SPECTRAL INVARIANCE FOR NORMAL OPERATORS UNDER TRACE CLASS PERTURBATIONS

A dissertation presented

by

Scott O'Hare

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

July, 1979

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

SCOTT O'HARE

We, the dissertation committee for the above candidate for the Ph.D. degree, hereby recommend acceptance of the dissertation.

Detlef Gromoll, Chairman

Ronald G. Douglas, First reader

Charles Berger, Second reader

Stephan Brown, Political Science Dept.

The dissertation is accepted by the Graduate School.

, Dean

Graduate School

Abstract of the Dissertation

SPECTRAL INVARIANCE FOR NORMAL OPERATORS

UNDER TRACE CLASS PERTURBATIONS

by

Scott O'Hare

Doctor of Philosophy

in

MATHEMATICS

State University of New York at Stony Brook

1979

In this paper we generalize an important result of Perturbation Theory known as the Kato-Rosenblum Theorem. This theorem asserts that the absolutely continuous parts of two self-adjoint operators whose difference is trace class are unitarily equivalent. We shall obtain the same conclusion for a broad class of normal operators, at the same time developing some rather interesting techniques for the treatment of spectral invariance problems.

The Direct Integral representation for normal operators is used throughout; unitary equivalence of the absolutely continuous parts (with respect to a σ-finite Borel measure μ) translates in this context to the μ-essential equality of the spectral multiplicity functions. For a subset Γ of the complex plane, and μ as above, we say the K-R result holds for (Γ,μ)

provided the spectral multiplicity functions of normal operators N_1, N_2 are μ-essentially equal whenever $N_2 - N_1$ is trace class, and the spectra of N_1, N_2 lie in Γ.

The new techniques of this paper give a particularly simple proof of the K-R result in the unitary case (when Γ is the unit circle and μ is Lebesgue measure). A noticeable feature of the method is that it is necessary to examine only the geometric and measure theoretic properties of Γ. As we discover, the conditions imposed on Γ need not be especially strong; a rather large class of Jordan areas Γ can be defined (the class $AC_{\Psi}(I)$) for which the K-R result holds. This class contains the C^1 homeomorphic images of the unit arc I, together with the convex Jordan arcs (those which form part of the boundary of a convex region). We also consider homomorphic transformations of Γ under which the problem is invariant, and conclude from the existence of certain such maps that Γ need only be "locally $AC_{\Psi}(I)$" order for the K-R result to hold. Thus we obtain our main result (Theorem 7.2). Since no boundedness conditions on Γ are imposed, there is no such condition on the normal operators either. Finally, we discuss a natural analogue of the Kuroda Hypothesis for the problem, and show that, just as in the self-adjoint case, the unitary invariance of absolutely continuous parts still holds.

Dedication

To Jeannette O'Hare,

my mother.

Table of Contents

LIST OF SYMBOLS

LIST OF DIAGRAMS

ACKNOWLEDGEMENTS

I would like to express my thanks to all those whose assistance made this project possible. Special thanks go to Professor Charles Berger for his commentary and support, and of course to Professor Ronald G. Douglas, thesis advisor and Operator Theorist Extraordinaire, for his dedication and unfailing insight.

A special debt of gratitude is due to my own wife, Sylvia, for her tremendous accomplishments on my behalf, and finally, to Mrs. Estella Shivers for her competent and speedy preparation of the manuscript.

§0. INTRODUCTION

The "diagonalization" results of Weyl (1909 [19]) and von-Neumann (1935 [18]) may be said to have initiated one of the major investigations in Operator Theory. Given a self-adjoint operator A in a separable Hilbert space $\mathcal{H}$, we may find by the Weyl Theorem a compact operator K such that $\tilde{A} = A + K$ is diagonal, that is, there exists an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors for $\tilde{A}$. Moreover, we may take K so that

i) $\|K\|_\infty = \sup \{|\lambda| : \lambda \text{ is an eigenvalue } (K^*K)^{\frac{1}{2}})$

is arbitrarily small. The von-Neumann Theorem asserts that K may be a Hilbert-Schmidt operator ($K \in \mathcal{J}_2$) with Schmidt norm $\|K\|_2$ arbitrarily small. In 1958 Kuroda [11] showed that K may be chosen to lie in the p^{th} Schatten class $\mathcal{J}_p$, with $\|K\|_p$ as small as desired, for $p > 1$.

However, the case $p = 1$ is special, as usual. In 1957, T. Kato and M. Rosenblum published results which show that the absolutely continuous part of a self-adjoint operator A in $\mathcal{H}$ is invariant, up to unitary equivalence, under trace class ($\mathcal{J}_1$) perturbations (see Theorem 1.1). It is therefore impossible to "diagonalize" A with such a perturbation.

An analogous theory, initiated in 1971 by the important papers of Berg [3] and Sikonia [21], has been developed for normal operators. It is presently known (cf. Voiculescu [16]) that a normal operator may in fact be diagonalized by a Hilbert-

Schmidt perturbation. Also, in 1974, it was pointed out in a Doctoral Dissertation by J. Voigt that the absolutely continuous part of a normal operator, with respect to planar Lebesgue measure m_2, is unitarily invariant under trace class perturbations [16].

However, this latter theorem falls short of adequately generalizing the Kato-Rosenblum results; in fact, it is easily seen that there is a major class of normal perturbation problems about which nothing at all is said. Suppose for instance that

$$m_2(\sigma(N_1) \cup \sigma(N_2)) = 0$$

for some normal operators N_1, N_2 in $\mathcal{H}$; then the equivalence of absolutely continuous parts is a triviality (Proposition 2.1). It is not hard to find important types of Normal Operators that fall into this category.

One senses that we need expect only "zero dimensional" differences in the spectra when $N_2 - N_1$ is trace class; it is this sort of generalization of the Kato-Rosenblum Theorem (1.1) that seems most appropriate. In the context of the result cited above, this means that not only should we have invariance for the absolutely continuous part (with respect to m_2), but for a good deal of the "singular" part as well.

In the present paper we go a long way toward substantiating this conjecture.

We begin by taking $\Gamma \subset \mathbb{C}$ with μ a positive σ-finite Borel measure on Γ, and N a normal operator in $\mathcal{H}$ with $\sigma(N) \subset \Gamma$. One can define in a natural way the absolutely continuous subspace for N with respect to μ (Proposition 2.1). The absolutely continuous part of N is then the restriction of N to that subspace. Given a pair of normal operators N_1, N_2 with $\sigma(N_1) \cup \sigma(N_2) \subset \Gamma$, we show that the absolutely continuous parts with respect to μ are equivalent to iff the spectral multiplicity functions δ_{N_1}, δ_{N_2} are equal a.e. (μ). Then we ask what conditions on (Γ,μ) are sufficient to guarantee that $\delta_{N_1} = \delta_{N_2}$ a.e. (μ), whenever N_2, N_1 are normal operators, as above, with $N_2 - N_1 \in \mathcal{J}_1$. Under such conditions we say "the K-R result holds" for (Γ,μ).

In §4 a method is introduced which gives a rather elegant proof of the K-R result for the unitary case $(\mathbb{T},m)$ ($\mathbb{T}$ is the unit circle). These methods are then formalized into a set of sufficient conditions for the general case (Proposition 4.6). An important ingredient here is the fact that the trace class perturbation remains invariant under the holomorphic functional calculus (at least when N_1, N_2 are bounded). This turns out to be a special case of results developed by Birman and Solomyak [20].

Sections 5 and 6 consist of geometric and measure-theor tic analysis: Convex rectifiable Jordan Curves in section 5, and a more general class of curves, called $AC_\Psi(I)$, in section 6. The key properties turn out to be local; thus in section 7 we see that (Γ,μ) need only be "essentially locally" like a curve in $AC_\Psi(I)$. In particular Γ need not be bounded (thus N_1, N_2 need not be bounded either). Also a rather broad class of C^1 curves are covered by this result. We conclude by extending our result under the so-called "Kuroda Hypothesis" (§7.4). All of the spaces (Γ,μ) have the property that $m_2(\Gamma) = 0$; that is, the measure μ is singular with respect to m_2.

Where does this leave us with respect to the diagonalization problem? One suspects that the parallels with the self-adjoint case will continue to hold. Thus we might anticipate that normal operators of the kind discussed here may be diagonalized by operators $K \in \mathcal{J}_p$ with $1 < p < 2$, and $\|K\|_p$ arbitrarily small. Techniques from the present investigation might combine with the methods of Kuroda's paper [11] to deal with this possibility.

Finally, we must mention the following: The classical proof of the Kato-Rosenblum Theorem involves demonstrating existence of certain strong operator limits known as wave operators. These objects are themselves of substantial interest, particularly in that branch of quantum dynamics known as Scattering Theory. However, we shall not be concerned with wave operators here, confining ourselves instead to the spectral invariance problem.

Before we begin, a few brief notes. For a topological space X, we shall let $\mathcal{B}_X$ denote the Borel subsets of X. $\mathcal{H}$ shall be a seperable Hilbert space throughout, with Γ is a subspace of $\mathbb{C}$, and μ will be a positive σ-finite Borel measure on Γ. The bounded operators on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$; the bounded linear transforms from a Hilbert space $\mathcal{H}$ to a second Hilbert space $\mathcal{H}'$ will be written $\mathcal{B}(\mathcal{H},\mathcal{H}')$. The compact operators in $\mathcal{B}(\mathcal{H})$ are denoted by $\mathcal{J}_\infty$.

Each $K \in \mathcal{J}_\infty$ can be represented in the form

$$K = \sum_{n=1}^{\infty} s_n(\cdot,\omega_n)\theta_n$$

where the collections $\{\omega_n\}$ and $\{\theta_n\}$ are orthonormal systems in $\mathcal{H}$, and the sequence $\{s_n\}$ tends monotonically to zero. The s_n are the eigenvalues of the positive operator $|K| = (KK^*)^{\frac{1}{2}}$. K is said to lie in the Schatten class $\mathcal{J}_p$ for $1 \le p < \infty$ provided

$$\|K\|_p = \{\sum_{n=1}^{\infty} |s_n|^p\}^{\frac{1}{p}} < \infty$$

($\|K\|_\infty$ is defined by (1)). The Schatten classes $\mathcal{J}_p$ are ideals in $\mathcal{B}(\mathcal{H})$, $1 \le p \le \infty$, and are also Banach spaces with their respective norms $\|\cdot\|_p$. The relations

$$\|AKB\|_p \le \|A\| \ \|K\|_p \ \|B\|$$

are also well known, for $A,B \in \mathcal{B}(\mathcal{H})$, $K \in \mathcal{J}_p$. $\mathcal{J}_1$ is called the trace class, and $\mathcal{J}_2$ is called the collection of Hilbert-Schmidt operators. (See Reed and Simon [13]).

§1. Spectral Theory

Let N be a normal operator with domain $\mathfrak{D}(N)$ dense in a seperable Hilbert space $\mathcal{H}$. A unique, finitely additive, projection-valued measure E is associated with N, defined on the Borel subsets $\mathfrak{B}$ of $\mathbb{C}$ and supported on $\sigma(N)$, with respect to which $N = \int \lambda \, dE(\lambda)$. Integration dE is a $*$-algebra isomorphism from the class of E - essentially bounded functions $L^\infty(E)$ on $\sigma(N)$ to a W^*-algebra of bounded operators on $\mathcal{H}$ (generated by N if N is bounded).

For $x,y \in \mathcal{H}$ the relation $E_{x,y}(\omega) = (E(\omega)x,y)$ for $\omega \in \mathfrak{B}$, defines a complex measure $E_{x,y}$ on $\mathbb{C}$ satisfying

$$(\phi(N)x,y) = \int \phi \, dE_{x,y} \qquad \phi \in L^\infty(E).$$

For $x \in \mathcal{H}$, $\nu_x = E_{x,x}$ is called the <u>spectral measure</u> associated with x; $\mathfrak{D}(N)$ consists of those $x \in \mathcal{H}$ for which $\int |\lambda|^2 \, d\nu_x < \infty$. E is called the <u>spectral resolution</u> for N. (cf. Rudin, Functional Analysis [14]).

For a self-adjoint operator A in $\mathcal{H}$ the collection

$$\mathcal{H}_{ac}(A) = \mathcal{H}^m_{ac}(A) = \{x \in \mathcal{H} : \nu_x << m\}$$

is a reducing subspace for A, called the <u>absolutely continuous subspace</u>. We write $A^{ac} = A|\mathcal{H}_{ac}$. The following result (Kato [9], p.542) provides this paper with its main point of departure.

1.1. <u>Theorem (Kato/Rosenblum)</u> Let A_1, A_2 be self-adjoint operators in $\mathcal{H}$ such that $A_2 - A_1 = T$ is trace class. Then there is a partial isometry $V \in \mathcal{B}(\mathcal{H})$ with initial space $\mathcal{H}_{ac}(A_1)$ and final space $\mathcal{H}_{ac}(A_2)$ such that $VA_1 = A_2V$.

One proof of 1.1 proceeds by establishing existence under the hypotheses of the strong limit.

$$W_+ = W_+(A_2, A_1) = s - \lim_{t\to\infty} e^{itA_2} e^{-itA_1} P$$

where P is orthogonal projection $\mathcal{H} \to \mathcal{H}_{ac}(A_1)$. This is done first for finite rank perturbations T, then various estimates are used in passing to the limit in trace norm. The <u>wave operator</u> W_+ is then the desired partial isometry.

