GROUPS OF CONFORMAL AND ANTICONFORMAL SELF MAPS OF RIEMANN SURFACES

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Abstract of the Dissertation

Groups of Conformal and Anticonformal

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We show the existence of linearizations for groups of conformal and anticonformal homeomorphisms of Riemann Surfaces. Finitely generated groups acting in the plane may be classified in terms of specific linearizations. These classifying groups are called Koebe groups.

The existence of linearizations is applied to prove the existence of canonical representatives for plane domains called pseudocircle domains. In special cases this gives a solution to the Kreisnormierungs problem.

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INTRODUCTION

The first problem we consider is concerned with generalizing several results about finite genus uniformizations of Riemann Surfaces to treat uniformization questions associated with Klein Surfaces. The main theorem here is that any group of conformal and anticonformal autohomeomorphisms of a plane domain can be linearized. This generalizes Maskit's results for the conformal case [M2]. Our proofs make use of several other of Maskit's results which taken together give a detailed description of the linearized groups. An important class of these groups called Koebe groups may be extended to include groups with anticonformal maps. An existence and uniqueness theorem for Koebe groups is shown to hold for this larger class of groups.

The second problem we consider may also be called a uniformization problem. Classically it is a conformal mapping problem, originally posed by Koebe and often referred to as the Koebe Kreisnormierungs problem. The question here is whether a plane domain may be mapped conformally and homeomorphically onto a domain whose boundary components are either circles or single points.

Koebe was able to answer this question in the affirmative for domains of finite connectivity and for domains of infinite connectivity satisfying a symmetry condition. A more recent history of the problem may be found in a paper of R. J. Sibner's [S].

We develop a theory of prime ends for domains of infinite connectivity which allows us to talk about isolated pieces of the boundary. By linearizing a group constructed according to information derived from a given plane domain it is shown that the domain may be mapped onto a circle domain modulo limiting behavior. This limiting behavior is dealt with by considering a wider class of domains called pseudocircle domains. We prove Koebe's conjecture for pseudocircle domains, thereby solving the Pseudokreisnormierungs problem.

CHAPTER I

§0.

As background we present the basic theory of Kleinian and extended Kleinian groups, sometimes from the more general viewpoint of groups acting on Riemann surfaces. Several results of a more advanced nature are also included. The natural generalizations of these results to the cases where the groups under consideration contain anticonformal maps are stated and proven in Chapter II.

§1.

The group of conformal and anticonformal homeomorphisms of the Riemann Sphere, $\hat{\mathbb{C}}$, will be denoted by Möb. The conformal mappings in Möb, also known as the Linear Fractional Transformations, are those of the form $g(z) = \frac{az+b}{cz+d}$ where a,b,c,d are complex numbers, and $ad-bc \neq 0$. The anticonformal mappings in Möb, or the extended linear fractional Transformations, are of the form $\gamma(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ again where a,b,c,d are complex numbers, and $ad-bc \neq 0$.

The subgroup of Möb whose elements are the Linear

Fractional Transformations will be referred to as Möb$^+$. In general, if G is a group of homeomorphisms of an oriented manifold then G^+ will be the subgroup of index two in G whose elements preserve orientation. If G is a subgroup of Möb then $G^+ = G \cap \text{Möb}^+$.

There is a surjective homeomorphism ϕ: SL(2,$\mathbb{C}$) $\to$ Möb$^+$, where $\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$. In this way elements of Möb$^+$ may be represented as 2x2 matrices of determinant one. The square of the trace of a Linear Fractional Transformation is well defined in terms of this representation. The kernal of ϕ is the subgroup $\{\pm id\}$ of SL(2,$\mathbb{C}$) where $id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence Mob$^+$ may be identified with the complex Lie Group PSL(2,$\mathbb{C}$) = SL(2,$\mathbb{C}$)/$\{\pm id\}$.

The elements of Mob may be classified by choosing a set of transformations in normal form so that every element of the group is conjugate to exactly one of these transformations. More precisely, every element of Mob$^+$ is conjugate in Mob$^+$ to exactly one of the transformations of the form;

1. $z \mapsto z+1$
2. $z \mapsto kz$, $|k| = 1$, $k \neq 1$
3. $z \mapsto kz$, $|k| > 1$

Every anticonformal element of Möb is conjugate in Möb to exactly one of the transformations of the form;

4. $z \mapsto \bar{z}$

5. $z \mapsto \bar{z}+1$

6. $z \mapsto k\bar{z}$, $k>1$

7. $z \mapsto e^{\pi i \sigma}\bar{z}^{-1}$, $0<\sigma<1$

The first three types of transformations are called respectively *parabolic*, *elliptic*, and *loxodromic*. A loxodromic element for which k is real and greater than one is called *hyperbolic*.

Since the trace of a matrix is invariant under conjugation, a simple computation reveals that for $g \in Mob^{+}$, $g \neq id$,

a. g is parabolic if and only if $tr^2 g = 4$

b. g is elliptic if and only if $0 \leq tr^2 g < 4$

c. g is hyperbolic if and only if $tr^2 g > 4$

d. if $tr^2 g$ satisfies none of the above then g is loxodromic.

The transformations of type (4), which are called *reflections*, are the only mappings in Möb that have more than two fixed points in $\hat{\mathbb{C}}$. The fixed point set of a reflection is always a circle or a straight line (which is a circle passing through ∞). This fact plays a crucial role in the proof of Theorem III.4 towards the end of Chapter III.

§2.

We now consider how a group acts on a surface. Let S be a Riemann Surface and G a group of conformal and anticonformal homeomorphisms of S onto itself. Given a set of points $X \subset S$ we denote by G(X) the stability subgroup of X in G; that is, $G(X) = \{g \in G | g(X) = X\}$.

The group G is said to act discontinuously at a point x in S if

1. G(x) is finite, and there is an open neighborhood U of x in S so that
2. $G(U) = G(x)$, and
3. $g(U) \cap U = \phi$ for all $g \in G - G(x)$

Denote by $\Omega(G)$ the set of points in S at which G acts discontinuously. $\Omega(G)$ is open in S and we call it the regular set. The complement of $\Omega(G)$ in S is called the limit set and is written $\Lambda(G)$. It is easily shown that $\Omega(G) = \Omega(G^+)$.

G is a discontinuous group, or is said to act discontinuously on S, if $\Omega(G) \neq \phi$. A discontinuous subgroup of Möb is called a Kleinian group if $G = G^+$, and is called an extended Kleinian group if $G \neq G^+$.

If S is a compact surface of genus $g > 1$ then it is well known that any group G, as above, is finite, and therefore a discontinuous group; in fact, $\Omega(G) = S$.

If G is Kleinian or extended Kleinian then it can be shown [M5] that $\Lambda(G)$ contains 0, 1, 2 or uncountably many points. Those G with $\Lambda(G)$ finite are called elementary groups.

A special class of Kleinian groups are the Fuchsian groups. These are groups F for which there exists an open disc Δ in $\mathbb{C}$ (including half-planes in $\mathbb{C}$) so that $F(\Delta) = F$. $\Lambda(F)$ must then lie on the boundary of the disc Δ. If $\Lambda(F)$ is the entire circle then F is called Fuchsian of the first kind; otherwise, it is of the second kind. Extended Fuchsian groups are defined analogously.

We return now to the general case of a group G acting on a Riemann surface S. Two points x and y in $\Omega(G)$ are said to be equivalent mod G if there is a $g \in G$ with $g(x) = y$. Mod G is an equivalence relation, and we denote by $\Omega(G)/G$ the space of equivalence classes.

In this thesis we will be primarily concerned with the equivalence relation induced by the conformal part G^+ of the group G. There is a naturally defined projection map $\Pi\colon \Omega(G^+) \to \Omega(G^+)/G^+$ where $\Pi(x)$ is the equivalence class of x. It is well known [M5] that one may define a topology and, a unique conformal structure on $\Omega(G^+)/G^+$ so that the projection map Π is a holomorphic branched regular covering; that is, Π is holomorphic and it is a regular covering if one omits a discrete set of points on $\Omega(G^+)$ and their images on $\Omega(G^+)/G^+$. In some local coordinate about an omitted point the projection map has the form $z \mapsto z^n$ for some positive integer n. The omitted points in $\Omega(G^+)$ are the fixed points of finite order elements in G^+. With the induced conformal structure $\Omega(G^+)/G^+$ is a countable collection of Riemann surfaces.

