Algebraic Cycles in a Certain Fiber Variety

A Dissertation presented

by

Mouw-ching Tjiok

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We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of the dissertation.

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Abstract of the Dissertation

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In this paper, we study the algebraic cycles in a fiber variety as defined by M. Kuga. We shall consider the algebraic cycles in a fiber variety V parametrized by a quotient of a bounded symmetric domain X by a discrete subgroup $\Gamma \subset Aut(X)$, where X is the product of N copies of the upper half-plane $H = \{z \in \mathbb{C}, \mid Im\, z > 0\}$, i.e.,

$$X = H \times \ldots \times H \qquad (N \text{ copies}).$$

Generally, in a non-singular complex projective variety, an algebraic cycle is also a topological cycle, and so it determines a rational homology class, and by Poincare duality, also a cohomology class. Rational cohomology classes on a non-singular complex projective variety, let us denote it by $\mathfrak{X}$, determined in this way are called algebraic cohomology classes.

Let us denote the subspace of $H^{2r}(\mathfrak{X}, \mathbb{Q})$ spanned by algebraic cohomology classes by $c(\mathfrak{A}^r(\mathfrak{X}))$ or by $c(\mathfrak{A}_{m-r}(\mathfrak{X}))$, where $m = \dim \mathfrak{X}$.

Our main result is the theorem (8.1.1): Let V be the total space V of the family $V \xrightarrow{\pi} U$ of abelian varieties over $U = \Gamma\backslash H^N$, defined by a totally indefinite quaternion algebra $\mathfrak{B}$ with a totally real number field k of degree N as center. For this variety V,

$$H^{2k}(V, \mathbb{Q}) = c(\mathfrak{A}^k(V))$$

for $2k < N$.

This implies the Hodge conjecture for $H^{2k}(V, \mathbb{Q})$ with $2k < N$ automatically.

There are three mathods of construction of algebraic (co-) cycles in V:

(A-1) Take algebraic cycle y in the base space U, and make the full inverse $\pi^{-1}(y)$ by the projection $\pi : V \longrightarrow U$.

(A-2) Take algebraic cycle Z in the generic fiber $F_p = \pi^{-1}(p)$, (p is a generic point over a definition field k), such that Z is algebraic over k(p), and make the union of all specializations of Z over k. We denote this by Locus (Z/k).

(A-3) Make intersections $\pi^{-1}(y)\cdot$Locus (Z/k) of $\pi^{-1}(y)$ of

(A-1) and Locus (Z/k) of (A-2).

Actually in low codimensional cases of $2k < N$, all algebraic cycles are spanned by those of (A-1,2,3). Moreover we can see that, if $2k < N$, all of these algebraic cycles of the type of (A-1,2,3) span the whole cohomology group $H^{2k}(V,\mathbb{Q})$, by essentially dimension calculations.

My mother, in memoriam

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INTRODUCTION

The purpose of this paper is to study algebraic cycles in a fiber variety as defined by M. Kuga in [6],[7]. More precisely, we shall consider algebraic cycles in a fiber variety V parametrized by a quotient of a bounded symmetric domain X by a discrete subgroup $\Gamma \subset \mathrm{Aut}(X)$, where X is the product of N copies of the upper half-plane $H = \{z \in \mathbb{C}, \mathrm{Im}\, z > 0\}$, i.e.,

$$X = H \times H \times \ldots \times H \qquad \text{(N copies)}.$$

As our main result, we shall show that the Hodge conjecture is true, up to a certain codimension, in this particular case.

We formulate the content of the Hodge conjecture in a rather general context.

Let $\mathfrak{X}$ be a non-singular complex projective variety. An algebraic cycle on $\mathfrak{X}$ of codimension r, which we shall denote by Z, is also a topological cycle on $\mathfrak{X}$, and so it determines a rational homology class, thus by Poincare duality, a cohomology class $c(Z) \in H^{2r}(\mathfrak{X}, \mathbb{Q})$. Rational cohomology classes on $\mathfrak{X}$ determined in this way are called algebraic cohomology classes. Let us denote the space of $\mathbb{Q}$-linear combinations of algebraic cycles on $\mathfrak{X}$ of codimension r by $\mathfrak{A}^r(\mathfrak{X}, \mathbb{Q})$, or by

$\mathfrak{A}^r(\mathcal{X})$, and also denote $\mathfrak{A}^r(\mathcal{X},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R}$ by $\mathfrak{A}^r(\mathcal{X},\mathbb{R})$, and $\mathfrak{A}^r(\mathcal{X},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$ by $\mathfrak{A}^r(\mathcal{X},\mathbb{C})$. A cohomology class in the r-th de Rham cohomology group $H^r(\mathcal{X})$ is said to be of type (p,q) if it can be represented by an r-form which, in each local coordinate system, is a sum of forms of the type $f\cdot dz_I\wedge d\bar{z}_J$, where $|I|=p$, $|J|=q$, and $p+q=r$. Let $H^{p,q}$ be the subspace of $H^r(\mathcal{X})$ consisting of cohomology classes of type (p,q). Lefschetz has proved that

$$c(\mathfrak{A}^1(\mathcal{X})) = H^{1,1}\cap H^2(\mathcal{X},\mathbb{Q}).$$

In general, it is the case that

$$c(\mathfrak{A}^p(\mathcal{X}))\subset H^{p,p}\cap H^{2p}(\mathcal{X},\mathbb{Q}),$$

and Hodge has conjectured $c(\mathfrak{A}^p(\mathcal{X})) = H^{p,p}\cap H^{2p}(\mathcal{X},\mathbb{Q})$ in each codimension p.

We shall now formulate our main result.

<u>Theorem</u> (8.1.1): For a family of abelian varieties $V\xrightarrow{\pi}U$, which is parametrized by a quotient of a bounded symmetric domain X by a discrete subgroup $\Gamma\subset \mathrm{Aut}(X)$, where X is the product of N copies of the upper half-plane. Then

$$\mathfrak{A}^r(V,\mathbb{Q}) = H^{2r}(V,\mathbb{Q})$$

for $2r < N$. Therefore the Hodge conjecture for $H^{2r}(V)$ with $2r < N$ is true.

This paper is divided into eight chapters. We now give a brief description of this paper.

In Chapter I, we reproduce the construction of family of abelian varieties $V \xrightarrow{\pi} U$, as given by Kuga in [6]. We first define $V \xrightarrow{\pi} U$ as torus bundle, with torus as fiber; then we show that this is indeed a fiber bundle. The complex structure of V is introduced by means of symplectic triple. The existence and uniqueness of family of abelian varieties is proved by Kuga in [6], and in this lecture-notes Kuga also showed, that V is biholomorphically isomorphic to a non-singular projective algebraic variety. In 1.4, we describe a particular case of family of abelian varieties $V \xrightarrow{\pi} U$, where $U = \Gamma \backslash X$, and $X = H \times \ldots \times H$ is the product of N copies of the upper half-plane, and $G = SL(2,\mathbb{R}) \times \ldots \times SL(2,\mathbb{R})$; then we describe the construction using results of Satake. In 1.5, we showed that this particular example can indeed be constructed out of totally indefinite quaternion algebra over a totally real number field. In the case $N = 1$, this was treated in Kugas' notes [6].

Chapter II deals with the cohomology group on V, after a short account of Matsushima-Murakami's theory of vector-bundle-valued forms, we formulate the result of Matsushima-Shimura, which is crucial to our proof. In this chapter, we study the cohomology groups of V, beginning with an isomorphism, which is obtained by Kuga, (see 2.3(1))

$$H^p(V, \mathbb{C}) = H^p(\Gamma x L, \mathbb{C}),$$

using Leray spectral sequence and Hochschild-Serre spectral sequence, the E_2-terms of both sides turn out to be cohomology groups of Matsushima-Murakami type; and this can also be described as an eigenspace of certain strtching operator defined by Kuga in [6]. The most important results of the last section of chapter II are Theorem (2.4.1) (which is due to Kuga) and the subsequent decomposition (1).

In Chapter III, we study the algebraic cycles in a generic fiber. Here we make two important assumptions on algebraic cycles, which enable us to describe algebraic cycles as intersections of divisors. Later, we shall prove that these assumptions are indeed true in our particular case, this was done in Chapter IV.

In Chapter V, we describe the algebraic cycles in the total space V of the family of abelain varieties $V \xrightarrow{\pi} U$, generated by specializations of an algebraic cycle Z in a generic fiber F_p (which is defined over a generic point p in U over k). In the following two sections, we study the harmonic forms on V, and the description of algebraic cycles as differential forms. In particular, we show that the cohomology group $H^{\langle 2N, 2r\rangle}(V, \mathbb{Q})$ as subspace of $H^{2N+2r}(V, \mathbb{Q})$ is spanned by algebraic cycles; and so does the cohomology group $H^{\langle o, 2r\rangle}(V, \mathbb{Q})$, as a subspace of $H^{2r}(V, \mathbb{Q})$, is spanned by algebraic cycles.

In Chapter VI, we describe the algebraic cycles in the base space U, using the Chern class of a certain line bundle over U. Our main result is Cor. (6.1.4), which states that, if $2r \neq N$, then the cohomology group $H^{\langle 2r, o\rangle}(V, \mathbb{Q})$ is spanned by algebraic cycles of type $\frac{1}{\pi^r}\omega_{i_1} \wedge \ldots \wedge \omega_{i_r}$, where π means the number $3.1415926\ldots$. We want to apologize for the double use of the symbol π in this chapter, in most cases, the π's occurred in the formulas are the number $3.1415926\ldots$.

Chapter VII is devoted to the study of cohomology groups $H^{\langle 2p, 2r\rangle}(V, \mathbb{Q})$. We show that, for $2p \neq N$, the subgroup $H^{\langle 2p, 2r\rangle}(V, \mathbb{Q})$ of $H^{2p+2r}(V, \mathbb{Q})$ is spanned by algebraic cycles.

Finally, in Chapter VIII, we give the proof of our main theorem, utilizing the results of previous chapters.

In fact, using Theorem (4.3.9), we can determine the dimension of $H^{2r}(V,\mathbb{C})$ explicitly.

CHAPTER I. THE VARIETY V

1.1 Torus bundle $V \longrightarrow U$

Let G be a connected real semi-simple Lie group with finite center, and K be a maximal compact subgroup of G. The space $X = G/_K$ is therefore a symmetric space; and X is of purely non-compact type (i.e., all simple factors of X are non-compact) if G is not compact. If G is compact, then X is a point.

Let $(\tilde{F},\rho)$ be a finite dimensional representation of the group G over $\mathbb{R}$; i.e., $\tilde{F}$ is a finite dimensional linear space over $\mathbb{R}$, and ρ is a homomorphism of G to the group $GL(\tilde{F}/\mathbb{R})$ of all linear automorphisms of the vector space $\tilde{F}$:

$$\rho : G \longrightarrow GL(\tilde{F}/\mathbb{R}).$$

We form the semi-direct product $G\times\tilde{F}$ by defining

$$(g,w)(g',w') = (gg',\rho(g)w'+w),$$

for $(g,w),(g',w') \in G\times\tilde{F}$, and with this multiplication law $G\times\tilde{F}$ is given a group structure. In this group $G\times\tilde{F}$, $\{1\}\times\tilde{F}$ is obviously a normal subgroup, and

$$G\times\tilde{F}/_{\{1\}\times\tilde{F}} \cong G.$$

Furthermore, $G \times \tilde{F}$ acts on the product space $X \times \tilde{F}$ in a natural way, i.e.,

$$(g,w)(x,u) = (g(x), \rho(g)u+w),$$

for $(g,w) \in G \times \tilde{F}$, and $(x,u) \in X \times \tilde{F}$. This action is transitive; and the isotropy subgroup of a point (x_o, o) is a compact group $K \times \{o\}$, where $x_o = \nu(1)$ is the image of the unit element $1 \in G$ by the natural mapping $\nu: G \longrightarrow X = G_{/K}$, so that we have a natural identification

$$X \times \tilde{F} = G \times \tilde{F}_{/K \times \{o\}} .$$

Assume that there exists a lattice L in $\tilde{F}$, (i.e., a discrete subgroup of $\tilde{F}$, with compact quotient ${}_L\backslash \tilde{F}$) and a discrete cocompact subgroup Γ in G with no finite order element except 1; such that

(1) $\rho(\Gamma) \subset GL(L) = \{ g \in GL(\tilde{F}) \mid gL = L \}$.

From the data $\{G, K, X, \tilde{F}, \rho, L, \Gamma\}$ satisfying (1), we are going to construct a manifold V which is a torus bundle over $U = {}_\Gamma\backslash X$.

By the assumption (1) which means $\rho(\gamma)L = L$, for all $\gamma \in \Gamma$, $\rho(\gamma)$ induces an automorphism of the torus $F = {}_L\backslash \tilde{F}$. We shall denote the induced automorphism by $\rho(\gamma)$:

(2) $\rho(\gamma) : F \longrightarrow F$.

The facts that Γ is a discrete subgroup of G, and L is a lattice in $\tilde{F}$, together with assumption (1), show that $\Gamma \times L$ in $G \times \tilde{F}$ is a discrete subgroup, which acts on $X \times \tilde{F}$ properly discontinuously and without fixed points. The quotient $\Gamma \times L \backslash X \times \tilde{F}$ is then a manifold. We shall denote it by V. Since Γ is cocompact, V is compact.

Our next step is to define the fibering structure of V over a locally symmetric space $U = {}_{\Gamma}\backslash X$.

$X \times F$ is simply connected, and $\Gamma \times L$ operating on $X \times \tilde{F}$ freely, therefore, the space $X \times \tilde{F}$ is the universal covering space of V, and the covering transformation group $\Gamma \times L$ is isomorphic to the fundamental group of V. Now, since $\{1\} \times L$ is a normal subgroup of $\times L$, this group $\{1\} \times L$ corresponds to a normal covering space ${}_{\{1\} \times L}\backslash X \times \tilde{F}$, which is naturally identified with $X \times F$, where $F = {}_{L}\backslash \tilde{F}$. The covering transformation group of $X \times F$ over V is ${}_{\{1\} \times L}\backslash \Gamma \times L$, which is canonically identified with . Therefore, the space V can be considered as quotient space

$$V = {}_{\Gamma}\backslash X \times F$$

of $X \times F$ by Γ. Here, the action of Γ on $X \times F$ is given by

$$\gamma(x,u) = (\gamma(x), \rho(\gamma)u),$$

for every $\gamma \in \Gamma$, $(x,u) \in X \times F$, where $\rho(\gamma)$ is the operation of γ on $F = {}_{L}\backslash\tilde{F}$ defined in (2).

Now consider the projection $\tilde{\pi}_1$ of $X \times F$ onto X; then $\tilde{\pi}_1$ commutes with the operation of Γ; so that the following diagram is commutative for all $\gamma \in \Gamma$:

$$\begin{array}{ccc} X \times F & \longrightarrow & X \times F \\ {\scriptstyle \tilde{\pi}_1}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_1} \\ X & \longrightarrow & X \end{array} .$$

Therefore, $\tilde{\pi}_1$ induces naturally a projection π of V onto $U = {}_{\Gamma}\backslash X$. And again we have the following commutative diagram

$$\begin{array}{ccc} V & \longleftarrow & X \times F \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_1} \\ U = {}_{\Gamma}\backslash X & \longleftarrow & X \end{array} .$$

Since Γ has no fixed points on X, we can prove that the inverse image $\pi^{-1}(x)$ of any point x of U by π is a torus isomorphic to $F = {}_{L}\backslash\tilde{F}$. In the sequel, we shall denote $\pi^{-1}(x)$ by F_x.

Summarizing, by our above construction, our triple $\{V, \pi, U\}$ is a fiber bundle

1. whose structure group is Γ, and the standard fiber is the torus F;

2. which is associated with the covering $X \xrightarrow{p} \Gamma\backslash X = U$;

3. and such that the operation of the structure group Γ on the fiber F is defined by (2).

1.2 Symplectic triple (β, σ, J) and the symplectic group

Let $\tilde{F}$ be a vector space over $\mathbb{R}$ of even dimension $2m$. A triple (β, σ, J) is called a symplectic triple if the following are satisfied:

(S-1) β is an alternating bilinear form on $\tilde{F}$,

(S-2) σ is a positive definite symmetric bilinear form on $\tilde{F}$,

(S-3) $J \in GL(\tilde{F})$ such that $J^2 = -1$,

(S-4) $\beta(x, Jy) = \sigma(x, y)$.

Note that two of (β, σ, J) determine the third factor; i.e., if (β, σ, J) and (β, σ, J') are both symplectic triples, then $J' = J$. Or, if (β, σ, J) and (β, σ', J) are symplectic triples, then $\sigma' = \sigma$.

Also note that, by taking a basis $(Z_1, \ldots, Z_{2m})$ of

$\tilde{F}$, a symplectic triple (β, σ, J) is represented by three matrices (B, S, J) which satisfy

(BS-1) $\quad {}^tB = -B$,

(BS-2) $\quad {}^tS = S > 0$,

(BS-3) $\quad J^2 = -1$, $\qquad$ (this is equivalent to $BS^{-1}B = -S$)

(BS-4) $\quad BJ = S$;

and the matrix pair (B, S) is the "symplectic pair" in the sense of Kuga's notes [6].

On $\mathbb{R}^{2m}$ we define the standard triple $(j, 1, \mathbb{J})$ by

$$j(x,y) = {}^tx \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix} y = \sum_{i=1}^{m} (x_i y_{m+i} - x_{m+i} y_i) ,$$

$$1(x,y) = {}^tx \cdot y = \sum_{i=1}^{2m} x_i y_i ,$$

$$\mathbb{J} x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x ,$$

for $x = \begin{pmatrix} x_1 \\ \vdots \\ x_{2m} \end{pmatrix} \in \mathbb{R}^{2m}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_{2m} \end{pmatrix} \in \mathbb{R}^{2m}$.

All symplectic triples (β, σ, J) on $\tilde{F}$ are isomorphic to the standard triple. Namely, for a given triple (β, σ, J) on $\tilde{F}$, there exists a linear isomorphism T of $\mathbb{R}^{2m}$ to $\tilde{F}$, such that

$$\beta(Tx, Ty) = j(x, y),$$

$$\sigma(Tx, Ty) = 1(x, y),$$

$$\mathbb{J}(Tx) = T(\mathbb{J}x).$$

In particular, the triple $(-j, 1, -\mathbb{J})$ is isomorphic to $(j, 1, \mathbb{J})$; i.e., there is a matrix $T \in GL(\mathbb{R}^{2m})$ such that

$$^{t}TjT = -j, \qquad \text{where} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$^{t}TT = 1_{2m},$$

$$^{t}T\mathbb{J}T = -\mathbb{J} \qquad \text{where} \quad \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a given symplectic triple $t = (\beta, \sigma, J)$, define

$$G_t = Sp(\tilde{F}, \beta) = \{ g \in GL(\tilde{F}) \mid \beta(gx, gy) = \beta(x, y) \},$$

$$SO_t = SO(\tilde{F}, \sigma) = \{ g \in SL(\tilde{F}) \mid \sigma(gx, gy) = \sigma(x, y) \},$$

and

$$K_t = G_t \cap SO_t.$$

Then G_t is a real Lie group isomorphic to $Sp(2m,\mathbb{R})$ and K_t is a maximal compact subgroup. Furthermore, in the homogeneous space $X_t = G_{t}/_{K_t}$, we shall define a G_t-invariant complex structure J_t in the following way.

The Lie algebra $\underline{g}_t$ of G_t is identified with the Lie algebra

$$\underline{g}_t = \left\{ M \in \mathrm{End}_{\mathbb{R}}(\tilde{F}) \;\middle|\; \beta(Mx,y) + \beta(x,My) = 0 \right\},$$

and the Lie algebra $\underline{k}_t$ of K_t is

$$\underline{k}_t = \left\{ M \in \underline{g}_t \;\middle|\; \sigma(Mx,y) + \sigma(x,My) = 0 \right\}.$$

Let

$$\underline{p}_t = \left\{ M \in \underline{g}_t \;\middle|\; \sigma(Mx,y) - \sigma(x,My) = 0 \right\}.$$

Then $\underline{g}_t = \underline{k}_t \oplus \underline{p}_t$ is a Cartan decomposition of $\underline{g}_t$.

The natural projection map $\nu = \nu_t$ from G_t to $X_t = G_{t/K_t}$ induces an isomorphism $\nu_*|\underline{p}_t$ of $\underline{p}_t$ to $T_o(X_t)$, where $o = \nu(1)$.

The third member J of our triple $t = (\beta,\sigma,J)$ is a linear endomorphism of $\tilde{F}$, moreover, $J \in \underline{g}_t$, this can be shown in the following manner:

$$\beta(Jx,y) \underset{(1)}{=} -\beta(Jx,-y)$$

$$\underset{(2)}{=} -\beta(Jx, JJy) \qquad \text{(since } J^2 = -1 \text{)}$$

$$\underset{(3)}{=} -\sigma(Jx, Jy) \qquad \text{(by (S-4))}$$

$$\underset{(4)}{=} -\sigma(Jy, Jx) \qquad \text{(since } \sigma \text{ is symmetric)}$$

$$= \beta(Jy, x) \qquad \text{(by the same way as (3),(2), (1))}$$

$$= -\beta(x, Jy) \qquad \text{(since } \beta \text{ is alternating);}$$

so, we obtain

(§) $\beta(Jx, y) + \beta(x, Jy) = 0,$

hence $J \in \underset{\sim}{g}_t$.

Moreover, $J \in \underset{\sim}{k}_t$. Again this can be verified as follows.

$$\sigma(Jx, y) = \beta(Jx, Jy) \qquad \text{(by (S-4))}$$

$$= -\beta(x, JJy) \qquad \text{(by (§))}$$

$$= -\sigma(x, Jy) \qquad \text{(by (S-4))}$$

therefore,

(§§) $\sigma(Jx, y) + \sigma(x, Jy) = 0,$

hence $J \in \underset{\sim}{k}_t$.

Finally J belongs to the center of $\underset{\sim}{k}_t$. To show this, take an arbitrary $M \in \underset{\sim}{k}_t$. Then

$$\beta(MJx,y) = -\beta(Jx,My) = \beta(x,JMy) = \sigma(x,My)$$
$$= -\sigma(Mx,y) = -\sigma(JMx,Jy)$$
$$= +\beta(JMx,y),$$

therefore,

$$\beta((MJ - JM)x,y) = 0$$

for every $x,y \in \tilde{F}$. Since β is non-degenerate, this means

$$(MJ - JM)x = 0$$

for all x, i.e., $MJ - JM = 0$, hence $[M,J] = 0$ for all $M \in \underline{k}_t$. QED.

Now put

$$j_t = \exp(\pi/4\ J) \in K_t.$$

Since $Ad(K_t)\underline{p}_t = \underline{p}_t$, $Ad(j_t)\underline{p}_t = \underline{p}_t$. Also,

$$[Ad(j_t)|_{\underline{p}_t}](Z) = JZ$$

for $Z \in \underline{p}_t$. To prove this, we need several lemmas.

Lemma (1.2.1): For $Z \in \underline{p}_t$, $JZ + ZJ = 0$.

