(x)$,

$\tilde{d}(\tilde{x}, \pi^{-1}(\partial C)) = \tilde{d}(\tilde{x}, \partial\widetilde{C}) = d(x, \partial C) \geq r$,

because of the equidistance of fibers in a riemannian fibration.

Let $C(1) = \bigcap C^r$, $\widetilde{C}(1) = \bigcap \widetilde{C}^r$, then C(1) is also compact, totally convex in M, with $\dim C(1) < \dim C$, $\widetilde{C}(1)$ is closed, convex in R^n, $\dim\tilde{C}(1) = \dim C(1) + k$. Clearly, $\pi^{-1}(C(1)) = \tilde{C}(1)$. If $\dim C(1) \neq 0$, then by the above, $\partial\tilde{C}(1) \neq \emptyset$ and $\partial C(1) \neq \emptyset$, where ∂ means taking the intrinsic boundary. Then by iterating the argument, we obtain a sequence of subsets :

$$C(1) \supset C(2) \supset C(3) \supset \cdots \text{, and}$$

$$\tilde{C}(1) \supset \tilde{C}(2) \supset \tilde{C}(3) \supset \cdots \quad ,$$

with $\dim C(1) > \dim C(2) > \cdots$. Hence for some i, $\dim C(i) = 0$, i.e. $C(i)$ is a point and thus $\tilde{C}(i)$ is a flat fiber, since $\tilde{C}(i) = \pi^{-1}(C(i))$ is convex and $\partial C(i) = \emptyset$. So we finally prove the existence of a totally geodesic fibre.

Remark 3.9

C(i) is the "soul" of the manifold M, in the terminology of [CG] which is a single point in this case. It will follow from our later discussion that this point is actually a pole, which provides a particularly simple argument that M is diffeomorphic to R^{n-k}. Note that a point soul in general is not a pole.

Now we can prove our main result.

Theorem 3.10

Let $\pi : R^n \to M^{n-k}$ be a riemannian fibration of flat euclidean space R^n with connected fibers. Then π is "homogeneous", i.e. there exists a free action of R^k on R^n by euclidean motions leaving π invariant. So, M is the orbit space of this action.

Proof : By Theorem 3.1 we find $p \in M$, such that F_p is a totally geodesic flat fiber. We construct the action of R^k in two steps.

First, we derive a necessary (and sufficient) condition for the tensor A along F_p .

Let X_i be an orthonormal frame at p. Recall that X is a horizontal lift iff $\nabla_V X = A_X V + T_V X$, where V is any vertical vector field. In our case, we restrict to the fiber F_p , which is totally geodesic, i.e., T = 0. So the equation reduces to $\nabla_V X = A_X V$. The horizontal vector fields X_i , $1 \leq i \leq n-k$, defined along the fiber F_p are the horizontal lifts of X_i iff they satisfy the system of equations $\nabla_V X_i = A_{X_i} V$ for all vertical vector field V. Since F_p is totally geodesic in R^n , we can choose a globally parallel orthonormal frame V_j , $1 \leq j \leq k$ on F_p . Then the system of equations becomes

$$(*) \quad \nabla_{V_l} X_i = A_{X_i} V_l = \sum_{j=1}^{n-k} \langle A_{X_i} V_l, X_j \rangle X_j$$

$$= \sum_{j=1}^{n-k} - \langle A_{X_i} X_j, V_l \rangle X_j$$

$A_{X_i} X_j$ when restricted to the totally geodesic fiber F_p is a Killing field. At the same time, $\| A_{X_i} X_j \|$ is constant along the fiber. Thus $\langle A_{X_i} X_j, V_l \rangle$ must be constant on the fiber. Let $a^l_{ij} =: \langle A_{X_i} X_j , V_l \rangle$, then the matrix $A_l =: (a^l_{ij})$ is skew symmetric in i,j. By the theory of differential equation, the system (*) is solvable iff the following integrability conditions are satisfied :

$$(**) \qquad \nabla_{V_l} \nabla_{V_m} X_i - \nabla_{V_m} \nabla_{V_l} X_i = 0 \ , \quad 1 \leq i \leq n-k \ , \quad 1 \leq l,m \leq k \ ,$$