The subgroup G^+ is of index two in the group G and therefore G^+ is a normal subgroup of G. Consequently, for g an anticonformal element of G, there is an anticonformal homeomorphism γ: $\Omega(G^+)/G^+$ so that $\gamma \circ \Pi = \Pi \circ g$; that is γ makes the diagram commute:

$$\begin{array}{ccc} \Omega(G^+) & \xrightarrow{g} & \Omega(G^+) \\ \Pi\downarrow & & \downarrow\Pi \\ \Omega(G^+)/G^+ & \xrightarrow{\gamma} & \Omega(G^+)/G^+. \end{array}$$

Clearly, the map γ is independent of the anticonformal element g chosen, and γ^2 is the identity. We call γ the anticonformal involution of S induced by the anticonformal half of G.

§3.0

The theorems that will be proven in Chapter II are generalizations of several theorems of B. Maskit's from the realm of Kleinian to Extended Kleinian groups. In this section we will state these theorems of Maskit's and define several notions necessary for an understanding of the theorems.

Theorem I.1. Let D be a plane domain, and let G be the group of all conformal homeomorphisms of D onto itself. Then there exists a univalent function ϕ, mapping D onto some other domain D', so that every element of $G' = \phi G \phi^{-1}$ is a Linear Fractional Transformation.

Proof. [M2].

If G acts discontinuously on D then G' is a Kleinian group. The domain D' must lie in a connected component Δ of $\Omega(G')$ which is mapped onto itself by every element of G'. G' is said to be a Kleinian group with an invariant component Δ.

If G does not act discontinuously on D then Theorem I.1 may be easily deduced as a consequence of several classical theorems. This is outlined in Chapter II.

§3.1

A Riemann Surface S is said to be of finite genus g if S embeds conformally in a compact surface of genus g and not in one of lower genus. Let S be a surface of genus g and let $i: S \to S^*$ be a conformal embedding of S into a compact surface S^* of genus g. S is called a finite Riemann Surface if $S^*-i(S)$ consists of a finite number of points. S is called a surface of finite type if $S^*-i(S)$ consists of a finite number of connected components.

Theorem I.1 has a generalization to surfaces of finite genus:

Theorem I.2. Let S be a Riemann Surface of genus g. Then there is a closed Riemann Surface S^* of genus g and a conformal embedding of S into S^* so that, under this embedding, every conformal self map of S is the restriction of a conformal self map of S^*.

Proof. [M2]

§3.2

An isomorphism $\phi: G \to G^*$ between Kleinian groups is called type preserving if

1. ϕ preserves the square of the trace of elliptic elements, and

2. ϕ and ϕ^{-1} take parabolic elements to parabolic elements.

We denote by C_1 the set of finitely generated Kleinian groups with invariant components.

Let G and G* be groups in C_1 with invariant components Δ and Δ^*. An orientation preserving homeomorphism $f: \Delta \to \Delta^*$ which induces an isomorphism f_* of G onto G* by $g \mapsto f \circ g \circ f^{-1}$ is called a _weak similarity_. If the isomorphism f_* is type preserving then f is a _similarity_.

Let g be a parabolic element of a group G in C_1. g is called _accidental_ if there is a weak similarity ϕ between G and another group G* in C_1 so that $\phi \circ g \circ \phi^{-1}$ is not parabolic.

A group G in C_1 is called _basic_ if

1. the invariant component Δ of G is simply connected, and

2. G contains no accidental parabolic elements.

A basic group G is degenerate if the invariant component $\Delta = \Omega(G)$.

Let G be a group in C_1 with invariant component Δ. A _structure_ subgroup H of G is a subgroup which satisfies the following:

1. H is a basic group
2. if a fixed point of a parabolic element $g \in G$ lies in $\Lambda(H)$ then $g \in H$.
3. H is a maximal subgroup of G satisfying (1) and (2).

C_0 is the set of groups in C_1 none of whose structure subgroups are degenerate. A group G in C_1 is called a Koebe group if every structure subgroup of G is either Fuchsian or elementary. Clearly all Koebe groups are in the set C_0. An Extended Koebe group is an Extended Kleinian group G with an invariant component and with G^+ a Koebe group.

Theorem I.3. Let G be a finitely generated Kleinian group with an invariant component. There is a unique Koebe group G*, and there is a unique (up to elements of $PSL(2,\mathbb{C})$) conformal similarity between G and G*.

Proof. [M3].

CHAPTER II

§0.

Let D be a plane domain and let G be a group of conformal and anticonformal homeomorphisms of D onto itself. We say that G can be linearized if there is a conformal homeomorphism f mapping D onto a domain D' in the complex plane so that $f\,G\,f^{-1}$ is a group of Extended Linear Fractional Transformations.

In these terms Maskit's Theorem I.1 may be stated: If G is a group of conformal homeomorphisms of a plane domain D onto itself, then G can be linearized. This Chapter will be devoted to proving the following extension of Theorem I.1:

Theorem II.1. Let D be a plane domain and let G be a group of conformal and anticonformal homeomorphisms of D onto itself. Then G can be linearized.

In certain cases this theorem, as Maskit notes in [M2], may be deduced from classical theorems. If D is simply connected, then it is a simple exercise using the Riemann mapping theorem, and Schwarz lemma. If D has a non-trivial cyclic fundamental group then it is well known that D is conformally equivalent to either the punctured plane, the punctured disc, or an annulus. In all of these cases a conformal or anticonformal self map is an Extended Linear Fractional Transformation.

If the fundamental group of D is not cyclic, then it may be deduced from the Uniformaization Theorem that the holomorphic universal covering space of D is a conformal disc. It then follows from a well known result in the Theory of Fuchsian groups [K, p. 48], that a group G^+ of conformal self-maps of D acts discontinuously throughout D. As was observed in I.2, the coset space D/G^+ has a complex structure which makes the projection map holomorphic. The anticonformal maps in G then project to a unique involution on D/G^+.

Assume that the fundamental group of D is not infinite cyclic. Let F be the set of points in D fixed by non-trivial elements of G^+.

Lemma II.1. F is a discrete set in D.

Proof. Let $\{z_i\}_{i=1}^{\infty}$ be a sequence of points in F with $\lim_{i\to\infty} z_i = z$ where $z \in D$. Clearly, G^+ cannot act discontinuously at z, which is a contradiction. ∎

We know from elementary complex analysis that an isolated singularity of a conformal homeomorphism is removable. In light of lemma II.1, there is therefore no loss of generality in assuming, for the proof of Theorem II.1, that G^+ acts freely throughout D.

We will also assume, for the remainder of this chapter, that the fundamental group of D is not cyclic.

§1.

We will first prove a "finite" version of Theorem II.1 which is analogous to Maskit's Theorem 12 in [M1]. It should be noted that whenever Theorem I.1 [M2, Theorem A] is cited, we could just as well use the weaker Theorem 12.

Lemma II.2. Let D be a plane domain as above, and let H be a finitely generated group of conformal homeomorphisms of D onto itself. Then there is a conformal homeomorphism ϕ of D onto a domain D' so that the group $H' = \phi H \phi^{-1}$ is a Koebe group. Moreover, this can be done so that the group H' does not contain any accidental parabolic elements.

Proof. By Maskit's Theorem I.1 H can be linearized. Let f be a conformal homeomorphism of D onto a domain

D_f so that $H_f = f\,H\,f^{-1}$ is a group of linear fractional transformations. As we saw in I.3.0, H_f is a Kleinian group with an invariant component Δ_f of $\Omega(H_f)$, and $D_f \subset \Delta_f$.

Maskit's Theorem I.3 states the existence of a conformal homeomorphism h of Δ_f into the plane so that $h\,H_f\,h = H^*$ is a Koebe group. Then $\phi = h \circ f$ gives the first conclusion of the Lemma.