Proof: $\beta(ZJx,y) = -\beta(Jx,Zy) = \beta(x,JZy) = \sigma(x,Zy)$

$$= \sigma(Zx,y) = \beta(Zx,Jy) = -\beta(JZx,y)$$

for $J \in \underline{g}_t$, therefore,

$$\beta((ZJ + JZ)x,y) = 0$$

for all x,y; hence, $ZJ + JZ = 0$. QED

Lemma (1.2.2): For $Z \in \underline{p}_t$, we have

$$(\text{ad } J)(Z) = [J,Z] = JZ - ZJ = 2JZ.$$

Proof: $JZ - ZJ = -(JZ + ZJ) + 2JZ = 0 + 2JZ.$ QED

Lemma (1.2.3): For $Z \in \underline{p}_t$, we have

$$(\text{ad } J)^n(Z) = 2^n J^n Z.$$

Proof:

$$\begin{aligned}(\text{ad } J)^2 Z &= [J,[J,Z]] = [J, 2JZ] \\ &= 2J^2Z - (-2JZJ) \\ &= 2J^2Z + 2J^2Z \\ &= 4J^2Z,\end{aligned}$$

$$\begin{aligned}(\text{ad } J)^3 Z &= [J, 4J^2Z] \\ &= 4J^3Z - (4J^2Z)J \\ &= 4J^3Z - ((-4J^3Z)) \qquad (ZJ = -JZ) \\ &= 8J^3Z,\end{aligned}$$

...

$$(\text{ad } J)^n(Z) = 2^n J^n Z.$$ QED

<u>Lemma</u> (1.2.4): For $Z \in \underline{p}_t$,

$$\text{Ad}\,(\exp(\Theta J))(Z) = (\cos(2\Theta)1 - \sin(2\Theta)J)(Z).$$

<u>Proof</u>:

$$(\text{Ad}\,(\exp(\Theta J))(Z) = (\exp\,(\text{ad}(\Theta J)))(Z)$$

$$= (1 + \Theta(\text{ad } J) + \Theta^2/_{2!}(\text{ad } J)^2 + \ldots)(Z)$$

$$= \sum_{n=o} \frac{\Theta^n}{n!} (\text{ad } J)^n (Z)$$

$$= \sum_{n=o} \frac{\Theta^n}{n!} (\ 2^n) J^n (Z)$$

$$= (\sum_{n=1} \frac{(\ 2\Theta)^n}{n!} J^n)(Z)$$

$$= (1 + ((\ 2\Theta)J) - \frac{(\ 2\Theta)^2}{2!} 1 - \frac{(\ 2\Theta)^3}{3!} J + \frac{(\ 2\Theta)^4}{4!} 1$$

$$- \frac{(\ 2\Theta)^5}{5!} J - \frac{(\ 2\Theta)^6}{6!} 1 - \ldots)(Z)$$

$$= (1 - \frac{(\ 2\Theta)^2}{2!} + \frac{(\ 2\Theta)^4}{4!} - \ldots)1(Z)$$

$$+ ((\ 2\Theta) - \frac{(\ 2\Theta)^3}{3!} + \frac{(\ 2\Theta)^5}{5!} - \ldots)J(Z)$$

$$= (\cos(\ 2\Theta)1 + \sin(\ 2\Theta)J)(Z)$$

$$= (\cos(2\Theta)1 + \sin(2\Theta)J(Z).$$

Using the facts $J^2 = -1$, $J^3 = -J$, $J^4 = 1$, etc. QED

Lemma (1.2.5): For $Z \in \underline{p}_t \subset \mathrm{End}(\tilde{F})$, we have

$$\mathrm{ad}(j_t)(Z) = JZ.$$

Proof:

$$\mathrm{ad}(j_t)(Z) = \mathrm{Ad}(\exp(-\pi/_4\ J))(Z)$$

$$= (\cos(2\tfrac{\pi}{4}) + \sin(2\tfrac{\pi}{4}))J(Z)$$

$$= JZ.$$

QED

Therefore $\mathrm{ad}(j_t)$ is a complex structure on $\underline{p}_t$. Since j_t belongs to the center of K_t, $\mathrm{ad}(j_t)$ is an $\mathrm{Ad}(K_t)$-invariant complex structure on $\underline{p}_t$, and thus induces a G_t-invariant complex structure on $X_t = G_{t/K_t}$. From now on, for a given symplectic triple $t = (\beta, \sigma, t)$ the homogeneous space X_t is always understood as the complex manifold with respect to this complex structure J_t, defined by $\mathrm{Ad}(j_t)$ as above. In particular, $\mathrm{Ad}(\exp(\pi/_4\ J))$ induces an $Sp(J)$-invariant complex structure on $X_1 = Sp(J)_{/O(1) \cap Sp(J)}$, and with this complex structure X_1 is holomorphically equivalent to $\mathfrak{h}^m = \{Z \in M(m,\mathbb{C}) : {}^tZ = Z$ and $\mathrm{Im}\, Z > 0\}$. The equivalence is induced by the mapping

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \ni Sp(J) \xrightarrow{\ \Phi\ } \frac{Ai + B}{Ci + D} \in \mathfrak{h}^m .$$

Therefore the map $\mathfrak{F}_T : X_t \longrightarrow \mathfrak{h}^m$ in the diagram

$$\begin{array}{ccccc} G_t & \longrightarrow & Sp(J) & \searrow & \\ \downarrow & & \downarrow & & \\ X_t & \longrightarrow & X_1 & \longrightarrow & \mathfrak{h}^m \end{array} \qquad (X_t \xrightarrow{\ \mathfrak{F}_T\ } \mathfrak{h}^m)$$

is biholomorphic.

1.3 Complex structures on V and U

A torus $_L\backslash\tilde{F}$ is a complex torus, when $\tilde{F}$ has a structure of the complex linear space with a complex structure J. The complex torus $(_L\backslash\tilde{F}, J)$ is an abelian variety, if and only if, there is a real-valued bilinear form $\beta(-,-)$ on $\tilde{F}\times\tilde{F}$ satisfying

(i) $\beta(u,v) = -\beta(v,u)$; i.e., β is an alternating form,

(ii) $\beta(u,Jv)$ is a positive definite symmetric bilinear form of u,v.

(iii) $\beta(u,v)$ takes integral values of $L\times L$.

(Such a bilinear form β is called a polarization of $F = {}_L\backslash\tilde{F}$.)

Following Kuga, we make the following assumption, which he called the Integrality Assumption 2 in [6], (Chap. II, §5). Let $\{G, K, X, \tilde{F}, \rho, L, \Gamma\}$ be the datas chosen as in 1.1.

Integrality Assumption 2: There exists a non-degenerate alternating bilinear form $\beta(u,v)$ on $\tilde{F}\times\tilde{F}$, such that

(i) $\beta(\rho(g)u,\rho(g)v) = \beta(u,v)$ for all $u,v \in \tilde{F}$ and all $g \in G$,

(ii) $\beta(u,v) \in \mathbb{Z}$ if $(u,v) \in L\times L$.

Note: This form β is represented by an integral skew-symmetric matrix

$$B = (\beta(Z_i,Z_j))$$

with respect to the canonical basis $(Z_1, \ldots, Z_{2m})$ of $\tilde{F}$. The matrix B satisfies obviously the following conditions:

(1) ${}^t\rho(g)B\rho(g) = B$ for all $g \in G$,

(2) B is integral,

(3) ${}^tB = -B$,

(4) $\det B \neq 0$, i.e., B is the matrix discussed in [6].

By a Lemma of Kuga ([6], Lemma I-4-1), under the Integrality Assumption 2, the bilinear form β is extendable to a symplectic triple $t = (\beta,\sigma,J)$ such that

(5) $\rho(K) \subset K_t = Sp(\tilde{F},\beta) \cap SO(\tilde{F},\sigma)$,

such a triple $t = (\beta,\sigma,J)$ will be called a very admissible symplectic triple with respect to $\{G, K, \rho, \beta\}$.

Now take a very admissible symplectic triple $t = (\beta, \sigma,J)$ with respect to the system $\{G, K, \rho, \beta\}$, and define $GL(\tilde{F})$-valued function

(6) $J(x) = J(g) = -\rho(g)J\rho(g)^{-1}$

for $x = \nu(g)$. It is easy to see $J(gk) = J(g)$ for $k \in K$. So, $J(g)$ is actually a function of $x = \nu(g)$. Then

(7) $J(g(x)) = \rho(g)J(x)\rho(g)^{-1}$ for all $x \in X$ and all $g \in G$,

(8) $J(\gamma(x)) = \rho(\gamma)J(x)\rho(\gamma)^{-1}$ for all $x \in X$ and all $\gamma \in \Gamma$,

(9) $J(x)^2 = -1$.

Note that $J(1) = -\rho(1)J\rho(1)^{-1} = -J$ is not the third member J of $t = (\beta,\sigma,J)$; it is the negative $-J$ of J. Also define symmetric bilinear forms $A(x) = A(g)$ on $\tilde{F}\times\tilde{F}$ parametrized by $x = \nu(g) \in X$, by $A(x)(u,v) = \beta(u,J(x)u)$. Then $(-\beta,A(x), J(x))$ is a symplectic triple for any $x \in X$.

Property (9) asserts that for a fixed $x \in X$, $J(x)$ defines a complex structure on the fiber $\tilde{F}_x$ in $X\times\tilde{F}$, and on the torus $\tilde{F}_x$ in $X\times F$ (with $F = {}_L\backslash\tilde{F}$), therefore $(\tilde{F}_x$,

$J(x))$ is a complex torus. Since (8) and $\rho(\gamma): L \longrightarrow L$, $J(x)$ and $J(\gamma(x))$ give the same complex structure on the fiber F in V. Furthermore, by means of Integrality Assumption 2, F is also an abelian variety.

The Integrality Assumption 2 asserts that the representation ρ is a homomorphism of G into $G_t = Sp(\tilde{F},\beta)$, namely

$$\rho(G) \subset G_t = Sp(\tilde{F},\beta).$$

The admissibility of $t = (\beta,\sigma,J)$ for $\{G, K, \rho, \beta\}$ implies that ρ sends the compact group K into the compact group $K_t = Sp(\tilde{F},\beta) \cap O(\sigma)$;

$$\rho(K) \subset K_t = Sp(\tilde{F},\beta) \cap O(\sigma).$$

Hence the homomorphism ρ induces a mapping τ of the quotient space $X = G_{/K}$ into the quotient space $X_t = G_{t/K_t}$, which makes the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & G_t = Sp(\tilde{F},\beta) \\ \nu \downarrow & & \downarrow \nu_t \\ X & \xrightarrow{\tau} & X_t = G_{t/K_t}. \end{array}$$

The induced mapping τ will be called an Eichler map.

We shall consider the case where $X = G_{/K}$ is a bounded symmetric domain, it possesses then a G-invariant complex structure J_X. The quotient space $U = \Gamma\backslash X$ maintains naturally the structure of the complex analytic manifold with the induced complex structure J_U induced from J_X. According to Kodaira, (see e.g., [14]) U is a Hodge variety with respect to a Kähler metric ds_o^2, where ds_o^2 is a hermitian metric of U induced from a G-invariant metric ds_o^2 of X, therefore, by a Theorem of Kodaira, U is a non-singular projective algebraic variety.

Finally, we want to define the complex structure of the total space V. We begin with the following

Definition: A complex structure J_V of V is admissible if it satisfies the following conditions:

(A-1) the underlying real analytic structure of J_V coincides with the one V already possesses,

(A-2) the projection map $\pi: V \longrightarrow U$ is holomorphic with respect to J_V and J_U,

(A-3) the restriction of J_V on each fiber F_Q coincides with the J_Q (and this coincides with the complex structure defined by $J(x) = \rho(g)(-J)\rho(g)^{-1}$ for $x = \nu(g)$.)

Assume that (V, J_V) be a complex manifold with an admissible complex structure J_V. Lift this complex structure

J_V of V to the complex structure $\tilde{J}$ of $X \times \tilde{F}$. Thus, $X \times \tilde{F}$ is a complex-analytic manifold with the complex structure $\tilde{J}$; — we denote this by $(X \times \tilde{F}, \tilde{J})$ — and $\tilde{p}: X \times \tilde{F} \longrightarrow V$ is holomorphic; and operation of every element (γ, a) in $\Gamma \times L$ on $X \times \tilde{F}$ is biholomorphic. The condition (A-2) implies that the projection $\tilde{\pi}_2: X \times \tilde{F} \longrightarrow X$ is holomorphic, so that every fiber $\tilde{\pi}_2^{-1}(x) = \{x\} \times \tilde{F} = \tilde{F}_x$ is a complex submanifold. The submanifold $\tilde{F}_x$ has a structure of complex linear space of which complex structure is given by

$$J(x) = \rho(g)(-J)\rho(g)^{-1}.$$

The complex structure $\tilde{J}|_{\tilde{F}_x}$ of the complex submanifold $\tilde{F}_x$ coincides with $J(x)$, a fact which is implied by (A-3).

On the complex structure J_V (or $\tilde{J}$) we assume

(A-4) (i) For any two holomorphic local sections

$$s_1 : \mathcal{U} \longrightarrow X \times \tilde{F},$$
$$s_2 : \mathcal{U} \longrightarrow X \times \tilde{F}$$

defined on the same open set $\mathcal{U}$ in X, the mapping

$s_1 + s_2 : \mathcal{U} \ni x \longrightarrow s_1(x) + s_2(x)$ is again a holomorphic local section.

(A-4) (ii) For any holomorphic local section

$$s : \mathcal{U} \longrightarrow X \times \tilde{F},$$

the mapping $\mathcal{U}\times\mathbb{C} \ni (x,\zeta) \longrightarrow \zeta\cdot s(x) \in \tilde{F}_x \subset \tilde{\pi}_2^{-1}(\mathcal{U})$ of $\mathcal{U}\times\mathbb{C}$ into $X\times\tilde{F}$ is holomorphic.

If an admissible J_V (or $\tilde{J}$) satisfies (A-4) (i), (ii), then J_V (or $\tilde{J}$) will be called very admissible. Very admissibility implies that the section $\tilde{\Phi} : X \ni x \longrightarrow (x,o) \in X\times\tilde{F}$ is holomorphic, so that the section of origin $\Phi : U \longrightarrow V$ must be holomorphic. In [6], Kuga proved that V has a very admissible complex structure, if and only if, the Eichler mapping $\tau: X \longrightarrow X_t$ is holomorphic, and if J_V exists, then it is unique ([6], Theorem II-6-3).

If the Eichler map τ is holomorphic, denote the unique very admissible complex structure of V by J_V. Then the Riemannian metric

$$(10) \quad ds^2 = ds_o^2 + A(x)(d\xi(u), d\xi(u))$$

is a Kähler metric with respect to J_V; here ds_o^2 is the Kähler metric of $U = \Gamma\backslash X$, and

$$\xi(u) = \begin{pmatrix} \xi^1(u) \\ \cdot \\ \cdot \\ \cdot \\ \xi^{2m}(u) \end{pmatrix}$$

are the real coordinates on $\tilde{F} = \mathbb{R}^{2m}$ with respect to the standard basis

$$Z_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} , \ldots , Z_{2m} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

of $\mathbb{R}^{2m}$. The complex manifold V with the complex structure J_V is a Hodge variety with respect to ds^2. Hence by a Theorem of Kodaira, V is biholomorphically isomorphic to a non-singular projective algebraic variety.

1.4 The case of a product of upper half-planes

In this section, we discuss an example in which X is the product of N-copies of the upper half-plane $H = \{ z = x+iy \in \mathbb{C} \ \ \mathrm{Im}\, z > 0 \}$.

Let G be the product of N-copies of $SL(2,\mathbb{R})$:

$$G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times \ldots \times SL(2,\mathbb{R}) \qquad (N\text{-times}).$$

The i-th component of the product will be denoted by G_i; so

$$G = G_1 \times G_2 \times \ldots \times G_N$$

$$G_i = SL(2,\mathbb{R}).$$

This product G operates on the product

$$X = H \times H \times \dots \times H$$

of the N-copies of the upper half-plane $H = \{ z = x+iy,$ $\operatorname{Im} z > 0 \}$, by the fractional linear transformations

$$z \longmapsto \frac{az + b}{cz + d},$$

applied component-wise, i.e., the action of $g = (g_1, g_2, \dots, g_N) \in G$ on $z = (z_1, z_2, \dots, z_N) \in X$ is $g(z) = (g_1(z_1), g_2(z_2), \dots, g_N(z_N))$; where

$$g_i(z_i) = \frac{a_i z_i + b_i}{c_i z_i + d_i}, \qquad \text{for} \quad g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2,\mathbb{R}).$$

This action is transitive, and the isotropy group K at the point $I = (i, i, \dots, i)$ is the compact subgroup $K = SO(2) \times \dots \times SO(2)$. Therefore, X is identified with G/K:

$$g(i) \longleftrightarrow gK.$$

Consider a discrete subgroup Γ in G, without finite subgroup except $\{1\}$, and $\Gamma\backslash G$ is compact. Then Γ acts on X properly discontinuously, and the quotient space $U = \Gamma\backslash X$ is a compact manifold. Moreover, since the actions of elements g of G on X are bi-holomorphic, U becomes a complex manifold, with induced complex structure from X; and actually U is embeddable in a projective space, and is an projective algebraic variety.

We are going to construct families of abelian varieties $V \xrightarrow{\pi} U$ over such $U = \Gamma\backslash X$ with a good choice of Γ. For this purpose, we have to discuss representations of our group G.

The trivial representation of $SL(2,\mathbb{R})$ will be denoted by $\mathbb{1}$. Since $SL(2,\mathbb{R})$ is a group of matrix, the identity mapping id which sends $g \in SL(2,\mathbb{R})$ to itself g is a matrix representation, which we denote by id. The representation space $\mathcal{V}_{id}$ of id is $\mathbb{R}^2$. The symmetric tensor representation of $SL(2,\mathbb{R})$ of the degree m is denoted by $\gamma^{(m)}$; which is defined as follows.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, define a matrix $\gamma^{(m)}(g) \in SL(m+1,\mathbb{R})$, by

$$\begin{pmatrix} u_1^m \\ u_1^{m-1} v_1 \\ u_1^{m-2} v_1^2 \\ \cdot \\ \cdot \\ \cdot \\ v_1^m \end{pmatrix} = \gamma^{(m)}(g) \begin{pmatrix} u^m \\ u^{m-1} v \\ u^{m-2} v^2 \\ \cdot \\ \cdot \\ \cdot \\ v^m \end{pmatrix},$$

where u, v are variables and u_1, v_1 are another pair of variables related with u, v by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then the mapping $\gamma^{(m)} : g \longrightarrow \gamma^{(m)}(g)$ of $SL(2,\mathbb{R})$ to $SL(m+1,\mathbb{R})$ is obviously a representation; this is called the symmetric tensor representation of the degree m. The representation space $V_{\gamma^{(m)}}$ of $\gamma^{(m)}$ is $\mathbb{R}^{m+1}$, which we also denote by $V^{(m)}$. Obviously, $\gamma^{(0)} = 1$, $\gamma^{(1)} = \mathrm{id}$. $\gamma^{(m)}$ for $m = 0,1,2,\ldots$ are all irreducible, and they exhaust all irreducible (continuous) representation of $SL(2,\mathbb{R})$. The representation $\mathrm{id} = (V^{(1)}, \gamma^{(1)})$ is particularly important in this investigation. Denote the symmetric matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by $S_{\sqrt{-1}}$, which is defined on $V^{(1)} \cong \mathbb{R}^2$, the symmetric bilinear form $S_{\sqrt{-1}}(x,y) = {}^t x S_{\sqrt{-1}} y = x_1 y_1 + x_2 y_2$, for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

The orthogonal group $SO(V^{(1)}, S_{\sqrt{-1}})$ is exactly $SO(2,\mathbb{R})$, which is the isotropy subgroup of the point $\sqrt{-1} \in H$, in $SL(2,\mathbb{R})$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, put $z = g(\sqrt{-1}) = \dfrac{(a\sqrt{-1} + b)}{(c\sqrt{-1} + d)}$, define S_z by

$$S_z = {}^t g^{-1} S_{\sqrt{-1}} g^{-1},$$

this is again a positive definite symmetric matrix. S_z is well defined, i.e., for two g_1, g_2 with $g_1(\sqrt{-1}) = g_2(\sqrt{-1})$, then ${}^t g_1{}^{-1} S_{\sqrt{-1}} g_1{}^{-1} = {}^t g_2{}^{-1} S_{\sqrt{-1}} g_2{}^{-1}$. The orthogonal group $SO(\mathcal{V}^{(1)}, S_z) = \{k \in SL(2,\mathbb{R}) \mid {}^t k S_z k = S_z\}$ is the isotropy group of the point $z \in H$.

The matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is denoted by $J_{\sqrt{-1}}$. Since $(J_{\sqrt{-1}})^2 = -1$, $J_{\sqrt{-1}}$ defines a complex linear structure on $\mathcal{V}^{(1)} = \mathbb{R}^2$; and $(\mathcal{V}^{(1)}, J_{\sqrt{-1}}) = \mathbb{C}$. The matrices k in $SL(2,\mathbb{R})$ which commutes with this $J_{\sqrt{-1}}$ form the subgroup $SO(2,\mathbb{R})$. For an arbitrary element $g \in SL(2,\mathbb{R})$, put $z = g(\sqrt{-1})$, and define J_z by

$$J_z = g J_{\sqrt{-1}} g^{-1}.$$

J_z is well defined, i.e., for $g_1, g_2 \in SL(2,\mathbb{R})$ with $g_1(\sqrt{-1}) = g_2(\sqrt{-1})$, then $g_1 J_{\sqrt{-1}} g_1{}^{-1} = g_2 J_{\sqrt{-1}} g_2{}^{-1}$. And the elements k of $SL(2,\mathbb{R})$ which commute with J_z form the orthogonal group

$SO(\mathcal{V}^{(1)}, S_z)$. J_z is also a complex linear structure of $\mathcal{V}^{(1)}$ $= \mathbb{R}^2$; $(\mathcal{V}^{(1)}, J_z) = \mathbb{C}$.

Let $G = G_1 \times G_2 \times \dots \times G_N = SL(2,\mathbb{R}) \times \dots \times SL(2,\mathbb{R})$ be our group. The projection of G to the i-th component G_i $= SL(2,\mathbb{R})$ will be denoted by $proj_i$. The representation $\gamma^{(m)}$ of $G_i = SL(2,\mathbb{R})$ combined with $proj_i$ is a representation of G which we denote by $\gamma_i^{(m)}$; i.e.,

$$\gamma_i^{(m)} = \gamma^{(m)} \circ proj_i : G \longrightarrow SL(m+1,\mathbb{R}).$$

The representation space of $\gamma_i^{(m)}$ is denoted by $\mathcal{V}_i^{(m)}$.

The representation $\gamma_1^{(n_1)} \otimes \gamma_2^{(n_2)} \otimes \dots \otimes \gamma_N^{(n_N)}$ with the representation space $\mathcal{V}_1^{(n_1)} \otimes \mathcal{V}_2^{(n_2)} \otimes \dots \otimes \mathcal{V}_N^{(n_N)}$ is simply denoted as $\gamma^{(n_1, n_2, \dots, n_N)}$ and $\mathcal{V}^{(n_1, n_2, \dots, n_N)}$ respectively. Obviously

$$\gamma_i^{(n)} = \gamma^{(0,\dots,\underset{\text{i-th place}}{n},0,\dots,0)}.$$

It is known that all irreducible (continuous) representation of G are $\gamma^{(n_1, n_2, \dots, n_N)}$, $(n_i = 0,1,2,\dots)$.

The Satake's list of admissible representations, i.e., those give rise to a family of abelian varieties, tells us that the admissible representations of our $G = SL(2,\mathbb{R}) \times \dots \times SL(2,\mathbb{R})$ are only sums of trivial representations and $\gamma_i^{(1)}$ $(i = 1,\dots,N)$ with multiplicities. Namely, if $(\tilde{F},P)$ is an admissible representation of $G = SL(2,\mathbb{R}) \times \dots \times SL(2,\mathbb{R})$, then

$$\tilde{F} = m_1 V_1^{(1)} \oplus m_2 V_2^{(1)} \oplus \dots \oplus m_N V_N^{(1)} \oplus m(1) ,$$

$$P = m_1 \gamma_1^{(1)} \oplus m_2 \gamma_2^{(1)} \oplus \dots \oplus m_N \gamma_N^{(1)} \oplus m(1) .$$

From now on, we shall discuss only the case without trivial factor; i.e., the case with $m = 0$. So, let

$$P = m_1 \gamma_1^{(1)} \oplus m_2 \gamma_2^{(1)} \oplus \dots \oplus m_N \gamma_N^{(1)} ,$$

$$\begin{aligned} F &= m_1 V_1^{(1)} \oplus m_2 V_2^{(1)} \oplus \dots \oplus m_N V_N^{(1)} \\ &= (V_1^{(1)} \underbrace{\oplus \dots \oplus}_{m_1} V_1^{(1)}) \oplus (V_2^{(1)} \underbrace{\oplus \dots \oplus}_{m_2} V_2^{(1)}) \oplus \dots \\ &\oplus (V_N^{(1)} \underbrace{\oplus \dots \oplus}_{m_N} V_N^{(1)}) . \end{aligned}$$

For a point $z = (z_1,z_2,\dots,z_N) \in X = H^N$, define a complex linear structure J_z on $\tilde{F}$, as follows:

$$J_Z = (J_{z_1} \oplus \dots \oplus J_{z_1}) \oplus (J_{z_2} \oplus \dots \oplus J_{z_2}) \oplus \dots \oplus$$

$$(J_{z_N} \oplus \dots \oplus J_{z_N}).$$

Then $(\tilde{F}, J_Z)$ is a $\mathbb{C}$-linear space of $\mathbb{C}$-dimension $\Sigma_{i=1}^{N} m_i$: $(\tilde{F}, J_Z) = \mathbb{C}^{(\Sigma_{i=1}^{N} m_i)}$.