A second method produces W_+ as the implicit solution to the Friedrichs integral equation

$$W_+ = P_1 + i \int_0^\infty e^{itA_1} T W_+ e^{-itA_1} dt$$

where the integral again is evaluated as a strong limit. The operation $\Gamma^+ B = s - \lim_{t\to\infty} \int_0^t e^{itA_1} B e^{-itA_1} dt$ which is defined on a linear submanifold $\mathcal{D}(\Gamma^+)$ of $\mathcal{B}(\mathcal{H})$ is an inverse for commutation with A_1, in the sense that

$$Bu = [(\Gamma^+ B)A_1 - A_1 \Gamma^+(B)]u; \ u \in \mathcal{D}(A_1), \ B \in \mathcal{D}(\Gamma^+).$$

Both methods can be found in full detail in [9]. We shall give a new and rather different proof in §3.

One of the major distinct forms of the Spectral Theorem asserts that each normal operator in $\mathcal{H}$ acts as multiplication by the identity on a suitable "continuous direct sum" or direct integral of separable Hilbert spaces over $\sigma(N)$. This version of the Spectral Theorem generates all the others, for once it is established, the various results known collectively as "the Spectral Theorem" follow as easy corollaries. In the context of the Direct Integral Representation, as this version is known, the Kato-Rosenblum Theorem undergoes a definite simplification. Although we shall not prove the Direct Integral Representation Theorem, we shall sketch in some detail the theoretical background. In so doing, we shall develop machinery which will subsequently prove useful. Our outline follows Dixmier's presentation [5].

A field of separable Hilbert spaces on a set X is an assignment $\xi \to \mathcal{H}_\xi$ of a separable Hilbert space $\mathcal{H}_\xi$ to each $\xi \in X$. An element $x = \{x(\xi)\}$ of $\Pi\, \mathcal{H}_\xi$ is called a vector field on X; these form a complex vector space.

Suppose X is endowed with a positive, σ-finite measure μ. The field $\{\mathcal{H}_\xi\}$ is called μ-measurable if there is a subspace V of vector fields such that:

a) $\xi \to (x(\xi), y(\xi))$ is a μ-measurable function for every $x, y \in V$.

b) If y is a vector field and $\xi \to (x(\xi), y(\xi))$ is μ-measurable $\forall\, x \in V$, then $y \in V$.

c) There is a sequence $y_1, y_2, \cdots$ of vector fields such that $\{y_k(\xi)\}_{k=1}^{\infty}$ generates $\mathcal{H}_\xi$ for each ξ.

Under these circumstances, the multiplicity function $\delta(\xi) = \dim \mathcal{H}_\xi$ is also μ-measurable. Further, an analogue of the Gram-Schmidt method allows c) to be strengthened so that, for each $\xi \in X$, the sequence $\{y_k(\xi)\}_{k=1}^{\delta(\xi)}$ actually forms an orthonormal basis for $\mathcal{H}_\xi$. The collection $\{y_k\}_{k=1}^{\infty}$ is then called a field of orthonormal bases for $\prod_{\xi \in X} \mathcal{H}_\xi$; V is the collection of μ-measurable vector fields. Given a field $\{y_k\}_{k=1}^{\infty}$ of orthonormal bases each $x \in V$ has the expansion

$$\text{i)} \quad x(\xi) = \sum_{k=1}^{\infty} f_k(\xi) y_k(\xi)$$

where $f_k(\xi) = (x(\xi), y_k(\xi))$ is a μ-measurable function on X.

Consider now those $x = \{x(\xi)\} \in V$ which are "square integrable", that is, for which

$$\|x\|^2 = \int_X \|x(\xi)\|^2 \, d\mu(\xi) < \infty .$$

These obviously form a linear subspace of V. If we factor out the closed subspace $N = \{x \in V : \|x\| = 0\}$ we obtain a Hilbert space $\tilde{\mathcal{H}}$ whose inner product is given by

$$(x, y) = \int_X (x(\xi), y(\xi)) \, d\mu(\xi)$$

(Verification proceeds precisely as in the proof that $L^2(\mu)$ is a Hilbert space. With one or two exceptions we shall allow the identification of an equivalence class in $\mathfrak{H}$ with its representative.)

$\mathfrak{H}$ is called the Direct Integral Hilbert Space (or Hilbert Integral) for the field $\{\mathcal{H}_\xi\}$ and measure μ, symbolically

$$\mathfrak{H} = \int_X^\oplus \mathcal{H}_\xi \, d\mu(\xi) .$$

Also the notation $\int_X^\oplus x(\xi)d\mu(\xi)$ denotes those $x = \{x(\xi)\}$ which are elements of $\mathfrak{H}$. When μ is counting measure we have

$$\mathfrak{H} = \bigoplus_{\xi \in X} \mathcal{H}_\xi$$

(this accounts for Naimark's term "continuous direct sum" for $\mathfrak{H}$ referred to earlier). Another noteworthy case occurs when $\delta(\xi) \equiv 1$ on X; then $\mathfrak{H} = L^2(\mu)$. If $\{y_k\}_{k=1}^{\infty}$ is a field of orthonormal bases, observe that the y_k lie in $\mathfrak{H}$ iff the measure μ is finite. On the other hand, for $x \in \mathfrak{H}$, the partial sums of the "Fourier expansion" given by i) are square integrable vector fields which converge to x in $\mathfrak{H}$.

Suppose now we are given two (μ) measurable fields $\{\mathcal{H}_\xi\}$, $\{\mathcal{H}'_\xi\}$ of separable Hilbert spaces on (X,μ) with Hilbert integrals $\mathfrak{H}$, $\mathfrak{H}'$ respectively. We define a measurable field of bounded linear transformations to be an assignment of some $T_\xi \in \mathcal{B}(\mathcal{H}_\xi,\mathcal{H}'_\xi)$ to each $\xi \in X$, with the property that $\{T_\xi x(\xi)\}$

is a measurable vector field whenever $\{x(\xi)\}$ is. The function $\delta(\xi) = \|T(\xi)\|$ is then measurable; if $s \in L^{\infty}(\mu)$ the field $\{T_\xi\}$ is called <u>essentially bounded</u> and the relation

ii) $\quad (Tx)(\xi) = T_\xi x(\xi) \qquad \xi \in X$

defines a bounded linear transformation $T : \mathfrak{H} \to \mathfrak{H}'$, symbolically,

iii) $\quad T = \int_X^{\oplus} T_\xi \, d\mu(\xi).$

In the event $s \notin L^{\infty}(\mu)$, iii) defines an unbounded transformation T whose domain is some proper subspace of $\mathfrak{H}$. Linear transformations arising from measurable fields $\{T_\xi\}$ in this way are called <u>decomposable</u>. The field $\{T_\xi\}$ corresponding to a decomposable transformation is μ-essentially unique. If

$$S = \int_X^{\oplus} S_\xi \, d\mu(\xi), \quad T = \int_X^{\oplus} T_\xi \, d\mu(\xi),$$

$$T' = \int_X^{\oplus} T'_\xi \, d\mu(\xi)$$

are decomposable transformations with $S,\ T \in \mathfrak{B}(\mathfrak{H}, \mathfrak{H}')$, $T' \in \mathfrak{B}(\mathfrak{H}', \mathfrak{H})$, $\alpha, \beta \in \mathbb{C}$ then it is easily verified that $\alpha S + \beta T$, $T'S$, and T^* are decomposable transformations in $\beta(\mathfrak{H}, \mathfrak{H}')$, $\mathfrak{B}(\mathfrak{H})$, and $\mathfrak{B}(\mathfrak{H}', \mathfrak{H})$, respectively with

$$\alpha S + \beta T = \int_X^{\oplus} (\alpha S_\xi + \beta T_\xi) \, d\mu(\xi), \quad T'S = \int_X^{\oplus} T'_\xi S_\xi \, d\mu(\xi)$$

$$T^* = \int_X^{\oplus} T^*_\xi \, d\mu(\xi).$$

A special class of decomposable operators in $\mathcal{B}(\widetilde{\mathcal{H}})$ are the diagonal operators; these are the multiplication operators M_f for $f \in L^\infty(\mu)$ defined by

$$(M_f x)(\xi) = f(\xi)x(\xi), \quad x \in \widetilde{\mathcal{H}}.$$

Let $\widetilde{\mathcal{H}}' = \int_X^{\oplus} \mathcal{H}'_\xi \, d\mu(\xi)$ as above, and let the diagonal operators in $\mathcal{B}(\widetilde{\mathcal{H}}')$ be written M'_f, $f \in L^\infty(\mu)$. We then have the following characterization of decomposable operators in $\mathcal{B}(\widetilde{\mathcal{H}}, \widetilde{\mathcal{H}}')$:

1.2. Theorem: $T \in \mathcal{B}(\widetilde{\mathcal{H}}, \widetilde{\mathcal{H}}')$ is decomposable if $TM_f = M'_f T$ for every $f \in L^\infty(\mu)$.

If X is a subset of the complex plane and f is the identity function on X we write simply M for M_f (even if $f \notin L^\infty(\mu)$). We now state the Direct Integral Representation Theorem for normal operators.

1.3. Theorem: Let N be a normal operator in H. Then there is a finite positive Borel measure ν supported on $\sigma(N)$, a ν-measurable field $\{\mathcal{H}_\zeta\}$ of separable Hilbert spaces on $\sigma(N)$, and a unitary transformation $U : \mathcal{H} \to \widetilde{\mathcal{H}} = \int_{\sigma(N)}^{\oplus} \mathcal{H}_\zeta d\nu(\zeta)$ for which $UN = MU$. The measure ν is unique up to mutual absolute continuity, and the multiplicity function $\delta_N(\zeta) = \dim \mathcal{H}_\zeta$ is ν-essentially unique.

In this representation the spectral resolution is given by

$$E(\omega) = \int^{\oplus} \chi_\omega(\zeta) I_\zeta d\nu(\zeta) \qquad \omega \in \mathcal{B}$$

when I_ζ is the identity on $\mathcal{H}_\zeta$. It follows that the spectral measures $\nu_x = E_{x,x}$ satisfy

$$\text{iv)} \quad \nu_x(\omega) = \int_\omega^\oplus \|x(\zeta)\|^2 d\nu(\zeta) \qquad \omega \in \mathcal{B}$$

i.e. $$d\nu_x = \|x(\zeta)\|^2 d\nu$$

(we shall frequently identify $x \in \mathcal{H}$ with its image $Ux \in \tilde{\mathcal{H}}$).

§2. Absolute Continuity and Spectral Multiplicity

In this section we make the simplification of the Kato-Rosenblum Theorem promised earlier, and develop appropriate definitions for the normal case.

Let N be a normal operator in $\mathcal{H}$, and let μ be a positive, σ-finite Borel measure in the complex plane. Let $\tilde{\mathcal{H}} = \int^{\oplus} \mathcal{H}_{\xi} d\nu(\xi)$ be the direct integral representation space for N, $\delta(\xi)$ the multiplicity function, and $U : \mathcal{H} \to \tilde{\mathcal{H}}$ the unitary which implements the representation. Also let ν_{ac}, ν_s be the absolutely continuous and singular parts respectively of ν in its Jordan Decomposition with respect to μ.

Finally, set

$$\mathcal{H}^{\mu}_{ac}(N) = \{x \in \mathcal{H} : \nu_x << \mu\}$$

where ν_x is the spectral measure associated with x.

2.1. Proposition: $\mathcal{H}^{\mu}_{ac}(N)$ is a reducing subspace for N, in fact, $U\mathcal{H}^{\mu}_{ac}(N) = \int^{\oplus} \mathcal{H}_{\xi} d\nu_{ac}(\xi)$, and this latter space is unitarily equivalent to $\int^{\oplus} \mathcal{H}_{\xi} d\mu(\xi)$ via a decomposable transformation.

Proof: That $U\mathcal{H}^{\mu}_{ac}(N) = \int^{\oplus} \mathcal{H}_{\xi} d\nu_{ac}(\xi)$ is immediate from iv), and it is evident from this and the representation heorem $\mathcal{H}^{\mu}_{ac}(N)$ is a reducing subspace for N.

Now set $S = \{\xi : \frac{d\nu}{d\mu}(\xi) > 0\}$; then $\exists\ S' \subset \mathbb{C} \setminus S$ with $\mu(S) = \nu_s(\mathbb{C} \setminus S') = 0$. Since $\nu(\mathbb{C} \setminus (S \cup S')) = 0$ we may set $\mathcal{H}_{\xi} = \{0\}$ for $\xi \notin S \cup S'$ without changing $\tilde{\mathcal{H}}$. We then have

v) $\int_{\mathbb{C}}^{\oplus} \mathcal{H}_\xi \, d\mu(\xi) = \int_S^{\oplus} \mathcal{H}_\xi \, d\mu(\xi)$.

Write $\tilde{\mathcal{H}}_\mu = \int_S^{\oplus} \mathcal{H}_\xi \, d\mu(\xi)$, $\tilde{\mathcal{H}}_{ac}^{\mu} = \int_S^{\oplus} \mathcal{H}_\xi \, \frac{d\nu}{d\mu}(\xi) \, d\mu(\xi)$. Define $V_\xi \in \mathcal{B}(\mathcal{H}_\xi)$ for $\xi \in S$ by $V_\xi x = \frac{d\nu}{d\mu}(\xi)^{-\frac{1}{2}} x$, and $V : \tilde{\mathcal{H}}_\mu \to \tilde{\mathcal{H}}_{ac}^{\mu}$ by

vi) $(Vx)(\xi) = V_\xi(x(\xi)) = \frac{d\nu}{d\mu}(\xi)^{-\frac{1}{2}} x(\xi)$ for

$x = \{x(\xi)\} \in \tilde{\mathcal{H}}_\mu$. Then

$$(\|Vx\|_{\tilde{\mathcal{H}}_{ac}^{\mu}})^2 = \int_S^{\oplus} \|(Vx)(\xi)\|^2 \, \frac{d\nu}{d\mu}(\xi) \, d\mu(\xi)$$

$$= \int_S^{\oplus} \|x(\xi)\|^2 \, d\mu(\xi) = (\|x\|_{\tilde{\mathcal{H}}_\mu})^2$$

so that V is isometric. Since V^{-1} may be given explicitly by the formula $(V^{-1}y)(\xi) = V_\xi^{-1} y(\xi) = \frac{d\nu}{d\mu}(\xi) y(\xi)$ for $y \in \tilde{\mathcal{H}}_{ac}^{\mu}$ $= \int_S^{\oplus} \mathcal{H}_\xi d\nu_{ac}(\xi)$, we have shown that V is unitary. Finally, it is obvious from vi) that V is decomposable. ∎

$\mathcal{H}_{ac}^{\mu}(N)$ is called <u>absolutely continuous subspace</u> for N, with respect to μ.