The fact that H^* can be found without accidental parabolics is an easy consequence of Theorem 1 in [M4]. More specifically, using this theorem one may specify that every parabolic in H^* is doubly cusped and corresponds to at least one puncture on Δ/H^*, where Δ is the invariant component of H^*. ■

With the normalization permitted by Lemma II.2, an easy proof of our "finite" version of Theorem II.1 is possible.

<u>Theorem II.2</u>. Let D be a plane domain and let G be a group of conformal and anticonformal homeomorphisms of D onto itself so that G^+ acts freely on D and so that $S = D/G^+$ is a finite Riemann surface. Then G can be linearized.

Proof. We first show that G^+ is finitely generated. Since G^+ acts freely, the projection $p: D \to S$ is a regular covering, and the induced homomorphism $p_*: \pi_1(D) \to \pi_1(S)$ is a monomorphism. G^+ acts as the group of deck transformations for the regular covering and is therefore isomorphic to the quotient group $\pi_1(S)/p_*(\pi_1(D))$. Since S is of finite type the group $\pi_1(S)$ is finitely generated; hence, $\pi_1(S)/p_*(\pi_1(D)) \cong G^+$ is also finitely generated.

By lemma II.2 there is a conformal homeomorphism $\phi: D \to D'$ in $\mathbb{C}$ so that $\Gamma^+ = \phi \, G^+ \, \phi^{-1}$ is a Koebe group, Γ^+ does not contain accidental parabolic elements, and D' is contained in the invariant component Δ of $\Omega(\Gamma^+)$.

The anticonformal maps in $\Gamma = \phi \, G \, \phi^{-1}$ act a priori only on D', but this action extends naturally to all of Δ. To see this, notice that D'/Γ^+ is a subset of Δ/Γ^+ whose complement consists of a finite set of points on Δ/Γ^+. This set is the image of $\Delta - D'$ under the projection map. The points in $\Delta - D'$ are therefore isolated from one another. This shows that they are removable singularities of the anticonformal mappings.

Let γ be an anticonformal element of Γ. We will show that γ is an extended linear fractional transformation. This will complete the proof of Theorem II.2.

γ induces an automorphism $\gamma_*: \Gamma^+ \to \Gamma^+$ by $\gamma_*(g) = \gamma \circ g \circ \gamma^{-1}$. Since Γ^+ does not contain accidental parabolic elements, γ_* and γ_*^{-1} must take parabolic elements to parabolic elements. In order to see that γ_* is type preserving we must also show that it preserves the square of the trace of elliptic elements, or equivalently, that it preserves the minimal geometric generators of finite cyclic subgroups of Γ^+. The minimal geometric generators of a finite cyclic group are those elements which in normal form (see I.1) look like $z \mapsto e^{i\sigma}z$ where $|\sigma|$ is minimal over the cyclic group. The fact that minimal generators are preserved is clearly a consequence of γ_* having been induced by a homeomorphism of Δ.

Let $j: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be complex conjugation; that is, $j(z) = \bar{z}$. j induces, by conjugation, an isomorphism j_* taking Γ^+ onto another Koebe group Γ^* with invariant component $\Delta^* = j(\Delta)$. Since j_* is induced by a global homeomorphism it is type preserving.

The map $j \circ \gamma = \phi: \Delta \to \Delta^*$ is a conformal homeomorphism. It induces an isomorphism $\phi_*: \Gamma^+ \to \Gamma^*$ by $\phi_*(g) = \phi \circ g \circ \phi^{-1}$. ϕ_* is type preserving, since it may be written as $\phi_* = j_* \circ \gamma_*$, a composition of type preserving isomorphisms; hence, ϕ is a conformal similarity. Consequently, by Theorme I.3, ϕ is a linear fractional transformation, and $\gamma = j \circ \phi$ is an extended linear fractional transformation. ∎

In proving Theorem II.2 we have also generalized Maskit's Theorem I.3.

<u>Corollary II.1</u>. Let G be a finitely generated extended Kleinian group with an invariant component. Then there is a unique extended Koebe group G^*, and a unique conformal similarity between G and G^* (unique up to elements of Möb^+).

Proof. All that remains to be proven is the uniqueness. Let Δ be the invariant component of G. Suppose that G_1 and G_2 are extended Koebe groups with invariant components Δ_1 and Δ_2, conformally similar to G by similarities ϕ_1 and ϕ_2. By Maskit's Theorem I.3 G_1^+ is conjugate to G_2^+ in Möb^+ and the maps ϕ_1 and ϕ_2 differ by an element of Möb^+. Hence, it suffices to assume that $G_1^+ = G_2^+$ and that the restrictions of the isomorphisms induced by ϕ_1 and ϕ_2 on G^+ are identical.

It follows immediately from Theorem I.3 that ϕ_1 and ϕ_2 differ only by an element of Möb^+ and therefore we may further assume that $\phi_1 = \phi_2$.

We will show that $G_1 = G_2$. The quotient spaces Δ/G^+, Δ_1/G_1^+, and Δ_2/G_2^+ all represent the same Riemann surface S and the anticonformal parts of the groups all project to the same anticonformal involution γ of S. Clearly, the set of all lifts of γ to $\Delta_1 = \Delta_2$ gives the unique extension of $G_1^+ = G_2^+$ to $G_1 = G_2$. ∎

Proof of Theorem II.1.

The proof will be presented in two stages. In the first the hypotheses of Theorem II.2 are weakened by requiring the surface $S = D/G^+$ to be of finite type. Recall that a Riemann surface of finite type is a Riemann surface of finite genus which is the result of removing a finite number of points and conformal discs from a closed surface. We will be following, with only minor modification, Maskit's arguments in [M2].

Suppose $S = D/G^+$ is a surface of finite type. Denote by γ the anticonformal involution of S which is induced by the anticonformal part of G. Let $p: D \to S$ be the holomorphic covering projection.

Let S* be the surface gotten from S by glueing in punctured discs along the boundary contours of S. There is a naturally defined conformal embedding $f: S \to S^*$ so that the complement of f(S) in S* consists of a disjoint union of punctured discs.

Lemma II.3. S^* may be chosen so that the involution $f \circ \gamma \circ f^{-1}$ of $f(S)$ extends to an anticonformal involution γ^* on S^*.

Proof. A small neighborhood of a non-point boundary contour of S is conformally an annulus. Choose a collection of disjoint conformal annuli $\{A_i\}_{i=1}^N$ on S, one bounding each non-point boundary contour, so that either $\gamma(A_i) = A_i$ or $\gamma(A_i) = A_j$ for $i \neq j$.

Given an open A_i we glue on a punctured disc as follows: map A_i onto a standard annulus $\{z \mid 1<|z|<K\}$ in $\mathbb{C}$. Then we may consider A_i as a subdomain of the punctured disc $\{z \mid 0<|z|<K\}=\Delta^*$. This is exactly the appropriate complex structure on $A_i \cup \Delta^*$, which agrees with the structure on all of S.

We now observe that γ extends automatically when S^* is constructed in this fashion. If $\gamma(A_i) = A_i$ then the involution induced by γ on $\{z \mid 1<|z|<K\}$ is the restriction of an anticonformal reflection of $\mathbb{C}$ in a line through the origin. This defines γ^* on the punctured disc. If $\gamma(A_i) = A_j$ $i \neq j$ then representing A_j as the same standard annulus the map induced by γ is again a reflection.

<u>Lemma II.4</u>. Lef f, S, and S* be as in lemma II.3. There is a homeomorphism h: S → S* which is homotopically equivalent to f and shich satisfies $h \circ \gamma = \gamma^* \circ h$.