Now, for the discontinuous group Γ, if there is a lattice $L \subset \tilde{F}$, such that $\rho(\Gamma)L = L$ and a bilinear form B on $\tilde{F}$, satisfying

(i) $B(L,L) \subset \mathbb{Z}$,

(ii) $B(x, J_Z y)$ is symmetric and positive definite,

(iii) $B(\rho(g)x, \rho(g)y) = B(x,y)$ for every $g \in G$;

then we can construct a family of abelian varieties $V \xrightarrow{\pi} U = \Gamma \backslash X$.

1.5 Case of quaternion algebra

This section is the continuation of 1.4, and we are going to discuss cases in which the lattice L and the bilinear form B are actually constructable.

Let k be a totally real algebraic number field of degree N, i.e., $[k:Q] = N$; and let $\mathfrak{B}$ be a totally indefinitely quaternion algebra with center k.

Take a (maximal) order $\mathfrak{O}$ in $\mathfrak{B}$, and the group $\Gamma(\mathfrak{O},1)$ of all units $\gamma\,(\in \mathfrak{O}^{\times})$ of $\mathfrak{O}$ with the reduced norm $\nu(\gamma) = 1$. Let Γ be a subgroup of $\Gamma(\mathfrak{O},1)$ with finite index and without finite subgroup except $\{1\}$.

Take an element β of $\mathfrak{B}$, such that

(β-1) $\beta \in \mathfrak{O}$,

(β-2) $\beta^{\iota} = -\beta$, where ι is the canonical involution of the quaternion algebra $\mathfrak{B}$,

(β-3) the reduced norm $\nu(\beta)$ of β is a totally positive element of k.

Make the bilinear form $f_{\beta}(x,y)$ on $\mathfrak{B}$ by

$$f_{\beta}(x,y) = \mathrm{tr}\,(x\beta y^{\iota}), \qquad \text{for } x,y \in \mathfrak{B},$$

where tr means the reduced trace of $\mathfrak{B}$. $f_{\beta}(x,y)$ is a bilinear form on $\mathfrak{B}_{/k}$, with values in k. $f_{\beta}(x,y)$ satisfies the following properties:

(f-β-1) $f_{\beta}(x,y)$ are integers in k, if $x,y \in \mathfrak{O}$,

(f-β-2) $f_{\beta}(y,x) = -f_{\beta}(x,y)$,

(f-β-3) f_{β} is non-degenerate,

(f-β-4) $f_{\beta}(\ x,\ y) = f_{\beta}(x,y)$ for all $\gamma \in \Gamma$.

(<u>Proof</u>: (1) if $x,y \in \mathfrak{O}$, then $x\beta y^{\iota} \in \mathfrak{O}$, so the value of tr is an integer in k.

(2) $f_\beta(y,x) = \mathrm{tr}\,(y\beta x\ ) = \mathrm{tr}\,((y\beta x^t)^t\ ) = \mathrm{tr}\,(x\beta^t y^t)$

$= \mathrm{tr}\,(x(-\beta)y^t) = -\mathrm{tr}\,(x\beta y^t) = -f_\beta(x,y)$,

since $\mathrm{tr}\,(z^t) = \mathrm{tr}\,(z)$.

(3) Since $\mathrm{tr}\,(xy)$ is non-degenerate and β is invertible.

(4) $f_\beta(\gamma x,\gamma y) = \mathrm{tr}\,(\gamma x\beta(\gamma y)^t\ ) = \mathrm{tr}\,(\gamma x\beta y^t\gamma^t)$

$= \mathrm{tr}\,(\gamma^t\gamma x\beta y^t\ ) = \mathrm{tr}\,(\nu(\gamma)x\beta y^t)$

$= \mathrm{tr}\,(x\beta y^t) = f_\beta(x,y)$,

since $\gamma^t\gamma = \nu(\gamma)$ = the reduced norm of $\gamma = 1$. QED)

We form a direct sum of m copies of the algebra $\mathfrak{B}$, denote it by $\tilde{F}$, i.e.,

$$\tilde{F} = \underbrace{\mathfrak{B} \oplus \ldots \oplus \mathfrak{B}}_{m} .$$

On $\tilde{F}$, the group Γ acts (on the left) through the left multiplication:

$$\left.\begin{array}{l}\Gamma \ni \gamma \\ \tilde{F} \ni (b_1,b_2,\ldots,b_m)\end{array}\right\} \longmapsto (\gamma b_1,\gamma b_2,\ldots,\gamma b_m) .$$

This action is denoted as ρ $(= \rho^{(m)})$,

(§) $\rho : \Gamma \longrightarrow GL(\tilde{F}/k)$.

The lattice

$$L = \mathfrak{O} \oplus \mathfrak{O} \oplus \cdots \oplus \mathfrak{O} \subset \tilde{F}$$

in $\tilde{F}$ is obviously Γ-invariant.

Define a (k-valued) bilinear form $f_\beta^{(m)}$ on the k-linear space $\tilde{F}$ by

$$f_\beta^{(m)}(x,y) = \sum_{i=1}^{m} f_\beta(x_i, y_i)$$

where

$$x = (x_1, x_2, \ldots, x_m) \in \tilde{F} = \mathfrak{B}^m,$$

$$y = (y_1, y_2, \ldots, y_m) \in \tilde{F} = \mathfrak{B}^m.$$

Then we have obviously

(f-m-β-1) $f_\beta^{(m)}(x,y)$ are integers in k, if $x, y \in L$,

(f-m-β-2) $f_\beta^{(m)}(y,x) = -f_\beta^{(m)}(x,y)$,

(f-m-β-3) $f_\beta^{(m)}$ is non-degenerate,

(f-m-β-4) $f_\beta^{(m)}(\rho(\gamma)x, \rho(\gamma)y) = f_\beta^{(m)}(x,y)$ for all $\gamma \in \Gamma$.

We denote the trace map of the field k to $\mathbb{Q}$ by $tr_{k/\mathbb{Q}}$; and let

$$F_\beta^{(m)}(x,y) = tr_{k/\mathbb{Q}}(f_\beta^{(m)}(x,y)).$$

Then

(F-m-β-0) $F_\beta^{(m)}$ is a $\mathbb{Q}$-bilinear form on the $\mathbb{Q}$-vector space $F/\mathbb{Q}$.

(F-m-β-1) $F_\beta^{(m)}(x,y) \in \mathbb{Z}$ if $x, y \in L$,

(F-m-β-2) $F_\beta^{(m)}(y,x) = -F_\beta^{(m)}(x,y)$,

(F-m-β-3) $F_\beta^{(m)}$ is non-degenerate,

(F-m-β-4) $F_\beta^{(m)}(\rho(\gamma)x, \rho(\gamma)y) = F_\beta^{(m)}(x,y)$ for all $\gamma \in \Gamma$.

Since $\beta^\iota = -\beta$, $\beta^2 = -\beta\beta^\iota = -\nu(\beta)$, let this element $\nu(\beta)$ be $d \in k$; by the condition (β-3) d is a totally positive element of k.

The totally indefinite quaternion algebra $\mathfrak{B}$ over the center k, which is totally real of the degree N, has N-distinct representation classes into $M_2(\mathbb{R})$. We denote them by $[\Psi_1], \ldots, [\Psi_n]$ and take a representation $\Psi_i : \mathfrak{B} \quad M_2(\mathbb{R})$, from each class $[\Psi_i]$, $(i = 1, \ldots, N)$. Other representations in the class $[\Psi_i]$ are

$$\left.\begin{array}{rl} A\Psi_i A^{-1} : \mathfrak{B} & \longrightarrow M_2(\mathbb{R}) \\ b & \longmapsto A\Psi_i(b)A^{-1} \end{array}\right\},$$

where $A \in GL_2(\mathbb{R})$. The restriction of Ψ_i to the center $k = k1$, is

$$k \ni a \xrightarrow{\Psi_i} \Psi_i(a1) = \begin{pmatrix} \varphi_i(a) & 0 \\ 0 & \varphi_i(a) \end{pmatrix} ,$$

where $\varphi_j : k \longrightarrow \mathbb{R}$ is an injection of k to $\mathbb{R}$. Moreover, φ_i is independent of the choice of Ψ_i from $[\Psi_i]$, and therefore determined by the class $[\Psi_i]$; and if $[\Psi_i] \neq [\Psi_j]$ then $\varphi_i \neq \varphi_j$, $(i, j = 1, 2, \ldots, N)$. So $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ are the set of all distinct injections of k into $\mathbb{R}$, and φ_i characterize $[\Psi_i]$.

Define an involution ι of $M_2(\mathbb{R})$, by

$$\iota \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} ,$$

then the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Psi_i} & M_2(\mathbb{R}) \\ \iota \downarrow & & \downarrow \iota \\ \mathcal{B} & \xrightarrow{\Psi_i} & M_2(\mathbb{R}) \end{array}$$

is commutative; and

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Psi_i} & M_2(\mathbb{R}) \\ \mathrm{tr} \downarrow & & \downarrow \mathrm{tr} \\ k & \xrightarrow{\varphi_i} & \mathbb{R} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Psi_i} & M_2(\mathbb{R}) \\ \nu \downarrow & & \downarrow \det \\ k & \xrightarrow{\varphi_i} & \mathbb{R} \end{array}$$

are also commutative.

Now the matrix $\Psi_i(\beta) \in M_2(\mathbb{R})$ has $tr\,(\Psi_i(\beta)) = \varphi_i(tr(\beta)) = \varphi_i(0) = 0$ and $\det\,(\Psi_i(\beta)) = \varphi_i(\nu(\beta)) = \varphi_i(d) > 0$. Put $\sqrt{\varphi_i(d)} = \lambda_i \in \mathbb{R}$, and consider the matrix

$$\Lambda_i = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix},$$

then

$$tr\,\Lambda_i = 0\,,$$

$$\det \Lambda_i = \lambda_i^{\,2} = \varphi_i(d)\,.$$

So both $\Psi_i(\beta)$ and Λ_i are both real semi-simple matrices with the same trace, and same determinant, i.e., with the same characteristic equation $X^2 + \varphi_i(d) = 0$; i.e., with the same characteristic roots $\pm\sqrt{-\varphi_i(d)}$. So there is a real matrix $A_i \in GL_2(\mathbb{R})$ such that

$$A_i \psi_i(\beta) A_i^{-1} = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}.$$

Now denote the representations $A_i \psi_i A_i^{-1}$ $(i = 1, \ldots, N)$ by ϕ_i $(\in [\psi_i])$. Then

$$\phi_i(a) = \varphi_i(a)1, \qquad \text{for} \quad a \in k,$$

$$\phi_i(\beta) = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$$

and the diagrams

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\phi_i} & M_2(\mathbb{R}) \\ \iota \big\downarrow & & \big\downarrow \iota \\ \mathfrak{B} & \xrightarrow{\phi_i} & M_2(\mathbb{R}) \end{array},$$

and

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\phi_i} & M_2(\mathbb{R}) \\ \text{tr} \big\downarrow \nu & \text{tr} & \big\downarrow \det \\ k & \xrightarrow{\varphi_i} & \mathbb{R} \end{array}$$

are commutative.

Now, since $\mathfrak{B}$ is totally indefinite, $\mathfrak{B} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $M_2(\mathbb{R}) \oplus \ldots \oplus M_2(\mathbb{R})$, (N-times). An identification I of $\mathfrak{B}\otimes_{\mathbb{Q}}\mathbb{R}$ with $M_2(\mathbb{R})^N$,

$$\mathfrak{B}\otimes_{\mathbb{Q}}\mathbb{R} \underset{I}{=} M_2(\mathbb{R}) \oplus \cdots \oplus M_2(\mathbb{R})$$

is defined in such a way that $\alpha \in \mathfrak{B}$ goes to:

$$I(\alpha \otimes 1) = (\phi_1(\alpha), \phi_2(\alpha), \ldots, \phi_N(\alpha));$$

we have

$$\begin{array}{ccccc} \mathfrak{B} & \longrightarrow & \mathfrak{B}\otimes_{\mathbb{Q}}\mathbb{R} & \xrightarrow{I} & M_2(\mathbb{R}) \oplus \cdots \oplus M_2(\mathbb{R}) \\ \cup\!\!| & & & & \\ \alpha & \longmapsto & & & (\phi_1(\alpha), \ldots, \phi_N(\alpha)). \end{array}$$

From now on, we identify $\mathfrak{B}\otimes_{\mathbb{Q}}\mathbb{R}$ with $M_2(\mathbb{R})^N$ by this identification I:

$$\mathfrak{B}\otimes_{\mathbb{Q}}\mathbb{R} \underset{I}{=} M_2(\mathbb{R}) \oplus \cdots \oplus M_2(\mathbb{R}) .$$

Define $\mathbb{R}$-bilinear forms on the $\mathbb{R}$-linear space $M_2(\mathbb{R})$ by

$$f_i(X,Y) = \mathrm{tr}\left(X \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix} Y^t\right) \qquad (i = 1,\ldots,N).$$

Then

(f-0) $\quad f_i(X,Y) = \lambda_i(-x_{12}y_{22} + x_{22}y_{12} - x_{11}y_{21} + x_{21}y_{11})$

for $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, $Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$,

(f-1) $f_i(X,Y) = -f_i(Y,X)$,

(f-2) f_i is non-degenerate,

(f-3) $f_i(gX,gY) = \det(g)\, f_i(X,Y)$ for $g \in GL_2(\mathbb{R})$,

(f-4) $f_i(X,J_z Y) = \lambda_i\, S_z\left[\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix},\begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix}\right] + S_z\left[\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix},\begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}\right]$

so this is symmetric and positive definite.

Proof: (f-0) can be proved by simple calculation, and (f-1) by (f-0).

(f-3): $f_i(gX,gY) = \operatorname{tr}\left(gX\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}(gY)^{\iota}\right)$

$$= \operatorname{tr}\left(gX\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix} Y^{\iota} g^{\iota}\right) = \operatorname{tr}\left(g^{\iota} gX\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix} Y^{\iota}\right)$$

$$= \det(g)\, \operatorname{tr}\left(X\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix} Y^{\iota}\right) = \det(g) f_i(X,Y),$$

since $g^{\iota} g = \det(g) 1_2$.

For the proof of (f-4), we need two lemmas.

Lemma (1.5.1): $\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y\right)^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y^t$.

Proof:

$$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y\right)^t = Y^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^t = Y^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -y_{12} & -y_{22} \\ y_{11} & y_{21} \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{pmatrix} = \begin{pmatrix} -y_{12} & -y_{22} \\ y_{11} & y_{21} \end{pmatrix}.$$

Lemma (1.5.2): $f_i(X, J_{\sqrt{-1}} Y) = \lambda_i \operatorname{tr}(XY^t)$

$$= \lambda_i \quad S_{\sqrt{-1}} \left[\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} \right] + S_{\sqrt{-1}} \left[\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}, \begin{pmatrix} y_{12} \\ y_{22} \end{pmatrix} \right] ,$$

where $S_{\sqrt{-1}}$ is the quadratic form defined in 1.4.

Proof:

$$f_i(X, J_{\sqrt{-1}} Y) = \operatorname{tr}\left(X \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y\right)^t)$$

$$= \operatorname{tr}\left(X \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y^t\right) = \lambda_i \operatorname{tr}(XY^t)$$

$$= \lambda_i \operatorname{tr}\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{pmatrix} \right)$$

$$= \lambda_i \operatorname{tr} \begin{pmatrix} x_{11}y_{11}+x_{12}y_{12} & * \\ * & x_{21}y_{21}+x_{22}y_{22} \end{pmatrix}$$

$$= \lambda_i [x_{11}y_{11} + x_{12}y_{12} + x_{21}y_{21} + x_{22}y_{22}]$$

$$= \lambda_i \left(S_{\sqrt{-1}} \left[\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} \right] + S_{\sqrt{-1}} \left[\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}, \begin{pmatrix} y_{12} \\ y_{22} \end{pmatrix} \right] \right).$$

So $f_i(X, J_{\sqrt{-1}}Y)$ is positive definite, symmetric bilinear form. In particular, $f_i(X,Y)$ are non-degenerate, this proves (f-2).

Let $J_z = gJ_{\sqrt{-1}}g^{-1}$, for $g \in SL(2,\mathbb{R})$. Then

$$\begin{aligned} f_i(X, J_zY) &= f_i(X, gJ_{\sqrt{-1}}g^{-1}Y) \\ &= f_i(gg^{-1}X, gJ_{\sqrt{-1}}g^{-1}Y) \\ &= \det g)\, f_i(g^{-1}X, J_{\sqrt{-1}}g^{-1}Y) \\ &= \lambda_i \operatorname{tr}(g^{-1}X(g^{-1}Y)^t) \end{aligned}$$

(by Lemma 1.5.2)

$$= \lambda_i \, (S_{\sqrt{-1}}(\text{1st column of } g^{-1}X), (\text{1st column of } g^{-1}Y))$$

$$+ S_{\sqrt{-1}}(\text{2nd column of } g^{-1}X), (\text{2nd column of } g^{-1}Y)))$$

$$= \lambda_i \, (S_z \, (\text{ist column of } X, \text{ 1st column of } Y)$$

$$+ S_z \, (\text{2nd column of } X, \text{ 2nd column of } Y))$$

$$= \lambda_i \, (S_z(\left[\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix}\right]) + S_z(\left[\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}, \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}\right]).$$

This concludes the proof of (f-4).

Let

(1) $\tilde{F} = \mathfrak{B}^m$.

Then by our previous identification, we have

(2) $\tilde{F} \otimes_{\mathbb{Q}} \mathbb{R} = (\mathfrak{B} \otimes_{\mathbb{Q}} \mathbb{R})^m = (M_2(\mathbb{R})^N)^m = M_2(\mathbb{R})^{Nm}$

(2') $= \underbrace{\underbrace{(M_2(\mathbb{R}) \oplus \ldots \oplus M_2(\mathbb{R}))}_{m} \oplus \ldots \oplus \underbrace{(M_2(\mathbb{R}) \oplus \ldots \oplus M_2(\mathbb{R}))}_{m}}_{N}$.

The image of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathfrak{B}^m$ in $M_2(\mathbb{R})^{Nm}$ under $\alpha \xmapsto{\ast \otimes 1} \alpha \otimes 1$ is then:

$$(3)\quad (\phi_1(\alpha), \ldots, \phi_N(\alpha)) \oplus \ldots \oplus (\phi_1(\alpha), \ldots, \phi_N(\alpha)).$$

m-times

Define a bilinear form $\mathfrak{f}(X,Y)$ on $M_2(\mathbb{R})^{Nm}$ by

$$(4)\quad \mathfrak{f} = (f_1 \oplus \ldots \oplus f_N) \oplus \ldots \oplus (f_1 \oplus \ldots \oplus f_N)$$

m-times

where f_i are those occurring in (f-0) to (f-4). Then the pull back of f under the injection

$$\tilde{F} \xrightarrow{\ *\otimes 1\ } M_2(\mathbb{R})^{Nm}$$

is the bilinear form $F_\beta^{(m)}$ occurring in (F-m-β-0) to (F-m-β-4); i.e.,

$$\mathfrak{f}(X\otimes 1, Y\otimes 1) = F_\beta^{(m)}(X,Y).$$

From now on, we denote this $\mathfrak{f}$ by $\mathfrak{f}_\beta^{(m)}$.

Identifying $M_2(\mathbb{R})$ with $\mathbb{R}^2 \oplus \mathbb{R}^2$ by

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \oplus \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix},$$

then, we have

$$(5)\quad M_2(\mathbb{R})^N = (\mathbb{R}^2 \oplus \mathbb{R}^2) \oplus (\mathbb{R}^2 \oplus \mathbb{R}^2) \oplus \ldots \oplus (\mathbb{R}^2 \oplus \mathbb{R}^2)$$

N-times

and

$$(6) \quad M_2(\mathbb{R})^{Nm} = \underbrace{M_2(\mathbb{R})^N \oplus \dots \oplus M_2(\mathbb{R})^N}_{m\text{-times}} .$$

Define a complex structure $J_z^{(1)}$ on $M_2(\mathbb{R})^N$ for $z = (z_1, z_2, \dots, z_N) \in X$, by

$$(7) \quad J_z^{(1)} = (J_{z_1} \oplus J_{z_1}) \oplus (J_{z_2} \oplus J_{z_2}) \oplus \dots \oplus (J_{z_N} \oplus J_{z_N})$$

according to the decomposition (5); and a complex structure $J_z^{(m)}$ on $M_2(\mathbb{R})^{Nm}$ by

$$(8) \quad J_z^{(m)} = J_z^{(1)} \oplus J_z^{(1)} \oplus \dots \oplus J_z^{(1)}$$

according to the decomposition (6). Therefore,

$$(9) \quad J_z^{(m)} = \underbrace{((J_{z_1} \oplus J_{z_1}) \oplus \dots \oplus (J_{z_N} \oplus J_{z_N})) \oplus \dots \oplus ((J_{z_1} \oplus J_{z_1}) \oplus \dots \oplus (J_{z_N} \oplus J_{z_N}))}_{m\text{-times}}$$

according to the decomposition (2').

Also define a positive definite quadratic form $S_{\beta,z}^{(1)}$ on $M_2(\mathbb{R})^N$ by

$$(10)\quad S_{\beta,z}^{(1)} = (\lambda_1 S_{z_1} \oplus \lambda_1 S_{z_1}) \oplus (\lambda_2 S_{z_2} \oplus \lambda_2 S_{z_2}) \oplus \cdots \oplus (\lambda_N S_{z_N} \oplus \lambda_N S_{z_N})$$

according to the decomposition (5), and define $S_{\beta,z}^{(m)}$ on $M_2(\mathbb{R})^{Nm}$ by

$$(11)\quad S_{\beta,z}^{(m)} = S_{\beta,z}^{(1)} \oplus \cdots \oplus S_{\beta,z}^{(1)}$$

according to the decomposition (6).

The representation $\rho^{(1)}$ of $G = SL(2,\mathbb{R})^N$ on $M_2(\mathbb{R})^N$ is defined by

$$(12)\quad \rho^{(1)} = (\gamma^{(1)}\circ \mathrm{proj}_1 \oplus \gamma^{(1)}\circ \mathrm{proj}_1) \oplus (\gamma^{(1)}\circ \mathrm{proj}_2 \oplus \gamma^{(1)}\circ \mathrm{proj}_2) \oplus \cdots \oplus (\gamma^{(1)}\circ \mathrm{proj}_N \oplus \gamma^{(1)}\circ \mathrm{proj}_N)$$

according to the decomposition (5). This can be written in matrix form

$$(13)\quad \rho^{(1)}(g) = \begin{pmatrix} g_1 & & & & & & 0 \\ & g_1 & & & & & \\ & & g_2 & & & & \\ & & & g_2 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & \cdot \end{pmatrix}$$

The representation $\rho^{(m)}$ of G on $M_2(\mathbb{R})^{Nm}$ is defined by

(14) $\rho^{(m)} = \underbrace{\rho^{(1)} \oplus \rho^{(1)} \oplus \dots \oplus \rho^{(1)}}_{m\text{-times}}$

according to the decomposition (6). Then the inclusion

(15) $\mathcal{B} \hookrightarrow \mathcal{B} \otimes_{\mathbb{Q}} \mathbb{R} = M_2(\mathbb{R})^N$

induces the inclusion

(16) $\Gamma \hookrightarrow SL(2,\mathbb{R})^N = G,$

and the representation ρ defined by (§), of Γ into $GL(\tilde{F}/k)$, and the $\rho^{(m)}$ of G on $GL(M_2(\mathbb{R})^{Nm}) = GL_{4mN}(\mathbb{R})$ is compatible. Here $\tilde{F}$ is identified with $\tilde{F} \otimes 1$ in $\tilde{F} \otimes_{\mathbb{Q}} \mathbb{R} = M_2(\mathbb{R})^{Nm}$.