2.2. <u>Theorem</u>: Let N_1, N_2 be normal operators in $\mathcal{H}$, μ a σ-finite positive Borel measure on $\mathbb{C}$. Then the following are equivalent:

a) The spectral multiplicity functions for N_1, N_2 are equal a.e. (μ) in $\mathbb{C}$.

b) There is a unitary transformation $W : \mathcal{H}_{ac}^{\mu}(N_1) \to \mathcal{H}_{ac}^{\mu}(N_2)$ such that $W\tilde{N}_1 = \tilde{N}_2 W$; where $\tilde{N}_1$ is the restriction to

$\mathfrak{H}^{\mu}_{ac}(N_i)$ of N_i for $i = 1,2$.

<u>Proof</u>: Let $\int^{\oplus}\mathfrak{H}^i_\xi \, d\nu_i(\xi) = \tilde{\mathfrak{H}}^i$ be the direct integral representation space for N_i, $i = 1,2$. Given a) it suffices, in view of Proposition 2.1, to exhibit a decomposable unitary transformation $U : \tilde{\mathfrak{H}}^1_\mu \to \tilde{\mathfrak{H}}^2_\mu$, where $\tilde{\mathfrak{H}}^i_\mu = \int^{\oplus}\mathfrak{H}^i_\xi \, d\mu(\xi)$. Therefore let $\{y^i_k\}$ be a field of orthonormal bases for $\{\mathfrak{H}^i_\xi\}$, $i = 1,2$, and define $U_\xi : \mathfrak{H}^1_\xi \to \mathfrak{H}^2_\xi$ by $U_\xi(y^1_k(\xi)) = y^2_k(\xi)$ for all k, ξ. Since $\dim \mathfrak{H}^1_\xi = \mathfrak{H}^2_\xi$ a.e. (μ) we have U_ξ unitary a.e. (μ). Therefore $U = \int^{\oplus} U_\xi \, d\mu(\xi)$ is unitary.

Now suppose b) is true. Applying Proposition 2.1 and theorem 1.3 we obtain a unitary $U : \tilde{\mathfrak{H}}^1_\mu \to \tilde{\mathfrak{H}}^2_\mu$ satisfying $UM_1 = M_2U$ (M_i is multiplication by the identity in $\tilde{\mathfrak{H}}^i_\mu$, $i = 1,2$). Let E_1, E_2 be the spectral resolution for M_1, M_2 respectively. Given $\omega \in \mathfrak{B}$ it is easily verified that $U^*E_2(\omega)UM_1 = M_1U^*E_2(\omega)U$, and therefore (cf Rudin [14] Theorem 13.33) $U^*E_2(\omega)U$ commutes with $E_1(\omega')$ for all $\omega' \in \mathfrak{B}$.

Take $B \subset C$ compact, $x \in \int^{\oplus}_B \mathfrak{H}^1_\xi \, d\mu(\xi) = \operatorname{ran} E_1(B)$, and let $B_{x,U} = \{\xi : \|Ux(\xi)\| > 0\}$. If $\mu(B_{x,U}\setminus B) > 0$, then $\exists\, z_0 \in C$, $\rho > 0$ such that $B_\rho(z_0) \cap K = \emptyset$ and $\mu(\omega) > 0$, where $\omega = B_{\rho/2}(z_0) \cap B_{x,U}$. Set $y = U^*E_2(\omega)Ux$; from the foregoing we have $y \in \int^{\oplus}_K \mathfrak{H}^1_\xi \, d\mu(\xi)$, therefore

$$\|(z_0 - M_1)y\|^2 = \int_B |z_0 - \xi|^2 \, \|y(\xi)\|^2 \, d\mu(\xi) \geq \rho^2 \, \|y\|^2.$$

On the other hand,

$$\|(z_0 - M_1)y\|^2 = \|U(z_0 - M_1)y\|^2 = \|(z_0 - M_2)Uy\|^2$$

$$= \|(z_0 - M_2)\ E_2(\omega)Ux\|^2 = \|E_2(\omega)(z_0 - M_2)E_2(\omega)Ux\|^2$$

$$= \int_\omega |z_0 - \xi|^2\ \|E_2(\omega)Ux(\xi)\|^2\ d\mu(\xi) \le \frac{\rho^2}{4}\ \|E_2(\omega)Ux\|^2$$

$= \frac{\rho^2}{4}\ \|y\|^2$. This contradiction shows that $\mu(B_{x,U}\backslash B) = 0$; it follows that

vii) $E_2(B)UE_1(B) = UE_1(B)$ for compact $B \subset \mathbb{C}$.

Now take arbitrary $B \in \mathfrak{B}$. By regularity and σ-finiteness of μ we can find subsets ω_i of B, $i = 1,2,\ldots$ each of which is the countable union of compacts, with $\mu(B - \omega_i) \to 0$ as $i \to \infty$. Fix $x \in \operatorname{ran} E_1(B)$ and let $B_{x,U} = \{\xi : \|Ux(\xi)\| > 0\}$, $\tilde{\omega} = B_{x,U}\backslash B$. Then obviously $\|E_1(B\backslash\omega_j)X\| \to 0$ as $i \to \infty$; also by vii) we have $E_2(\tilde{\omega})UE(\omega_j) = 0$. Therefore

$\|E_2(\tilde{\omega})Ux\| = \|E_2(\tilde{\omega})UE_1(B - \omega_j)x\| \le \|E_1(B\backslash\omega_j)x\|$, so that $E_2(\tilde{\omega}) = 0$, $\mu(\tilde{\omega}) = 0$, and vii) holds for arbitrary $B \in \mathfrak{B}$; therefore

$$E_2(B)U = E_2(B)U(E_1(B) + E_1(\mathbb{C}\backslash B))$$

$$= E_2(B)U\ E_1(B) + E_2(B)U\ E_1(\mathbb{C}\backslash B)$$

$$= U\ E_1(B) + E_2(B)E_2(\mathbb{C}\backslash B)U\ E_1(\mathbb{C}\backslash B)$$

$$= U\ E_1(B).$$

In other words, U intertwines multiplication by characteristic functions. Since simple functions are norm dense in $L^\infty(\mu)$ (even in the unbounded case) we have

$$M_f U = U M_f \qquad f \in L^\infty(\mu) \qquad\qquad f \in L^\infty(\mu)$$

so that U is decomposable, $U = \int^\oplus U_\xi \, d\mu(\xi)$, by Theorem 1.2. Since U is unitary, U_ξ must be unitary for μ-almost all ξ; consequently $\dim \mathcal{H}^1_\xi = \dim \mathcal{H}^2_\xi$ a.e. (μ). ∎

The Kato-Rosenblum Theorem 1.1 may now be stated in the following equivalent form. As we shall see, this form of the theorem is well suited to generalization.

2.2. Theorem: Let A_1, A_2 be self-adjoint operators with $T = A_2 - A_1$ in trace class. Then the spectral multiplicity functions for A_1, A_2 are equal almost everywhere (m) in $\mathbb{R}$.

The last result of this section gives a "factorization" in $\tilde{\mathcal{H}}$ for Hilbert Schmidt Operators on $\mathcal{H}$.

2.4. Proposition: Let $\mathcal{H}$ be separable Hilbert space and $\{\mathcal{H}_\xi\}$ a field of separable Hilbert spaces over a measure space (X,μ), where μ is σ-finite and positive. Suppose $U : \mathcal{H} \to \tilde{\mathcal{H}} = \int_X^\oplus \mathcal{H}_\xi \, d\mu(\xi)$ is a unitary transformation, and $K \in \mathcal{B}(\mathcal{H})$ is Hilbert Schmidt; then a representative may be chosen from each element of $\tilde{\mathcal{H}}$ so that the equation

viii) $K_\xi x = (UKx)(\xi) \qquad x \in \mathcal{H}$

defines a Hilbert-Schmidt transformation $K_\xi : \mathfrak{H} \to \mathfrak{H}_\xi$ for each ξ.

Proof. We first choose a countable orthonormal basis $\{e_n\}$ for $\mathfrak{H}$ and let Y be the set of finite linear combinations of these with complex rational (C_Q) coefficients. Y is countable, $Y = \{y_1, y_2, \ldots\}$, and dense in $\mathfrak{H}$; also Y is a linear manifold over C_Q. Now fix a representative u_{y_k} for each $y_k \in Y$. Given an ordered 4-tuple $\alpha = (r,s,q_1,q_2)$, where r,s are positive integers and q_1, q_2 complex rationals, define

$$w_\alpha = \{\xi \in X : u_{q_1 y_r + q_2 y_s} \neq q_1 u_{y_r}(\xi) + q_2 u_{y_s}(\xi)\}$$

Since vector space operations are well defined on equivalence classes in $\mathfrak{H}$ we must have $\mu(w_\alpha) = 0$ for each α. But the α are countable, therefore $\mu(B) = 0$ where $B = \bigcup_\alpha w_\alpha$. If we now redefine the u_{y_K} to be zero on N, then viii) defines linear transformations $K_\xi : Y \to \mathfrak{H}_\xi$, $\xi \in X$.

By Monotone Convergence,

$$\text{ix)} \quad \int_X \left(\sum_{n=1}^{\infty} \|K_\xi e_n\|^2\right) d\mu(\xi) = \sum_{n=1}^{\infty} \int_X \|K_\xi e_n\|^2 \, d\mu(\xi)$$

$$= \sum_{n=1}^{\infty} \int_X \|(UKe_n)(\xi)\|^2 \, d\mu(\xi) = \sum_{n=1}^{\infty} \|UKe_n\|^2$$

$$= \|K\|_2^2 < \infty$$

(where $\|\cdot\|_2$ is the Hilbert-Schmidt norm). It follows that the quantity $\|K_\xi\|_{2,\xi} = (\sum_{n=1}^{\infty} \|K_\xi e_n\|^2)^{\frac{1}{2}}$ satisfies

x) $\|K_\xi\|_{2,\xi} < \infty$

for μ-almost all $\xi \in X$. By suitably expanding the set N on which the u_{y_K} vanish we obtain x) for all $\xi \in X$. Then for

$y = \sum_{n=1}^{k} \alpha_n e_n \in Y$,

xi) $\|K_\xi y\| \le \sum_{n=1}^{k} |\alpha_n| \; \|K_\xi e_n\| \le (\sum_{n=1}^{k} |\alpha_n|^2)^{\frac{1}{2}} (\sum_{n=1}^{k} \|K_\xi e_n\|^2)^{\frac{1}{2}}$

$\le \|y\| \cdot \|K_\xi\|_{2,\xi}$

so that each K_ξ extends in unique fashion to a bounded operator on $\mathcal{H}$, with

xii) $\|K_\xi\| \le \|K_\xi\|_{2,\xi}$.