Proof. γ and γ^* induce equivalence relations on S and S* respectively. Let $S/_{\langle\gamma\rangle}$ and $S^*/_{\langle\gamma^*\rangle}$ be the quotient spaces. They are both manifolds, possibly with boundary. f projects to a unique mapping $\tilde{f}: S/_{\langle\gamma\rangle} \to S^*/_{\langle\gamma^*\rangle}$. Let $\tilde{h}: S/_{\langle\gamma\rangle} \to S^*/_{\langle\gamma^*\rangle}$ be a homeomorphism in the same homotopy class as f. Then the orientation preserving lift of $\tilde{h}$ to S is a homeomorphism h: S → S* which is in the same homotopy class as f. Notice that this lifting may be done in the sense of covering spaces away from the fixed point set of γ and then extended canonically. ∎

The projection map p: D → S is a holomorphic regular covering. Since h is a homeomorphism the composition $p^* = h \circ p$: D → S* is a regular covering. By pulling back the complex structure on S* to D via p* we get a new complex structure on D which, in this context, we will call D*. D* is topologically a planar surface; hence, by application of the uniformization theorem, we may assume that it is a plane domain. Then p*: D* → S* is a holomorphic regular covering. The group G on D becomes G* on D* and the anticonformal part of G* projects to γ^* on S^*. S^* is a finite Riemann surface, so by Theorem II.2 we may assume that D^* is a domain in the plane and G^* is a group of extended linear fractional transformations.

Since f and h are in the same homotopy class of maps, $f: S \to S^*$ lifts to a conformal homeomorphism $\tilde{f}$ taking D into D^*. Then $\tilde{f} \circ G \circ \tilde{f}^{-1} = G^*$ acting on $\tilde{f}(D)$ is a group of extended linear fractional transformations.

That completes the first step in the proof.

§3.1

We now proceed without any restrictions on the Riemann surface $S = D/G^+$.

Normalize D so that the unit disc Δ is contained in D, and so that $g(\Delta) \cap \Delta = \phi$ for all $g \in G$. Set $M_0 = p(\Delta)$. There exists an exhaustion of S by surfaces $\{M_m\}$, where each M_m is of finite type [A+S].

Let k be the smallest number for which $M_k \cap \gamma(M_k) \neq \emptyset$. We will work with the exhaustion of S by γ-invariant surfaces $\{S_n\}_{n=0}^{\infty}$ where $S_n = M_{n+k} \cup \gamma(M_{n+k})$ $n=0,1,2,\ldots$ Clearly, the S_n are also finite type and $S_0 \supset p(\Delta)$.

For each non-negative integer n let D_n be the component of $p^{-1}(M_n)$ containing Δ. G_n is defined as the subgroup of G which leaves D_n invariant; that is, $Gn = \{g \in G \mid g(D_n) = D_n\}$.

Given a point z in D let w be a path from z to 0. For n large enough $p(w) \subset S_n$, so w and hence z lie in D_n. It follows that $D = \bigcup_{n=0}^{\infty} D_n$. In a similar fashion, if $g \in G$ then there is an integer $n > 0$ that $g(0) = z$ is in D_n. Since D_n is either mapped onto itself or completely away from itself by $g \in G$ it must be that $g \in G_n$; hence, $\bigcup_{n=0}^{\infty} G_n = G$.

Since the surfaces S_n are of finite type, we may conclude from the results in section 3.0 that, the groups G_n acting on the domains D_n may be linearized. For each positive integer n let $\phi_n \colon D_n \to \mathbb{C}$ be a conformal homeomorphism for which $\phi_n G_n \phi_n^{-1}$ is a group of extended linear fractional transformations. We may, without loss of generality, assume that the maps ϕ_n have been normalized so the near the origin $\phi_n(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m$ and so that $0 \notin \phi_n(\Delta)$.

Let $\Delta_k = \{z \mid |z| < k\}$, the open disc of radius k centered at the origin.

<u>Lemma II.5</u> ($\frac{1}{4}$-theorem): Let $F\colon \Delta_k \to \hat{\mathbb{C}}-0$ be a conformal homeomorphism normalized so that $F(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m$ for $z \in \Delta_k$. Then $F(\Delta_k) \supset \hat{\mathbb{C}} - \Delta_{4/k}$.

Proof. Consider a function $f(z) = z+a_2z^2+\cdots$ which is one-to-one and holomorphic on Δ_k. We will show that $f(\Delta_k) \supset \Delta_{k/4}$. When k=1 this is a well known corollary of the area theorem, called the 1/4-theorem [R1, 14.14, pg. 278].

Let $\lambda(z) = kz$. Then $H(z) = \lambda^{-1}\circ f\circ\lambda(z)$ is a conformal homeomorphism taking Δ into $\mathbb{C}$. We compute:

$$H(z) = \frac{1}{k}[f(kz)] = \frac{1}{k}[kz+a_2k^2a^2+\cdots] = z+a_2kz^2+\cdots+a_mk^{m-1}z^m+\cdots$$

H therefore satisfies the hypothesis of the 1/4-theorem. We may conclude that $H(\Delta) \supset \Delta_{1/4}$. It follows immediately that $f\circ\lambda(\Delta) \supset \Delta_{k/4}$, or $f(\Delta_k) \supset \Delta_{k/4}$.

To see how this implies the lemma set $f(z) = \frac{1}{F(z)}$.

Then $f(z) = \dfrac{1}{\frac{1}{z}+a_1z+\cdots} = \dfrac{z}{1+a_1z^2+\cdots} = z$ + higher order terms. We may conclude, as above, that $f(\Delta_k) \supset \Delta_{k/4}$, and consequently that $F(\Delta_k) \supset \mathbb{C}-\overline{\Delta}_{4/k}$. ■

Let K be a compact subset of D-{0}. Then for some integer $m > 0$, $K \subset D-\Delta_{1/m}$, and so, by lemma 5, $|\phi_n(z)| < 4m$ for all $n > 0$ and for all $z \in D_n \cap K$. In other words, the functions ϕ_n are uniformly bounded on compact subsets of D-{0}.

The Arzela-Ascoli Theorem implies the existence of a convergent subsequence of the ϕ_n, which we will again call ϕ_n, converging uniformly to a function ϕ in compact subsets of D-{0}. Then by Hurwitz's theorem either ϕ is a conformal homeomorphism or $\phi(z) = \infty$ for all $z \in D-\{0\}$. We conclude from lemma 5 and the normalization that ϕ is a conformal homeomorphism on D-{0}, and consequently on all of D.

It remains to be proven that $\phi\, G\, \phi^{-1}$ is a group of extended linear fractional transformations. This will follow by showing that for a given $g \in G$ the sequence $\phi_n \circ g \circ \phi_n^{-1}$ converges uniformly to $\phi \circ g \circ \phi^{-1}$ on an open subset of Δ. This clearly implies convergence of the sequence $\{\phi_n \circ g \circ \phi_n^{-1}\}$, in Möb, to $\phi \circ g \circ \phi^{-1}$.

We will prove convergence when g is anticonformal. The proof is analogous for g conformal and may be found in [M2].

Let $\Delta_0 = \{z \,|\, |z-\frac{1}{2}|<\frac{1}{4}\}$, and let $\Delta_1 = \{z \,|\, |z-\frac{1}{2}|<\frac{1}{8}\}$. Set $\Delta_1^* = \phi(\Delta_1)$. Convergence of the ϕ_n implies that for large values of n, $\phi_n^{-1}(\Delta_1^*) \subset \Delta_0$. Things were arranged so that $g(\Delta) \cap \Delta = \phi$. Hence, for $z \in \Delta_0$, $|g(z)|>1$. By the normalization and lemma II.5 we may conclude that $|\phi_n \circ g(z)|<4$ for $z \in \Delta_0$. In other words $\phi_n(g(\Delta_0)) \subset \Delta_4$. Consequently, $|\frac{d}{d\bar{z}}(\phi_n \circ g)|$ are uniformly bounded in Δ_0.