Summarizing, we have

(I) $(J_z^{(m)})^2 = -1;$

(II) $\mathcal{f}_\beta^{(m)}(X, J_z^{(m)}Y) = S_{\beta,z}^{(m)}(X,Y);$

(III) $\mathcal{f}_\beta^{(m)}(X,Y) \in \mathbb{Z}$ if $X,Y \in L$; where $L \subset \tilde{F} = \tilde{F} \otimes 1 \subset M_2(\mathbb{R})^{Nm}$ is the lattice defined above;

(IV) $\mathcal{f}_\beta^{(m)}$ is alternating, and non-degenerate;

(V) $\mathcal{f}_\beta^{(m)}(\rho(g)X, \rho(g)Y) = \mathcal{f}_\beta^{(m)}(X,Y)$ for $g \in G$.

Then, from the following data

$$G = SL(2,\mathbb{R})^N$$

$$K = SO(2,\mathbb{R})^N$$

$$X = H \times \cdots \times H = H^N$$

$$\Gamma \subset \Gamma(\mathfrak{O},1) \subset G$$

$$\tilde{F} = M_2(\mathbb{R})^{Nm} = \tilde{F} \otimes_{\mathbb{Q}} \mathbb{R} = (\mathfrak{B}^m) \otimes \mathbb{R} = \mathbb{R}^{4Nm}$$

$$\rho = \rho^{(m)} : G \longrightarrow GL_{4Nm}(\mathbb{R})$$

$$L = \mathfrak{O}^m \subset \mathfrak{B}^m \subset \tilde{F}$$

$$\mathcal{S}_\beta^{(m)}$$

$$J_z^{(m)}$$

$$S_z^{(m)}$$

we define the family of abelian varieties $V_\beta^{(m)} \xrightarrow{\pi_\beta^{(m)}} U = \Gamma\backslash X$.

The purpose of this paper is to investigate algebraic cycles in the algebraic varieties $V_\beta^{(m)}$.

CHAPTER II. COHOMOLOGY GROUPS OF V

In 1.1, we have defined $V = {}_{\Gamma \times L}\backslash X \times \tilde{F}$, the quotient space of a contractible space $X \times \tilde{F}$ by a discontinuous subgroup $\Gamma \times L$ ($\subset G \times \tilde{F}$). Following Kuga and Matsushima-Murakami, we shall define the de Rham cohomology group of V.

2.1 The Matsushima-Murakami's theory

For the moment, we let G be a group of automorphisms of a bounded symmetric domain X, on which G acts naturally, and Γ be a discrete subgroup of G with compact quotient ${}_{\Gamma}\backslash G$, and Γ acts on X without fixed points; and let P be a representation of G in a complex vector space, say E. Let $A^r(\Gamma, X, P)$ be the complex vector space of all E-valued r-forms ω defined on X such that

(1) $\omega \circ L_\gamma = P(\gamma)\omega$

for all $\gamma \in \Gamma$, where L_γ denotes the transformations of X defined by γ, and $\omega \circ L_\gamma$ denotes the r-forms obtained by transforming ω by L_γ. The exterior differentiation d sends forms in $A^r(\Gamma, X, P)$ to $A^{r+1}(\Gamma, X, P)$. Therefore, d defines a coboundary operators of the graded module $A^*(\Gamma, X, P) = \sum_r A^r(\Gamma, X, P)$. The r-th cohomology group of the complex

$(A^*(\Gamma, X, P), d)$ is defined by

$$H^r(\Gamma, X, P) = \ker\{d: A^r(\Gamma, X, P) \to A^{r+1}(\Gamma, X, P)\} / dA^{r-1}(\Gamma, X, P) .$$

The cohomology group $H^r(\Gamma, X, P)$ is also definable as de Rham cohomology group with vector-bundle-valued differential forms.

Let M be a compact manifold, and $\mathbb{E} \xrightarrow{\pi} M$ is a locally constant vector bundle over M. The vector bundle $\mathbb{E} \to M$ is locally constant means that, $\mathbb{E}$ is defined by a system of transition functions $\{f_{UV}; \mathcal{U}\}$ with respect to an open covering $\mathcal{U}$ of M such that each

$$f_{UV} : U \cap V \longrightarrow GL_n(\mathbb{R})$$

is a constant mapping. Equivalently, $\mathbb{E}$ is a vector bundle associated to the universal covering $\tilde{M} \xrightarrow{\tilde{\omega}} M$ as its principal bundle. Then the de Rham cohomology group $H^r{}_{DR}(M, \mathbb{E})$ is defined as

$$H^r{}_{DR}(M, \mathbb{E}) = \frac{\ker\{d: \Gamma(M, \mathbb{E}\otimes\Lambda^r(T^*)) \to \Gamma(M, \mathbb{E}\otimes\Lambda^{r+1}(T^*))\}}{\operatorname{im}\{d: \Gamma(M, \mathbb{E}\otimes\Lambda^{r-1}(T^*)) \to \Gamma(M, \mathbb{E}\otimes\Lambda^r(T^*))\}} .$$

Here T^* is the cotangent bundle of M, $\Lambda^r(T^*)$ is the r-th exterior power of T^*, $\mathbb{E} \otimes \Lambda^r(T^*)$ is the tensor product as bundles over M, and $\Gamma(M, -)$ means the space of sections.

The exterior operator d means $1 \otimes d$ in $\mathbb{E} \otimes \Lambda^r(T^*)$; it is well defined, since $\mathbb{E}$ is locally constant.

Let $\mathcal{C}(\mathbb{E})$ be the sheaf of germs of locally constant sections of $\mathbb{E}$ over M. Then the cohomology $H^r(M, \mathcal{C}(\mathbb{E}))$ of M with coefficients in $\mathcal{C}(\mathbb{E})$ is canonically isomorphic to the de Rham cohomology $H^r_{DR}(M, \mathbb{E})$:

(2) $H^r(M, \mathcal{C}(\mathbb{E})) \cong H^r_{DR}(M, \mathbb{E})$.

To interpret our cohomology group $H^r(\Gamma, X, P)$ in terms of $H^r_{DR}(M, \mathbb{E})$, we construct a locally constant vector bundle $\mathbb{E}_P$ over $M = \Gamma\backslash X$ ($= U$ in our previous notation, in chap. I).

The representation space $\mathcal{V}_P$ of our representation P has been denoted by E. Construct the product space $X \times E$; and on which we let the group Γ acts as

$$\left.\begin{array}{r}\Gamma \ni \gamma \\ X \times E \ni (x, u)\end{array}\right\} \longmapsto (\gamma(x), P(\gamma)u) \in X \times E$$

The action of Γ on $X \times E$ is properly discontinuous, so let the quotient $\Gamma\backslash X \times E$ be denoted by $\mathbb{E}_P$. $\mathbb{E}_P$ with the natural mapping $\pi: \mathbb{E}_P \longrightarrow M$ making

$$\begin{array}{ccc} \Gamma\backslash X\times E = \mathbb{E}_P & \longleftarrow & X\times E \\ \pi \downarrow & & \downarrow \mathrm{proj}_1 \\ \Gamma\backslash X = M & \longleftarrow & X \end{array}$$

a commutative diagram, is a vector bundle over M, associated to the universal covering $X \longrightarrow M$.

The space of sections $\Gamma(M, \mathbb{E}_P \otimes \Lambda^r(T^*))$ is canonically isomorphic with $A^r(\Gamma, X, P)$, and therefore we have

(3) $H^r(\Gamma, X, P) \cong H^r_{DR}(M, \mathbb{E}_P)$.

Since X is diffeomorphic to an Euclidean space, then we have

(4) $H^r(\Gamma, X, P) \cong H^r(\Gamma; E)$,

where $H^r(\Gamma; E)$ is the r-th cohomology group of the abstract group Γ with coefficients in the Γ-module E. In summary,

(5) $H^r(\Gamma, X, P) \cong H^r_{DR}(M, \mathbb{E}_P) \cong H^r(M, \mathcal{C}(E_P)) \cong H^r(\Gamma; E)$.

2.2 The main result of Matsushima-Shimura

Let X be the product of N copies of the upper half-plane, i.e., $X = H \times \dots \times H$ (N copies) with $H = \{ z \in \mathbb{C};\ \mathrm{Im}\, z > 0 \}$. Let $G = SL(2,\mathbb{R}) \times \dots \times SL(2,\mathbb{R})$ (N copies), and let $K = SO(2,\mathbb{R}) \times \dots \times SO(2,\mathbb{R})$ (N copies). Let Γ be a discrete subgroup of G with compact quotient space $\Gamma\backslash G$, acting on X without fixed points. Finally, let P be a representation of G in a complex vector vector space $\mathbb{E}$. We retain our notations in 2.1.

The complex vector space of all $\mathbb{E}$-valued r-forms on X has been denoted by $A^r(\Gamma,X,P)$, and its r-th cohomology group by $H^r(\Gamma,X,P)$.

Now we define a hermitian inner product on the vector space $A^r(\Gamma,X,P)$. For this let $\underline{g}$ be the Lie algebra of G, let $\underline{k}$ be the Lie algebra of K, and let $\underline{p}$ be the orthogonal complement of $\underline{k}$ in $\underline{g}$ with respect to the Cartan decomposition.

Let $\mathbb{E}_P \longrightarrow X$ be the vector bundle associated to $X \longrightarrow \Gamma\backslash X$ by the representation $P|_\Gamma$ of Γ. By a lemma of Matsushima-Murakami [13], we know that E_P is also associated to the bundle $\Gamma\backslash G \longrightarrow \Gamma\backslash X$ by the representation $P|_K$ of K. Now E being a complex vector space, there exists a positive definite hermitian inner product $(u,v)_E$ on E such that

$$(P(Y)u,\ v)_E = -(u,\ P(Y)v)_E \qquad \text{for } Y \in \underline{k},$$

$$(P(Y)u,\ v)_E = (u,\ P(Y)v)_E \qquad \text{for } Y \in \underline{p}.$$

The representation P of $\underline{g}$ being extended over $\underline{g}_{\mathbb{C}}$, the complexified Lie algebra of $\underline{g}$, it follows therefore

$$(P(Y)u,\ v)_E = -(u,\ P(Y)v)_E \qquad \text{for } Y \in \underline{k}_{\mathbb{C}} = \underline{k} \otimes_{\mathbb{R}} \mathbb{C},$$

$$(P(Y)u,\ v)_E = (u,\ P(Y)v)_E \qquad \text{for } Y \in \underline{p}_{\mathbb{C}} = \underline{p} \otimes_{\mathbb{R}} \mathbb{C}.$$

The first condition is equivalent to say that $P(k)$ $(k \in K)$ are unitary operators with respect to the inner product $(u,v)_E$.

Therefore, this inner product defines hermitian metrics in the fibers of E_P. Using this and Riemannian metric g in $\Gamma\backslash X$, we may define successively inner product among E_P-valued forms.

With respect to this positive-definite inner product, we define the adjoint operator δ to d, and thus the Laplacian operator $\Delta = d\delta + \delta d$. A form $\omega \in A^r(\Gamma, X, P)$ is called harmonic, if $\Delta\omega = 0$; a result of Matsushima-Murakami [11] asserts that every cohomology class of $H^r(\Gamma, X, P)$ is representable by a unique form. Denote the space of all harmonic r-forms on $A^r(\Gamma, X, P)$ by $\mathcal{H}^r(\Gamma, X, P)$, then we have

(1) $H^r(\Gamma, X, P) \cong \mathcal{H}^r(\Gamma, X, P)$.

The G-invariant complex structure on X also gives rise to a bigraded complex $A^{p,q}(\Gamma,X,P)$, and the direct sum decomposition

$$(2) \quad A^r(\Gamma,X,P) = \sum_{p+q=r} A^{p,q}(\Gamma,X,P),$$

where $A^{p,q}(\Gamma,X,P)$ is the complex vector space of forms of type (p,q). And we have the corresponding decompositions for $H^r(\Gamma,X,P)$ and $\mathcal{H}^r(\Gamma,X,P)$

$$(3) \quad H^r(\Gamma,X,P) = \sum_{p+q=r} H^{p,q}(\Gamma,X,P),$$

and

$$(4) \quad \mathcal{H}^r(\Gamma,X,P) = \sum_{p+q=r} \mathcal{H}^{p,q}(\Gamma,X,P).$$

Denote by $b^r(\Gamma,X,P)$ the dimension of $\mathcal{H}^r(\Gamma,X,P)$, and $h^{p,q}(\Gamma,X,P)$ the dimension of $\mathcal{H}^{p,q}(\Gamma,X,P)$. Then by the decomposition of Hodge type, we have

$$(5) \quad \dim_{\mathbb{C}} H^r(\Gamma,X,P) = b^r(\Gamma,X,P) = \sum_{p+q=r} h^{p,q}(\Gamma,X,P).$$

If the representation P of G is reducible, then P can be written as the direct sum of the representations $P_1, \ldots,$ P_s, and the cohomology group $H^r(\Gamma, X, P)$ decomposes accordingly into direct sum of the cohomology groups $H^r(\Gamma, X, P_i)$, where P_i $(i = 1, \ldots, s)$ are the absolutely irreducible representations of G.

Now let $G = G_1 \times \ldots \times G_N$, with $G_i = SL(2,\mathbb{R})$ $(i = 1, \ldots, N)$. The representation of G in the vector space $\tilde{F} = \mathcal{V}_{m_1} \otimes \ldots \otimes \mathcal{V}_{m_N}$ is defined by

$$(6) \qquad \gamma^{(m_1 \ldots m_N)} = \gamma^{m_1} \otimes \gamma^{m_2} \otimes \ldots \otimes \gamma^{m_N}$$

with $m_i \geq 0$, where γ^{m_i} $(m_i \geq 1)$ denotes the representation of $SL(2,\mathbb{R})$ in the complex vector space $\mathcal{V}_{m_i}$ of all symmetric tensors of order m_i constructed over $\mathbb{C}$, as defined in 1.4. Then every absolutely irreducible representation of G is of the type $\gamma^{m_1 \ldots m_N}$.

Now we formulate the result of Matsushima-Shimura [14].

Theorem (Matsushima-Shimura) (2.2.1): Let us retain the notations above. Then we have

(i) $h^{p,q}(\Gamma, X, \gamma^{m_1 \cdots m_N}) = 0$ if $p \neq q$ and $p+q \neq N$.

(ii) If $\gamma^{m_1 \cdots m_N} = \gamma^{0,0,\ldots,0}$ = the trivial representation, then

$$h^{p,p}(\Gamma, X, \gamma^{m_1 \cdots m_N}) = \binom{N}{p} \quad \text{for} \quad 2p \neq N;$$

if $\gamma^{m_1 \cdots m_N} \neq \gamma^{0,0,\ldots,0}$, then

$$h^{p,p}(\Gamma, X, \gamma^{m_1 \cdots m_N}) = 0 \quad \text{for} \quad 2p \neq N$$

Proof: See Matsushima-Shimura ([14], page 445).

So, we have

Corollary (2.2.2):

$$\dim_R H^j(U, \mathbb{R}) = \dim_{\mathbb{R}} H^j(\Gamma, X, \text{trivial})$$

$$= \sum_{p+q=j} h^{(p,q)}(\Gamma, X, \gamma^{(0\ldots 0)})$$

$$= \begin{cases} 0 & \text{if } j = \text{odd}, \quad j \neq N, \\ \binom{N}{p} & \text{if } j = 2p \neq N. \end{cases}$$

Corollary (2.2.3): $H^{2p}(U,\mathbb{C}) = H^{p,p}(U,\mathbb{C})$ for $2p \neq N$.

Corollary (2.2.4): If $2p \neq N$, then $H^{2p}(U,\mathbb{C})$ is spanned by $\omega_{i_1} \wedge \ldots \wedge \omega_{i_p}$, where $\omega_i = (dx_i \wedge dy_i)/y^2$.

Since these are exactly $\binom{N}{p}$ harmonic differential forms $\omega_{i_1} \wedge \ldots \wedge \omega_{i_p}$, $i_1 < i_2 < \ldots < i_p$, and they are linearly independent.

Corollary (2.2.5): $H^r(\Gamma, X, \gamma^{m_1 \ldots m_N}) = 0$ for any $(m_1, \ldots, m_N)$ if r is odd and $r \neq N$.

Since our group G is $SL(2,\mathbb{R})^N$ without compact factor; any representation (V_p, P) of G is completely reducible (over $\mathbb{C}$) into a direct sum of absolutely irreducible representations $(\mathcal{V}^{(m_1 \ldots m_N)}, \gamma^{(m_1 \ldots m_N)})$:

$$(\mathcal{V}_p, P) = \oplus\, \mu_{(m_1 \ldots m_N)} (\mathcal{V}^{(m_1 \ldots m_N)}, \gamma^{(m_1 \ldots m_N)})$$

where $\mu_{(m_1 \ldots m_N)}$ are multiplicities. Among irreducible components, $\gamma^{(0 \ldots 0)}$ is trivial and all others are non-trivial. The part of $\mathcal{V}_p$, which corresponds to the trivial part $\mu_{(0 \ldots 0)} \gamma^{(0 \ldots 0)}$ is denoted by $\mathcal{V}_p^{\,G}$, and the restriction of P on $\mathcal{V}_p^{\,G}$, which is $\mu_{(0 \ldots 0)}$-fold of trivial representations is denoted by P^G. Similarly, the part of $\mathcal{V}_p$, corresponding to the sum of other component:

$$\sum_{(m_1 \ldots m_N) \neq (0 \ldots 0)} \mu_{(m_1 \ldots m_N)} \mathcal{V}^{(m_1 \ldots m_N)}$$

is denoted by $\mathcal{V}_p^{\,var}$, and will be called the variant part of $\mathcal{V}_p$; the restriction of P to $\mathcal{V}_p^{\,var}$ is denoted by P^{var}. Then

$$(\mathcal{V}_p, P) = (\mathcal{V}_p^{\,G}, P^G) \oplus (\mathcal{V}_p^{\,var}, P^{var}).$$

Taking cohomology:

$$H^{(p,q)}(\Gamma, X, \mathcal{V}_p) = H^{(p,q)}(\Gamma, X, \mathcal{V}_p^G) \oplus H^{(p,q)}(\Gamma, X, \mathcal{V}_p^{var}).$$

By Theorem (2.2.1), we immediately have

(7) $H^{(p,q)}(\Gamma, X, \mathcal{V}_p^{var}) = 0$

if $p+q \neq N$.

Now, since $(\mathcal{V}_p^G, P^G) \cong (C^\mu, \mu\text{-fold of } \gamma^{(0\ldots0)})$ where $\mu = \mu^{(0\ldots0)}$; we have

(8) $H^{(p,q)}(\Gamma, X, \mathcal{V}_p^G) \cong \mu^{(0\ldots0)} H^{(p,q)}(\Gamma, X, \gamma^{(0,\ldots0)})$

$$\cong H^{(p,q)}(\Gamma, X, \gamma^{(0\ldots0)}) \otimes_{\mathbb{C}} \mathcal{V}_p^G.$$

Combining this with Theorem (2.2.1), we have

(9) $H^{(p,q)}(\Gamma, X, \mathcal{V}_p^G) = 0$ if $p \neq q$, $p+q \neq N$;

and

(10) $H^{(p,p)}(\Gamma, X, \mathcal{V}_p^G) \cong \mathbb{C}^{\binom{N}{p}} \otimes \mathcal{V}_p^G = (\mathcal{V}_p^G)^{\binom{N}{p}}$

for $2p \neq N$; and as corollaries, we have

Corollary (2.2.6):

$$H^r(\Gamma, X, \mathcal{V}_p^G) \cong \begin{cases} 0 & \text{for } r \text{ odd}, \ r \neq N, \\ (\mathcal{V}_p^G)^{\binom{N}{p}} & \text{for } r = 2p \neq N. \end{cases}$$

Corollary (2.2.7):

$$H^{2r}(\Gamma, X, \mathcal{V}_p) = H^{2r}(\Gamma, X, \mathcal{V}_p^G) = H^{(r,r)}(\Gamma, X, \mathcal{V}_p^G).$$

Corollary (2.2.8): If a is odd and $a \neq N$, then

$$H^a(\Gamma, X, \mathcal{V}_p) = 0$$

for any representation $(\mathcal{V}_p, P)$ of G.

2.3 Cohomology groups of V

In this section, we shall apply the de Rham theorem of 2.1 to the case $X \times \tilde{F}$, $\Gamma \times L$, and the trivial representation of $\Gamma \times L$. Where $\Gamma \times L$ is the semi-direct product defined in 1.1 with the representation ρ. Then the de Rham cohomology group $H^p_{DR}(V, \mathbb{C})$ of V is canonically isomorphic to the abstract cohomology group $H^p(\Gamma \times L, \mathbb{C})$ of $\Gamma \times L$ with trivial representation:

(1) $H^p_{DR}(V,\mathbb{C}) = H^p(\Gamma \times L,\mathbb{C})$.

Our next step is to study the two cohomology groups of this isomorphism. First, following Kuga, we use a theorem of Hochschild-Serre to investigate the cohomology group $H^p(\Gamma \times L,\mathbb{C})$. Now, $\{1\}\times L$ is a commutative normal subgroup of $\Gamma \times L$, and ${}_{\{1\}\times L}\backslash^{\Gamma \times L} \cong \Gamma$. The following theorem of Hochschild-Serre gives a relationship between the cohomology group of a group A and the cohomology group of a quotient group ${}_B\backslash^A$ of A by a normal subgroup B.

Theorem (Hochschild-Serre) (2.3.1): Let A be a group, B a commutative normal subgroup and $\Gamma = {}_B\backslash^A$; $H^q(B,\mathbb{C})$ is considered as a Γ-module. Then there exists a spectral sequence $\{ \Sigma_{p,q} E_r^{p,q} , d_r\}_{r=2}^{\infty}$ such that

$$E_2^{p,q} = H^p(\Gamma, H^q(B,\mathbb{C})) ,$$

$$\Sigma_{p+q=m} E^{p,q} \cong H^m(A,\mathbb{C}) .$$

In order to apply this theorem for $A = \Gamma \times L$, $B = \{1\}\times L = L$ and $\Gamma = \Gamma$, to get $H^p(\Gamma \times L,\mathbb{C})$, we have to determine $H^q(L,\mathbb{C})$ and the action of Γ on it.

Since L is a lattice group $(= \mathbb{Z}^{2m})$ of rank $2m$, the cohomology group $H^q(L,\mathbb{C})$ is determined in the following way.

Consider the dual space $\Lambda^q(L\otimes\mathbb{C})^*$ of the q-th homogeneous part $\Lambda^q(L\otimes\mathbb{C})$ of the exterior algebra $\Lambda(L\otimes C)$ over $L\otimes C$. For an element f $^q(L\otimes C)^*$, we can define a q-cocycle c_f of L by

$$c_f(u_o,u_1,\dots,u_q) = \sum_{i=o}^{q} (-1)^i f(u_o\wedge\dots\wedge \hat{u}_i \wedge \dots\wedge u_q)$$

$$= f((u_1-u_o)\wedge(u_2-u_o)\wedge\dots\wedge(u_q-u_o)),$$

and we can prove that the mapping $f \longrightarrow c_f$ induces an isomorphism of $\Lambda^q(L\otimes\mathbb{C})^*$ onto $H^q(L,\mathbb{C})$. Thus, we have obtained a canonical identification

$$H^q(L,\mathbb{C}) \cong \Lambda^q(L\otimes\mathbb{C})^* ,$$

which associates a class c_f of $H^q(L,\mathbb{C})$ to an element f of $\Lambda^q(L\otimes\mathbb{C})^*$. The operation of $\gamma\in\Gamma$ $(= {}_{\{1\}xL}\backslash\Gamma xL)$ on the normal subgroup $\{1\}xL \leq L \cong \mathbb{Z}^{2m}$ is just the matrix-multiplication of $\rho(\gamma)^{-1}$ from the left. In fact, taking a representative $(\gamma,u)\in\Gamma xL$ of $\gamma\in\Gamma$, and letting $u^\gamma = \gamma^{-1}u\gamma$, we have

$$u^\gamma = (1,u)^\gamma = (\gamma,o)^{-1}(1,u)(\gamma,o) = (\gamma^{-1},o)(\gamma,u) = (1,\rho(\gamma)^{-1}u).$$

Therefore, the operation of γ on the cocycle c_f is given by

$$\gamma(c_f)(u_o,u_1,\ldots,u_q) = c_f(\rho(\gamma)^{-1}u_o, \ldots, \rho(\gamma)^{-1}u_q)$$

$$= f(\rho(\gamma)^{-1}(u_1-u_o)\wedge\rho(\gamma)^{-1}(u_2-u_o)\wedge\ldots\wedge\rho(\gamma)^{-1}(u_q-u_o))$$

$$= (\Lambda^q(\rho(\gamma))^* f)((u_1-u_o)\wedge\ldots\wedge(u_q-u_o))$$

$$= c_{{}^t(\Lambda^q\rho(\gamma))^{-1}f}(u_o,u_1,\ldots,u_q).$$

Hence the operation of $\gamma\in\Gamma$ on $H^q(1\times L,\mathbb{C}) = \Lambda^q(L\otimes\mathbb{C})^*$ is just multiplication by the matrix

$$(\Lambda^q\circ\rho)^*(\gamma) = {}^t\Lambda^q(\rho(\gamma)^{-1}) .$$

Namely, $H^q(1\times L,\mathbb{C})$ is the representation space of Γ, whose matrix representation is $(\Lambda^q\circ\rho)^*$, so that

(2) $H^q(\Gamma, H^q(1\times L,\mathbb{C})) \cong H^p(\Gamma, (\Lambda^q\circ\rho)^*)$.