Moreover, x) shows that each K_ξ is Hilbert-Schmidt, with Schmidt norm $\|K_\xi\|_{2,\xi}$. It remains only to verify that, for $h \in \mathcal{H}\backslash Y$, the vector field $u_h(\xi)$ defined by $u_h(\xi) = K_\xi h$ is in fact a representative of the equivalence class UKh. We take a sequence $\{y_n\} \subset Y$ with $y_n \to h$ in $\mathcal{H}$ and let $u = \{u(\xi)\}$ be any representative of UKh; the idea is to show $\|u - u_h\|_{\mathfrak{H}} = 0$. Now for each n,

$\int_X \|u(\xi) - u_h(\xi)\|^2$

$\le \int_X (\|u(\xi) - u_{y_n}(\xi)\| + \|u_{y_n}(\xi) - u_h(\xi)\|)^2 d\mu(\xi)$

(continued)

$$\leq 2 \int_X \|u(\xi) - u_{y_n}(\xi)\|^2 \, d\mu(\xi) + 2 \int_X \|u_{y_n}(\xi) - u_h(\xi)\|^2 \, d\mu(\xi)$$

$$= 2 \, \|UKh - UKy_n\|^2 + 2 \int_X \|K_\xi(y_n - h)\|^2 \, d\mu(\xi)$$

$$\leq 2 \, \|K\|^2 \|y_n - h\|^2 + 2 \, \|y_n - h\|^2 \int_X \|K_\xi\|^2 \, d\mu(\xi) \; .$$

But $\int_X \|K_\xi\|^2 \, d\mu(\xi) \leq \int_X \|K_\xi\|^2_{2,\xi} \, d\mu(\xi) = \|K\|_2^2$ by ix); therefore

$$(\|u - u_h\|_{\tilde{\mathcal{H}}})^2 \leq 4 \, \|K\|_2^2 \, \|y_n - h\|^2.$$

Now take limits as $n \to \infty$. ∎

Observe that iii) may be rewritten as

$$\text{xiii)} \quad UKx = \int_X^{\oplus} K_\xi x \, d\mu(\xi), \qquad x \in \mathcal{H}.$$

The adjoint maps $K_\xi^* : \mathcal{H}_\xi \to \mathcal{H}$ are all Hilbert-Schmidt with Schmidt norm $\|K_\xi^*\|_{2,\xi} = \|K_\xi\|_{2,\xi}$. Analogous to xiii) we must also have

$$\text{xiii)'} \quad K^*U^*\tilde{x} = \int_X K_\xi^* \tilde{x}(\xi) \, d\mu(\xi), \qquad \tilde{x} \in \tilde{\mathcal{H}}$$

To see this, observe first of all that

$$\int_X \|K_\xi^* \tilde{x}(\xi)\| \, d\mu(\xi) \leq \int_X \|K_\xi^*\| \, \|\tilde{x}(\xi)\| \, d\mu(\xi)$$

$$\leq \left(\int_X \|K_\xi^*\|^2 \, d\mu(\xi)\right)^{\frac{1}{2}} \left(\int_X \|\tilde{x}(\xi)\|^2 \, d\mu(\xi)\right)^{\frac{1}{2}}$$

$$\leq \left(\int_X \|K_\xi\|^2_{2,\xi} \, d\mu(\xi)\right)^{\frac{1}{2}} \|\tilde{x}\| = \|K\|_2 \, \|\tilde{x}\|$$

(Here we have used $\|K_\xi^*\|_{2,\xi} \leq \|K_\xi^*\|_{2,\xi} = \|K_\xi\|_{2,\xi}$). Since $\mathcal{H}$ is separable, this implies the integral $\int_X K_\xi^* \tilde{x}(\xi)\, d\mu(\xi)$ exists in $\mathcal{H}$. Now let $y \in \mathcal{H}$ be arbitrary; then

$$\left(\int_X K_\xi^* \tilde{x}(\xi)\, d\mu(\xi), y\right) = \int_X (K_\xi^* \tilde{x}(\xi), y)\, d\mu(\xi)$$

$$= \int_X (\tilde{x}(\xi), K_\xi y)\, d\mu(\xi) = (\tilde{x}, UKy)_{\tilde{\mathcal{H}}},$$

which proves xiii).

§3. A New Proof of the Kato-Rosenblum Theorem

We proceed to give a direct proof of 2.3 which, as we have already shown, is an equivalent formulation of the Kato-Rosenblum Theorem 1.1. The method employed here is from an unpublished sketch of Brown and Douglas, dated 1973.

To begin with, we shall assume the trace class perturbation $T = A_2 - A_1$ is positive, since by elementary spectral theory T is the difference of positive trace class operators. We shall also let K be a Hilbert-Schmidt operator with $T = KK^*$.

3.1. Lemma. Let M be a reducing subspace for A_1 containing ran T. Then M reduces A_2, and $A_1 = A_2$ in $M^\perp$.

Proof: Let P_M be orthogonal projection onto M; we must show that $P_M(\mathcal{D}(A_2)) \subset \mathcal{D}(A_2)$ and $A_2(\mathcal{D}(A_2) \cap M) \subset M$. The first relation is obvious, since $\mathcal{D}(A_1) = \mathcal{D}(A_2)$ and M reduces A_1. But then also

$$A_2(\mathcal{D}(A_2) \cap M) = (A_1 + T)(\mathcal{D}(A_1) \cap M) \subset M + \operatorname{ran} T = M.$$

Finally, $M^\perp \subset \ker T$ by self-adjointness of T so $A_1 = A_2$ in $M^\perp$.

From this it is clear we lose no generality in taking $\mathcal{H}$ to be the smallest reducing subspace for A_1 which contains ran T (given the existence of such a subspace, which is obvious). Now for $\xi \in \mathbb{C}$, $\operatorname{Im} \xi \neq 0$ we define

$$\Phi_j(\xi) = K^*(A_j - \xi)^{-1}K \qquad j = 1,2.$$

Then

$$\Phi_1(\xi) - \Phi_2(\xi) = K^*(A_1 - \xi)^{-1} [(A_2 - \xi)] - (A_1 - \xi)](A_2 - \xi)^{-1}K$$

$$= K^*(A_1 - \xi)^{-1}KK^*(A_2 - \xi)^{-1}K$$

$$= \Phi_1(\xi)\ \Phi_2(\xi)$$

so that

xiv) $(I + \Phi_1(\xi))(I - \Phi_2(\xi)) = I.$

Likewise, a straightforward calculation gives

xv) $(I + \Phi_1(\xi))\mathrm{Im}\ \Phi_2(\xi)(I + \Phi_1(\xi)^*) = \mathrm{Im}\ \Phi_1(\xi).$

Let $\mathfrak{H}_i = \int_{\mathbb{R}}^{\oplus} \mathfrak{H}_\lambda^i \cdot d\nu(\lambda)$ be the direct integral representation space for A_i, $i = 1,2$. Since the ν_i are finite measures we have

xvi) $\int_{-\infty}^{\infty} \frac{d\nu_i(t)}{1 + t^2} < \infty$.

Define the Hilbert-Schmidt transformations $K_\lambda^i : \mathfrak{H} \to \mathfrak{H}_\lambda^i$ for each i as in Proposition 2.4. Since $\mathfrak{H}$ is the smallest reducing subspace for A_i containing ran T, $i = 1,2$, the range of K_λ^i must be dense in $\mathfrak{H}_\lambda^i$ for ν_i - almost all $\lambda \in \mathbb{R}$, $i = 1,2$. Otherwise, it is fairly simple to construct a proper reducing subspace for A_i containing ran K, and hence ran T. This fact will be crucial at the very end of the proof.

3.2. <u>Lemma</u>: For each i, $\Phi_i(\xi)$ is the integral, with respect to ν_i, of the trace class valued function $\lambda \to (K^i_\lambda)^*(\lambda - \xi)^{-1}K^i_\lambda$. That is,

$$\text{xvii)} \quad \Phi_i(\xi) = \int_{-\infty}^{\infty} \frac{(K^i_\lambda)^* K^i_\lambda}{\lambda - \xi} \, d\nu_i(\lambda) \qquad i = 1,2; \ \xi \notin \mathbb{R} .$$

<u>Proof</u>: We shall suppress the index i. If $\|\cdot\|_1$ is the trace norm in $\mathcal{J}_1(\mathcal{H})$, then we obtain, using ix),

$$\int_{-\infty}^{\infty} \left\|\frac{K^*_\lambda K_\lambda}{\lambda - \xi}\right\|_1 d\nu(\lambda) = \int_{-\infty}^{\infty} |\lambda - \xi|^{-1} \|K_\lambda\|_2^2 \, d\nu(\lambda) \le \|K\|_2^2 \, \mathcal{I}m\xi.$$

This, together with the fact that $(\mathcal{J}_1(\mathcal{H}), \|\cdot\|_1)$ is a separable Banach space, gives existence of the integral in xvii). For $x \in \mathcal{H}$ we may write

$$\Phi(\xi)x = K^*U^*U(A - \xi)^{-1} U^*UKx.$$

Now apply xiii), xiii)', and the fact that

$$U(A - \xi)^{-1}U^* = M_{(\lambda-\xi)^{-1}} = \int_{\mathbb{R}}^{\oplus} \frac{1}{\lambda - \xi} I_\lambda \, d\nu(\lambda)$$

to obtain xvii). ∎

Let us continue to suppress index i. From xvii) it follows that

$$\mathcal{I}m\ \Phi(x + iy) = \int_{-\infty}^{\infty} \frac{yK^*_\lambda K_\lambda}{(x - \lambda)^2 + y^2} \, d\nu(\lambda) = \pi\int_{-\infty}^{\infty} P_y(x - \lambda)K^*_\lambda K_\lambda d\nu(\lambda),$$

where $P_y(x) = \dfrac{y}{\pi(x^2 + y^2)}$ is the Poisson Kernel for the upper half-plane. Therefore we claim

xviii) $\lim_{y \to 0^+} \mathcal{I}m\, \Phi(x + iy) = \Phi(x + i0^+) = \pi \frac{d\nu}{d\lambda}(x) K_x^* K_x$

where the limit is taken in $(\mathcal{J}_1, \|\cdot\|_1)$, for m - almost all $x \in \mathbb{R}$. This is a result of the following generalization of Fatou's Theorem:

3.3. Theorem: Let μ be a positive Borel measure on $\mathbb{R}$ with

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1 + t^2} < \infty .$$

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space, and $\varphi: \mathbb{B} \to \mathbb{B}$ a Bochner Integrable function with respect to μ. If we form the Poisson Integrals

$$\varphi_y(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} P_y(x - \lambda)\varphi(\lambda)\, d\mu(\lambda) \qquad y > 0$$

then for those x for which $\frac{d\mu}{dm}(x) = \mu'(x)$ exists, we have

$$\lim_{y \to 0^+} \|\varphi_y(x) - \varphi(x)\mu'(x)\| = 0.$$

Proof: We assume that the scalar case $\mathbb{B} = \mathbb{C}$ (Fatou's Theorem) is known (Hoffman [8]). But the general case reduces to this, for if μ_s is the singular part of μ with respect to m, then

$$\|\varphi(x)\mu'(x) - \varphi_y(x)\varphi\|$$

$$\leq \|\varphi(x)\mu'(x) - \int_{\mathbb{R}} P_y(x - \lambda)\varphi(x)\mu'(x)\, d\lambda\|$$

$$+\|\int_{\mathbb{R}} P_y(x - \lambda)\varphi(\lambda)\, d\mu_s(\lambda)\|$$

$$\leq \int_{\mathbb{R}} P_y(x - \lambda)\, \|\varphi(x)\mu'(x) - \varphi(\lambda)\mu'(\lambda)\|\, d\lambda$$

$$+ \int_{\mathbb{R}} P_y(x - \lambda)\, \|\varphi(\lambda)\|\, d\mu_s(\lambda).$$

Now apply the Scalar case to the last two integrals to see that the limit is zero whenever $\mu'(x)$ exists. *(It should be remarked that the result here has not been stated in its maximal generality; it holds, for instance, with nontangential limits, as well as with $\mathbb{B}$-valued measures μ satisfying $\int_{-\infty}^{\infty} \frac{d\|\mu\|(t)}{1 + t^2} < \infty$.)

In any event, xviii) has been established. Another well-known result from Analytic Function Theory can be generalized to the operator valued context as follows:

3.4. <u>Theorem</u>: Let $\mathcal{H}$ be a separable Hilbert space, and $\varphi(\xi) = \varphi(x + iy)$ a bounded holomorphic $\mathcal{B}(\mathcal{H})$-valued function in the half-plane $y = \mathcal{I}m\ \xi > 0$. Then $\lim_{y\to 0^+} \phi(x + iy)$ exists in the strong operator topology for m - almost all $x \in \mathbb{R}$.

The proof of this result, which is fairly straightforward, may be found in the excellent treatise <u>Harmonic Analysis of Operators on Hilbert space</u>, by Sz-Nagy and Foias [6]. Observe now that

$$e^{i\Phi(\xi)} = e^{iRe\Phi(\xi)}e^{-Im\Phi(\xi)}$$

satisfies the hypotheses of 3.5, for if we apply the spectral theorem to the self-adjoint operators Re $\Phi(\xi)$ and $\mathcal{I}m\ \Phi(\xi)$ we obtain, respectively, $\|e^{iRe\Phi(\xi)}\| \le 1$ and $\|e^{-\mathcal{I}m\Phi(\xi)}\| \le 1$ (the latter because $\mathcal{I}m\ \Phi(\xi) > 0$). Thus by the above,

$$s - \lim_{y\to 0^+} e^{i\Phi(x+iy)}$$

exists for m - almost $x \in \mathbb{R}$; moreover, it is clear this limit can be zero only for x comprising a set of Lebesgue measure zero, in view of xviii). Thus we conclude that

$$\text{xix)} \quad s - \lim_{y \to 0^+} \Phi(x + iy) = \Phi(x + i0^+)$$

exists for m - almost $x \in \mathbb{R}$. The same can be said for $\Phi(x + i0^+)^* = s - \lim_{y \to 0^+} \Phi(x + iy)^*$, since

$$\Phi(\xi)^* = K^*((A - \xi)^{-1})^* = K^*(A - \overline{\xi})^{-1}K = \Phi(\overline{\xi}),$$

and we apply the results for the lower half plane to $\Phi(\xi)$.

Now take a sequence of positive numbers $y_1, y_2, \ldots$ decreasing to zero, and set $\xi_n = x + iy_n$. We may rewrite xix) and the corresponding result for adjoints, as

$$s - \lim_{n \to \infty} \Phi(\xi_n) = \Phi(x + i0^+)$$

$$s - \lim_{n \to \infty} \Phi(\xi_n)^* = \Phi(x + i0^+)^* \qquad \text{a.e.}(m).$$

Since for sequences, the product of s-limits equals the s-limit of products (Reed and Simon [13]), we may pass to the strong limit in xiv) and xv) to obtain

$$(I + \Phi_1(x + i0^+))(I - \Phi_2(x + i0^+)) = I$$

and

$$(I + \Phi_1(x + i0^+)) \frac{d\nu_2}{dm}(x)(K_x^2)^* K_x^2(I + \Phi_1(x + i0^+)^*) = \frac{d\nu_1}{dm}(x)(K_1^1)^* K_x^1$$

respectively, for m - almost all $x \in \mathbb{R}$. We recall that K_x^i

has dense range for almost all x; thus the corresponding transformations $(K_x^1)^*$ are injective. Since $I + \Phi_1(x + i0^+)$ and $I + \Phi_1(x + i0^+)^*$ are invertible we obtain from the last equation

$$\dim(\operatorname{ran} K_x^2) = \dim(\operatorname{ran} K_x^1)$$

for m - almost all x. By separability of the H_x^1 we have

$$\dim(\operatorname{ran} K_x^1) = \dim H_x^1$$

whenever $\operatorname{ran} K_x^1$ is dense in H_x^1, and therefore

$$\dim H_x^1 = \dim H_x^2$$

for m - almost all $x \in \mathbb{R}$. ∎

§4. The K-R Result for $(\mathbb{T},m)$

4.1. Definition: Given a subset Γ of $\mathbb{C}$, with μ a positive, σ-finite Borel measure on Γ, we shall say that the K-R result holds for (Γ,μ) if for any normal operators N_1, N_2 in $\mathcal{H}$ such that $N_2 - N_1 \in \mathcal{J}_1$ and $\sigma(N_1) \cup \sigma(N_2) \subset \Gamma$, the relation

$$\delta_{N_1}(\xi) = \delta_{N_2}(\xi)$$

holds for μ - almost all $\xi \in \Gamma$, where δ_{N_i} is the spectral multiplicity function for N_i, $i = 1,2$.