Consider the inequality

$$|\phi \circ g \circ \phi^{-1}(z) - \phi_n \circ g \circ \phi_n^{-1}(z)| \leq |\phi \circ g \circ \phi^{-1}(z) - \phi_n \circ g \circ \phi^{-1}(z)| + |\phi_n \circ g \circ \phi^{-1}(z) - \phi_n \circ g \circ \phi_n^{-1}(z)|.$$

We will show that the two expressions on the right hand side approach zero uniformly for $z \in \Delta_1^*$ as n approaches infinity. For the first expression this is an obvious consequence of the uniform convergence of the ϕ_n to ϕ on D-{0}. Given $\varepsilon > 0$ the uniform boundedness of the derivatives $|\frac{d}{d\bar{z}}(\phi_n \circ g)|$ on Δ_0 implies the existence of a $\delta > 0$ so that if $|z-w| < \delta$ for z and w in Δ_0 then $|\phi_n \circ g(z) - \phi_n \circ g(w)| < \varepsilon$. Then we may choose n large enough that $\phi_n^{-1}(\Delta_1^*) \subset \Delta_0$ and $|\phi_n^{-1}(z) - \phi^{-1}(z)| < \delta$ for all $z \in \Delta_1^*$. That proves that the second term in the inequality approaches zero, and completes the proof of Theorem II.1. ■

Theorem II.3. Let S be a Riemann surface of genus g. Then there is a closed Riemann urface S* of genus g and a conformal embedding of S into S* so that, under the embedding, every conformal and anticonformal self map of S is the restriction of a conformal or anticonformal self map of S*.

Proof. Maskit's proof of Theorem I.1, in the last section of [M2], goes through without modification in light of Theorem II.1. ■

CHAPTER III

§0.

As an application of Theorem II.1 we will prove the existence of certain canonical representations in the conformal equivalence class of a plane domain. These are called pseudocircle domains and are characterized by their boundary behavior. For a large class of plane domains the representative pseudocircle domains are actually circle domains; thus, in those cases, this application provides a solution to the Koebe Kreisnormierungs problem.

The statement of the main theorem occurs in section 5 with the proof following. We begin by investigating some properties of the boundaries of plane domains and their behavior under conformal mappings.

§1.

Let D be a plane domain. By the boundary of D we mean the complement of D in its closure. Sometimes we will refer to the boundary of D as ∂D. A connected component of ∂D is called a boundary component.

Lemma III.1. Let D and D' be plane domains and $f: D \to D'$ a homeomorphism. Then f induces a one-to-one correspondence between the boundary components of D and the boundary components of D'. In particular, given a boundary component b of D then there is a boundary component b' of D' so that if $\{z_i\}_{i=1}^{\infty}$ is any sequence of points in D accumulating exactly at points of b then the image sequence $\{f(z_i)\}_{i=1}^{\infty}$ accumulates exactly at points of b'.

Proof. Clearly the sequence $\{f(z_i)\}_{i=1}^{\infty}$ in D' must accumulate only at points of $\partial D'$. Suppose there are subsequences $\{f(w_i)\}_{i=1}^{\infty}$ and $\{f(x_i)\}_{i=1}^{\infty}$ accumulating at distinct boundary components b'_1 and b'_2 of D'.

By a classical theorem in plane topology [W; Cor. 3.11, pg 35] there is a Jordan curve σ in D' so that b'_1 and b'_2 lie in different components of the complement of σ in $\mathbb{C}$. Without loss of generality we may assume that the sequences $\{f(w_i)\}_{i=1}^{\infty}$ and $\{f(x_i)\}_{i=1}^{\infty}$ lie in distinct components of the complement of σ in D'. Hence the sequences $\{w_i\}_{i=1}^{\infty}$ and $\{x_i\}_{i=1}^{\infty}$ lie in distinct components of the complement of $f^{-1}(\sigma)$ in D. The boundary component b lies in one component of the complement of $f^{-1}(\sigma)$ in D; therefore one of the subsequences, lying in a different component, is bounded away from b.

This contradicts the assumption that $\{z_i\}_{i=1}^{\infty}$ accumulates exactly at the boundary component b, and therefore $\{f(z_i)\}_{i=1}^{\infty}$ must accumulate at exactly one boundary component b' of D'. ∎

§2.

A point z in a boundary component b is isolated if it is not a point of accumulation of other boundary components.

Each boundary component lies in a unique connected component of the complement of D in $\hat{\mathbb{C}}$. Let b be a boundary component of D and let B be the component of $\hat{\mathbb{C}}$-D containing b. $\hat{\mathbb{C}}$-B is simply connected. If b is not a single point then there is a Riemann map ϕ: $\hat{\mathbb{C}}-B \to \Delta$, where Δ denotes the unit disc. The unit circle $\partial\Delta$ is the boundary component of the domain $\phi(D)$ corresponding to b under the map ϕ. We say that the boundary component b is spacious if there is an isolated point on the boundary component $\partial\Delta$ of the domain $\phi(D)$. This definition does not depend on the Riemann map chosen.

Notice that for a point z to be an isolated boundary point of a circular boundary component C there must be an open arc about z in C consisting entirely of isolated boundary points. Consequently, the set of all isolated boundary points of a circular boundary component must be a countable union of open arcs on the circle.

The role of spaciousness in what follows is to provide a gross measure for determining whether a component of the boundary of a domain contains "isolated pieces." This notion will be refined and made more precise in later sections.

We will now prove two technical lemmas which together imply the invariance of spacious boundary components under conformal mappings.

Lemma III.1. Let $f: D \to D'$ be a conformal homeomorphism between the domains D and D', with the boundary component b of D corresponding to the boundary component b' of D' under f. If b is spacious then b' is not a single point.

Proof. We assume that b' is a point; without loss of generality that point may be chosen to be the origin.

Let B be the component in the complement of D containing b, and let $\phi: \hat{\mathbb{C}}-B \to \Delta$ be a Riemann map. Let I be the set of isolated points of $\phi(D)$ on $\partial\Delta$. Let $R(z) = \frac{1}{\bar{z}}$, reflection in $\partial\Delta$. The set $\omega = \phi(D) \cup I \cup R(\phi(D))$

is a domain in $\mathbb{C}$, and $R(\omega) = \omega$.

The map $h(z) = f \circ \phi^{-1}(z)$ is a conformal homeomorphism taking $\phi(D)$ onto D'. Define the map $H: \omega \to \mathbb{C}$ by

$$H(z) = \begin{cases} h(z) \text{ for } z \in \phi(D) \\ 0 \text{ for } z \in I \\ R \circ h \circ R(z) \text{ for } z \in R(\phi(D)). \end{cases}$$

By the reflection principle H is holomorphic, but this is impossible since $H(I) = 0$; hence, b' contains more than one point. ■

Suppose now that b is spacious. Then we can choose Riemann maps $\phi: \hat{\mathbb{C}} - B \to \Delta$ and $\Psi: \hat{\mathbb{C}} - B' \to \Delta$, where B and B' are respectively the complementary components of D and D' containing b and b'. Let I again be the set of isolated points on the boundary component $\partial\Delta$ of $\phi(D)$, and let I' be the set of isolated points on the boundary component $\partial\Delta$ of $\Psi(D')$.

<u>Lemma III.3</u>. The map $g = \Psi \circ f \circ \phi^{-1}$ taking $\phi(D)$ onto $\Psi(D')$ extends to a homeomorphism of $\phi(D) \cup I$ onto $\Psi(D') \cup I'$.

Proof. Again let $\omega = \phi(D) \cup I \cup R(\phi(D))$. As a consequence of the reflection principle there is a holomorphic function G defined on ω with $G(z) = g(z)$ on $\phi(D)$ and $G(z) = R \circ g \circ R(z)$ on $R(\phi(D))$. G is clearly a homeomorphism of $\phi(D) \cup R(\phi(D))$ onto $\Psi(D) \cup R(\Psi(D'))$, and hence must be a homeomorphism on all of ω. It is evident that the restriction of G to $\phi(D) \cup I$ is a homeomorphism onto $\Psi(D') \cup I'$ extending g. ∎

These lemmas have the obvious

Corollary: Spacious boundary components correspond to spacious boundary components under conformal homeomorphisms. ∎

§3.

We present a brief summary of basic prime end theory. For a more detailed exposition and proofs of the theorems consult [C+L].

Throughout this section it will be assumed that D is a simply connected domain in $\hat{\mathbb{C}}$ and that $\hat{\mathbb{C}}-D$ contains more than two points.

A simple closed Jordan arc whose interior lies in D and both of whose end points lie in $\overline{D}$ is called a

<u>cross cut</u> in D.