Hence, we have determined the E_2-terms of the spectral sequence of the theorem (2.3.1), i.e., $E_2^{p,q} \cong H^p(\Gamma, (\Lambda^q\circ\rho)^*)$. Actually, we know that $E_2 = E_\infty$ in our case. There are several proof of this fact. Kuga proved this by using harmonic forms on V. Deligne and Satake showed it by fiber stretching operators Θ_N^* (see next section).

Therefore we have, by the theorem (2.3.1),

Theorem (2.3.2):

$$H^m(V, \mathbb{C}) \cong \bigoplus_{p+q=m} H^p(\Gamma, (\Lambda^q \circ \rho)^*).$$

The Hochschild-Serre's spectral sequence $\{E_r^{p,q}\}$ for the group triple $\{A, B, \Gamma\} = \{\Gamma \times L, 1 \times L, \Gamma\}$ can also be described by means of the Leray's spectral sequence of the fibering structure $V \xrightarrow{\pi} U$. We shall describe it briefly.

Suppose given topological spaces X, Y with a continuous mapping $f: X \longrightarrow Y$ and a sheaf $\mathfrak{F}$ over X. The q-th direct image sheaf is the sheaf $R_f{}^q(\mathfrak{F})$ on Y associated to the presheaf

$$\mathcal{U} \longrightarrow H^q(f^{-1}(\mathcal{U}), \mathfrak{F}),$$

where $\mathcal{U}$ is some open set in Y. The Leray spectral sequence is a spectral sequence $\{E_s\}$ with

(3)
$$E_\infty \Longrightarrow H^*(X, \mathfrak{F}),$$
$$E_2^{p,q} = H^p(Y, R_f{}^q(\mathfrak{F})).$$

In our case, $V \xrightarrow{\pi} U$ is a C^∞-fiber bundle with compact fiber F. Thus V, U and F are manifolds, π is a C^{∞}-mapping, and

$$\pi^{-1}\mathcal{U} = \mathcal{U} \times F$$

for sufficiently small open sets $\mathcal{U} \subset U$. For the constant sheaf $\mathbb{Q}$ on V, by the Künneth formula,

$$(4) \quad H^b(\pi^{-1}(\mathcal{U}), \mathbb{Q}) \cong H^b(F, \mathbb{Q}) .$$

This shows that $R_{\pi}{}^b(\mathbb{Q}) \cong H^b(F, \mathbb{Q})$ as the locally constant sheaves. The action of the fundamental group $\pi_1(U,p) \cong \Gamma$ on $H^b(F_p, \mathbb{Q})$ is known, and it is the same as the representation $(\Lambda \circ \rho)^*$ of Γ. From these facts and the fact that the space $X = H^N$ is contractible, we have

$$(5) \quad E_2^{p,q} = H^a(U, H^b(F_p, \mathbb{C})) \cong H^a(\Gamma, (\Lambda^b \circ \rho)^*).$$

Namely, the E_2-terms of both (Hochschild-Serre's and Leray's) spectral sequences coincide. By this isomorphism, using the following theorem of Deligne, we can have another proof of the degeneration $E_2 = E_{\infty}$ of the Hochschild-Serre spectral sequence of the group triple $\{ \Gamma \times L, 1 \times L, \Gamma \}$.

Remembering that V and U are compact Kähler manifolds and

$$\pi : V \longrightarrow U$$

is a surjective, holomorphic mapping of maximal rank. Moreover the fibers $\pi^{-1}(p) = F_p$ are also compact Kähler.

Theorem (Deligne [1],[3]) (2.3.3): In this situation, the Leray spectral sequence of $V \xrightarrow{\pi} U$ degenerates at E_2, i.e.,

$$E_2 \cong E_\infty ,$$

so that

$$(6) \quad H^r(V, \mathbb{Q}) \cong \bigoplus_{a+b=r} H^a(U, R_\pi{}^*(\mathbb{Q})).$$

On the other hand, since $X \times \tilde{F}$ is contractible, the cohomology group $H^r(V, \mathbb{Q})$ of $V = {}_{\Gamma \times L}\backslash(X \times \tilde{F})$ is isomorphic to the group cohomology $H^r(\Gamma \times L)$. Combining all these, we have

$$(7) \quad H^r(\Gamma \times L) \cong \bigoplus_{a+b=r} H^a(\Gamma, (\Lambda^b \circ \rho)^*).$$

This show the degeneracy of the Hochschild-Serre's spectral sequence.

2.4 We now define the fiber-stretching operator $\textcircled{H}_n$ $(n = 0, \pm 1, \pm 2, \ldots)$ of V onto V. First, we do the N-multiplication of torus $(N \in \mathbb{Z})$. For this, we consider the following diagram

$$\begin{array}{ccc} \tilde{F} & \longrightarrow & \tilde{F} \\ \downarrow & & \downarrow \\ L\backslash\tilde{F} & \xrightarrow{\ \theta_N\ } & L\backslash\tilde{F} \end{array},$$

Since $NL \subset L$, multiplication by N in $\tilde{F}$ defines the map $\theta_N : {}_{L}\backslash\tilde{F} \longrightarrow {}_{L}\backslash\tilde{F}$. If we do the N-multiplication on the fiber, we have

$$\begin{array}{ccc} V & \longleftarrow & X \times ({}_{L}\backslash\tilde{F}) \\ \textcircled{H}_N \downarrow & & \downarrow \\ V & \longleftarrow & X \times ({}_{L}\backslash\tilde{F}) \end{array}.$$

The right-hand-side vertical map sends (x,u) onto $(x,\theta_n u)$. Thus we obtain a holomorphic map of V onto V (which is not one-to-one, because θ_n is not). $\textcircled{H}_N$ is a holomorphic map, which is compayible with fiber-structure:

$$\begin{array}{ccc} V & \xrightarrow{\Theta_N} & V \\ \pi \downarrow & & \downarrow \pi \\ U & \xrightarrow{id} & U \end{array} .$$

Furthermore, we have the induced map of cohomology groups

$$H^r(V, \mathbb{Q}) \xrightarrow{\Theta_N{}^*} H^r(V, \mathbb{Q})$$

(and respectively, over $\mathbb{R}$ and $\mathbb{C}$). The following theorem is due to Kuga.

<u>Theorem</u> (2.4.1): The eigenvalues of $\Theta_N{}^*$ on $H^r(V, \mathbb{Q})$ are $N^0, N^1, \ldots, N^r$. Put

$$\mathcal{H}^{\langle a,b \rangle}(V, \mathbb{Q}) = \{x \in H^r(V, \mathbb{Q}) \mid \Theta_N{}^*(x) = N^b x, \quad \text{for every } N\},$$

with $a = r-b$. Then

$$H^r(V, \mathbb{Q}) = \bigoplus_{a+b=r} \mathcal{H}^{\langle a,b \rangle}(V, \mathbb{Q}).$$

And

$$\mathcal{H}^{\langle a,b\rangle}(V,\mathbb{C}) = \mathcal{H}^{\langle a,b\rangle}(V,\mathbb{Q}) \otimes \mathbb{C} \cong E_2^{a,b} \quad \text{(canonically).}$$

Obviously,

$$(1) \quad H^r(V,\mathbb{C}) \cong \bigoplus_{a+b=r} \mathcal{H}^{\langle a,b\rangle}(V,\mathbb{C}).$$

On the other hand, since V is a Kähler manifold, $\mathcal{H}^r(V,\mathbb{C})$ can be decomposed in the following way, due to Hodge.

$$(2) \quad \mathcal{H}^r(V,\mathbb{C}) = \bigoplus_{p+q=r} \mathcal{H}^{(p,q)}(V),$$

where $\mathcal{H}^{(p,q)}(V)$ is the space of harmonic (p,q)-forms on V. The relation of decompositions just shown above is given by the following

<u>Lemma</u> (2.4.3): The decompositions (1) and (2) are compatible. For $a+b = p+q = r$, put

$$\mathcal{H}^{(p,q)}(V) \cap \mathcal{H}^{\langle a,b\rangle}(V) \cong \mathcal{H}^{\langle a,b;p,q)}(V).$$

Then

$$\mathcal{H}^{(p,q)}(V) = \bigoplus_{a+b=r} \mathcal{H}^{\langle a,b;p,q)}(V) ,$$

$$\mathcal{H}^{\langle a,b\rangle}(V) = \bigoplus_{p+q=r} \mathcal{H}^{\langle a,b;p,q)}(V) .$$

Proof: $\Theta_N : V \longrightarrow V$ is a holomorphic mapping. The restriction of Θ_N^* to $\mathcal{H}^{(p,q)}(V)$ maps $\mathcal{H}^{(p,q)}(V) \longrightarrow \mathcal{H}^{(p,q)}(V)$, namely the image of $\Theta_N^* : \omega^{(p,q)} \longrightarrow \Theta_N^*(\omega^{(p,q)})$ is again of type (p,q).

By means of stretching operators θ_n^*, Deligne and Satake proved the degeneracy $E_2 = E_\infty$ of the spectral sequence of $\{\Gamma x L, L, \Gamma\}$. Let us sketch that here.

Let "stretching operator" θ_n, operates on $\Gamma x L$ (semi-direct product through ρ), by

$$\begin{array}{ccc} \Gamma x L & \xrightarrow{\quad \theta_n \quad} & \Gamma x L \\ \Psi & & \Psi \\ (\gamma, d) & \longrightarrow & (\gamma, nd) \end{array} .$$

θ_n's $(n = 0, \pm 1, \pm 2, \ldots)$ are then endomorphisms of the groups xL. 1xL is the invariant subgroup of θ_n's, and θ_n's induce trivial maps on Γ. Θ_n induces homomorphism Θ_n^* of $H^r(\Gamma \times L, \mathbb{Q})$. Since the spectral sequence is constructed functorially, the operator Θ_n also induces operators Θ_n^* on each term $E_r^{a,b}$ of spectral sequence, commuting with d's. Moreover it is proved easily the action Θ_n^* on $E_2^{a,b}$ is the scalar multiplication of n^b. So we have the diagram:

$$\begin{array}{ccc} E_2^{a,b} & \xrightarrow{\quad d_2 \quad} & E_2^{a-1,b+2} \\ \big\downarrow n^b & & \big\downarrow n^{b+2} \\ E_2^{a,b} & \xrightarrow{\quad d_2 \quad} & E_2^{a-1,b+2} \end{array}$$

which is commutative:

$$(*) \quad n^{b+2}(d_2(x)) = d_2(n^b x) = n^b(d_2(x)),$$

for all $x \in E_2^{a,b}$.

Since $E_2^{p,q}$ are vector space over $\mathbb{Q}$, (*) implies $d_2(x) = 0$ for all x, i.e., $d_2 = 0$. This means:

$$E_2 \cong E_3 = \dots \cong E_\infty .$$

QED.

CHAPTER III. ALGEBRAIC CYCLES IN A GENERIC FIBER

In this chapter, we summarize a result of Kuga [8], on algebraic cycles in a generic fiber F_p of a family $V \xrightarrow{\pi} U$ of abelian varieties. (See also [9]). For this purpose, we have to investigate the action of $\pi_1(U,p) =$ on $H^r(F_p,\mathbb{R})$.

3.1 We recall that an algebraic cycle of codimension r on a compact smooth algebraic variety, say, $\mathfrak{X}$, is by definition a formal linear combination $Z = \Sigma_i \lambda_i Y_i$ of irreducible subvarieties Y_i of codimension r with rational numbers λ_i as coefficients. The group of all algebraic cycles of codimension r on $\mathfrak{X}$ is denoted by $\mathfrak{A}^r(\mathfrak{X}, \mathbb{Q})$, or simply $\mathfrak{A}^r$. Also we put $\mathfrak{A}^r(\mathfrak{X}, \mathbb{R}) = \mathfrak{A}^r(\mathfrak{X}, \mathbb{Q}) \otimes_{\mathbb{Q}}\mathbb{R}$, and $\mathfrak{A}^r(\mathfrak{X}, \mathbb{C}) = \mathfrak{A}^r(\mathfrak{X}, \mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$.

Let m be the complex dimension of the algebraic variety $\mathfrak{X}$. Algebraic cycle $Z = \Sigma_i \lambda_i Y_i$ of codimension r is also called algebraic cycle of dimension $m-r$, and $\mathfrak{A}^r(\mathfrak{X}, \mathbb{Q})$ is also denoted by $\mathfrak{A}_{m-r}(\mathfrak{X}, \mathbb{Q})$: $\mathfrak{A}^r(\mathfrak{X}, \mathbb{Q}) = \mathfrak{A}_{m-r}(\mathfrak{X}, \mathbb{Q})$. Also we use notations $\mathfrak{A}_{m-r}(\mathfrak{X}, \mathbb{R})$ and $\mathfrak{A}_{m-r}(\mathfrak{X}, \mathbb{C})$, which are actually $\mathfrak{A}^r(\mathfrak{X}, \mathbb{R})$ and $\mathfrak{A}^r(\mathfrak{X}, \mathbb{C})$

respectively. Also we use the notations $\mathfrak{A}^r_{\mathbb{R}} = \mathfrak{A}_{m-r,\mathbb{R}}$, and $\mathfrak{A}^r_{\mathbb{C}} = \mathfrak{A}_{m-r,\mathbb{C}}$ for $\mathfrak{A}^r(\mathfrak{X}, \mathbb{R})$ and $\mathfrak{A}^r(\mathfrak{X}, \mathbb{C})$ respectively.

If Y is a subvariety of codimension r in $\mathfrak{X}$, it defines a homology class on $\mathfrak{X}$. We shall denote it by $c(Y)$, and we have

$$c(Y) \in H_{2m-2r}(\mathfrak{X}, \mathbb{Q}).$$

This definition can be extended by linearity to give the homology class $c(Z)$ of any algebraic cycle Z on $\mathfrak{X}$. Therefore we have the following map c,

$$(1) \qquad \mathfrak{A}_{m-r} \xrightarrow{\ c\ } H_{2m-2r}(\mathfrak{X}, \mathbb{Q}).$$

This naturally induces a map

$$(2) \qquad \mathfrak{A}_{m-r,\mathbb{R}} \xrightarrow{\ c\ } H_{2m-2r}(\mathfrak{X}, \mathbb{R}),$$

which is also denoted by c. Also we consider

$$\mathfrak{A}_{m-r,\mathbb{C}} = \mathfrak{A}^r_{\mathbb{C}} \xrightarrow{\ c\ } H_{2m-2r}(\mathfrak{X}, \mathbb{C}) = H^{2r}(\mathfrak{X}, \mathbb{C}),$$

by tensoring with $\mathbb{C}$.

Let us identify the cohomology group with homology group,

$$H^{2r}(\mathfrak{X}, \mathbb{R}) = H_{2m-2r}(\mathfrak{X}, \mathbb{R}),$$

by Poincare duality. Then the above map c in (2) is also described as

$$\mathfrak{A}^r_{\mathbb{R}} \xrightarrow{\ c\ } H^{2r}(\mathfrak{X}, \mathbb{R}).$$

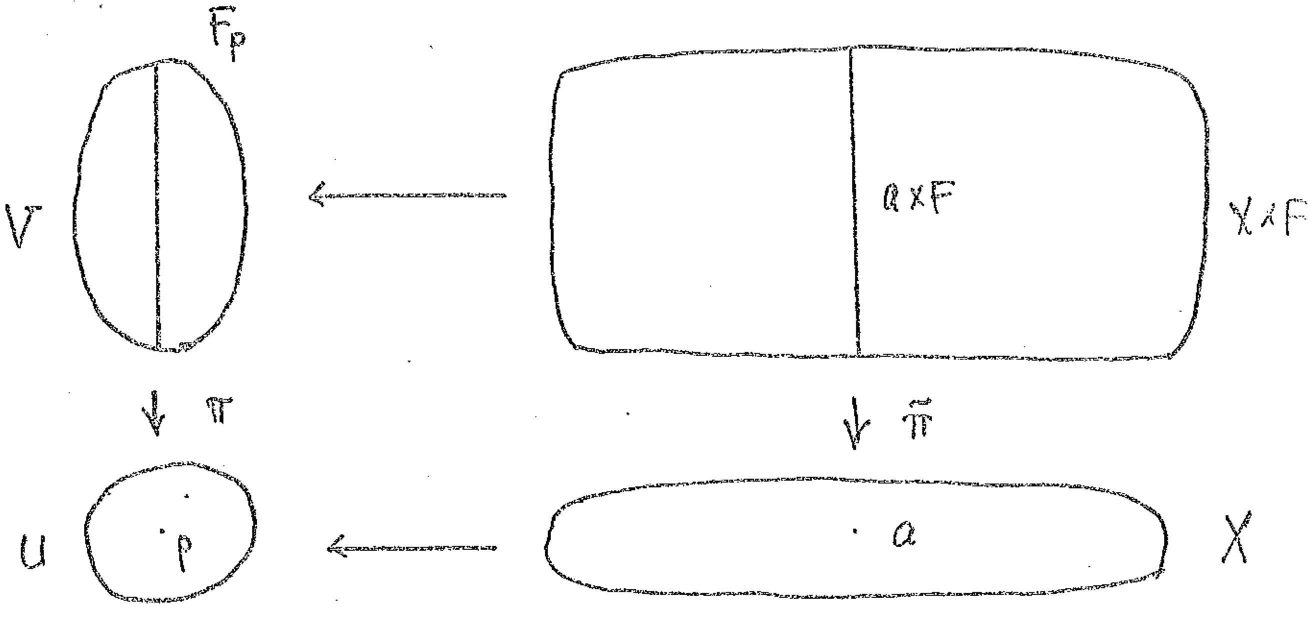
For an algebraic cycle Y, the (co-)homology class $c(Y) \in H^{2r}(\mathfrak{X}, \mathbb{Q}) = H_{2m-2r}(\mathfrak{X}, \mathbb{Q})$ is sometimes called also algebraic cycle or algebraic cocycle. Also an algebraic cycle $Y \in \mathfrak{A}^r(\mathfrak{X}, \mathbb{Q}) = \mathfrak{A}_{m-r}(\mathfrak{X}, \mathbb{Q})$ is sometimes called algebraic co-cycle; especially when its image $c(Y)$ is $H^{2r}(\mathfrak{X}, \mathbb{Q}) = H_{2m-2r}(\mathfrak{X}, \mathbb{Q})$ is considered as an cohomology class rather than homology class.

3.2 Take a base point p of U. The fundamental group $\pi_1(U,p)$ acts on the homology (cohomology) groups $H_b(F_p,\mathbb{R})$ ($H^b(F_p,\mathbb{R})$) of the fiber $F_p = \pi^{-1}(p)$ over $p \in U$ of $\{V,\pi,U\}$ by the "displacement" of (homological) cycles in the fibers over a path γ from p to p. The "displacement" induces an automorphism γ_* of $H^b(F_p, \mathbb{R})$ that depends only on the homotopy class of γ.

If we take a point $a \in X$, which covers the base point $p \in U$, then we can identify the fundamental group $\pi_1(U,p)$ with the discrete group Γ as follows. For an element $\gamma \in \Gamma$, consider the curve $C(\gamma)$ which connect $\gamma(a)$ to a in X, and consider the image of $C(\gamma)$ $(\subset X)$ in $U = \Gamma\backslash X$. The image of $C(\gamma)$ is a closed curve in U, defining a homotopy class in $\pi_1(U,p)$. Then the map

(1) $$\Gamma \ni \gamma \longmapsto (\text{the class of the image of } C(\gamma)) \in \pi_1(U,p)$$

is an isomorphism between Γ and $\pi_1(U,p)$. By this isomorphism we identify $\pi_1(U,p)$ with Γ. The isomorphism depends on the choice of $a \in X$.



Using the above choice of $a \in X$, we can identify the fiber $F_p = \pi^{-1}(p)$ over $p \in U$ with the torus $F = {}_L\backslash\tilde{F}$, as follows:

$$\begin{array}{ccccc} F & \longrightarrow & \{a\} \times F & \longrightarrow & F_p \\ \cup & & \cup & & \\ u & \longmapsto & (a,u) & \longmapsto & p_1(a,u). \end{array}$$

By this identification

(2) $F = F_p$,

the homology group of F_p is

(3) $H_1(F_p, \mathbb{Z}) = H_1(F, \mathbb{Z}) = L$,

and

(4) $H_1(F_p, \mathbb{R}) = H_1(F, \mathbb{R}) = L \otimes \mathbb{R} = \tilde{F}$.

Also with the identification (2) and the identification (1): $\Gamma = \pi_1(U,p)$, it is easy to see that the action of $\pi_1(U,p)$ on the homology group $H_1(F_p, \mathbb{R}) = \tilde{F}$ is identified with the action $\rho|_\Gamma$ of Γ

on F, defined in 1.1. Note that our $\rho_{|\Gamma}$ is a restriction of a symplectic representation ρ of the Lie group G. Namely the action of $\pi_1(U, p_o) = \Gamma$ on $H_1(F_{p_o}, \mathbb{R}) = \tilde{F}$ is extendable to an action of the Lie group G. Also note that an extension of a representation of Γ to G, if it exists, must be unique, by a theorem of A. Borel.

Since

$$H_r(\text{torus}, \mathbb{R}) = \Lambda^r(H_1(\text{torus}, \mathbb{R})),$$

$$H_r(F_{p_o}, \mathbb{R}) = \Lambda^r(\tilde{F}),$$

and the action of G on $H_r(F_{p_o}, \mathbb{R}) = \Lambda^r(\tilde{F})$ is equivalent to $\Lambda^r(\rho) = \rho \wedge \ldots \wedge \rho$ (r-times).

Analogously,

$$H^1(F_{p_o}, \mathbb{R}) = \tilde{F}^*,$$

$$H^r(F_{p_o}, \mathbb{R}) = \Lambda^r(\tilde{F}^*),$$

and the action of G on $H^r(F_{p_o}, \mathbb{R}) = \Lambda^r(\tilde{F}^*)$ is equivalent to $(\Lambda^r(\rho))^*$. So the action of the fundamental group

$\pi_1(U,p) = \Gamma$ on the cohomology group $H^r(F_p,\mathbb{R}) = \Lambda^r(\tilde{F}^*)$ is the restriction of the representation $(\Lambda^r(\rho))^*$ of the Lie group G.

3.3 Let k be a field of definition of the algebraic variety U. We assume also k is an algebraic closure of a finitely generated field over $\mathbb{Q}$. Let p be a generic point of $U_{/k}$, and $F_p = \pi^{-1}(p)$ the generic fiber, which is defined over k(p).

Now by Kuga [8], we have

<u>Theorem</u> (3.3.1): $c(\mathfrak{A}^r(F_p,\mathbb{C})) \hookrightarrow H^{2r}(F_p,\mathbb{C})^{\Gamma} = H^{2r}(F_p,\mathbb{C})^G$,

where

$$H^{2r}(F_p,\mathbb{C})^{\Gamma} = \{x \in H^{2r}(F_p,\mathbb{C}) \mid \gamma.x = x \quad \text{for all} \quad \gamma \in \Gamma\},$$

$$H^{2r}(F_p,\mathbb{C})^{G} = \{x \in H^{2r}(F_p,\mathbb{C}) \mid g.x = x \quad \text{for all} \quad g \in G\}.$$

<u>Proof</u>: The non-trivial equality is due to A. Borel. For the proof, see Kuga [8].