In terms of this definition, Theorem 2.3 asserts simply that the K-R result holds for $(\mathbb{R},m)$. The K-R result is also known for the unitary case $(\mathbb{T},m)$, that is, the absolutely continuous parts of two unitary operators whose difference is trace class must be unitarily equivalent. For a published proof one may consult Birman [4], although a much simpler approach involving Cayley Transforms can be formulated.

In addition, we intend to give our own proof of the K-R result for $(\mathbb{T},m)$. We do this to introduce a new technique which shall be applied subsequently under far more difficult circumstances. In the unitary case, very little resistance is encountered, and the resulting proof is quite simple. In the general situation it shall be necessary to develop a number of auxiliary techniques.

First some notation: Each angle φ determines the line

$$\Lambda_\varphi = \{re^{i\varphi} : -\infty < r < \infty\} \text{ in } \mathbb{C},$$

together with two open semicircles of the unit circle $\mathbb{T}$, namely,

$$C_\varphi^+ = \{e^{i(\varphi+\alpha)} : 0 < \alpha < \pi\}$$
$$C_\varphi^- = \{e^{i(\varphi+\alpha)} : \pi < \alpha < 2\pi\}.$$

The map $\pi_\varphi : \mathbb{T} \to \Lambda_\varphi$ denotes orthogonal projection, that is,

$$1) \quad \pi_\varphi(\zeta) = e^{i\varphi}\mathrm{Re}(e^{-i\varphi}\zeta) \qquad \zeta \in \mathbb{T}.$$

4.2. <u>Lemma</u>: Suppose $S \subset \mathbb{T}$, with $m(S) > 0$. Then we can find an angle φ and a subset E of Λ_φ with $m(E) > 0$ such that $\pi_\varphi^{-1}(E) \subset S$.

<u>Proof</u>: Let I be a subarc of $\mathbb{T}$ and take φ so that Λ_φ bisets I; we shall write $B^\pm = B \cap C_\varphi^\pm$, $B_I = B \cap I$, and $B^* = \{\overline{\xi} : \xi \in B\}$ for $B \subset \mathbb{T}$. If $m(e^{i\varphi}((e^{-i\varphi}S_I^+)^*) \cap S_I^-) = 0$ then, since $e^{i\varphi}((e^{-i\varphi}S_I^+)^*) \subset I^-$, we have

$$m(S_I) = m(S_I^+) + m(S_I^-) = m(e^{i\varphi}((e^{-i\varphi}S_I^+)^*)) + m(S_I^-)$$
$$\leq m(I^-) = \frac{1}{2}\, m(I).$$

Therefore, simply by choosing I so that $m(S_I) > \frac{1}{2}\, m(I)$ we obtain $m(\tilde{E}) > 0$, where $\tilde{E} = e^{i\varphi}((e^{-i\varphi}S_I^+)^*) \cap S_I^-$. Set $E = \pi_\varphi(\tilde{E}) \subset \Lambda_\varphi$. Then $m(E) > 0$; also

$$E \subset \pi_\varphi(e^{i\varphi}((e^{-i\varphi}S_I^+)^*)) = \pi_\varphi(S_I^+) \text{ by (i) above.}$$

Thus $E \subset \pi_\varphi(S_I^+) \cup \pi_\varphi(S_I^-)$, and it follows that $\pi_\varphi^{-1}(E) \subset S_I^+ \cup S_I^- \subset S$. ∎

Now let U_1, U_2 be unitaries with spectral multiplicity functions δ_{U_1}, δ_{U_2} respectively, and $T = U_2 - U_1 \in \mathcal{J}_1$. If $m\{\zeta \in \mathbb{T} : \delta_{U_1}(\zeta) \neq \delta_{U_2}(\zeta)\} > 0$ then there is a subset S of $\mathbb{T}$ with $m(S) > 0$ on which one multiplicity function is strictly greater than the other, say $\delta_{U_1} > \delta_{U_2}$ on S. We can apply Lemma 4.2 to S to obtain angle φ and subset $E \subset \Lambda_\varphi$ with $m(E) > 0$ and $\pi_\varphi^{-1}(E) \subset S$.

Consider now the operators $\pi_\varphi(U_k) = e^{i\varphi}\mathrm{Re}(e^{-i\varphi}U_k)$ defined through the functional calculus for $k = 1,2$. Each $A_k = \mathrm{Re}(e^{i\varphi}U_k)$ is self-adjoint; moreover,

$$A_2 - A_1 = \mathrm{Re}(e^{-i\varphi}(U_2 - U_1)) = \mathrm{Re}(e^{-i\varphi}T) \in \mathcal{J}_1,$$

so that by 2.3, $\delta_{A_2}(r) = \delta_{A_1}(r)$ for m - almost all $r \in \mathbb{R}$. Since $\delta_{\pi_\varphi(U_k)}(\eta) = \delta_{A_k}(e^{-i\varphi}\eta)$ for $\eta = re^{i\varphi} \in \Lambda_\varphi$, we see that

$$\delta_{\pi_\varphi(U_1)}(\eta) = \delta_{\pi_U(U_2)}(\eta) \text{ a.e. in } \Lambda_\varphi.$$

However, for $\eta \in E$ we have $\pi_\varphi^{-1}(\eta) \subset S$; therefore

$$\delta_{\pi_\varphi(U_1)}(\eta) = \sum_{\zeta\in\pi_\varphi^{-1}(\eta)} \delta_{U_1}(\zeta) > \sum_{\zeta\in\pi_\varphi^{-1}(\eta)} \delta_{U_2}(\zeta) = \delta_{\pi_\varphi(U_2)}(\eta) \qquad \eta \in E.$$

This contradiction shows that $m(S) = 0$; hence we have proven

4.3. <u>Theorem:</u> The K-R result holds for $(\mathbb{T}, m)$.

From here we go on to discuss more general pairs (Γ, μ). Suppose Lemma 4.2 could be proven exactly as written for (Γ, μ); that is, suppose for $S \subset \Gamma$, $\mu(S) > 0$, we can find φ and $E \subset \Lambda_\varphi$ with $m(E) > 0$, $\pi_\varphi^{-1}(E) \subset S$. Then the same argument used above would go through without a hitch. Unfortunately, even in the case of the convex rectifiable Jordan curve, which we consider next, Lemma 4.2 fails to hold. But as the following result indicates, we shall first be allowed to transform Γ with analytic maps, at least in the bounded case; this greatly increases the applicability of our method.

4.4. <u>Lemma:</u> Suppose that A and B are bounded operators with $T = B - A$ in trace class, and φ is some function holomorphic in a neighborhood of $\sigma(A) \cup \sigma(B)$. Then $\varphi(A) - \varphi(B)$ is also trace class*.

<u>Proof:</u> Let U be an open set containing $\sigma(A) \cup \sigma(B)$ such that φ is holomorphic in U; also let $V \subset U$ be open, with σ $\sigma(A) \cup \sigma(B) \subset V$ and $\gamma = \partial V$ the finite union of rectifiable Jordan curves (for existence, see [15]). Applying the Riesz Functional Calculus, we have

$$\varphi(A) - \varphi(B) = \int_\gamma \varphi(\zeta)(A - \zeta)^{-1} d\zeta - \int_\gamma \varphi(\zeta)(B - \zeta)^{-1} d\zeta$$

*For a rather general treatment of this sort of "perturbation invariance problem", see the paper by Birman and Solmyak [20].

$$= \int_\gamma \varphi(\zeta)[(A - \zeta)^{-1} - (B - \zeta)^{-1}]d\zeta$$

$$= \int_\gamma \varphi(\zeta)(A - \zeta)^{-1}\ T(B - \zeta)^{-1}d\zeta$$

where the integral is evaluated in $\mathcal{B}(\mathcal{H})$. However the integral is trace class for each $\varkappa$, and since

$$\int_\gamma \|\varphi(\zeta)(A - \zeta)^{-1}\ T(B - \zeta)^{-1}\|_1 d\zeta$$

$$\leq \int_\gamma |\varphi(\zeta)|\ \|(A - \zeta)^{-1}\ (B - \zeta)^{-1}\|\ \|T\|_1\ |d\zeta|$$

$$\leq \|\varphi\|_{\gamma,\infty}\ \|T_1\|\ d(\gamma, \sigma(A) \cup \sigma(B))^2 \int_\gamma |d\zeta|$$

The integral exists in $(\mathcal{J}_1, \|\cdot\|_1)$. This shows that $\varphi(A) - \varphi(B)$ is trace class with

$$\|\varphi(A) - \varphi(B)\|_1 \leq \|\varphi\|_{\gamma,\infty}\ \|T\|_1\ d(\gamma, \sigma(A) \cup \sigma(B))^2 \int_\gamma |d\zeta|. \quad \blacksquare$$

Observe that the trace class may be replaced in this proof by any ideal $\mathcal{J} \subset \mathcal{B}(\mathcal{H})$ endowed with a norm $\|\cdot\|_\mathcal{J}$ making it into a Banach space, and satisfying $\|AB\| \leq \|A\|\ \|B\|_\mathcal{J}$ for $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{J}$. For example, the compact operators with the $\mathcal{B}(\mathcal{H})$ norm, and all the Schatten classes $\mathcal{J}_p$, $1 \leq p \leq \infty$, fall into this category.

4.5. <u>Definition</u>: A bounded Borel function f on Γ will be said to <u>preserve</u> $\mathcal{J}_1$ - <u>perturbations</u> on Γ provided $f(N_2) - f(N_1) \in \mathcal{J}_1$ for every pair of normal operators N_1, N_2 in $\mathcal{H}$ (not necessarily bounded*) such that $N_2 - N_1 \in \mathcal{J}_1$ and $\sigma(N_1) \cup \sigma(N_2) \subset \Gamma$. For example, when Γ is bounded and φ is

*See Rudin [14] for a discussion of the functional calculus for unbounded normal operators.

holomorphic in a neighborhood of Γ, then φ preserves $\mathcal{J}_1$ perturbations on Γ by the preceding lemma.

We shall now state a sufficient condition for the K-R result on (Γ,μ); note the incorporation of a generalized form of 4.2 into the hypotheses.

4.6. <u>Proposition</u>: Suppose for each $S \subset \Gamma$, $\mu(S) > 0$, we can find a bounded measurable function Ψ which preserves $\mathcal{J}_1$-perturbations on Γ, together with a line Λ and subset E of Λ, $m(E) > 0$, such that $(\tilde{\pi}_\Lambda \Psi)^{-1}(\eta)$ is a finite subset of S for each $\eta \in E$ ($\tilde{\pi}_\Lambda : \Psi(\Gamma) \to \Lambda$ is orthog proj.). Then the K-R result holds for (Γ,μ).

<u>Proof</u>: Let N_1, N_2 be normal operators in $\mathcal{H}$ with $N_2 - N_1 \in \mathcal{J}_1$, $\sigma(N_1) \cup \sigma(N_2) \subset \Gamma$. By symmetry it suffices to show $\mu(S) = 0$ when $S = \{\zeta \in \Gamma : \delta_{N_1}(\zeta) > \delta_{N_2}(\zeta)\}$. It is known (cf Rudin [14], Theorem 13.24) that $\varphi(N_1)$, $\varphi(N_2)$ are normal and by 4.5, $\varphi(N_2) - \varphi(N_1) \in \mathcal{J}_1$. Hence $(\tilde{\pi}_\Lambda\varphi)(N_1) \in \mathcal{J}_1$, and applying 2.3 as in the proof of 4.3, we see that

$$\delta_{(\tilde{\pi}_\Lambda\varphi)(N_1)} = \delta_{(\tilde{\pi}_\Lambda\varphi)(N_2)} \quad \text{a.e. } (m) \text{ on } \Lambda.$$

But because $(\tilde{\pi}_\Lambda\varphi)^{-1}(\eta)$ is a finite subset of S for $\eta \in E$, we have (just as in the proof of Theorem 4.3),

$$\delta_{(\tilde{\pi}_\Lambda\varphi)(N_1)}(\eta) = \sum_{\zeta \in (\tilde{\pi}_\Lambda\varphi)^{-1}(\eta)} \delta_{N_1}(\zeta) > \sum_{\zeta \in (\tilde{\pi}_\Lambda\varphi)^{-1}(\eta)} \delta_{N_2}(\zeta) = \delta_{(\tilde{\pi}_\Lambda\varphi(N_2),}(\eta)$$

and again, the contradiction gives $\mu(S) = 0$. ∎

Perhaps the main significance of 4.6 is that it reduces the operator theoretic question to a problem of geometric/measure-theoretic analysis of the spectrum.