We have $\nabla_{V_l} X_i = -\sum_{j=1}^{n-k} a^l_{ij} X_j$

$$\nabla_{V_m} X_i = -\sum_{j=1}^{n-k} a^l_{ij} X_j$$

$$\nabla_{V_l} \nabla_{V_m} X_i = -\sum_{j=1}^{n-k} a^l_{ij} \nabla_{V_l} X_j = -\sum_{j,h=1}^{n-k} a^m_{ij} \, a^l_{jh} X_h \ ,$$

$$\nabla_{V_m} \nabla_{V_l} X_i = -\sum_{j=1}^{n-k} a^l_{ij} \nabla_{V_m} X_j = -\sum_{j,h=1}^{n-k} a^l_{ij} \, a^m_{jh} X_h \ .$$

Therefore (**) is equivalent to

$$(***) \qquad \sum_{j=1}^{n-k} (a^l_{ij} \, a^m_{jh} - a^m_{ij} \, a^l_{jh}) = 0 \ , \quad 1 \leq h \leq n-k \ .$$

Hence (***) is the condition for A such that the system (*) has global solutions.

We are now in a position to construct the π-invariant free action of R^k. If $\overline{X} = \sum a_i \overline{X}_i$ is a vector at p., then the horizontal lift of $\overline{X}$ along the fiber F_p is of the form $X = \sum a_i X_i$, where $\overline{X}_i$, X_i are as above. If $q \in M$, there is a minimal connection σ between p and q, parametrized by arc length, $\sigma(0) = p$, $\sigma(s) = q$, $X = \dot{\sigma}(0)$. The fiber over q is given by

$$\left\{ \tilde{p} + sX(\tilde{p}) \,\middle|\, \tilde{p} \in F_p \right\} \ .$$

X is the horizontal lift of $\bar{X}$ along F_p . In other words, every fiber can be considered as a graph of a function $F : R^k \longrightarrow R^{n-k}$ in R^n where F assigns to each point p of the totally geodesic fiber the vector $sX(p)$. Note that the length $\|F\|$ is constant.

Let us consider the matrices $\langle A_{X_i} X_j, V_l \rangle =: A_l$, $1 \leq l \leq k$. The integrability condition (***) of A is exactly the condition that all the matrices commute . Since the matrices A_l are skew symmetric, their exponentials $\mathrm{Exp}\, A_l$ are orthogonal. By (***) we also have that those matrices $\mathrm{Exp}\, A_l$ commute with each other. Hence, we have a group representation $\alpha : R^k \longrightarrow O(n-k)$ defined by $\alpha(v) = \mathrm{Exp} \langle A_{X_i} X_j, V \rangle$, where $v \in R^k$.
Clearly, α is a group homomorphism. Since R^k is abelian and connected, so is its image $\alpha(R^k)$. Actually $\alpha(R^k)$ is contained in the maximal torus of $SO(n-k)$. The maximal torus of $SO(n-k)$ is of dimension $r = [\frac{n-k}{2}]$, which implies the dimension of $\alpha(R^k)$ is not greater than $[\frac{n-k}{2}]$.

We conclude that each riemannian fibration $\pi : R^n \longrightarrow M^{n-k}$ gives rise to a group representation $\alpha : R^k \longrightarrow SO(n-k)$.

Let $q \in M$, σ as before, the minimal geodesic between p and q. $\bar{X} = s\,\dot{\sigma}(0)$, where $s = d(p,q)$. Let X be the horizontal lift of $\bar{X}$ along the fiber F_p . The points

$\tilde{p} + sX(\tilde{p})$ form the fiber F_q over q, $\tilde{p} \in F_p$. The lift X satisfies the equation $\nabla_V X = A_X V$ for every vertical field V. So, when X is restricted to a straight line L of F_p , the locus of the points $\{ \tilde{p} + sX(\tilde{p}) \}$ for $\tilde{p} \in L$ describe a helix lying on a "cylindrical surface" with that straight line L as axis. Hence, every fiber is orbit of the R^k-action β on R^n defined as follow : Let $v \in R^k$, $(a,b) \in R^k + R^{n-k} \cong R^n$, and

$\alpha : R^k \longrightarrow SO(n-k)$ the group representation constructed above.

Define $\beta(v)(a,b) = (a + v, \alpha(v)b)$.

So, β is acting on the first component in R^k by translation and on the second component in R^{n-k} by rotation, i.e., β acts on R^n by "glide rotation". It is clear from the construction that β leaves fibers invariant. This completes our proof.