A <u>chain</u> in D is a sequence $\{q_i\}$ of cross cuts in D satisfying

1. $q_i \cap q_j = \phi$ for $i \neq j$

2. q_n separates D into two open sets. One of these contains q_{n-1} and the other contains q_{n+1}.

3. Measured in the spherical metric the diameter of q_n approaches zero as n approaches infinity.

Of the two domains determined by q_n one, which we shall call d_n, contains all of the q_i for $i > n$.

Two chains $\{q_n\}$ and $\{q_n'\}$ are <u>equivalent</u> if for all positive numbers n the domains d_n contain all but a finite number of the cross cuts q_n' and the domains d_n' contain all but a finite number of the cross cuts q_n.

A <u>prime end</u> of D is an equivalence class of chains in D.

The <u>impression</u> of a prime end P is the set $I(P) = \cap \overline{d_n}$.

Let $f: D \to \mathbb{C}$ be a function. The <u>cluster set</u> of the function f at z_0 is the set $C(f,z_0) = \bigcap_{r>0} \overline{D_r}$ where $D_r = f(\Delta_r \cap (D-z_0))$ and Δ_r denotes the disc of radius r centered at z_0. This is the set of accumulation points of the image under f of sequences of points in D converging to z_0.

Theorem P1. A prime end contains a chain of cross-cuts that lie on a sequence of concentric circles whose radii tend to zero. ■

Theorem P2. Under a conformal homeomorphism f of the open unit disc onto a simply connected domain D, there exists a one-to-one correspondence between the points z on the unit circle and the prime ends P(z) of D. Moreover, $C(f,z) = I(P(z))$. ■

We say that a sequence of points $\{z_i\}_{i=1}^{\infty}$ in D converges to a prime end P if given any chain $\{q_i\}_{i=1}^{\infty}$ in P then each domain d_n contains all but a finite number of the z_i. Then we may describe the correspondence of Theorem P2, between a point z on $\partial\Delta$ and a prime end P(z) of D, as follows: a sequence of points $\{z_i\}_{i=1}^{\infty}$ in Δ converges to z if and only if the sequence $\{f(z_i)\}_{i=1}^{\infty}$ converges to the prime end P(z).

§4.

Now let D be an arbitrary plane domain and let b be a boundary component of D containing more than one point. The prime ends of b are defined to be the prime ends (in the sense of section 3) of the simply connected domain $\hat{\mathbb{C}}$-B, where B is the component of $\hat{\mathbb{C}}$-D containing b. This is not the most general definition possible and is perhaps especially unsatisfactory in that a chain may not even lie in D; however, it allows a precise characterization of the type of boundary behavior with which we are primarily concerned.

Let P be a prime end of a boundary component b of D, and let $\{q_i\}$ be a chain in P. As before a q_n determines two domains in $\hat{\mathbb{C}}$-B, and the one containing all q_i for $i > n$ shall be denoted by d_n.

Again, the impression of P is the set $I(P) = \cap \overline{d}_n$.

A prime end P of a boundary component b is isolated if there is a chain $\{q_i\}$ in P for which the domains d_n all lie in D; that is, the d_n do not contain any boundary points of D. Evidently if P is isolated then any chain $\{q_n'\}$ in P has the property that all but a finite number of the domains d_n' lie in D.

The following two examples should help to clarify these notions.

For each positive integer n define the line segments

$\ell_n = \{z \mid 0 \leq x \leq 2,\ y = \frac{1}{n}\}$. Also, set $\ell_\infty = \{z \mid 0 \leq x \leq 3,\ y = 0\}$ and $\ell_0 = \{z \mid x = 0,\ 0 \leq y \leq 2\}$. Then $E = \bigcup_{n=0}^{\infty} \ell_n$ is a closed connected subset of the plane.

Let N_k and M_k be the closed discs of radius $\frac{1}{2n(n+1)}$ centered at $1+i\frac{2n+1}{2n(n+1)}$ and $1-i\frac{2n+1}{2n(n+1)}$ respectively.

Let q_k be the line segment joining the points $2+\frac{i}{k}$ and $2+\frac{1}{k}$.

In the first example we consider the domain $D_1 = \hat{\mathbb{C}}-(E \cup \bigcup_{k=2}^{\infty} N_k)$ (see Fig. 9). The sequence of cross cuts $\{q_k\}$ determines a prime end P of the boundary component E of D_1. P is not an isolated prime end since each domain d_k contains the boundary circles of the discs N_n for $n \geq k$. The impression of P is the line segment $L = \{z \mid y = 0,\ 0 \leq x \leq 2\}$.

Notice that the point $\frac{1}{2}$ lies on the impression of P but is an isolated boundary point of D_1.

In the next example let $D_2 = \hat{\mathbb{C}}-(E \cup \bigcup_{k=2}^{\infty} M_k)$. Again the sequence of cross cuts $\{q_k\}$ determines a prime end P of the boundary component E of D_2, which has as its impression the line segment L. In this example P is an isolated boundary point of D_2.

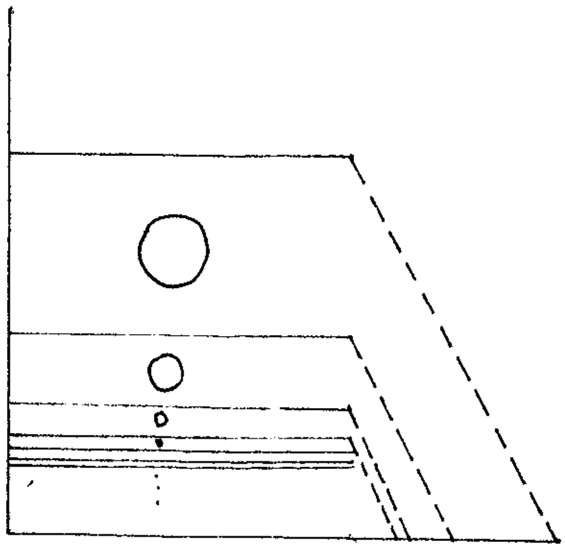

Figure 1

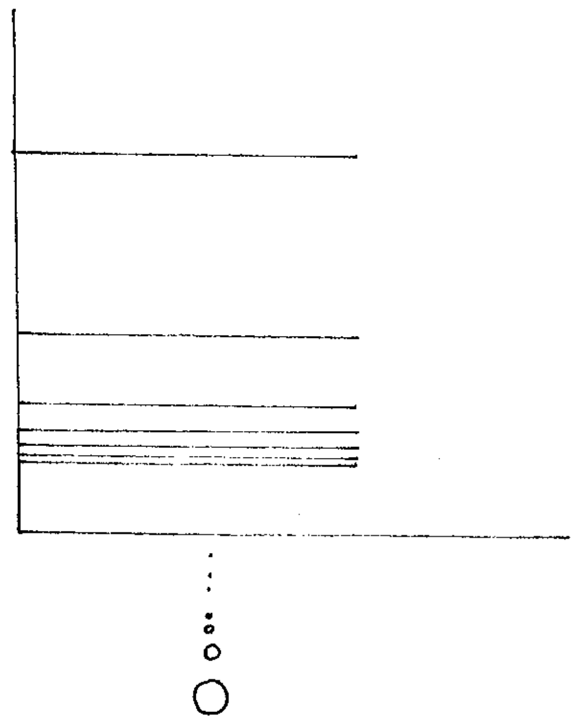

Figure 2

These examples illustrate the independence of the two notions: isolated points and isolated prime ends. In the special case where the boundary component b is a Jordan curve the two notions coincide. This is a consequence of an old theorem [R,14.19, p.281] of Caratheodory.

We may now deduce the following generalizations of Theorem P2.

Proposition III.1. Let b be a spacious component of ∂D contained in the component B of $\hat{\mathbb{C}}$-D. Let ϕ: $\hat{\mathbb{C}}$-B $\to \Delta$

be a Riemann map. Then there is a one-to-one correspondence between the isolated boundary points z of the domain $\phi(D)$ lying on $\partial\Delta$ and the isolated prime ends $P(z)$ of the boundary component b.