3.4 We consider now when r = 1. Then

$$c(\mathfrak{A}^1(F_p,\mathbb{C})) \subset \Lambda^2(\tilde{F}_{\mathbb{C}}^*)^G .$$

$c(\mathfrak{A}^1(F_p,\mathbb{C}))$ is the space of all cohomology classes of $\mathbb{C}$-linear combinations of divisors. And we shall make the following Assumption

$$(C\text{-}1)\quad c(\mathfrak{A}^1(F_p,\mathbb{C})) = \Lambda^2(\tilde{F}_{\mathbb{C}}^{\,*})^G .$$

Indeed this assumption is true in many cases, in particular, for symplectic group G without compact factor. (See chap. IV).

From now on, denote the linear space $\tilde{F}_{\mathbb{C}}^{\,*}$ by $\mathfrak{F}$ for simplicity: so (C-1) is written as

$$(C\text{-}1)\quad \mathfrak{A}^1(F_p,\mathbb{C}) = \Lambda^2(\mathfrak{F})^G .$$

Now, consider the following inclusion

$$\Lambda^2(\tilde{F}_{\mathbb{C}}^{\,*}) = \Lambda^2(\mathfrak{F})^G \subset \Lambda^{even}(\mathfrak{F}),$$

and look at the G-invariant part; so we have

$$\Lambda^2(\mathfrak{F})^G \subset \Lambda^{even}(\mathfrak{F})^G \subset \Lambda^{even}(\mathfrak{F}).$$

Now let us make a second assumption

$$(C\text{-}2)\quad \Lambda^{2m}(\mathfrak{F})^G = \underbrace{\Lambda^2(\mathfrak{F})^G \wedge \Lambda^2(\mathfrak{F})^G \wedge \cdots \wedge \Lambda^2(\mathfrak{F})^G}_{m \text{ times}},$$

for $2m = 4, 6, \ldots$.

Therefore assumptions (C-1) and (C-2) imply

(1) $c(\mathfrak{A}^m(F_p, \mathbb{C})) = \Lambda^{2m}(\mathfrak{F})^G$,

for all $2m = 2, 4, 6, \ldots$. Moreover, all algebraic cycles are homologously intersection of divisors:

$$\vartheta \cdot \vartheta \cdot \vartheta \cdots \cdot \vartheta = \vartheta \wedge \vartheta \wedge \cdots \wedge \vartheta \quad \text{(m times)}.$$

In summary, we have the following

Theorem (3.4.1): Under the assumptions (C-1) and (C-2), we have

(2) $c(\mathfrak{A}^r(F_p, \mathbb{C})) = \Lambda^{2r}(\mathfrak{F})^G$

for all r, and moreover, all algebraic cycles in F_p are homologous to linear combinations of intersections of divisors.

Finally, we remark that the assumption (C-2) is true for $G = Sp(n, \mathbb{R})$.

3.5 In this section, we analyze the assumption (C-2).

Let G be a connected, semi-simple Lie group, and $\mathfrak{F}$ the linear space on which G acts. Let us consider the tensor algebra $T(\mathfrak{F})$, and let $\mathfrak{J}$ be the ideal of $T(\mathfrak{F})$, generated by $\{x\otimes y + y\otimes x \mid x, y \in T(\mathfrak{F})\}$, so that the Grassmannian algebra of $\mathfrak{F}$ is $\Lambda(\mathfrak{F}) = T(\mathfrak{F})/\mathfrak{J}$. Since G acts on $\mathfrak{F}$, there- G acts on $\mathfrak{J}$, $T(\mathfrak{F})$ and $\Lambda(\mathfrak{F})$. Taking the G-invariant part of the short exact sequence

$$(1) \quad 0 \longrightarrow \mathfrak{J} \xrightarrow{\ i\ } T(\mathfrak{F}) \xrightarrow{\ j\ } \Lambda(\mathfrak{F}) \longrightarrow 0,$$

we have (if we consider the case of degree m)

$$(2) \quad 0 \longrightarrow \mathfrak{J}^{mG} \xrightarrow{\ i_m\ } T^m(\mathfrak{F})^G \xrightarrow{\ j_m\ } \Lambda^m(\mathfrak{F})^G \longrightarrow 0.$$

(Note that arrows "$\rightarrow$" in (1) are degree preserving algebra homomorphism.)

(<u>Proof</u>: We form the longer exact sequence

$$0 \longrightarrow \mathfrak{J}^{mG} \xrightarrow{\ i_m\ } T^m(\mathfrak{F})^G \xrightarrow{\ j_m\ } \Lambda^m(\mathfrak{F})^G \longrightarrow H^1(G, \mathfrak{J}^m) \longrightarrow \cdots$$

Now, we have $H^1(G, \mathfrak{J}^m) = (0)$; this is so because G is semi-simple, and therefore $\mathcal{J}^m$ is completely reducible. Furthermore $(\mathcal{J}^m)^G$ may be identified with $H^0(G, \mathcal{J}^m)$, and $\Lambda^m(\mathfrak{F})^G$ with $H^0(G, \Lambda^m(\mathfrak{F}))$, etc.

By summing up (2) for all $m = 0,1,2,3, \ldots$, we have

$$(3) \quad 0 \longrightarrow \mathcal{J}^G \longrightarrow T(\mathfrak{F})^G \longrightarrow \Lambda(\mathfrak{F})^G \longrightarrow 0.$$

Taking even m's; we have

$$(4) \quad T^{even}(\mathfrak{F})^G \longrightarrow \Lambda^{even}(\mathfrak{F})^G \longrightarrow 0.$$

Consider (2) for $m = 2$, we have

$$(5) \quad T^2(\mathfrak{F})^G \longrightarrow \Lambda^2(\mathfrak{F})^G \longrightarrow 0.$$

Summarizing, we have

Corollary (3.5.1): If $T^{even}(\mathfrak{F})^G$ is generated by $T^2(\mathfrak{F})^G$, then $\Lambda^{even}(\mathfrak{F})^G$ is also generated by $\Lambda^2(\mathfrak{F})^G$.

CHAPTER IV. ALGEBRAIC CYCLES AND Γ-INVARIANT CYCLES IN F_p.

In this and proceeding chapters V, VI, VII, VIII, we investigate the space of algebraic cycles $c(\mathcal{M}^r(V,\mathbb{R}))$ in the cohomology group $H^{2r}(V,\mathbb{R})$ of the total space V of the family of abelian varieties $V \xrightarrow{\pi} U$, defined in

Chapter I. In such a family the base is the quotient $U = \Gamma\backslash X$ of the product $X = H^N$ of N copies of the upper half-plane, and the group G is $SL(2,\mathbb{R})^N$. Families $V \xrightarrow{\pi} U$ of this kind havebeen constructed in 1.5 of Chapter I out of totally indefinite quaternion algebra $\mathcal{B}$ over a totally real number field k of degree N over $\mathbb{Q}$. Here we shall not recall the details of the construction. We only need the facts, that such a family exists, and the representation (the Satake representation) $(\tilde{F},\mathbb{P})$ of G, defining our V, must be of the form

$$\tilde{F} = \sum_{i=1}^{N} m_i \mathcal{V}_i^{(1)},$$

$$P = \sum_{i=1}^{N} m_i \gamma_i^{(1)}, \qquad \text{where} \quad \gamma_i^{(1)} = \gamma^{(1)} \circ \mathrm{proj}_i,$$

for some integer $m_1, \dots, m_N$.

Moreover, in the quaternion case as described in 1.5 of Chapter I, these multiplicities $m_1, \ldots, m_N$ have to be equal:

$$m_1 = m_2 = \cdots = m_N.$$

In the case $N = 1$, i.e., in the family of abelian varieties $V \xrightarrow{\pi} U = \Gamma\backslash H$, algebraic cycles were investigated in Kuga-Hall [9]. There, all (co)-homology classes generated by algebraic cycles were completely determined.

In our case of $N > 1$, we cannot expect such a strong result, since the algebraic cycles in the base variety $U = \Gamma\backslash X$ are already very hard to determine. All we can do in our case here is to determine the cohomology classes of algebraic cycles in the relatively small codimensional cohomology groups $H^{2r}(V,\mathbb{R})$ for $2r < N$.

In 3.3, we have seen that

$$H^2(F_p,\mathbb{C})^{\Gamma} = H^2(F_p,\mathbb{C})^G \supset c(\mathcal{O}\!\mathcal{L}^1(F_p,\mathbb{C}))$$

for a generic fiber F_p of a family of abelian varieties $V \xrightarrow{\pi} U$. Moreover, we have seen that the assumptions

$$(C\text{-}1) : H^2(F_p, \mathbb{C})^G = c(\mathfrak{A}^1(F_p, \mathbb{C})) ,$$

and

$$(C\text{-}2) : \Lambda^{2m}(\mathfrak{F})^G = (\Lambda^2(\mathfrak{F})^G)^{\Lambda m},$$

imply

$$H^{2r}(F_p, \mathbb{C})^{\Gamma} = H^{2r}(F_p, \mathbb{C})^G = c(\mathfrak{A}^r(F_p, \mathbb{C})).$$

In this chapter, we shall prove that the assumptions (C-1) and (C-2) are true in our case of $G = SL(2,\mathbb{R})^N$ and $X = H^N$; hence that the space of Γ-invariant cocycles $H^{2r}(F_p, \mathbb{Q})$ in a generic fiber F_p is generated by algebraic cocycles.

4.1 The proof of (C-1) in the case of $G = SL(2,\mathbb{R})^N$, $X = H^N$.

Let $K = SO(2,\mathbb{R})^N$. This is a maximal compact subgroup of $G = SL(2,\mathbb{R})^N$. K is also commutative.

From $SO(2,\mathbb{R})$ to K, an isomorphism Δ is defined by

$$\Delta(t) = (t, t, \ldots, t) \in K$$

for $t \in SO(2,\mathbb{R})$. And from $\mathbb{R}$ to K the homomorphism φ is defined by

$$\varphi(\Theta) = \Delta\left(\begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}\right).$$

The complex torus F_z defined in 1.4 and 1.5 of Chapter I, is the torus $F = {}_L\backslash\tilde{F}$, with the complex structure J_z defined in 1.4. It is easy to see that

$$J_z = \rho(g\,\varphi(\tfrac{\pi}{2})g^{-1}), \qquad \text{for } z = \nu(g).$$

An important fact is that the J_z is in the image $\rho(G) \subset GL(\tilde{F})$, of our representation ρ.

The harmonic 2-form on the torus $F_z = ({}_L\backslash\tilde{F}, J_z)$ is lifted to a 2-form

$$\omega = \sum_{i,j} a_{ij} d\zeta_i \wedge d\zeta_j + \sum_{i,j} b_{ij} d\zeta_i \wedge d\bar{\zeta}_j + \sum_{i,j} c_{ij} d\bar{\zeta}_i \wedge d\bar{\zeta}_j$$

with constant coefficients a_{ij}, b_{ij}, c_{ij}; and the ζ_i's are $\mathbb{C}$-linear coordinates of the $\mathbb{C}$-linear space $(\tilde{F}, J_z)$. So

$$J_z^*(\zeta_i) = \sqrt{-1}\,\zeta_i,$$

$$J_z^*(\bar{\zeta}_i) = -\sqrt{-1}\,\bar{\zeta}_i \,.$$

Therefore, the action of J_z^* on ω is

$$\begin{aligned} J_z^*(\omega) &= \sum_{i,j} a_{ij}\, J_z^*(d\zeta_i) \wedge J_z^*(d\zeta_j) \\ &\quad + \sum_{i,j} b_{ij}\, J_z^*(d\zeta_i) \wedge J_z^*(d\bar{\zeta}_j) \\ &\quad + \sum_{i,j} c_{ij}\, J_z^*(d\bar{\zeta}_i) \wedge J_z^*(d\bar{\zeta}_j) \\ &= -\sum a_{ij}\, d\zeta_i \wedge d\zeta_j + \sum b_{ij}\, d\zeta_i \wedge d\bar{\zeta}_j \\ &\quad - \sum c_{ij}\, d\bar{\zeta}_i \wedge d\bar{\zeta}_j \,. \end{aligned}$$

Therefore, we have

Lemma (4.1.1): ω is a (1,1) form if and only if

$$J_z^*(\omega) = \omega \,.$$

Since $J_z \in \rho(G)$, this proves

Corollary (4.1.2): G- and therefore Γ-invariant harmonic 2-forms on F_z are automatically of the type (1,1); i.e.,

$$H^2(F_z,\mathbb{C})^\Gamma \subset \mathcal{H}^{(1,1)}(F_z).$$

Next, consider the structure of the linear space $\tilde{F}$ and $\mathfrak{f} = \tilde{F}^*$, by means of the lattice L and the dual lattice $L_* \subset \mathfrak{f}$. Namely, a vector x in $\tilde{F}$ (or in $\mathfrak{f}$) is $\mathbb{Q}$-rational if and only if $x \in L\mathbb{Q}$ (or $L_*\mathbb{Q}$). A k-dimensional linear subspace $W \subset \tilde{F}$ (or in $\mathfrak{f}$) is defined over $\mathbb{Q}$ if and only if $W \cap L$ contains k independent vectors; so that

$$W = (W \cap L).\mathbb{R} = (W \cap L\mathbb{Q}).\mathbb{R}$$

and $W \cap L\mathbb{Q}$ is a k-dimensional $\mathbb{Q}$-linear subspace of $L\mathbb{Q}$. This $\mathbb{Q}$-rational structure of $\tilde{F}$ (or $\mathfrak{f}$) automatically defines $\mathbb{Q}$-rational structures in $\Lambda^r(\tilde{F})$ (or $\Lambda^r(\mathfrak{f})$). By the identification $H^r(F_{p_o},\mathbb{R}) = \Lambda^r(\mathfrak{f})$, the subspace $\Lambda^r(\mathfrak{f})_{\mathbb{Q}}$ of all $\mathbb{Q}$-vectors in $\Lambda^r(\mathfrak{f})$ is identified with $H^r(F_{p_o},\mathbb{Q})$:

$$\Lambda^r(\mathfrak{f})_{\mathbb{Q}} = H^r(F_{p_o},\mathbb{Q}).$$

The action $\Lambda^r(\rho^*)(\gamma)$ of $\gamma \in \Gamma = \pi_1(U,p_o)$ on $H^r(F_{p_o},\mathbb{R})$ sends $H^r(F_{p_o},\mathbb{Z})$ onto itself. Therefore it sends $\Lambda^r(\mathfrak{f})_{\mathbb{Q}}$ onto itself, i.e., the linear automorphism

$\Lambda^r(\rho^*)(\gamma)$ of $\Lambda^r(\mathfrak{g})$ is defined over $\mathbb{Q}$. Therefore, if an eigenvalue λ of $\Lambda^r(\rho^*)(\gamma)$ is rational, the corresponding eigenspace

$$\{x \in \Lambda^r(\mathfrak{g}) \mid \Lambda^r(\rho^*)(\gamma)x = \lambda x\} = \ker(\lambda 1 - \Lambda^r(\rho^*)(\gamma))$$

is defined over $\mathbb{Q}$. We have

<u>Lemma</u> (4.1.3): $H^r(F_{p_o},\mathbb{C})^\Gamma = \bigcap_{\gamma\in\Gamma} \ker(1 - \Lambda^r(\rho^*)(\gamma))$ is therefore defined over $\mathbb{Q}$.

Now, recall the following theorem of Lefschetz,

<u>Theorem</u> (Lefschetz): For a projective algebraic variety $W \subset P^N(\mathbb{C})$,

$$H^{(1,1)}(W) \cap H^2(W,\mathbb{Q}) = c(\mathfrak{A}^1(W,\mathbb{Q})).$$

<u>Corollary</u> (4.1.4): If a cohomology class $x \in H^{(1,1)}(W) \subset H^2(W,C)$ is defined over $\mathbb{Q}$, then $x \in c(\mathfrak{A}^1(W,\mathbb{C}))$.

Now using the lemmas (4.1.2), (4.1.3), and corollary (4.1.4), $H^2(F_{p_o},\mathbb{C})^\Gamma$ has a basis of $\mathbb{Q}$-rational vectors; they are in $H^{(1,1)}(F_{p_o})$, and therefore in $c(\mathfrak{A}^1(F_{p_o},\mathbb{C}))$. Hence $H^2(F_{p_o},\mathbb{C})^\Gamma \subset c(\mathfrak{A}^1(F_{p_o},\mathbb{C}))$.

On the other hand, we already know that

$$H^2(F_{p_o},\mathbb{C})^\Gamma \supset c(\mathfrak{A}^1(F_{p_o},\mathbb{C})).$$

Thus we have

Theorem (4.1.5): The assumption

$$(C\text{-}1) \quad H^2(F_{p_o},\mathbb{C})^\Gamma = c(\mathfrak{A}^1(F_{p_o},\mathbb{C})),$$

is true in our case of $G = SL(2,\mathbb{R})^N$, $X = H^N$.

4.2 SL(2,R)-invariant tensors

Recall that the representation $(\mathbb{R}^2, id)$ of $SL(2,\mathbb{R})$ has been denoted by $(\mathcal{V}^{(1)}, \gamma^{(1)})$.

In the book [19] of H. Weyl, $T(\mathbb{R}^2)^{SL(2,\mathbb{R})}$ is shown to be generated by $T^2(\mathbb{R}^2)^{SL(2,\mathbb{R})}$; i.e.,

$$T^\mu(\mathcal{V}^{(1)})^{SL(2,\mathbb{R})} = \begin{cases} 0, & (\text{if } \mu \equiv 1 \ (2)) \\ \left(T^2(\mathcal{V}^{(1)})^{SL(2,\mathbb{R})}\right)^{\otimes n} & (\text{if } \mu = 2n). \end{cases}$$

Actually, in $T^2(\mathcal{V}^{(1)}) = \mathcal{V}^{(1)} \otimes \mathcal{V}^{(1)}$, the one-dimensional subspace

$$\Lambda^2(\mathcal{V}^{(1)}) = \{x \otimes y - y \otimes x \mid x, y \in \mathcal{V}^{(1)}\}$$

is the $SL(2,R)$-invariant part, i.e.,

$$\Lambda^2(\mathcal{V}^{(1)}) = (T^2(\mathcal{V}^{(1)}))^{SL(2,\mathbb{R})}.$$

Then, for $\mu = 2n$,

$$T^{\mu}(\mathcal{V}^{(1)})^{SL(2,\mathbb{R})} = \{(x_1 \wedge y_1) \otimes (x_2 \wedge y_2) \otimes \dots \otimes (x_n \wedge y_n)\},$$

where $x \wedge y$ means $x \otimes y - y \otimes x$.

Now we consider the representation

$$(m\mathcal{V}^{(1)}, m\gamma^{(1)})$$

of $SL(2,\mathbb{R})$.

Since

$$T(m\mathcal{V}^{(1)}) = T(\mathcal{V}^{(1)} \oplus \dots \oplus \mathcal{V}^{(1)})$$

$$= T(\mathcal{V}^{(1)}) \otimes T(\mathcal{V}^{(1)}) \otimes \dots \otimes T(\mathcal{V}^{(1)})$$

as $SL(2,\mathbb{R})$-modules, we have

$$T^{(\mu)}(m\,V^{(1)}) = \bigoplus_{\mu=\nu_1+\nu_2+\cdots+\nu_m} T^{(\nu_1)}(V^{(1)}) \otimes \cdots \otimes T^{(\nu_m)}(V^{(1)})$$

$$\cong \otimes\, T^{(\nu_1+\cdots+\nu_m)}(V^{(1)})$$

$$\cong p^{(\mu,m)}\; T^{(\mu)}(V^{(1)})$$

as $SL(2,\mathbb{R})$-modules. Here $P(\mu,m)$ is the number of solutions

$(\nu_1, \nu_2, \ldots, \nu_m)$

of the diophantine equation:

$\nu_1 + \nu_2 + \cdots + \nu_m = \mu,$

$\nu_i \in \mathbb{Z}$

$0 \leq \nu_i$.

<u>Corollary</u> (4.2.1): $T(m\,V^{(1)})^{SL(2,\mathbb{R})}$ is generated by $T^2(m\,V^{(1)})^{SL(2,\mathbb{R})}$.

So by the Corollary (3.5.1), we have

<u>Corollary</u> (4.2.2): $\Lambda(m\,V^{(1)})^{SL(2,\mathbb{R})}$ is generated by

$\Lambda^2(m\mathcal{V}^{(1)})^{SL(2,\mathbb{R})}$, i.e.,

$$\Lambda^{\mu}(m\mathcal{V}^{(1)})^{SL(2,\mathbb{R})} = \begin{cases} 0 & \text{for } \mu \equiv 1\ (2), \\ \left((\Lambda^2(m\mathcal{V}^{(1)})^{SL(2,\mathbb{R})}\right)^{\wedge n} & \text{if } \mu = 2n. \end{cases}$$

The dimension of $\Lambda^{\mu}(m\mathcal{V}^{(1)})^{SL(2,\mathbb{R})}$ is determined in [6]; where $\dim \Lambda^{\mu}(m\mathcal{V}^{(1)})^{SL(2,\mathbb{R})}$ is denoted by $a(m,\mu,o)$, and

$$a(m,\mu,o) = \begin{cases} \binom{m}{\mu/2}^2 - \binom{m}{\mu/2+1}\binom{m}{\mu/2-1} & \text{for } \mu \equiv 0\ (2). \\ 0 & \text{for } \mu \equiv 1\ (2). \end{cases}$$

4.3 $SL(2,\mathbb{R})^N$-invariant tensors and the proof of (C-2) in the case $G = SL(2,\mathbb{R})$

We begin with some easy fundamental lemmas.

For a representation (F,P) of a group G, i.e., F is a vector space (over some field k), and P is a homomorphism of G to $GL(F)$, we denote by F^G the set of all vectors v in F, such that $P(g)v = v$ for all $g \in G$. We call F^G the "invariant part" of F, or "trivial part" of F.

The first two lemmas are obvious.

Lemma (4.3.1): Let G be a group and (F_i, P_i) $(i = 1, \ldots, k)$ representations of G. Then

$$(F_1 \otimes \ldots \otimes F_k)^G = F_1^G \otimes \ldots \otimes F_k^G.$$

Lemma (4.3.2): Let (F_i, P_i) $(i = 1,2)$ be two representations of G, such that the operation of G on (F_2, P_2) is trivial; i.e., $P_2(g) = id_{F_2}$ for all $g \in G$. Then in the representation

$$(F_1 \otimes F_2, P_1 \otimes P_2)$$

of G, the invariant part is

$$(F_1 \otimes F_2)^G = F_1^G \otimes F_2 .$$

Let G_1, G_2 be two groups, and let (F_i, P_i) be a representation of G_i $(i = 1,2)$. Then $(F_1 \otimes F_2, P_1 \otimes P_2)$ is a representation of $G = G_1 \times G_2$; which is gotten by identifying P_i with $P_i \circ proj_i$ $(i = 1,2)$, so that

$$(P_1 \otimes P_2)(g) = P_1(g_1) \otimes P_2(g_2)$$

for $g = (g_1, g_2)$ $G_1 \times G_2 = G$.

Now, in this situation, we have

Lemma (4.3.3): $(F_1 \otimes F_2)^G = F_1^{G_1} \otimes F_2^{G_2}$.

Proof: By identifying G_1 with $G_1 \times \{1\}$, G_2 with $\{1\} \times G_2$, G_1, G_2 are normal subgroups of $G = G_1 \times G_2$. Elements in G_1, and in G_2 are mutually commutative, and a vector is G-invariant, if and only if, it is G_1-invariant and G_2-invariant. Therefore,

$$(F_1 \otimes F_2)^G = ((F_1 \otimes F_2)^{G_1})^{G_2}$$

$$= (F_1^{G_1} \otimes F_2)^{G_2} = F_1^{G_1} \otimes F_2^{G_2}. \quad \text{QED}$$

Lemma (4.3.4): Let G_1, G_2 be two groups, and let (F_i, P_i) be a representation of G_i $(i = 1,2)$. Then $(F_1 \otimes F_2, P_1 \otimes P_2)$ is a representation of $G_1 \times G_2$, identifying P_i with $P_i \circ proj_i$ $(i = 1,2)$, and

$$(F_1 \oplus F_2)^G = F_1^{G_1} \oplus F_2^{G_2},$$

$$T(F_1 \oplus F_2)^G = T(F_1)^{G_1} \otimes T(F_2)^{G_2},$$

$$\Lambda(F_1 \oplus F_2)^G = \Lambda(F_1)^{G_1} \otimes \Lambda(F_2)^{G_2}.$$

Proof: Since $T(F_1 \oplus F_2) = T(F_1) \otimes T(F_2)$, and

$$\left(T^{(M)}(F_1 \oplus F_2)\right)^G = \bigoplus_{N_1+N_2=M} (T^{(N_1)}(F_1) \otimes T^{(N_2)}(F_2))^G$$

$$= \bigoplus_{N_1+N_2=M} T^{(N_1)}(F_1)^{G_1} \otimes T^{(N_2)}(F_2)^{G_2}$$

(by Lemma (4.3.3)).