Conversely, if we are given a group representation $\alpha : R^k \longrightarrow SO(n-k)$, we obtain a riemannian fibration of R^n with k-dimensional fibers. Let $\alpha : R^k \longrightarrow SO(n-k)$ be given. We define the R^k-action β on R^n as above :

$$v \in R^k , \ (a,b) \in R^k + R^{n-k} \cong R^n$$

$$\beta(v)(a,b) = (a+v, \alpha(v)b) .$$

This is a free action, since it is already free on the first factor R^k. Let π be the projection $R^n \longrightarrow M^{n-k}$, where M is the quotient manifold. As β acts by isometries on R^n, the flat metric of R^n projects to a riemannian metric for M with respect to which π becomes a riemannian fibration. Therefore, we have the complete classification result :

Theorem 3.11

Up to congruence in R^n , riemannian fibrations $\pi : R^n \longrightarrow M^{n-k}$ are in 1-1 correspondence with equivalence classes of (not necessarily faithful) representation

$\alpha : R^k \longrightarrow SO(n-k)$.

Remark 3.12

Of course, the equivalence classes of such homomorphisms α are essentially just the homomorphisms of R^k into a maximal torus of SO(n-k), up to the action of the Weyl group.

Remark 3.13

It follows from Theorem 3.10 that all fibers of any riemannian fibration $\pi : R^n \longrightarrow M^{n-k}$ are flat in the induced metric, (but usually not flat euclidean subspaces), since R^k is an abelian Lie group. Geometrically, they look like generalized helices.

Remark 3.14

The riemannian fibration $\pi : R^n \longrightarrow M^{n-k}$ is metrically trivial if and only if the representation $\alpha : R^k \longrightarrow SO(n-k)$ is trivial .

We conclude this section with some examples.

1. $\pi : R^3 \longrightarrow M^2$.

Since the fiber is one dimensional, the fibers are flow lines of a non-zero Killing field, which are helices in R^3. The homomorphism $\alpha : R \longrightarrow SO(2) = S^1$ is determined by some angle of rotation. If α is non-trivial, M^2 is a convex surface of revolution asymptotic to a cylinder.

2. $\pi : R^4 \longrightarrow M^3$.

Fiber dimension is one, $\alpha : R \longrightarrow SO(3)$ has a fixed point, since S^1 is the maximal torus in $SO(3)$. Again, the fibers are flow lines of a Killing field (non-vanishing), and $M^3 = M^2 \times R$ isometrically.

3. $\pi : R^4 \longrightarrow M^2$.

Fiber dimension is two. $\alpha : R^2 \longrightarrow SO(2) = S^1$ is either trivial or a projection followed by a covering map. In the latter case, $\ker\alpha = R \times T^1$, where T^1 is the 1-lattice. By choosing the base in R^2, we have two linearly independent vectors $u, v \in R^2$ such that $\alpha(tu) = id \in SO(2)$ for all $t \in R$ and $\alpha(nv) = id$ for $n \in Z$. The fiber of π is a "ruled surface" which is generated by a straight line in u-direction moving along the curve

$c(t) = \{(a+tv, \alpha(tv)b) : t \in R\}$ in R^4.

4. $\pi : R^5 \longrightarrow M^3$.

The maximal torus in $SO(3)$ is S^1, so this is similar to the Example 3. The fiber is a "ruled surface".

5. $\pi : R^n \longrightarrow M^{n-1}$.

The fiber dimension is one. $\alpha : R \longrightarrow SO(n-k)$. The fibers are flow lines of a Killing field (non-vanishing), i.e. " helices" in R^n. If n is even, M^{n-1} will split off a line.

6. $\pi : R^n \longrightarrow M^{n-2}$.

Fiber dimension is two. If $\alpha : R^2 \longrightarrow SO(n-2)$ has a kernel, then the fibers are "ruled surfaces" as in Example 3.

7. $\pi : R^n \longrightarrow M^1 = R$ is always trivial by Remark 3.14 .

4. One Dimensional Riemannian Foliations of R^n

In the previous section we have obtained a classification of all riemannian fibrations of R^n making very much use of the structure theory of complete manifolds of nonnegative sectional curvature. It turns out that the case of fiber dimension one can be dealt with more easily and directly in a way that does not depend on a global quotient manifold and thus generalizes to riemannian foliation of R^n (which are understood to be foliations of R^n that are locally riemannian fibrations). We will discuss this case in this section.