Proof. According to Theorem P2 there is a one-to-one correspondence between the prime ends of the simply connected domain $\hat{\mathbb{C}}-B$ and the points on $\partial\Delta$. Let z be a point on $\partial\Delta$ corresponding in this manner to the prime end $P(z)$.

We need to show that z is an isolated point of the boundary component $\partial\Delta$ of $\phi(D)$ if and only if $P(z)$ is an isolated prime end of the boundary component b of D. Suppose z is an isolated point but $P(z)$ is not an isolated prime end. Then there is a sequence $\{z_i\}_{i=1}^{\infty}$ of points in ∂D so that if $\{q_i\}$ is a chain in $P(z)$ then for all $n > 0, d_n \cap \{z_i\} \neq \phi$. By Theorem P2 $\lim_{i\to\infty} \phi^{-1}(z_i) = z$, which contradicts the assumption that z is isolated. The reverse implication follows in an analogous fashion. ■

Proposition 1 generalizes further to give

<u>Theorem III.3</u>. Let f: D → D' be a conformal homeomorphism between plane domains. Then f induces a one-to-one correspondence between the isolated prime ends of boundary components of D and the isolated prime ends of boundary components of D'.

Proof. Let P be an isolated prime end of a boundary component b of D. There is a boundary component b' of D' corresponding to b under f. Since P is isolated b and b' must be spacious. Let B and B' be the complementary components of the domains containing b and b' respectively. Choose Riemann maps $\phi: \hat{\mathbb{C}}-B \to \Delta$ and $\psi: \hat{\mathbb{C}}-B' \to \Delta$. As in the notation of lemma 3, let I and I' be the isolated points of the domains $\phi(D)$ and $\psi(D')$ respectively, on $\partial\Delta$.

By lemma 3 $\psi\circ f\circ\phi^{-1}$ extends to a homeomorphism of $\phi(D) \cup I$ onto $\psi(D') \cup I'$. This gives a one-to-one correspondence between points in I and points in I'. By Proposition 1, ϕ induces a one-to-one correspondence between the isolated prime ends of b and the points in I. Similarly, ψ induces a one-to-one correspondence between the isolated prime ends of b' and the points in I'.

Following these correspondences from an isolated prime end of b, to a point in I, to a point in I', to an isolated prime end of b', and back again gives the correspondence of Theorem 3. ∎

It should be observed that the isolated prime ends are perhaps the largest class of prime ends for which such a correspondence can generally be shown to exist. The following example will illustrate: Let D be the complement in $\hat{\mathbb{C}}$ of the collection of lines $\{\ell_n\}_{n=-\infty}^{\infty}$ where for $n \neq 0$, $\ell_n = \{x+iy \mid 1 \leq y \leq 1,\ x = \frac{\operatorname{sgn} n}{2^{|n|}}\}$ and

$\ell_0 = \{x+iy \mid 1 \leq y \leq -1,\ x=0\}$. It is easily shown, using an old theorem of Koebe's on domains with reflections [S2], that there is a conformal map f: D → D' under which the boundary component ℓ_0 corresponds to a single point boundary component ζ. Although there are an uncountable number of distinct prime ends of the boundary component ℓ_0 of D, the boundary component ζ of D' is without any prime ends to which these may correspond.

§5.

A boundary component b of a plane domain D is called a pseudocircle if there is a circle C_b and an open disc Δ_b in $\hat{\mathbb{C}}$ bounded by C_b so that

1. $\overline{D} \cap \Delta_b = \phi$ and

2. if P is an isolated prime end of b then I(P) lies on C_b.

It should be evident that the impressions of the isolated prime ends of a pseudocircle are exactly the isolated boundary points on the circle; hence, in this setting the isolated prime ends do correspond to isolated boundary points.

A boundary component none of whose prime ends is isolated is called a limit boundary component.

A plane domain D is a pseudocircle domain if $\partial\Delta$ consists of pseudocircles, single point boundary components, and limit boundary components--in other words, every spacious boundary component is a pseudocircle.

Theorem III.4. Every plane domain is conformally equivalent to a pseudocircle domain.

§6.

We first prove

Lemma 4. A plane domain has a countable number of spacious boundary components.

Proof. We suppose that D has at least one spacious boundary component a. Let A be the complementary component of D containing a. Choose a Riemann map $\phi: \mathbb{C}-A \to \Delta$. Then since spacious boundary components of D are in a one-to-one correspondence with spacious boundary components of $\phi(D)$, we can assume for the proof that D is a bounded domain in $\mathbb{C}$.

On each spacious boundary component b of D we may choose an isolated prime end P(b). By Theorem P1 there is a chain $\{q_i(b)\}_{i=1}^{\infty}$ in P(b) the elements of which lie on concentric circles about a point z(b). Since P(b) is isolated we may further stipulate that the domains $d_n(b)$, determined by the cross cuts $q_n(b)$, lie entirely inside D.

Let $\delta(b)$ be the radius of the circle containing $q_1(b)$. Choose a cross cut $q_k(b)$ which lies on a circle of radius less than $\delta(b)/4$. Call this cross cut q(b) and the associated domain d(b).

For distinct boundary components b and b' we show that $d(b) \cap d(b') = \phi$. If not then, without loss of generality, suppose that q(b) lies on a circle of radius greater than or equal to the radius of the circle containing q(b'). Then z(b'), which is a boundary point of D, lies in $d_1(b)$. This contradicts our previous assumptions.

Since the domains d(b) are all disjoint their number must be countable. ■

§7.0

Starting with a plane domain D we construct a domain $\mathcal{D}$ which contains D and is gotten by gluing together reflected pieces of D along isolated pieces of the boundary of D.

Before describing the general construction we will illustrate the procedure with a simple example. Let Δ_1 and Δ_2 be two disjoint open discs in $\mathbb{C}$ with disjoint closures. Let R_1 and R_2 denote respectively, reflection in the circles $\partial\Delta_1$ and $\partial\Delta_2$. Let D be the domain $\hat{\mathbb{C}}-(\overline{\Delta}_1 \cup \overline{\Delta}_2)$. Let G be the group generated by R_1 and R_2. G is an extended Kleinian group, and $\Omega(G)$ will correspond to the domain $\mathcal{D}$. Alternately, $\mathcal{D}$ may be gotten by gluing together all of the pieces $g(\overline{D})$ for $g \in G$ along their shared boundaries.

We now proceed with the construction. Let B = $\{b_i\}_{i=1}^{K}$, where K is a positive integer or ∞, be an ordered set of all spacious boundary components of D. Each b_i is contained in a component B_i of $\hat{\mathbb{C}}$-D. For each integer n, $1 \le n < \infty$, choose a Riemann map ϕ_n: $\hat{\mathbb{C}}-B_n \to \Delta$. Let I_n be the set of isolated boundary points of $\phi_n(D)$

on $\partial\Delta$. I_n is a countable union of open intervals on $\partial\Delta$.

$\mathcal{D}$ is constructed by gluing together pieces of the form $D^* = D \cup (\bigcup_{n=1}^{K} I_n)$ and $\overline{D}^* = \overline{D}\ (\bigcup_{n=1}^{K} I_n)$. (Here — denotes complex conjugation.) The pieces are first indexed by elements of a group which corresponds, in the above example, to the group generated by "reflections" in the spacious boundary components.

7.1

Let A be a set. A reduced word on the elements of A is a finite sequence $\{a_i\}_{i=1}^{N}$ of elements of the set A with the additional property that $a_j \neq a_{j+1}$ for all $j=1,\ldots,N-1$. We include the empty sequence as a word.