Therefore, summing over M we have

$$T(F_1 \oplus F_2)^G = T(F_1)^{G_1} \otimes T(F_2)^{G_2}.$$

Similarly,

$$\Lambda(F_1 \oplus F_2)^G = \Lambda(F_1)^{G_1} \otimes \Lambda(F_2)^{G_2}.$$

Corollary (4.3.5): Let $G_1, G_2, \ldots, G_N$ be N groups, and let (F_i, P_i) be a representation of G_i $(i = 1, \ldots, N)$. Then in the representation $(F_1 \oplus \cdots \oplus F_N, P_1 \oplus \cdots \oplus P_N)$ of $G = G_1 \times \cdots \times G_N$, (identifying P_i with $P_i \circ \mathrm{proj}_i$),

$$T(F_1 \oplus \cdots \oplus F_N)^G = \bigotimes_{i=1}^N T(F_i)^{G_i},$$

$$\Lambda(F_1 \oplus \cdots \oplus F_N)^G = \bigotimes_{i=1}^N \Lambda(F_i)^{G_i}.$$

Let $G = SL(2,\mathbb{R})^N = G_1 \times \cdots \times G_N$, $G_i = SL(2,\mathbb{R})$, be the same as in 1.4 and 1.5 of Chapter I, and let $(\tilde{F},P)$ be the Satake representation (as in 1.4); i.e.,

$$\tilde{F} = \tilde{F}_1 \oplus \cdots \oplus \tilde{F}_N,$$

$$P = P_1 \oplus \cdots \oplus P_N,$$

where

$$\tilde{F}_i \cong m_i \mathcal{V}_i^{(1)},$$

$$P_i \cong m_i \gamma_i^{(1)}, \qquad \gamma_i^{(1)} = \gamma^{(1)} \circ \mathrm{proj}_i.$$

Therefore, $(\tilde{F}_i, P_i)$ is essentially a representation of the i-th component G_i of G.

In this section, we shall investigate the structures of $T(\tilde{F})^G$ and $\Lambda(\tilde{F})^G$.

By corollary (4.3.5), we have

(1) $T(F)^G = T(F_1)^{G_1} \otimes T(F_2)^{G_2} \otimes \dots \otimes T(F_N)^{G_N}$.

Now, since $(\tilde{F}_i, P_i) \cong (m_i V^{(1)}, m_i \gamma^{(1)})$ as a representation of $G_i = SL(2,\mathbb{R})$, by the result in 4.2,

$$T(\tilde{F}_i)^G = T(\tilde{F}_i)^{G_i} \cong T(m_i V^{(1)})^{SL(2,\mathbb{R})}$$

is generated by

$$T^2(\tilde{F}_i)^{G_i} \cong T^2(m_i V^{(1)})^{SL(2,\mathbb{R})}.$$

Thus we have

<u>Theorem</u> (4.3.7): $T(\tilde{F})^G = \bigotimes_{i=1}^{N} T(\tilde{F}_i)^{G_i}$ is generated by

$$\bigoplus_{i=1}^{N} T^2(\tilde{F}_i)^{G_i}.$$

By the last corollary of 3.5, we have

<u>Theorem</u> (4.3.8): $\Lambda(\tilde{F})^G = \bigotimes_{i=1}^{N} \Lambda(\tilde{F}_i)^{G_i}$ is generated by

$$\bigoplus_{i=1}^{N} \Lambda^2(\tilde{F}_i)^{G_i}.$$

Namely, the assumption (C-2) is true for our case of $G = SL(2,\mathbb{R})^N$, $X = H^N$, $P =$ the Satake representation.

Considering the M-th degree part in $\Lambda(\tilde{F})^G = \bigotimes_{i=1}^{N} \Lambda(\tilde{F}_i)^{G_i}$, we have

$$\Lambda^M(\tilde{F})^G = \bigoplus_{\mu_1+\mu_2+\cdots+\mu_N=M} \Lambda^{\mu_1}(\tilde{F}_1)^{G_1} \otimes \cdots \otimes \Lambda^{\mu_N}(\tilde{F}_N)^{G_N}.$$

By taking dimension, and 4.2 (1), we have

<u>Theorem</u> (4.3.9): $\dim \Lambda^M(\tilde{F})^G = \sum_{\mu_1+\cdots+\mu_N=M} \left(\prod_{i=1}^{N} a(m_i,\mu_i,o)\right)$,

where

$$a(m,\mu,o) = \begin{cases} \binom{m}{\mu/2}^2 - \binom{m}{\mu/2+1}\binom{m}{\mu/2-1} & \mu \equiv 0 \quad (2), \\ 0 & \mu \equiv 1 \quad (2). \end{cases}$$

4.4. Summary of our results in 4.1 – 4.3

Theorem (4.4.1): Let $V \xrightarrow{\pi} U$ be the family of abelian varieties defined by

$$G = SL(2,\mathbb{R})^N$$

$$K = SO(2,\mathbb{R})^N$$

$$X = H \times \dots \times H = H^N$$

$$F = \bigoplus_{i=1}^{N} (m_i \, \mathcal{V}_i^{(1)})$$

$$P = \bigoplus_{i=1}^{N} (m_i \, \gamma_i^{(1)})$$

with Γ, L, β, σ appropriately chosen. Then in a generic fiber $F_p = \pi^{-1}(p)$, (where p is a generic point of U over a certain field of definition k), the space of algebraic cycles is the space of Γ-invariant cycles and they are homologically equivalent with linear combinations of homology classes of intersections of divisors, i.e.,

$$c(\mathcal{A}^r(F_p,\mathbb{Q})) = H^{2r}(F_p,\mathbb{Q})^{\Gamma} \quad ,$$

$$c(\mathfrak{A}^r(F_p, \mathbb{Q})) = \underbrace{\vartheta \wedge \vartheta \wedge \ldots \wedge \vartheta}_{r}$$

where $\vartheta = c(\mathfrak{A}^1(F_p, \mathbb{Q}))$.

Proof: (C-1) and (C-2) are true.

Theorem (4.4.2): $\dim_{\mathbb{Q}} c(\mathfrak{A}^r(F_p, \mathbb{Q})) = \dim \Lambda^{2r}(\mathfrak{f})^G$

$$= \sum_{\lambda_1 + \lambda_2 + \ldots + \lambda_N = r} \prod_{i=1}^{N} a(m_i, 2\lambda_i, o) .$$

Proof: $\tilde{F} = \mathfrak{f}$ as G-spaces, so the formula is equivalent to the formula in Theorem (4.3.9).

CHAPTER V. ALGEBRAIC CYCLES GENERATED BY INVARIANT CYCLES

5.1 In this section, we shall describe algebraic cycles W in the total space V of the family of abelian varieties $V \xrightarrow{\pi} U$, generated by specializations of algebraic cycle Z in a generic fiber F_p. See [8], [9].

Take a finitely generated field k_1 ($\subset \mathbb{C}$) of definition for V, U, π ; and denote by k the algebraic closure $\bar{k}_1$ of k_1. $k = \bar{k}_1$ is also a field of definition for V, U, π.

Let p be a generic point in U over k, and let $F_p = \pi^{-1}(p)$ be the corresponding generic fiber in V. Then F_p is defined over $k(p)$. In a given homology class C in $c(\mathfrak{A}^r(F_p, \mathbb{Q}))$, choose an algebraic cycle $Z \in \mathfrak{A}^r(F_p, \mathbb{Q})$, such that $C = c(Z)$. By Chow's moving lemma, there is an algebraic cycle $Z' \in \mathfrak{A}^r(F_p, \mathbb{Q})$ such that

(i) Z' is rationally equivalent to Z, therefore homologically equivalent to Z; $c(Z') = c(Z) = C$.

(ii) Z' is defined over some algebraic extension of $k(p)$.

Therefore, we may assume Z is defined over $\overline{k(p)}$ (the algebraic closure of the field $k(p)$).

First assume that Z is an irreducible algebraic subvariety of F_p. Let K be the smallest field of definition for Z containing $k(p)$. Then K is an algebraic extension of $k(p)$. Let $\{\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_d\}$; $d = [K:k(p)]$ be the set of all isomorphisms of K over $k(p)$ into the universal domain $\mathbb{C}$. Then $\{Z^{\sigma_1} = Z, Z^{\sigma_2}, \ldots, Z^{\sigma_d}\}$ is the complete set of conjugates of Z over $k(p)$, and therefore the cycle

$$\mathfrak{Z} = Z^{\sigma_1} + Z^{\sigma_2} + \ldots + Z^{\sigma_d}$$

is a prime rational cycle in F_p over $k(p)$. Since p is a generic point of U over k, by a theorem of A. Weil [17] (Theorem 6, p. 226), there exists a unique prime rational cycle W in V over k, such that

$$W \cdot F = \mathfrak{Z}.$$

Since k is algebraically closed, W is an irreducible subvariety of V. Moreover, the projection of W to U is surjective, and

$$\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \mathfrak{Z} + \dim_{\mathbb{C}} U$$

$$= \dim_{\mathbb{C}} Z + \dim_{\mathbb{C}} U.$$

Let us denote this variety W by

$$W = \mathrm{Locus}\,(\mathfrak{Z}) = \mathrm{Locus}\,(\mathfrak{Z}/k) = \mathrm{Locus}\,(Z/k)\ ;$$

and call it the locus of $\mathfrak{z}$ (or of Z) over k. Since the codimension of W in V is $\dim V - \dim W = (\dim U + \dim F_p) - (\dim Z + \dim U) = \dim F_p - \dim Z =$ the codimension of Z in F_p, and hence equal to r, $W \in \mathfrak{O}^r(V, \mathbb{Q})$.

For an algebraic cycle

$$Z = \sum_i n_i Z_i \in \mathfrak{O}^r(F_p, \mathbb{Q}),$$

where Z_i's are irreducible subvarieties defined over $\overline{k(p)}$, and $n_i \in \mathbb{Q}$, we define

$$\text{Locus } (Z/k) = \sum_i n_i \text{ Locus } (Z_i/k),$$

so that Locus $(Z/k) \in \mathfrak{O}^r(V, \mathbb{Q})$.

Denote by $\mathfrak{O}^r(F_{p/\overline{k(p)}}, \mathbb{Q})$ the space of $\mathbb{Q}$-linear combinations $\sum_i n_i Z_i$ $(n_i \in \mathbb{Q})$ of algebraic subvarieties Z_i of codimension r in F_p, defined over $\overline{k(p)}$; and by $\mathfrak{O}^r(V/_k, \mathbb{Q})$ the space of $\mathbb{Q}$-algebraic cycles rational over k, then the "Locus" is a map of $\mathfrak{O}^r(F_p/_{\overline{k(p)}}, \mathbb{Q})$ to $\mathfrak{O}^r(V/_k, \mathbb{Q})$.

Now we are going to determine the relation of homology class $c(Z) \in H^{2r}(F_p, \mathbb{Q})$ of $Z \in \mathfrak{O}^r(F_p/_{\overline{k(p)}}, \mathbb{Q})$ and $c(W) \in$

$H^{2r}(V,\mathbb{Q})$ of W = Locus $(Z/k) \in \mathfrak{A}^r(V/_k,\mathbb{Q})$.

First we assume that Z is irreducible subvariety. Let $\mathfrak{z} = Z^{\sigma_1} + Z^{\sigma_2} + \dots + Z^{\sigma_d}$ be the prime rational cycle in F_p over $k(p)$, then the proof of the Theorem in [8], [10], shows that

$$c(Z) = c(Z^{\sigma_1}) = c(Z^{\sigma_2}) = \dots = c(Z^{\sigma_d}) ,$$

hence

$$c(\mathfrak{z}) = d.c(Z).$$

5.2 Harmonic forms on V

In this section, we retain our notations in 1.4 and 1.5 of Chapter I, in particular, $G = SL(2,\mathbb{R})^N$, $X = H^N$.

By Theorem (2.4.1), we have

$$H^r(V,\mathbb{C}) = \bigoplus_{a+b=r} H^{\langle a,b\rangle}(V) ,$$

and

$$H^{\langle a,b\rangle}(V) = \{ x \in H^r(V) \mid \Theta_n x = n^b x \}$$

$$\cong H^a(\Gamma, \Lambda^b(\mathfrak{g})).$$

In order to adjust different notations used in [6], [11], and [14], we introduce the following convention.

The standard complex coordinates $(z_1, z_2, \ldots, z_N) \in H^N$ give the associated real coordinates

$$(x_1, y_1, x_2, y_2, \ldots, x_N, y_N).$$

Putting

$$x^1 = x_1,\ x^2 = x_2, \ldots, x^N = x_N,\ x^{N+1} = y_1,\ x^{N+2} = y_2,$$

$$\ldots, x^{2N} = y_N,$$

we introduce a real coordinate system

$$x = (x^1, x^2, \ldots, x^{2N}).$$

This x also denotes the corresponding point in X.

We use both real coordinate systems

$$(x^1, x^2, \ldots, x^{2N}) \text{ and } (x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$$

or

$$(z_1, z_2, \ldots, z_N)$$

for our convenience.

Put

$$\omega_i = \frac{dx_i \wedge dy_i}{y_i^{\,2}} \qquad (i = 1, \ldots, N).$$

They are G-invariant differential 2-forms on $X = H^N$, so they also represent 2-forms on $U = \Gamma\backslash X$. $\omega_{i_1} \wedge \ldots \wedge \omega_{i_a}$ are also considered as differential 2a-forms on X, as well as on U. In particular, $\omega_1 \wedge \ldots \wedge \omega_N$ is a 2N-form. We denote by vol(U) the total integral value:

$$(1) \quad \mathrm{vol}(U) = \int_{\Gamma\backslash X} \omega_1 \wedge \ldots \wedge \omega_N .$$

Also, we put

$$(2) \quad \Omega_U = \frac{1}{\mathrm{vol}(U)} \omega_1 \wedge \ldots \wedge \omega_N .$$

Let $\xi^1, \xi^2, \ldots, \xi^{2m}$ be the $\mathbb{R}$-linear coordinate system of $\tilde{F}$; where $2m = \dim_{\mathbb{R}} \tilde{F}$, with a set of generators $[e_1, \ldots, e_{2m}]$ of L as the coordinate basis. A vector u in $\tilde{F}$ has coordinate $(\xi^1, \ldots, \xi^{2m})$ if

$$u = \sum_i \xi^i e_i ,$$

and $u \in L$ if and only if $\xi^1, \ldots, \xi^{2m} \in \mathbb{Z}$. Therefore,

(3) $$\int_{L\backslash\tilde{F}} d\xi^1 \wedge \ldots \wedge d\xi^{2m} = 1,$$

where we consider $d\xi^1 \wedge \ldots \wedge d\xi^{2m}$ as an 2m-form on the torus ${}_L\backslash\tilde{F}$.

Similarly, $\eta = \sum c_{i_1 \ldots i_b} d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_b}$ is considered as a b-form on the torus ${}_L\backslash\tilde{F} = F$. Hence, it will also be considered as a b-form on the product $X \times F$. Therefore, if $\eta = \sum c_{i_1 \ldots i_b} d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_b}$ is moreover Γ-invariant, η may be considered as a b-form on ${}_\Gamma\backslash X \times F = V$. The set of all such b-forms on V is denoted by $\mathcal{H}^{\langle o,b \rangle}$:

(4) $$\mathcal{H}^{\langle o,b \rangle} = \left\{ \eta = \sum c_{i_1 \ldots i_b} d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_b} \ , \ \eta \text{ is } \Gamma\text{-invariant} \right\}.$$

Also, via the projection $V \xrightarrow{\pi} U$, a-forms $\omega = \sum f_A(x)\, dx^A$ are considered as forms ω on V; i.e., we identify $\pi^* \omega$ with ω. In Kuga's note [6], it is shown that if ω is a harmonic form on U with respect to the metric ds_o^2, $\pi^*\omega = \omega$ is also harmonic on V with respect to the metric $ds^2 = ds_o^2 + A(x)(d\xi, d\xi)$. The set of such harmonic forms on V is denoted by $\mathcal{H}^{\langle a,o\rangle}$.

Kuga also showed in [6], that

(i) $\mathcal{H}^{\langle a,o\rangle}$ is the space of harmonic forms representing the subspace $H^{\langle a,o\rangle}(V)$:

(5) $\mathcal{H}^{\langle a,o\rangle} = H^{\langle a,o\rangle}(V)$

by the identification of harmonic forms with cohomology classes. Therefore,

(6) $\mathcal{H}^{\langle a,o\rangle}(V) \cong H^a(U)$.

(ii) $\mathcal{H}^{\langle o,b\rangle}$ is the space of harmonic forms representing the subspace $H^{\langle o,b\rangle}(V)$;

(7) $\mathcal{H}^{\langle o,b\rangle} = H^{\langle o,b\rangle}(V)$.

<u>Proposition</u> (5.2.1): By the identification of homology groups $H_s(V,\mathbb{Q})$ with cohomology groups $H^{2N+2m-s}(V,\mathbb{Q})$ via Poincaré duality,

$$H_s(V,\mathbb{Q}) \overset{\wp}{\Longrightarrow} H^{2N+2m-s}(V,\mathbb{Q}).$$

We have

(i) The (point) $\in H_o(V,\mathbb{Q})$ goes to

$$(\text{point}) \overset{\wp}{\Rightarrow} \Omega_U \wedge d\xi^1 \wedge \dots \wedge d\xi^{2m}$$

$$= \frac{1}{\text{vol}(U)} \omega_1 \wedge \dots \wedge \omega_N \wedge d\xi^1 \wedge \dots \wedge d\xi^{2m}.$$

(ii) The orientation class $[V] \in H_{2N+2m}(V,\mathbb{Q})$ goes to

$\wp([V]) = 1 \in H^0(V,\mathbb{Q})$.

For any point $Q \in U$, the fiber $F_Q \subset V$ is an 2m-dimensional cycle in V. Since F_{Q_1}, F_{Q_2} are homotopic in V, their homology classes are the same, therefore, $[F_Q] \in H_{2m}(V)$. Identifying the homology group $H_{2m}(V)$ with the cohomology group $H^{2N}(V)$, we shall determine the harmonic form representing the (co)-homology class $[F_Q] = c(F_Q)$.

<u>Proposition</u> (5.2.2): By the identification $\wp$ in Prop. (5.2.1), $[F_Q] \in H_{2m}(V)$ goes to

$$\wp([F_Q]) = \Omega_U = \frac{1}{\text{vol}(U)} \omega_1 \wedge \dots \wedge \omega_N \in \mathcal{H}^{\langle 2N,0\rangle}$$

$$= H^{\langle 2N,0\rangle}(V) \subset H^{2N}(V).$$

<u>Proof</u>: This is given implicitly in Kuga's Notes ([6], Chap. II).

The section s of zero : $U \xrightarrow{s} V$, defines a submanifold $s(U) \subset V$ which we often identify with U. $s(U) = U$ defines a homology class $[s(U)] = [U] \in H_{2N}(V)$.

<u>Proposition</u> (5.2.3): The identification β in Prop. (5.2.1) sends $[U]$ to

$$\beta([U]) = d\xi^1 \wedge d\xi^2 \wedge \dots \wedge d\xi^{2N} \in \mathcal{H}^{\langle o,2N\rangle} = H^{\langle o,2N\rangle}(V)$$

$$\subset H^{2N}(V).$$

<u>Proof</u>: See Kuga's Notes ([6], Chapter II).

5.3 <u>Description of algebraic cycles</u> $\mathcal{Z}$, W <u>as differential forms</u>

Take an algebraic cycle Z in the generic fiber F_p of codimension r. Represent the corresponding cohomology class $c(Z) \in H^{2r}(F_p, \mathbb{R}) = \Lambda^{2r}(\mathfrak{f})$ in terms of harmonic $2r$-forms ζ:

$$c(Z) = \zeta = \sum c_{(i_1 \dots i_r)} d\xi^{i_1} \wedge \dots \wedge d\xi^{i_r}.$$

Here $c_{(i_1 \dots i_r)}$ are constants, since ζ is harmonic. Moreover ζ is G-invariant, because $c(Z) \in \Lambda^{2r}(\mathfrak{f})^G$; therefore ζ is considered as a 2r-form on V.

Now the inclusion $F_p \overset{\iota}{\hookrightarrow} V$ induces an inclusion

$$H_*(F_p) \overset{\iota_*}{\hookrightarrow} H_*(V),$$

so the cycle $c(Z) \in H_{2m-2r}(F_p)$ goes to a cycle $\iota_* c(Z) \in H_{2m-2r}(V)$.

Proposition (5.3.1): By the identification β of homology and cohomology, $\iota_* c(Z)$ goes to

$$\beta(\iota_* c(Z)) = \Omega_U \wedge \zeta \in H^{2N+2r}(V).$$

Proof: For the proof, it is sufficient to show that the equality

$$(1) \quad \int_Z \omega = \int_V (\Omega_U \wedge \zeta) \wedge \omega$$

holds for any harmonic form ω of degree $2m-2r$.

Now Z is a cycle in F_p, and $c(Z) = \zeta$ as cohomology classes in $H^{2r}(F_p, \mathbb{R})$. This means that

$$(2) \quad \int_Z \xi = \int_{F_p} \zeta \wedge \xi$$

for any harmonic $(2m-2r)$-forms ξ on F_p.

Now in order to show (1), we have to determine all harmonic forms in V; this has been done by Kuga in [6]. All harmonic forms of $H^k(V,\mathbb{R})$ are of the form

$$(3) \qquad \omega = \sum_{|A|+|B|=k} f_{A,B}(x)\, dx^A \wedge d\xi^B ,$$

where $A = (i_1,\ldots,i_a)$ is an oriented subset of indices $\{1,2,\ldots,2N\}$, and $B = (j_1,\ldots,j_b)$ is an oriented subset of $\{1,2,\ldots,2m\}$, and dx^A stands for $dx^{i_1}\wedge\ldots\wedge dx^{i_a}$, and $d\xi^B$ for $d\xi^{j_1}\wedge\ldots\wedge d\xi^{j_b}$, and $f_{A,B}(x)$ is some real analytic function of the variables x.

Harmonic forms ω belonging to $H^{\langle a,b\rangle}(V)$ are of the form

$$(4) \qquad \omega = \sum_{\substack{|A|=a,\\ |B|=b}} f_{A,B}(x)\, dx^A \wedge d\xi^B .$$

Now, take a harmonic form $\omega \in H^{2m-2r}(V)$. Denote $\omega^{\langle a,b\rangle}$ to be the $\langle a,b\rangle$-part of ω according to the decomposition $H^{2m-2r}(V) = \bigoplus_{a+b=2m-2r} H^{\langle a,b\rangle}(V)$. Then

$$\omega = \sum \omega^{\langle a,b\rangle} .$$

Let

$$\omega^{\langle a,b\rangle} = \sum_{\substack{|A|=a,\\ |B|=b}} f_{A,B}(x)\ dx^A \wedge d\xi^B .$$

Now, the left-hand-side of 5.3 (1) is

$$\int_Z \omega = \sum_{a+b=2m-2r} \int_Z \omega^{\langle a,b\rangle} .$$

If $a > 0$, then

$$\int_Z \omega^{\langle a,b\rangle} = \sum \int_Z f_{A,B}(x)\ dx^A \wedge d\xi^B$$

is 0, since $Z \subset F_p$, therefore, $dx^A = 0$ along Z. We have

$$\int_Z \omega = \int_Z \omega^{\langle o,b\rangle} = \sum_B f_{\emptyset,B}(x) \int_Z d\xi^B$$

$$= \sum_{|B|=2m-2r} f_{\emptyset,B}(x) \int_{F_p} \zeta \wedge d\xi^B$$

(by (2)).