Our main result in this section is the following :

Theorem 4.1

Any 1-dimensional riemannian foliation in R^n is homogeneous, i.e. it is determined by a non-zero Killing field. In particular, $\mathcal{F}$ is necessarily a riemannian fibration.

Proof : Suppose we are given a riemannian foliation $\mathcal{F}$ on R^n with 1-dimensional leaves. As R^n is simply connected, $\mathcal{F}$ is given by a global smooth non-singular vector field T, which we may normalize to have unit length, i.e. $\|T\| = 1$. Let F_p be the leaf through some point $p \in R^n$. F_p can be naturally parametrized as an integral curve $c_o : R \longrightarrow R^n$ with $c_0(0) = p$. Notice that F_p is necessarily complete in the induced metric, but could be compact or a submanifold that is not a closed subset.

Rather than looking at local riemannian fibrations (which we can do) we proceed more directly as follows :

Basic vector fields X along c_o are given by the equation

$$\dot{X} = \mathcal{X}_X T + A_X T \quad ,$$

they need not define vector field along F_p . Here again $\mathcal{X}_X = \langle \nabla_T X, T \rangle$, and A_X is the O'Neill's form which is, of course, globally defined. Now let X_o , Y_o be basic along c_o . Extend these fields to fields X,Y along the surface $R^2 \longrightarrow R^n$

(**) $(t,s) \longmapsto c_s(t) = c_o(t) + sX_o(t)$

so that X is the tangent vector field of the horizontal geodesic in direction X and T is determined by $\nabla_X Y$ being vertical,(i.e. Y is related to a parallel field in a local quotient along the geodesic determined by X).

Let $\beta = \langle \frac{dc_s(t)}{dt}, T \rangle$, so $\frac{dc_s}{dt} = \beta T = \tilde{T}$,

and $\tilde{T}$ vanishes nowhere, since for fixed s, (**) defines locally a diffeomorphism from c_o to c_s . Furthermore, we have the Jacobi equation

$$\frac{d^2 T}{ds^2} = 0$$

Notice that $\frac{dc_s}{dt} = T$, X commute. Now define

$$\alpha =: \langle \nabla_X Y, T \rangle = \langle \nabla_X Y, \tilde{T} \rangle \langle \tilde{T}, \tilde{T} \rangle^{-\frac{1}{2}} = -\langle A_X T, Y \rangle \quad .$$

Then $X\alpha = -2\,\mathcal{X}_X \alpha$, since

$$X\alpha = X \langle \nabla_X Y, \tilde{T} \rangle \langle \tilde{T}, \tilde{T} \rangle^{-\frac{1}{2}}$$

$$= X \left\{ -\langle \nabla_X \tilde{T}, Y \rangle \langle \tilde{T}, \tilde{T} \rangle^{-\frac{1}{2}} \right\}$$

$$= -\langle \nabla_X \nabla_X \tilde{T}, Y\rangle\langle \tilde{T},\tilde{T}\rangle^{-\frac{1}{2}} - \langle \nabla_X \tilde{T}, \nabla_X Y\rangle\langle \tilde{T},\tilde{T}\rangle^{-\frac{1}{2}}$$

$$+\langle \nabla_X \tilde{T}, Y\rangle\langle \nabla_X \tilde{T},\tilde{T}\rangle\langle \tilde{T},\tilde{T}\rangle^{-3/2}$$

$$= -\mathcal{K}_x\alpha - \mathcal{K}_x\alpha$$

$$= -2\,\mathcal{K}_x\alpha.$$

Since α is (locally) constant along the leaf (by O'Neill's formula) we have $\tilde{T}\alpha = 0$. It follows

$$0 = X\tilde{T}\alpha = \tilde{T}X\alpha = -2(\tilde{T}\mathcal{K}_x)\alpha \;.$$

Hence if $\alpha \neq 0$ for some Y, $\mathcal{K}_x$ is constant along the leaf. In case $\alpha = 0$ for all Y, i.e. $A_X T = 0$ along c_o, consider the (parametrized) leaves $c_s(t)$. Then $T(t) = \dot{c}_s(t) = \dot{c}_o(t) + s\dot{X}(t)$ $= T_o + s\mathcal{K}_x T_o(t) + sA_x T_o(t) = (1 + s\,\mathcal{K}_x)T_o(t) \neq 0$ by the above for all s; here we use that the fibration is "horizontally complete". Thus $|1+s\,\mathcal{K}_x|$ never vanishes. This happens only when $\mathcal{K}_x = 0$ along c_o. So we conclude that $\mathcal{K}_x$ is constant along the leaves, for each basic X. The latter part of the argument is global in nature. The last conclusion is in general not true for local fibration, cf. section 2.