§5. Convex Rectifiable Jordan Curves

In this section Γ will denote a rectifiable Jordan curve in $\mathbb{C}$ bounding a region G. $F : \mathbb{D} \to G$ will be a Riemann map establishing conformal equivalence of the open unit disc $\mathbb{D}$, and G, and $\tilde{F}$ the homeomorphic extension of F to $\overline{\mathbb{D}}$. $\tilde{F}|_{\mathbb{T}}$ will be denoted by γ.

It is known (cf. Privalov [12] Ch.III for this paragraph) that γ must be absolutely continuous, so that arc-length measure μ on Γ is given by

$$\text{i)} \quad \mu(\gamma(E)) = \int_E |\gamma'(t)|dt$$

for (Lebsesque) measurable $E \subset \mathbb{T}$. Since $\gamma : \mathbb{T} \to \Gamma$ is a homeomorphism, the Borel sets in Γ are precisely the sets $\{\gamma(E)\}$ where E is Borel in $\mathbb{T}$. The map γ^{-1} is also absolutely continuous so that an equivalence of the measure spaces $(\mathbb{T},m)$ and (Γ,μ) is established.

In the case when Γ is convex (i.e. $\overline{G}$ is convex) we claim the K-R result holds for (Γ,μ). The goal of this section is to establish certain properties of Γ (particularly 5.8), whence it follows, in a latter section, that the hypotheses of 4.6 are satisfied. For now, however, we shall not assume convexity of Γ. Let $\mathbb{T}_1$ denote the subset of $\mathbb{T}$ where $\gamma'(t) = \frac{d}{dt}\gamma(e^{it})$ is defined and nonzero. Now $\gamma'(t)$ is defined almost everywhere while if $\gamma'(t) = 0$ on a set $N \subset \mathbb{T}$ with $m(N) > 0$, then $\mu(\gamma(N)) = \int_N |\gamma'| \, dm = 0$ which contradicts the measure space equivalence; therefore we have $m(\mathbb{T} - \mathbb{T}_1) = 0$. Thus we can define $\theta(t) = \arg \gamma'(t)$ for $t \in \mathbb{T}_1$; let this be done in such a way that $\theta(t) \in [0,2\pi)$.

We shall generally abbreviate t for e^{it} in $\mathbb{T}$, as is commonly done - more explicit notation can be employed when necessary. Arguments t will also take values only in $[0,2\pi)$, so inequalities involving arguments have a very explicit meaning; for instance, we have the rules

ii) $a + b \geq a$ if $b < 2\pi - a$

iii) If $a + b \geq a$ then $a + b' \geq a$ for $b' \in [0,b]$.

By contrast, equalities of arguments (mod 2π) require no special handling.

A related consideration involves limits. We must distinguish between the "Real" and "Circle" topologies on $[0,2\pi)$; these have different neighborhoods of 0. A useful fact here is the following:

iv) Let f be a continuous $\mathbb{T}$-valued function, with ran $f \subset [0,b]$ for some argument $b < 2\pi$. Then f is continuous as a real valued function.

5.1. <u>Lemma</u>: For $t \in \mathbb{T}_1$, $s \in (0,2\pi)$ define

$$\alpha_t(s) = \arg(\gamma(t+s) - \gamma(t)) \qquad \in [0,2\pi)$$

then $\lim_{s\to 0^+} \alpha_t(s) = \theta(t)$; $\lim_{s\to 2\pi^-} \alpha_{t_0}(s) = \theta(t_0) + \pi$,

in the circle topology.

<u>Proof</u>: $\arg : \mathbb{C}\setminus\{0\} \to \mathbb{T}$ is a continuous map. Therefore

$$\lim_{s\to 0^+} \alpha_t(s) = \lim_{s\to 0^+} \arg(\gamma(t+s) - \gamma(t))$$

$$= \lim_{s\to 0^+} \arg\left(\frac{\gamma(t+s) - \gamma(t)}{s}\right) = \arg \lim_{s\to 0^+}\left(\frac{\gamma(t+s) - \gamma(t)}{s}\right)$$

$$= \arg \gamma'(t) = \theta(t);$$

here the interchange of arg and lim is justified by the fact that $\gamma'(t)$ exists and is nonzero. Likewise

$$\lim_{s\to 2\pi^-} \arg(\gamma(t+s) - \gamma(t)) = \lim_{s\to 0^+} \arg\,(\gamma(t-s) - \gamma(t))$$

$$= \lim_{s\to 0^+} \arg\left(\frac{\gamma(t-s) - \gamma(t)}{s}\right) = \arg \lim_{s\to 0^+}\left(\frac{\gamma(t-s) - \gamma(t)}{s}\right)$$

$$= \arg(-\gamma'(t)) = \pi + \theta(t). \quad \blacksquare$$

We will assume, without loss of generality, that the orientation induced by γ on Γ agrees with the usual counter clockwise orientation. Intuitively, this means that a person standing at $\gamma(t)$, $t \in \mathbb{T}_1$, and facing in the direction $\theta(t)$ would have the region G on the left and the exterior region $\widetilde{G} = \hat{\mathbb{C}} \setminus \overline{G}$ on the right. ($\hat{\mathbb{C}}$ is the complex sphere). With this in effect we have

5.2. Lemma: Suppose $t \in \mathbb{T}_1$, $\pi/2 > \epsilon > 0$. Then $\exists\ \delta > 0$ such that

$$\gamma(t) + re^{i(\theta(t) + \theta')} \in G$$

$$\gamma(t) - re^{i(\theta(t) + \theta')} \in \widetilde{G}$$

for all $r \in (0,\delta)$, $\theta' \in (\epsilon, \pi - \epsilon)$.

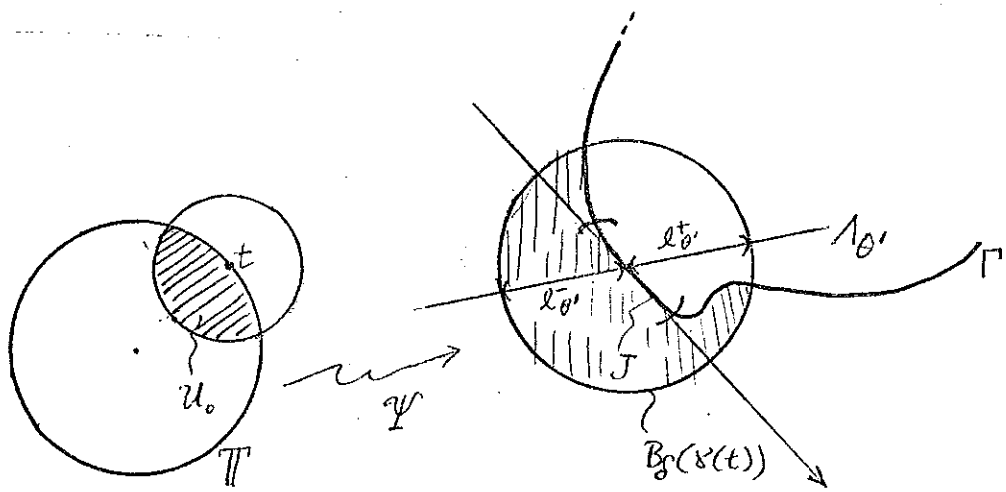
Proof: Let $\Lambda_{\theta'}$ be the line $\{\gamma(t) + re^{i(\theta(t)+\theta')} \mid -\infty < r < \infty\}$. By 5.1 $\exists$ open subarc J of Γ, $\gamma(t) \in J$, such that for every $\theta' \in (\epsilon, \pi - \epsilon)$, $J \cap \Lambda_{\theta'} = \{\gamma(t)\}$. J is open in the $\mathbb{C}$-subspace topology on Γ, so $\exists\ \delta > 0$ with $B_\delta(\gamma(t)) \cap \Gamma \setminus J = \{\gamma(t)\}$. For each $\theta' \in (\epsilon, \pi - \epsilon)$ the open segments $\ell^{\pm}_{\theta'}$ are disjoint* from Γ; hence connectedness, together with our choice of orientation shows that $\ell^{+}_{\theta'} \subset G$. It remains to show that $\ell^{-}_{\theta'} \subset \widetilde{G}$.

Assume the contrary, namely $\ell^{-}_{\theta'} \subset G$. Let $\overline{\Psi} : \overline{\mathbb{D}} \to C\ell(\widetilde{G})$ be the homeomorphism obtained by extending a conformal equivalence $\Psi : \mathbb{D} \to \hat{\mathbb{C}} \setminus G$ ($\hat{\mathbb{C}}$ is the complex sphere). Then $\exists\ \rho > 0$ with $U = B_\rho(t) \cap \overline{\mathbb{D}} \subset \Psi^{-1}(B_\delta(\gamma(t)) \cap \mathrm{Cl}\widetilde{G})$. $U \cap \mathbb{T}$ is an open subarc I_1 containing t; also, if $U_0 = U - I_1 = U \cap \mathbb{D}$ then $\Psi(U_0) = V$ is an open connected subset of $\widetilde{G}$, and $I_2 = \overline{\Psi}(I_1)$ is an open subarc of Γ with $\gamma(t) \subset I_2 \subset \partial V$. Setting $\ell_{\theta'} = \ell^{+}_{\theta'} \cup \{\gamma(t)\} \cup \ell^{-}_{\theta'}$ we then have $\ell_{\theta'} \cap B_\delta(\gamma(t)) \cap \widetilde{G} = \emptyset$

*Where we define $\ell^{\pm}_{\theta'} = \{\gamma(t) \pm re^{i(\theta(t)+\theta')} \mid r \in (0,\delta)\}$

and hence $\Lambda_{\theta'} \cap V = \emptyset$

Now V is connected, so $V \subset h$ for one of the two half planes h determined by $\Lambda_{\theta'}$, hence also $I_2 \subset \bar{h}$. But obviously no open subarc of Γ containing $\gamma(t)$ can have this property since $\Lambda_{\theta'}$ is not tangent to Γ at $\gamma(t)$. This condition shows $\ell^-_{\theta'} \subset \tilde{G}$, and the lemma follows. ∎



5.3. <u>Definition</u>: Recall that the rectifiable Jordan curve Γ is called <u>convex</u> provided it bounds a convex region. A subarc J of Γ is convex provided that, together with the line segment joining its end points, it bounds a convex region (possibly empty).

We also need a notation for subarcs of $\mathbb{T}$ and Γ. First, for $0 \le t_1 < t_2 \le 2\pi$ we set

$$(t_1, t_2) = \{e^{it} : t_1 < t < t_2\} \qquad (t_2 < 2\pi)$$

with appropriate modifications for square brackets. Using this we define, for $0 \le t_2 < t_1 \le 2\pi$

$$(t_1, t_2) = (t_1, 2\pi) \cup [0, t_2), \qquad \text{again}$$

with appropriate modifications. Finally for distinct $z_1, z_2 \in \Gamma$ we let

$$(z_1, z_2) = \gamma(\gamma^{-1}(z_1), \gamma^{-1}(z_2)).$$

The notation is consistent with our orientation on Γ.

5.4. <u>Theorem</u>: Γ is convex if for each $t \in \mathbb{T}_1$ the function

$$\tilde{\alpha}_t(s) = \alpha_t(s) - \theta(t) \in [0, 2\pi)$$

defined for $0 < s < 2\pi$, is nondecreasing.

<u>Proof</u>: Let Γ be convex, and choose $t_0 \in \mathbb{T}_1$. If $\tilde{\alpha}_{t_0}(s) > \pi$ for some $s \in (0, 2\pi)$ then

$$\alpha_{t_0}(s) = \theta(t_0) + \pi + \theta'$$

where $\theta' = \tilde{\alpha}_{t_0}(s) - \pi \in (0, \pi)$, so that for $\rho > 0$

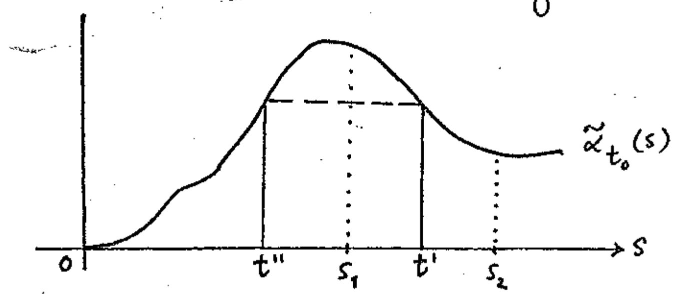
$$\gamma(t_0) + \rho e^{i\alpha_{t_0}(s)} = \gamma(t_0) - \rho e^{i(\theta(t_0) + \theta')}.$$

It follows from 5.2 that part of the segment joining $\gamma(t_0)$ and $\gamma(t_0 + s)$ lies in $\tilde{G}$, contradiction. Therefore

v) $\tilde{\alpha}_{t_0}(s) \le \pi, \qquad s \in (0, 2\pi)$

and $\tilde{\alpha}_{t_0}$ is continuous as a real-valued function by iv).

Suppose now that $\tilde{\alpha}_{t_0}(s_1) > \tilde{\alpha}_{t_0}(s_2)$ with $0 < s_1 < s_2 < 2\pi$. By continuity $\exists$ $t' \in (s_1, s_2)$ with $\tilde{\alpha}_{t_0}(s_1) > \tilde{\alpha}_{t_0}(t') > \tilde{\alpha}_{t_0}(s_2) \geq 0$, and since $\lim_{s \to 0^+} \tilde{\alpha}_{t_0}(s) = 0$ by lemma 5.1 we may, again using continuity, find $t'' \in (0, s_1)$ such that $\tilde{\alpha}_{t_0}(t'') = \tilde{\alpha}_{t_0}(t')$.