Now $\omega^{\langle o,2m-2r\rangle} \in H^{\langle o,2m-2r\rangle}(V) = \mathcal{H}^{\langle o,2m-2r\rangle}$

$= \{ \sum c_{j_1 \ldots j_{2m-2r}}\ d\xi^{j_1} \wedge \ldots \wedge d\xi^{j_{2m-2r}}, G\text{-invariant} \}$, by 5.2 (4), 5.2 (7), and the coefficients $f_{\emptyset,B}$ must be constants; $f_{\emptyset,B}(x) = c_{\emptyset,B}$. We may write

$$\int_Z \omega = \int_Z \omega^{\langle o,2m-2r\rangle} = \sum c_{\emptyset,B} \int_{F_p} \zeta \wedge d\xi^B$$

$$= \int_{F_p} \zeta \wedge (\sum_B c_{\emptyset,B} \, d\xi^B),$$

where $\omega^{\langle o,2m-2r\rangle} = \sum c_{\emptyset,B} \, d\xi^B$.

Secondly, the right-hand-side of (1) is

$$\int_V (\Omega_U \wedge \zeta) \wedge \omega = \int_V \Omega_U \wedge \zeta \wedge (\sum \omega^{\langle a,b\rangle}).$$

By 5.2 (2), we have

$$\Omega_U = \frac{1}{\mathrm{vol}(U)} \omega_1 \wedge \ldots \wedge \omega_N$$

$$= \frac{\pm 1}{\mathrm{vol}(U)} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{2N} / (y_1 y_2 \ldots y_N)^2 .$$

So, if $a \neq 0$, then

$$\Omega_U \wedge \omega^{\langle a,b\rangle} = \Omega_U \wedge (\sum_{\substack{|A|=a,\\ |B|=b}} f_{A,B} \, dx^A \wedge d\xi^B) = 0.$$

Therefore,

$$\int_V (\Omega_U \wedge \zeta) \wedge \omega = \int_V (\Omega_U \wedge \zeta) \wedge \omega^{\langle o, 2m-2r \rangle}.$$

Since $\omega^{\langle o, 2m-2r \rangle} = \sum_B c_{\emptyset, B}\, d\xi^B$ with constant coefficients $c_{\emptyset, B}$,

$$\int_V \Omega_U \wedge \zeta \wedge \omega = \sum_B c_{\emptyset, B} \int_V \Omega_U \wedge \zeta \wedge d\xi^B.$$

By Fubini's theorem, we have

$$\begin{aligned}
\int_V \Omega_U \wedge \zeta \wedge \omega &= \sum c_{\emptyset, B} \int_U \Omega_U(x) \int_{F_x} (\zeta \wedge d\xi^B) \\
&= \sum c_{\emptyset, B} \left(\int_U \Omega_U(x)\right)\left(\int_{L \backslash F} (\zeta \wedge d\xi^B)\right) \\
&= \sum c_{\emptyset, B} \int_{F_p} (\zeta \wedge d\xi^B) \\
&= \int_{F_p} \zeta \wedge \omega^{\langle o, 2m-2r \rangle}.
\end{aligned}$$

This is equal to the left-hand-side, so we have proved (1), hence Proposition (5.3.1).

Finally, we consider the cycle $W = \text{Locus}\,(\mathfrak{z}/k)$.

Since $c(\mathfrak{z}) = d.c(Z)$, $c(\mathfrak{z})$ is represented by the harmonic form $d.\zeta \in H^{2r}(F_p)$.

<u>Proposition</u> (5.3.2): By the identification $\wp$, $c(W) \in H_{2N+2m-2r}(V)$ goes to $\wp(c(W)) = d.\zeta \in H^{2r}(V)$.

<u>Proof</u>: To prove this, it is sufficient to see

$$(5) \qquad \int_W \omega = d \int_V \zeta \wedge \omega ,$$

for all harmonic forms ω of degree $2N+2m-2r$ in V.

For a point $x \in U$, denote $F_x . W = \mathfrak{z}_x$. Then for generic point $x \in U$, $\mathfrak{z}_x$, $\mathfrak{z} = \mathfrak{z}_p$ are homotopic to each other in V; therefore, they belong to the same homology class; i.e., cohomologous to $d.\zeta$ (via the identification $\wp$).

Let

$$\omega = \sum \omega^{\langle a,b \rangle} ,$$

$$\omega^{\langle a,b \rangle} = \sum_{\substack{|A| = a, \\ |B| = b}} f_{A,B}(x)\, dx^A \wedge d\xi^B .$$

Then the left-hand-side of (5) is

$$\int_W \omega = \sum \int_W f_{A,B}(x)\ dx^A \wedge d\zeta^B$$

$$= \sum_{a+b=2r} \sum_{\substack{|A|=a,\\ |B|=b}} \int_U f_{A,B}(x) \left(\int_{\gamma_x} d\zeta^B \right) dx^A$$

by Fubini's theorem.

If $a < 2N$, then the integral is 0, therefore

$$\int_W \omega = \int_W \omega^{\langle 2N, 2m-2r \rangle} .$$

$$= \int_U \sum_B f_{S,B}(x) \left(\int_{\gamma_x} d\zeta^B \right) dx^S$$

$$= d_\circ \int_U \sum_B f_{S,B}(x) \left(\int_{F_x} \zeta \wedge d\zeta^B \right) dx^S$$

$$= d_\circ \int_U \sum_B f_{S,B}(x)\, dx^S \left(\int_{L \setminus \tilde{F}} \zeta \wedge d\zeta^B \right) ,$$

where $S = \{1, 2, \ldots, 2N\}$.

Now, the right-hand-side of (5) is

$$d_\circ \int_V \zeta \wedge \omega = d_\circ \int_V \zeta \wedge \sum \omega^{\langle a, b \rangle} .$$

Since $\int_V \zeta \wedge \omega^{\langle a, b \rangle} = 0$ if $a \neq 2N$,

$$d.\int_V \zeta\wedge\omega = d.\int_V \zeta\wedge\omega^{\langle 2N,2m-2r\rangle}$$

$$= d.\int_V \zeta\wedge(\Sigma_B\, f_{S,B}(x)\ dx^S\wedge d\xi^B)$$

$$= d.\int_U f_{S,B}(x)\left(\int_{F_x}\zeta\wedge d\xi^B\right)dx^S$$

$$= d.\left(\int_U \Sigma_B\, f_{S,B}(x)\ dx^S\right)\int_{L\backslash\tilde{F}}(\zeta\wedge d\xi^B).$$

This is equal to the left-hand-side, and proves (5) and Proposition (5.3.2).

Corollary (5.3.3):

$$\beta(c(Z)) = \Omega_U\wedge\zeta \in H^{\langle 2N,2r\rangle}(V)\ ,$$

$$\beta(c(W)) = d.\zeta \in H^{\langle 0,2r\rangle}(V)\ .$$

The space of algebraic cycles $c(\mathfrak{A}^r(F_p,\mathbb{R}))$ is equal to $H^{2r}(F_p,\mathbb{R})^G = \Lambda^{2r}(\mathfrak{f})^G$ by Theorem (4.4.1).

Let $h = h_r = \dim_{\mathbb{R}}\Lambda^{2r}(\mathfrak{f})^G$ and take a basis $\zeta_1, \ldots, \zeta_h$ of $\Lambda^{2r}(\mathfrak{f})^G$. Take algebraic subvarieties $Z_1, \ldots, Z_h \in \mathfrak{A}^r(F_p)$, such that $c(Z_1), \ldots, c(Z_h) \in H^{\langle 2N,2r\rangle}(V)$ are represented by $\Omega_U\wedge\zeta_1, \ldots, \Omega_U\wedge\zeta_h$.

They are linearly independent, and we have

$$\dim \Lambda^{2r}(\mathfrak{g})^G = h = \dim (c(\mathfrak{A}^{N+r}(V)) \cap H^{\langle 2N,2r\rangle}(V))$$
$$\leq \dim (H^{\langle 2N,2r\rangle}(V)).$$

Since

$$\begin{aligned} H^{\langle 2N,2r\rangle}(V) &\cong H^{2N}(\Gamma, X, \Lambda^{2r}(\mathfrak{g})) \\ &\cong H^{2N}(\Gamma, X, \Lambda^{2r}(\mathfrak{g})^G) \qquad \text{(by Cor. (2.2.7))} \\ &\cong H^{N,N}(\Gamma, X, \Lambda^{2r}(\mathfrak{g})^G) \qquad \text{(by Cor. (2.2.7))} \\ &\cong (\Lambda^{2r}(\mathfrak{g})^G)^{\binom{N}{N}} \\ &= \Lambda^{2r}(\mathfrak{g})^G, \end{aligned}$$

since $2N \neq N$. Hence

$$\text{(6)} \quad \dim \Lambda^{2r}(\mathfrak{g})^G = \dim (c(\mathfrak{A}^{N+r}(V)) \cap H^{\langle 2N,2r\rangle}(V))$$
$$= \dim (H^{\langle 2N,2r\rangle}(V)).$$

From this, we have

<u>Corollary</u> (5.3.4): $c(\mathfrak{A}^{N+r}(V,\mathbb{Q})) \cap H^{\langle 2N,2r\rangle}(V,\mathbb{Q}) = H^{\langle 2N,2r\rangle}(V,\mathbb{Q})$, i.e., the subspace $H^{\langle 2N,2r\rangle}(V,\mathbb{Q})$ of $H^{2N+2r}(V,\mathbb{Q})$ is spanned

by algebraic (co)-cycles.

Finally, we consider $W_i = \text{Locus}(Z_{i/k})$. By Proposition (5.3.2), $\mathcal{P}(c(W_i)) = d_i \zeta_i$, where d_i are positive integers. Therefore $\mathcal{P}(c(W_i))$ $i = 1, \ldots, h$ are linearly independent algebraic cocycles in $H^{\langle o,2r\rangle}(V)$. We have the following inequalities,

$$h_r = \dim \Lambda^{2r}(\mathfrak{f})^G \leq \dim_{\mathbb{Q}} (c(\mathcal{O}\!\!\mathcal{l}^r(V,\mathbb{Q})) \cap H^{\langle o,2r\rangle}(V,\mathbb{Q}))$$

$$\leq \dim_{\mathbb{Q}} (H^{\langle o,2r\rangle}(V,\mathbb{Q})).$$

Since

$$H^{\langle o,2r\rangle}(V) \cong H^o(\Gamma,X, \Lambda^{2r}(\mathfrak{f})) = H^o(\Gamma,X, \Lambda^{2r}(\mathfrak{f})^G)$$

$$\cong (\Lambda^{2r}(\mathfrak{f})^G)^{\binom{N}{o}} \cong \Lambda^{2r}(\mathfrak{f})^G,$$

again, we have used Cor. (2.2.7), hence

$$\dim_{\mathbb{Q}} (H^{\langle o,2r\rangle}(V,\mathbb{Q})) = \dim_{\mathbb{R}} (H^{\langle o,2r\rangle}(V,\mathbb{R}))$$

$$= \dim (\Lambda^{2r}(\mathfrak{f})^G) = h_r.$$

And, we have

<u>Corollary</u> (5.3.5): $c(\mathcal{O}\!\!\mathcal{l}^r(V,\mathbb{Q}) \cap H^{\langle o,2r\rangle}(V,\mathbb{Q})) = H^{\langle o,2r\rangle}(V,\mathbb{Q})$,

i.e., the subspace $H^{\langle o,2r\rangle}(V,\mathbb{Q})$ of $H^{2r}(V,\mathbb{Q})$ is spanned by algebraic (co)-cycles; $\wp(c(W_i))$ $(i = 1,2,\ldots,h)$.

CHAPTER VI. ALGEBRAIC CYCLES WHICH COME FROM THE BASE SPACE U

6.1 ω_i $(i = 1, \ldots, N)$ on U

The factor of automorphy $J_i(\gamma, z) = (c_i z_i + d_i)$ defines a line bundle L_i on the variety $U = \Gamma \backslash X$, and $J_1^{a_1} \cdots J_N^{a_N} = \prod_{i=1}^{N} (c_i z_i + d_i)^{a_i}$ corresponds to the line bundle

$$a_1 L_1 + a_2 L_2 + \cdots + a_N L_N.$$

The space of sections to this line bundles is isomorphic to the space of automorphic forms:

$$\Gamma(U, \textstyle\sum a_i L_i) = \left\{ \varphi \mid \varphi(\gamma z) = \varphi(z) \prod (c_i z_i + d_i)^{a_i} \right\}$$

$$= \mathcal{G}_{(a_1, a_2, \ldots, a_N)}(\Gamma).$$

If a_i are all sufficiently large, then the line bundle is very ample, and the Chern class

$$\underline{c}(\textstyle\sum a_i L_i) = \sum a_i \, \underline{c}(L_i)$$

is the cohomology class $\beta(D)$ of the divisor D defined by

$$\varphi(z) = 0, \qquad \varphi \in \mathcal{S}_{(a_1 \ldots a_N)}(\Gamma), \qquad \varphi \neq 0.$$

On the other hand, the Chern class $\underline{\underline{c}}(\sum a_i L_i) \in H^2(U,\mathbb{R})$ is given as de Rham cohomology class of differential 2-forms

$$\underline{\underline{c}}(\sum a_i L_i) = \sum_{i=1}^{N} a_i \left(\frac{dz_i \wedge d\bar{z}_i}{y_i^2} \right) \left(\frac{1}{2\pi\sqrt{-1}} \right)$$

(see, for instance, Gunning [4], Chapter 7).

Now, since

$$\frac{dz \wedge d\bar{z}}{y^2} = \frac{(dx+idy) \wedge (dx-idy)}{y^2} = -2i \, \frac{dx \wedge dy}{y^2} \quad ;$$

therefore,

$$\underline{\underline{c}}(\sum a_i L_i) = - \left(\sum_{i=1}^{N} a_i \left(\frac{dx_i \wedge dy_i}{y_i^2} \right) \right) \frac{1}{\pi}$$

$$= - \frac{1}{\pi} \left(\sum_{i=1}^{N} a_i \omega_i \right).$$

We have,

<u>Corollary</u> (6.1.1): $\frac{1}{\pi}\omega_1, \ldots, \frac{1}{\pi}\omega_N \in c(\mathcal{O}\!\!\mathcal{l}^1(U,\mathbb{Q})) \subset H^2(U,\mathbb{Q})$.

Corollary (6.1.2): $\frac{1}{\pi^r}(\omega_{i_1}\wedge \dots \wedge \omega_{i_r}) \in c(\mathfrak{A}^r(U,\mathbb{Q})) \subset H^{2r}(U,\mathbb{Q})$.

By identifying $\pi^*(\omega)$ with ω, we have

$$\frac{1}{\pi^r}(\omega_{i_1}\wedge \dots \wedge \omega_{i_r}) \in c(\mathfrak{A}^r(V,\mathbb{Q})) \subset H^{2r}(V,\mathbb{Q}).$$

Since $\pi^*(H^{2r}(U,\mathbb{Q})) = H^{\langle 2r,o\rangle}(V,\mathbb{Q})$, because of 5.2 (6), hence

(1) $\frac{1}{\pi^r}(\omega_{i_1}\wedge \dots \wedge \omega_{i_r}) \in H^{\langle 2r;o\rangle}(V,\mathbb{Q})$.

These $\binom{N}{r}$-differential forms $\omega_{i_1}\wedge \dots \wedge \omega_{i_r}$ $(i_1<i_2< \dots <i_r)$ are linearly independent, and we have

Corollary (6.1.3): $\binom{N}{r} \leq \dim_{\mathbb{Q}} (c(\mathfrak{A}^r(V,\mathbb{Q}) \cap H^{\langle 2r,o\rangle}(V,\mathbb{Q}))$

$$\leq \dim_{\mathbb{Q}} (H^{\langle 2r,o\rangle}(V,\mathbb{Q})).$$

On the other hand, since

$$H^{\langle 2r,o\rangle}(V,\mathbb{R}) \subseteq H^{2r}(\Gamma,X,\Lambda^o\circ\rho^*) = H^{2r}(\Gamma,X,\text{trivial})$$

$$= H^{2r}(U,\mathbb{R}),$$

and by Cor. (2.2.2), $\dim H^{2r}(U,\mathbb{R}) = \binom{N}{r}$ for $2r \neq N$;

comparing the dimensions, we have

Corollary (6.1.4): If $2r \neq N$, then

$$c(\mathfrak{A}^r(V,\mathbb{Q}) \cap H^{\langle 2r,o\rangle}(V,\mathbb{Q})) = H^{\langle 2r,o\rangle}(V,\mathbb{Q}),$$

i.e., $H^{\langle 2r,o\rangle}(V,\mathbb{Q})$ is spanned by algebraic (co)-cycles of type $\frac{1}{\pi^r}\omega_{i_1}\wedge\ldots\wedge\omega_{i_r}$.

CHAPTER VII. THE COHOMOLOGY GROUPS $H^{\langle 2p,2r\rangle}(V,\mathbb{Q})$ FOR $2p \neq N$

7.1 By

$$H^{\langle 2p,2r\rangle}(V,\mathbb{R}) \cong H^{2p}(\Gamma, X, \Lambda^{2r}(\mathfrak{g})),$$

using Cor. (2.2.6), we have

$$H^{2p}(\Gamma, X, \Lambda^{2r}(\mathfrak{g})) = H^{2p}(\Gamma, X, \Lambda^{2r}(\mathfrak{g})^G) = \Lambda^{2r}(\mathfrak{g})^G \otimes \mathbb{C}^{\binom{N}{p}},$$

if $2p \neq N$. Now, consider differential forms

$$\frac{1}{\pi^p}(\omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_p}) \wedge \zeta_j$$

where $\zeta_1, \zeta_2, \ldots, \zeta_{h_r} \in \Lambda^{2r}(\mathfrak{g})^G$ are those forms representing $c(Z_i)$ as in 5.3. These $\binom{N}{p}\cdot h_r$ harmonic forms

$$\frac{1}{\pi^p}(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \wedge \zeta_j$$

(for $i_1 < i_2 < \cdots < i_p$, and $j = 1, \ldots, h$) are linearly independent, they belong to $H^{\langle 2p,2r\rangle}(V)$, and are algebraic cocycles. Therefore,

(1) $\binom{N}{p} h_r \leq \dim_{\mathbb{Q}} (c(\mathfrak{A}^{p+r}(V,\mathbb{Q})) \cap H^{\langle 2p,2r\rangle}(V))$

$$\leq \dim_{\mathbb{Q}} (H^{\langle 2p, 2r \rangle}(V, \mathbb{Q}))$$

$$= \dim (\Lambda^{2r}(\mathfrak{g})^G \otimes \mathbb{C}^{\binom{N}{p}})$$

$$= \binom{N}{p} \dim \Lambda^{2r}(\mathfrak{g})^G$$

$$= \binom{N}{p} h_r .$$

The inequalities in (1) are in fact equalities, hence

Proposition (7.1.1): $c(\mathfrak{A}^{p+r}(V, \mathbb{Q})) \cap H^{\langle 2p, 2r \rangle}(V, \mathbb{Q}) = H^{\langle 2p, 2r \rangle}(V, \mathbb{Q})$ if $2p \neq N$; i.e., if $2p \neq N$, the subgroup $H^{\langle 2p, 2r \rangle}(V, \mathbb{Q})$ of $H^{2p+2r}(V, \mathbb{Q})$ is spanned by algebraic cycles.

CHAPTER VIII. PROOF OF THE MAIN THEOREM

In this chapter, we finally determine algebraic cycles of codimension $2r \leq N$, in the total space V of the family of abelian varieties $V \xrightarrow{\pi} U$, with $U = \Gamma\backslash X$, $X = H^N$, $G = SL(2,\mathbb{R})^N$. We state our main result:

Theorem (8.1.1): $c(\mathcal{U}^r(V,\mathbb{Q})) = H^{2r}(V,\mathbb{Q})$ for $2r \leq N$.

7.1 By Cor. (5.3.4) and (5.3.5), cohomology classes of $H^{\langle 2N,2r\rangle}(V)$, $H^{\langle o,2r\rangle}(V)$ are generated by algebraic (co-)cycles ; and by Cor. (6.1.4), if $2p \neq N$, $H^{\langle 2p,o\rangle}(V)$ is also spanned by algebraic (co-)cycles. Further, by Prop. (7.1.1), if $2p \neq N$, $H^{\langle 2p,2r\rangle}(V,\mathbb{Q})$ is also spanned by algebraic (co-)cycles.

Now

$$H^k(V,\mathbb{Q}) = \bigoplus_{a+b=k} H^{\langle a,b\rangle}(V,\mathbb{Q});$$

and

$$H^{\langle a,b\rangle}(V,\mathbb{Q}) \cong H^a(\Gamma,X,\Lambda^b(\mathcal{J})).$$

This is equal to (0) if $a \equiv 1 \mod (2)$ and $a \neq N$.

Analogous to 7.1, we have

$$H^a(\Gamma, X, \Lambda^b(\mathfrak{g})) = \Lambda^b(\mathfrak{g})^G \otimes \mathbb{C}^{\binom{N}{p}} = \begin{cases} \{0\} & \text{if } b \equiv 1 \ (2) \\ \Lambda^{2r}(\mathfrak{g})^G \otimes \mathbb{C}^{\binom{N}{p}} & \text{if } b=2 \end{cases}$$

if $a = 2p \neq N$. If we rewrite the decomposition in the following way,

$$\begin{aligned} H^k(V,\mathbb{C}) &= \bigoplus_{a+b=k} H^{\langle a,b\rangle}(V,\mathbb{C}) \\ &= H^{\langle N,k-N\rangle}(V,\mathbb{C}) \qquad\qquad (\text{if } k \geq N) \\ &\quad \oplus \Big(\bigoplus_{\substack{a+b=k \\ a\neq N}} H^{\langle a,b\rangle}(V,\mathbb{C}) \Big) \\ &= H^{\langle N,k-N\rangle}(V,\mathbb{C}) \oplus \Big(\bigoplus_{p+r=k/2} \Lambda^{2r}(\mathfrak{g}_{\mathbb{C}})^G \otimes \mathbb{C}^{\binom{N}{p}} \Big). \end{aligned}$$

Here $H^{\langle N,k-N\rangle}(V,\mathbb{C}) = 0$ if $k < N$, and the second sum is 0 if k is odd. The dimension of $\Lambda^{2r}(\mathfrak{g})^G$ is given by Theorem (4.3.9). It is

$$\dim \Lambda^{2r}(\mathfrak{g})^G = \sum_{\substack{t_1+\dots+t_N=2r \\ t_i \text{ are even}}} \Big(\prod_{i=1}^{N} a(\mu_i, t_i, o) \Big)$$

where

$$a(\mu,t,o) = \begin{cases} \binom{\mu}{t/2}^2 - \binom{\mu}{t/2+1}\binom{\mu}{t/2-1} & t \equiv 0 \ (2) \\ 0 & t \equiv 1 \ (2) \end{cases}$$

Now, if $k < 2N$, then

(1) $H^k(V,\mathbb{Q}) = \{0\}$ for odd k;

(2) $H^k(V,\mathbb{C}) = \bigoplus_{r+p=M} (\Lambda^{2r}(\mathfrak{f})^G \otimes \mathbb{C}^{\binom{N}{p}}$ for even $k = 2M$.

8.2 Proof of Theorem (8.1.1)

Let $k = 2M < N$, then

$$H^{2M}(V,\mathbb{Q}) = \bigoplus_{a+b=2M} H^{a,b}(V,\mathbb{Q}) = \bigoplus_{p+r=M} H^{2p,2r}(V,\mathbb{Q}).$$

Because $a = 2M < N$, if a is odd, then $H^{a,b} = 0$; and if $a = 2p$, then

$$H^{\langle 2p,b\rangle}(V,\mathbb{C}) = H^{2p}(\Gamma, X, \Lambda^{b}(\mathfrak{g}_{\mathbb{C}})) = \Lambda^{b}(\mathfrak{g}_{\mathbb{C}})^{G} \otimes \mathbb{C}^{\binom{N}{p}},$$

and this is 0, if b is odd. Now, by Prop. (7.1.1), $H^{\langle 2p,2r\rangle}(V,\mathbb{Q})$ is spanned by algebraic cycles, since $2p = a < N$. Therefore, the total cohomology group $H^{2M}(V,\mathbb{Q})$ is spanned by algebraic cycles. QED.

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