Now consider the global vector field $X = T'$ on R^n. Taking an orthonormal frame X_i of basic fields along the (parametrized) leaf c_o, we see that $X = \sum_i \mathcal{K}_i X_i$ where $\mathcal{K}_i = \mathcal{K}_{X_i}$, are constant along c_o, so X is basic. We claim moreover, X is a gradient field on R^n. This is equivalent to showing that $\langle \nabla_A X, B\rangle$ is symmetric. It suffices to look at the following two cases locally :

(1) $A = T$ vertical ; $B = Y$ horizontal and basic along the leaves.

$$\langle \nabla_T X, Y \rangle = \langle \nabla_X T, Y \rangle = -\langle \nabla_X Y, T \rangle = \langle \nabla_Y X, T \rangle$$

Since X is basic, $[X,T]$ is vertical, $A_X Y = -A_Y X$.

(2) Both $A = Y$, $B = Z$ are horizontal and basic along the leaves;

$$\langle \nabla_Y X, Z \rangle = \langle \nabla_T \nabla_Y T, Z \rangle + \langle \nabla_{[Y,T]} T, Z \rangle$$

$$= T\langle \nabla_Y T, Z \rangle - \langle \nabla_Y T, \nabla_T Z \rangle + \langle \nabla_{[Y,T]} T, Z \rangle \quad .$$

Now, $\langle \nabla_Y T, Z \rangle = -\langle A_Y Z, T \rangle$ is constant along the leaves, so the first term vanishes. The bracket $[Y,T]$ is vertical, therefore $[Y,T] = \langle \nabla_Y T - \nabla_T Y, T \rangle T = -\mathcal{X}_Y T$, and the third term equals $-\mathcal{X}_Y \mathcal{X}_Z$, which is symmetric in Y,Z. Finally, since $\nabla_Y T$ is vertical, $\langle \nabla_Y T, \nabla_T Z \rangle = \langle \nabla_Y T, \nabla_Z T \rangle$ by the above, and the second term is symmetric in Y,Z.

Now we can define a smooth positive function L on R^n by $X = T' = \nabla \log L$. Note that $TL = 0$; i.e. L is constant along the leaves. Up to a constant, L measures locally the length of a piece of a leaf under horizontal geodesic displacement.

We finally claim that LT is a Killing field on R^n . This is equivalent to showing locally that :

(1) $\langle \nabla_T (LT), T \rangle = 0$

(2) $\langle \nabla_Y (LT), Y \rangle = 0$, and

(3) $\langle \nabla_T (LT), Y \rangle + \langle \nabla_Y (LT), T \rangle = 0$

for any basic horizontal field Y.

Proof : (1) $\langle \nabla_T(LT), T \rangle = L \langle \nabla_T T, T \rangle = 0$, since T has unit length.

(2) $\langle \nabla_Y(LT), Y \rangle = L \langle \nabla_Y T, Y \rangle = 0$ by skew symmetry of the O'Neill tensor.

(3) $\langle \nabla_T(LT), Y \rangle + \langle \nabla_Y(LT), T \rangle$

$= L\langle \nabla_T T, Y \rangle \quad + \quad YL\langle T, T \rangle \quad + L\langle T, \nabla_Y T \rangle$

$= -\mathcal{X}_Y L \quad + \mathcal{X}_Y L = 0.$

Hence we have shown that LT is a Killing field. Since the square of the length function of the Killing field in R^n is a quadratic function of the form $\|Bx+b\|^2$ with B skew symmetric; L assumes a non-zero absolute minimum in R^n. The leaf through that critical value is totally geodesic in R^n, hence it is a straight line, F_o .

Clearly, we have a global basic horizontal framing along the straight line F_o , which determines a riemannian fibration completely. Hence, we actually show that in our case, the one dimensional riemannian foliations of R^n are fibrations. Note that this not true for the compact case, for example; one dimensional riemannian foliation of S^3 need not be fibrations, cf. [GG] .

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