But this means that the three points $\gamma(t_0)$, $\gamma(t')$, $\gamma(t'')$ of Γ are co-linear. Convexity of Γ now demands that $\{\gamma(t_0 + s) : 0 \leq s \leq t'\}$ is a line segment, hence $\tilde{\alpha}_{t_0}(s) = \text{const}$ for $0 \leq s \leq t'$. But $s_1 < t'$ and $\tilde{\alpha}_{t_0}(s_1) > \tilde{\alpha}_{t_0}(t')$ contradiction. $\tilde{\alpha}_{t_0}(s)$ must therefore be a nondecreasing function.

Suppose now $\tilde{\alpha}_t(s)$ is nondecreasing, as a map into $[0, 2\pi)$ for each $t \in \mathbb{T}_1$. It follows by 5.1 that $\tilde{\alpha}_t(s) \leq \pi$ for $s \in (0, 2\pi)$. If Γ is not convex we may find points $\gamma_0 = \gamma(t_0)$, $\gamma_1 = \gamma(t_1) \in \Gamma$ such that some point ζ of the segment $\overline{\gamma_0\gamma_1}$ lies in $\tilde{G}$. Perturbing an endpoint if necessary, we may take $t_0 \in \mathbb{T}_1$. If the open segment $\overline{\gamma_0\zeta}$ does not lie entirely in $\tilde{G}$, then it intersects Γ at a point $z = \gamma(t_2)$ different from γ_0, γ_1. Set $s_1 = t_1 - t_0$, $s_2 = t_2 - t_0$. Since $z \in \overline{\gamma_0\gamma_1}$ we have $\tilde{\alpha}_{t_0}(s_1) = \tilde{\alpha}_{t_0}(s_2)$.

If $s_1 < s_2$ then $\tilde{\alpha}_{t_0}(s) \equiv \text{const}$ for $s_1 \geq s \geq s_2$; if $s_1 < s_2$

then $\tilde{\alpha}_{t_0}(s)$ const $s_2 \geq s \geq s_1$. In either event the entire segment $\overline{zY_1}$, which includes ζ, lies in Γ

Now suppose the entire open segment $\overline{Y_0, \xi}$ lies in $\tilde{G}$. Evidently this implies $\tilde{\alpha}_{t_0}(s_1) \notin (0,\pi)$ otherwise the preceding lemma (5.2) shows a small piece of $\overline{Y_0, \xi}$ lying in G. But $\tilde{\alpha}_{t_0}(s) \in [0,\pi]$ for all s; consequently $\tilde{\alpha}_{t_0}(s_1)$ is either 0 or π. If $\tilde{\alpha}_{t_0}(s_1) = 0$ then $\tilde{\alpha}_{t_0}(s) = 0$, $0 \leq s \leq s_1$, because $\tilde{\alpha}_{t_0}$ is nondecreasing. Likewise if $\tilde{\alpha}_{t_0}(s_1) = \pi$ we have $\tilde{\alpha}_{t_0}(s) = \pi$ for $s_1 \leq s < 2\pi$. In either case $\overline{Y_0 Y_1} \subset \Gamma$ and again we have a contradiction. Thus the theorem is proved. ∎

We should remark that an analogous result holds when Γ has the reverse orientation, namely, Γ is convex iff the $\tilde{\alpha}_t$ are <u>non-increasing</u>, $t \in \mathbb{T}_1$.

5.5. <u>Corollary</u>: Let Γ be convex $t_0 \in \mathbb{T}_1$. Then

a) $\tilde{\alpha}_{t_0}(s)$ is a continuous $\mathbb{R}$-valued function on $(0,2\pi)$, taking values in $[0,\pi]$.

b) $\tilde{\alpha}_{t_0}(s_0) = 0$ implies the subarc $[\gamma(t_0), \gamma(t_0 + s_0)]$ is a line segment. If $\tilde{\alpha}_{t_0}(s_0) = \pi$ then $[\gamma(t_0 + s_0), \gamma(t_0)]$ is a line segment.

c) Define $I_{t_0} = \{t \in \mathbb{T}_1 : \theta(t) = \theta(t_0)\}$ and $J_{t_0} = \gamma(I_{t_0})$. Then J_{t_0} is a line segment in Γ.

Proof: Part a) was established in the proof of 5.4. If $\tilde{\alpha}_{t_0}(s_0) = 0$, then $\tilde{\alpha}_{t_0}(s) = 0$ for $0 < s \le s_0$, and therefore $\arg(\gamma(t_0 + s) - \gamma(t_0)) = \text{const} = \theta(t_0)$ for $0 < s \le s_0$, which establishes the first assertion of b). The second is proved similarly. To prove c) suppose $t_1, t_2 \in \mathbb{T}_1$ with $\theta(t_1) = \theta(t_2)$; then the tangent lines ℓ_1, ℓ_2 at $\gamma(t_1), \gamma(t_2)$ respectively are parallel and have the same orientation. $\mathbb{C} \setminus \ell_i$ consists of two open half-planes, the "left" half-plane h^i_L, and the "right" half-plane h^i_R, $i = 1,2$ and 5.5a) says that $\Gamma \subset C\ell(h^i_L)$ for both i. Clearly this cannot be true if ℓ_1 and ℓ_2 are distinct. Thus, in particular, we have $\gamma(t_2) \in \ell_1$, whence $\tilde{\alpha}_{t_1}(t_2 - t_1) = 0$ or π, and the result follows by b).

Until further notice, Γ will also be assumed convex.

Write $J_{t_0} = \{\gamma_2, \gamma_1\}$ (the brackets { } indicate that the endpoints may or may not be included). If $\gamma(t_0)$ is not an endpoint we can find $\sigma_1, \sigma_2 \in (0, 2\pi)$, $\sigma_1 < \sigma_2$, with $\gamma_1 = \gamma(t_0 + \sigma_1)$, $\gamma_2 = \gamma(t_0 + \sigma_2)$. If $\gamma(t_0) = \gamma_1$ set $\sigma_1 = 0$, if $\gamma(t_0) = \gamma_2$ set $\sigma_2 = 2\pi$. We shall define a real valued function $\theta_{t_0}(s)$ by

$$\theta_{t_0}(s) = \begin{cases} 0 & \text{if } 0 < s \le \sigma_1 \\ \theta(t_0 + s) - \theta(t_0) & \text{if } \sigma_1 < s < \sigma_2 \text{ and } t_0 + s \in \mathbb{T}_1 \\ 2\pi & \text{if } \sigma_2 \le s < 2\pi. \end{cases}$$

Also we may abbreviate $I^1_{t_0} = (0, \sigma_1]$, $I^2_{t_0} = [\sigma_2, 2\pi)$, together with

$S_{t_0} = \{s \in (0,2\pi) : t_0 + s \in \mathbb{T}_1\}$ and $I^*_{t_0} = S_{t_0} \cap (\sigma_1,\sigma_2)$.

5.6. Lemma: For each $t_0 \in \mathbb{T}$ we have

a) $\tilde{\alpha}_{t_0}(s) = \theta_{t_0}(s)$ iff $s \in I^1_{t_0}$

b) $\tilde{\alpha}_{t_0}(s) + \pi = \theta_{t_0}(s)$ iff $s \in I^2_{t_0}$

c) $\tilde{\alpha}_{t_0}(s) < \theta_{t_0}(s) < \tilde{\alpha}_{t_0}(s) + \pi$ for $s \in I^*_{t_0}$

Proof: For $s_0 \in I^1_{t_0}$, $\overline{\gamma(t_0)\ \gamma\ (t_0 + s_0)}$ is a line segment so $\tilde{\alpha}_t(s)$ is constant for $0 < s \leq s_0$. However, $\lim_{s\to 0^+} \tilde{\alpha}_t(s) = 0$ by 5.1; thus $\tilde{\alpha}_t(s_0) = 0 = \theta_{t_0}(s_0)$, $s_0 \in I^1_{t_0}$. The identity

vi) $\alpha_{t_0}(s) = \pi + \alpha_{t_0+s}(2\pi - s)$

holds for all $t_0 \in \mathbb{T}_1$, $s \in (0,2\pi)$; therefore

vii) $\tilde{\alpha}_{t_0}(s) - \theta_{t_0}(s) = \alpha_{t_0}(s) - \theta(t_0 + s) = \pi + \tilde{\alpha}_{t_0+s}(2\pi - s)$

for $t_0 \in \mathbb{T}_1$, $s \in S_{t_0}$. Thus if $\tilde{\alpha}_{t_0}(\bar{s}) = \theta_{t_0}(\bar{s})$ for some $\bar{s} \in S_{t_0}$ we have, equivalently, $\tilde{\alpha}_{t_0+\bar{s}}(2\pi - \bar{s}) = \pi$. By 5.5b) $[\gamma(t_0), \gamma(t_0 + \bar{s})]$ is a line segment, and it follows that $\tilde{\alpha}_{t_0}(s) = \theta_{t_0}(s) = 0$, $s \in (0,\bar{s}]$. Thus a) is proven. The proof of b) proceeds in like fashion.

As for c), suppose $\bar{s} \in I^*_{t_0}$, $\tilde{\alpha}_{t_0}(\bar{s}) > \theta_{t_0}(\bar{s})$. Then $\pi > \tilde{\alpha}_{t_0}(\bar{s}) - \theta_{t_0}(\bar{s}) > 0$, which is impossible by vii). Similarly, if $\bar{s} \in I^*_{t_0}$, $\theta_{t_0}(\bar{s}) > \tilde{\alpha}_{t_0}(\bar{s}) + \pi$, then $2\pi > \theta_{t_0}(\bar{s}) - \tilde{\alpha}_{t_0}(\bar{s}) > \pi$, i.e. $\pi > \tilde{\alpha}_{t_0}(\bar{s}) - \theta_{t_0}(\bar{s}) > 0$ and the same contradiction results. This proves c), hence also the lemma. ∎

5.7. Theorem: For each $t_0 \in \mathbb{T}$, $\theta_{t_0}(s)$ is a nondecreasing function on S_{t_0}.

Proof: We must show for $s_1, s_2 \in S_{t_0}$, $s_1 < s_2$, that $\theta_{t_0}(s_1) \leq \theta_{t_0}(s_2)$, and it clearly suffices to do this for $s_1, s_2 \in I^*_{t_0}$. We shall set $t_1 = t_0 + s_1$, $t_2 = t_0 + s_2$, and $\bar{s} = s_2 - s_1$.

Now, if $\tilde{\alpha}_{t_2}(2\pi - \bar{s}) = \theta_{t_2}(2\pi - \bar{s})$ then $2\pi - \bar{s} \in I^1_{t_2}$ and the preceding lemma gives $\theta(t_2) = \theta(t_2 + 2\pi - \bar{s}) = \theta(t_1)$; thus $\theta_{t_0}(s_1) = \theta_{t_0}(s_2)$ and the theorem holds. Thus we may assume $\tilde{\alpha}_{t_2}(2\pi - \bar{s}) < \theta_{t_2}(2\pi - \bar{s})$. This gives

$$\tilde{\alpha}_{t_2}(2\pi - s_2) \leq \tilde{\alpha}_{t_2}(2\pi - \bar{s}) < \theta_{t_2}(2\pi - \bar{s})$$

$$= \theta(t_1) - \theta(t_2) = 2\pi - \theta_{t_1}(\bar{s})$$

and therefore, by ii),

$$\tilde{\alpha}_{t_2}(2\pi - s_2) + \theta_{t_1}(\bar{s}) > \theta_{t_1}(\bar{s}).$$

The preceding lemma, together with the identity vi), now implies

$$\tilde{\alpha}_{t_2}(2\pi - s_2) + \theta_{t_1}(\bar{s}) + \theta_{t_0}(s_1) = \alpha_{t_0+s_2}(2\pi - s_2) - \theta(t_0)$$

$$= \pi + \tilde{\alpha}_{t_0}(s_2) > \pi + \tilde{\alpha}_{t_0}(s_1)$$

$$> \theta_{t_0}(s_1).$$

Now apply iii) with $a = \theta_{t_0}(s_1)$, $b = \tilde{\alpha}_{t_2}(2\pi - s_2) + \theta_{t_1}(\bar{s})$, and $b' = \theta_{t_1}(\bar{s})$, to obtain

$$\theta_{t_0}(s_1) \le \theta_{t_0}(s_1) + \theta_{t_1}(\bar{s}) = \theta_{t_0}(s_2). \quad \blacksquare$$

Now, a bounded monotone function on $(0,2\pi)$ is differentiable at almost every point. θ_{t_0} is not defined on all of $(0,2\pi)$, but clearly a slightly modified form of the same result will hold. For instance, it is easily deduced from the preceding that s_{t_0} is "essentially" the domain of θ_{t_0}. We can easily extend θ_{t_0} to a monitone function $\tilde{\theta}_{t_0}$ on the whole interval by setting

$$\tilde{\theta}_{t_0}(s) = \sup \{\theta_{t_0}(\bar{s}) : \bar{s} < s, \bar{s} \in S_{t_0}\}$$

for $s \in (0,1) \setminus S_{t_0}$. $\tilde{\theta}_{t_0}(s)$ has a derivative $\tilde{\theta}'_{t_0}(s)$ almost everywhere in $(0,1)$, hence a.e. in S_{t_0}, and if $\tilde{\theta}'_{t_0}(s)$ exists for $s \in S_{t_0}$ then evidently

$$\tilde{\theta}'_{t_0}(s) = \lim_{\substack{\epsilon \to 0 \\ s+\epsilon \in S_{t_0}}} \frac{\theta_{t_0}(s+\epsilon) - \theta_{t_0}(s)}{s} .$$

In this sense, then, θ_{t_0} has a derivative almost everywhere; in the same sense we can also say θ_{t_0} is continuous. In addition it is not hard to see that the same result, in the same form, must hold for $\theta(t) = \arg \gamma'(t)$, which is defined on $\mathbb{T}_1$, and takes values in $[0,2\pi)$. For if we fix $t_0 \in \mathbb{T}_1$, then with the possible exception of endpoints we have

$$\theta(t) = \text{const} = \theta(t_0) \qquad \text{for } t - t_0 \in I'_{t_0} \cup I^2_{t_0}$$

$$\theta(t) = \theta_{t_0}(s) + \theta(t_0) \qquad \text{for } s = t - t_0 \in I^*_{t_0}$$

and therefore θ has essentially the same differentiability and continuity properties as θ_{t_0}. Thus we obtain the following corollary:

5.8. Corollary: The function $\theta(t) = \arg \gamma'(t)$, defined on $\mathbb{T}_1$ and taking values in $[0,2\pi)$, is differentiable almost every where in its domain. That is, for m - almost all t_0

$$\lim \frac{\theta(t) - \theta(t_0)}{t - t_0} \text{ exists, as } t \to t_0 \text{ in } [0,2\pi)$$

We shall soon see this property characterizes a much broader class of curves for which the K-R result can be obtained.