Let $\mathcal{B}'$ be the set whose elements are the letters β_i which correspond to the elements of the set $\mathcal{B}$ of boundary components. Let W be the set of reduced words on $\mathcal{B}'$. We define a binary operation on W which makes it into a group which we shall call G. The empty sequence will serve as the identity element and so we denote it by 1. Suppose w_1 and w_2 are two reduced words in W, $w_1 = \beta_{j_0},\ldots,\beta_{j_n}$, $w_2 = \beta_{k_0},\ldots,\beta_{k_m}$. Let ℓ be the smallest non-negative integer for which $\beta_{j_{n-\ell}} \neq \beta_{k_\ell}$. Then

$$w_1 \circ w_2 = \begin{cases} 1 & \text{if } n=m=\ell \\ \beta_{k_\ell}\beta_{k_{\ell+1}}\cdots\beta_{k_m} & \text{if } n=\ell<m \\ \beta_{j_o}\cdots\beta_{j_{n-\ell}} & \text{if } m=\ell<n \\ \beta_{j_o}\cdots\beta_{j_{n-\ell}}\beta_k\cdots\beta_{k_m} & \text{otherwise} \end{cases}$$

This corresponds to successively cancelling pairs of adjacent letters which are the same in the sequence gotten by following w_1 by w_2.

The group G is isomorphic to a free product of groups $\mathbb{Z}_2$ indexed by the set $\mathcal{B}'$. We will continue, where convenient, to represent elements of G by reduced words in W.

§7.2

For each $w \in G$ define

$$D_w = \begin{cases} D^* \times \{w\} & \text{if } w=1 \text{ or if the number of letters in } w \text{ is even} \\ \overline{D}^* \times \{w\} & \text{otherwise} \end{cases}$$

D_w is to be thought of as a copy of one of D^* or $\overline{D}^*$ with gluing information appended.

Let $\delta = \bigcup_{w \in G} D_w$. We define an equivalence relation on δ which accomplishes the gluing together of the pieces D_w along the "isolated" boundary curves $I_j \times \{w\}$. Two points (x,w) and (y,w') in δ are equivalent if and only if there is a positive integer n for which $x = y \in I_n$ and $w = \beta_n \circ w'$. We denote the equivalence by $\sim$.

Let $\mathcal{D}$ be the set of equivalence classes of δ with respect to this equivalence relation.

§7.3

Lemma III.4. $\mathcal{D}$ possesses canonically, the structure of a connected, planar Riemann surface.

Proof. We will define open sets about each point in $\mathcal{D}$ and coordinate functions on these open sets. By specifying that these functions be conformal homeomorphisms we define a topology and complex structure on $\mathcal{D}$.

Suppose d is a point in $\mathcal{D}$ which has a unique representative of the form (z,w) in δ, where z is a point in either D or $\overline{D}$. Let $\mathcal{U}$ be an open neighborhood of z in D (or $\overline{D}$). Then $\mathcal{U} \times \{w\}$ is a neighborhood of z in $\mathcal{D}$ and $\psi\colon \mathcal{U} \times \{w\} \to \mathbb{C}$ given by $\psi(\zeta,w) = \zeta$ is a coordinate chart about z.

If d is not as above then it must have two distinct representatives in δ: (x,w) and $(x,\beta_n \circ w)$, where for some positive integer n and $w \in G$, $x \in I_n$. Consider the domain $\omega_n = \phi_n(D) \cup I_n \cup R(\phi_n(D))$. There is a canonical bijection between ω_n and the subset $\{D_w - (\bigcup_{j \neq n} I_j \times \{w\})\} \cup \{D_{\beta_n \circ w} - (\bigcup_{j \neq n} I_j \times \{\beta_n \circ w\})\}/\sim$ of $\mathcal{D}$ where w may be chosen so that the two sets in brackets are respectively $(D \cup I_n) \times \{w\}$ and $(\bar{D} \cup I_n) \times \{\beta_n \circ w\}$. Open sets are defined via the bijection, and by making it a conformal homeomorphism we get coordinate charts about d which agree with one another and with those previously defined.

We will show that $\mathcal{D}$ is planar by demonstrating that any two Jordan curves in $\mathcal{D}$ have even intersection number. Suppose α and γ are two Jordian curves in $\mathcal{D}$ which we may assume have a finite number of intersections.

Clearly, α and γ lie in a connected open subset of $\mathcal{D}$ which is contained in the image of a finite number of the D_w in δ; that is, there exist distinct words $w_1, \ldots, w_n$ so that $\alpha \cup \gamma \subset (\bigcup_{i=1}^{n} D_{w_i})/\sim$. This is a consequence of the compactness of $\alpha \cup \gamma$.

It is evident that any subset of $\mathcal{D}$, constructed as above, out of a finite number of the D_w, may be assembled in the plane. Then $(\bigcup_{i=1}^{n} D_{w_i})/\sim$ is planar; hence, γ and α must have even intersection number.

We may further conclude that the Riemann surface $\mathcal{D}$ is conformally equivalent to a domain in $\mathbb{C}$. ∎

§7.4

Lemma III.5. G acts effectively as a group of conformal and anticonformal homeomorphisms of $\mathcal{D}$ onto itself.

Proof. We define the action on generators, and points in δ.

$$\beta_n(z,w) = \begin{cases} (\bar{z}, w\circ\beta_n) & \text{for } z\in D \text{ or } \bar{D} \\ (z, w\circ\beta_n) & \text{for } z\in I_K \text{ for a positive integer } K. \end{cases}$$

This is well defined on $\mathcal{D}$ since if $(x,w) \sim (x,\beta_K\circ w)$ for $x\in I_K$ then $(x,w\circ\beta\ ) \sim (x,\beta_K\circ w\circ\beta_n)$. The way $\mathcal{D}$ was defined it is evident that β_n acts as an anticonformal involution leaving fixed the points in $I_n \times \{1\}/\sim$.

Notice that $I_n \times \{1\}$ separates $\mathcal{D}$ into two components, and for $K \neq n$ $I_K \times \{1\}$ and $D \times \{1\}$ all lie in the same component. β_n acts by exchanging the two components; therefore, $\beta_n(D\times\{1\})$ lies in the component of $\mathcal{D}-(I_K\times\{1\})$ containing $D\times\{1\}$ for each $K\neq n$ and is disjoint from $D\times\{1\}$. It is clear then that $1\neq w\in G$ satisfies

$w(D\times\{1\}) \cap D\times\{1\} = \phi$. This shows that the action is effective. ■

§8.

We now complete the proof of Theorem III.4.

By Theorem II.1 there is a conformal homeomorphism $f\colon \mathcal{D} \to \hat{\mathbb{C}}$ so that $f\circ G\circ f^{-1} = G^*$ is a group of extended linear fractional transformations. Normalize so that for some β_K, $\infty \in f(D\times\{\beta_K\})$. This is to assure that $f(D\times\{1\})$ is a bounded domain in $\mathbb{C}$. Set $f\circ\beta_n\circ f^{-1} = \beta_n^* \in$ Möb. For a given n the map β_n^* leaves the set $f(I_n\times\{1\})$ pointwise fixed. Recalling the analysis in Chapter I, β_n^* must be a reflection in some circle c_n in $\mathbb{C}$ (straight lines are excluded by the normalization); hence, $f(I_n\times\{1\}) \subset c_n$ and $f(D\times\{1\})$ lies in the unbounded component in the complement of c_n.

Consider the map $i\colon D \to \mathcal{D}$ given as $i(z) = (z,1)$. i is a conformal homeomorphism. The composition $f\circ i\colon D \to \mathbb{C}$ maps D into the plane and by Theorem III.3 induces a one-to-one correspondence between the isolated prime ends of D and $f\circ i(D)$. In particular, if b is a spacious boundary component of D, B the component of $\hat{\mathbb{C}}$-D containg b, and $\phi\colon \hat{\mathbb{C}}-B \to \Delta$ the Riemann map selected in section 7, then $i\circ\phi^{-1}\colon \phi(D) \to D$ is a conformal homeomorphism that evidently extends to a homeomorphism of $\phi(D) \cup I_n$ onto $D\times\{1\} \cup I_n\times\{1\}/\sim$ in $\mathcal{D}$. The isolated prime ends of b correspond to isolated points on $\partial\Delta$ under ϕ; hence, they also correspond to isolated points on the boundary of i(D) in $\mathcal{D}$. This certainly is carried over under f. ∎

§9.

As a corollary we can immediately deduce a theorem of R. J. Sibner [S1].

Corollary III.1. Let D be a plane domain. Then D is conformally equivalent to a domain D', all of whose isolated boundary components are circles or points. ∎

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