§6. The Class $AC_\psi[a,b]$

6.1. Definition: $AC_\psi(I) = AC_\psi[a,b]$ will denote the class of absolutely continuous γ on $I = [a,b] \subset \mathbb{R}$ for which there exists a measurable subset I_0 of I (dependent on γ) satisfying

a) $m(I \setminus I_0) = 0$

b) γ' is defined and nonzero on I_0

c) $\theta = \arg \gamma'$ is a continuous map from I_0 (with the subspace topology) to $\mathbb{T}$.

We say that γ' is pseudocontinuous on I_0 if b) and c) hold; thus $AC_\psi(I)$ is the collection of absolutely continuous functions on I whose derivative is pseudocontinuous on subset of I having full measure. $AC_\psi(\mathbb{T})$, and $AC_\psi(J)$ for subarcs J of $\mathbb{T}$, are defined in the same way.

For example, if $\gamma \in C^1(I)$ and $\gamma'(t) \neq 0$ almost everywhere then obviously $\gamma \in AC_\psi(I)$. Additionally, in §5 we developed a class of functions in $AC_\psi(\mathbb{T})$, namely, functions γ which are boundary values of Riemann maps Φ from the unit disc onto bounded convex regions of the complex plane.

Throughout this section γ will denote an injective function in $AC_\psi(I)$ where $I = [0,1]$, and I_0 shall be as described above. (The injectivity will be substantially relaxed later on). By absolute continuity, arclength measure μ on $\Gamma = \gamma(I)$ is defined by formula 5(i); since γ is homeomorphic we see again that μ is a regular Borel measure on Γ. We shall prove in this section that (Γ,μ) satisfies the hypotheses of 4.6, and hence that the K-R result holds for (Γ,μ).

We begin with a local construction and analysis, which will be used throughout.

Write $I_0' = I_0 \cap (0,1)$ and choose a positive number $\alpha < \frac{\pi}{2}$. For $t_0 \in I_0'$ we find, by 6.1 c), an open subinterval I_α of $(0,1)$ containing t_0 with the property * that

i) $|\theta(t) - \theta(t_0)| < \alpha$ for $t \in I_0 \cap I_\alpha$.

Now $\gamma(I_\alpha)$ is open in the subspace topology on Γ; thus for each $t \in I_\alpha$ one can find an open disc D centered at $\gamma(t)$ such that $D \cap \Gamma \subset \gamma(I_\alpha)$. Let D be such a disc centered at $\gamma(t_0)$ and let $R = \text{rad } D$. When we refer to "local coordinates for α,R at $\gamma(t_0)$" we mean the coordinate system corresponding to a translation and rotation (together with a reflection, if Γ is negatively oriented) of $\mathbb{C}$ in which $\gamma(t_0)$ appears as the point iR, and $\theta(t_0) = 0$.

Suppose now that local coordinates for α,R have been established at $\gamma(t_0)$. Then i) becomes

i) $|\theta(t)| < \alpha$, $t \in I_0 \cap I_\alpha$.

For t_1, $t_2 \in I_\alpha$, $t_1 < t_2$, we have

---

* In keeping with our convention that all arguments take values in $[0,2\pi)$ we define $|\theta|$ to equal $\min\{\theta, 2\pi - \theta\}$.

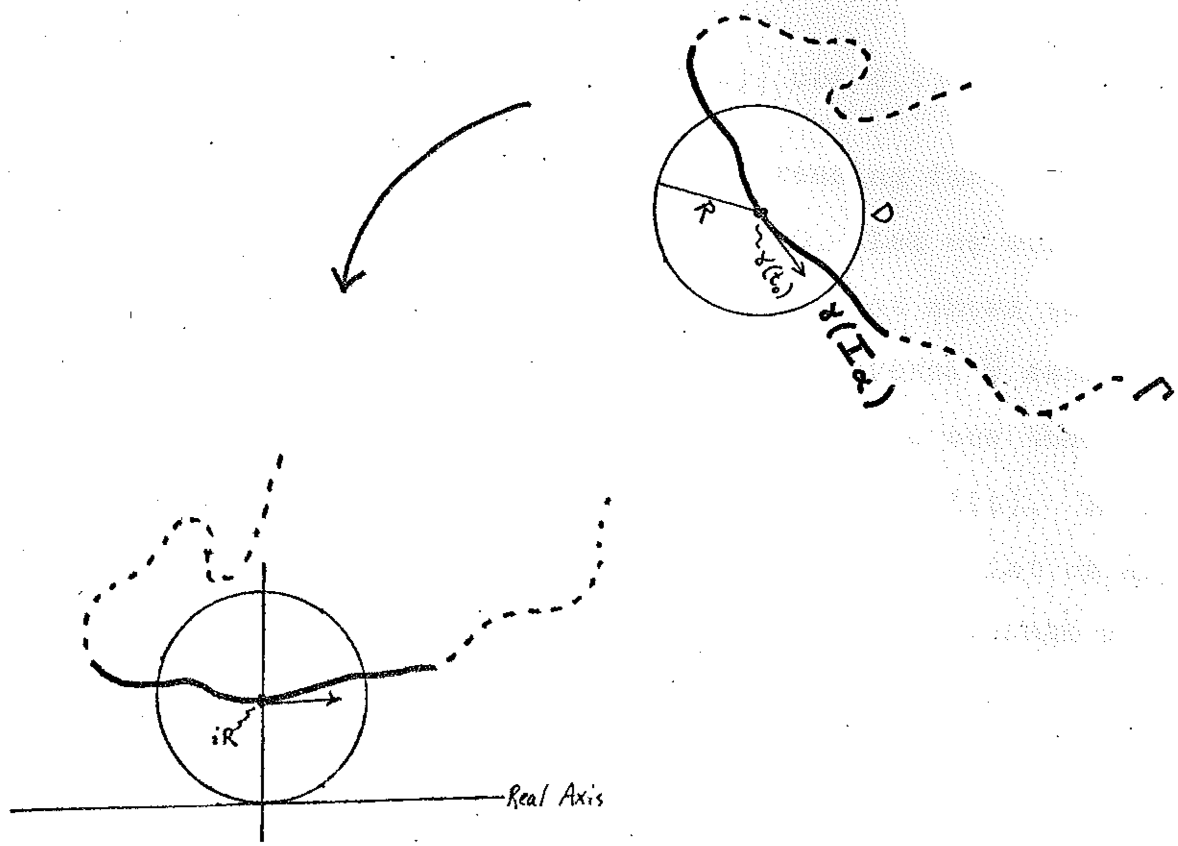


Diagram 1: Local coordinates for α, R at $\gamma(t_0)$

$$\operatorname{Re}\gamma(t_2) - \operatorname{Re}\gamma(t_1) = \int_{t_1}^{t_2} \operatorname{Re}\gamma'(t)dt$$

$$= \int_{t_1}^{t_2} |\gamma'(t)| \cos\theta(t)dt,$$

and it follows that $\operatorname{Re}\gamma(t)$ is strictly increasing for $t \in I_\alpha$, since for each t we have $|\gamma'(t)| \cos\theta(t) > 0$. Let us write $x = x(t) = \operatorname{Re}\gamma(t)$ for $t \in I_\alpha$. The above argument shows that $\operatorname{Re} : \gamma(I_\alpha) \to \operatorname{Re}\gamma(I_\alpha) = \tilde{I}_\alpha$ is invertible, with inverse $\tilde{\gamma}$ satisfying $\tilde{\gamma}(x(t)) = \operatorname{Re}\gamma(t)$, $x \in \tilde{I}_\alpha$. It is not hard to see that $x(t)$ is absolutely continuous, and that $x'(t) = \operatorname{Re}\gamma'(t)$. In particular, if $\tilde{I}_0 = \operatorname{Re}(\gamma(I_0 \cap I_\alpha))$ then $m(\tilde{I}_\alpha \setminus \tilde{I}_0) = 0$. Let us also define $\tilde{\theta}(x) = \theta(\gamma^{-1}\tilde{\gamma}(x))$ for $x \in \tilde{I}_\alpha \cap \tilde{I}_0$. Using the change of Variable Formula (Rudin, [15] p. 186) we see that, for Borel sets $B \subset I_\alpha$

ii) $$\mu(B) = \int_{\gamma^{-1}(B)} |\gamma'(t)|dt = \int_{\gamma^{-1}(B)} \sec\theta(t)\cdot\operatorname{Re}\gamma'(t)dt$$

$$= \int_{\operatorname{Re}(B)} \sec\tilde{\theta}(x)dx.$$

We therefore have the estimate

iii) $m(\operatorname{Re} B) \le \mu(B) \le m(\operatorname{Re} B)\sec\alpha$

Now take arbitrary $t \in I_\alpha = (\tau_1, \tau_2)$ and suppose we had originally chosen $\alpha < \pi/4$. Then $\operatorname{Re}\gamma'(s) > \operatorname{Im}\gamma'(s)$ for $s \in I_\alpha$. Let us observe what happens to $|\gamma(t+\epsilon) - \gamma(t)|$ as ϵ increases from an initial value of zero; we have

$$\frac{d}{d\epsilon}(|\gamma(t+\epsilon)-\gamma(t)|^2) = \frac{d}{d\epsilon}[\mathrm{Re}^2(\gamma(t+\epsilon)-\gamma) + \mathcal{I}m^2(\gamma(t+\epsilon)-\gamma(t))]$$

$$= 2\,\mathrm{Re}(\gamma(t+\epsilon-\gamma(t))\ \mathrm{Re}\ \gamma'(t+\epsilon) + 2\,\mathcal{I}m(\gamma(t+\epsilon)) - \gamma(t))\ \mathcal{I}m\ \gamma'(t+\epsilon)$$

$$= 2\,\mathrm{Re}\,\gamma'(t+\epsilon)\int_0^\epsilon \mathrm{Re}\ \gamma'(t+r)dr + 2\,\mathcal{I}m\ \gamma'(t+\epsilon)\int_0^\epsilon \mathcal{I}m\ \gamma'(t+r)dr,$$

and this latter quantity is strictly positive provided $t + \epsilon < \tau_2$. That is, if $t \in I_\alpha$, $|\gamma(t+\epsilon) - \gamma(t)|$ is a strictly increasing function for $0 \le \epsilon < \tau_2 - t$. In fact, $|\gamma(t+\epsilon) - \gamma(t)|$ is also strictly increasing as ϵ decreases from 0 to $\tau_1 - t$, by an identical argument.

Geometrically speaking, this tells us that if we find an open disc $D_{r_0} = D_{r_0}(\gamma(t))$ such that $\Gamma \cap D_{r_0} \subset \gamma(I_\alpha)$, then for $0 < r \le r_0$ the circles $C_r = C_r(\gamma(t)) = \partial D_r(\gamma(t))$ each interset $\gamma(I_\alpha)$ in exactly two points, and each $J_r = J_r(\gamma(t)) = \Gamma \cap D_r(\gamma(t))$ is an open subarc of $\gamma(I_\alpha)$.

Let us now briefly review some elementary concepts. A sequence $\{E_i\}$ of Borel sets in $\mathbb{R}^k$ <u>shrinks nicely</u> to $x_0 \in \mathbb{R}^k$ if there is a constant $c > 0$ and a sequence $\{r_i\}$ of positive numbers convergent to zero, such that $E_i \subset B_{r_i}(x_0)$ and $m(E_i) \ge c\, m(B_{r_i}(x_0))$ for each i. The <u>Lebesgue set</u> L_f of a function $f \in L^1(\mathbb{R}^k)$ is the collection of $x_0 \in \mathbb{R}^k$ such that

$$\text{iv)} \quad \lim_{i\to\infty} \frac{1}{m(E_i)} \int_{E_i} |f(x) - f(x_0)|dx = 0$$

for every sequence $\{E_i\}$ which shrinks nicely to x_0. A well known and important theorem asserts that the complement of L_f is a set of Lebesque measure zero for each $f \in L^1(\mathbb{R}^k)$ (see Rudin [15] p.).

If F is the characteristic function for a measurable set $S \subset \mathbb{R}^k$, $m(S) > 0$, then the points of $S \cap L_F$ are called <u>points of density</u> for S with respect to Lebesque measure. In this case, (iv) becomes

$$\text{v)} \quad \lim_{i\to\infty} \frac{m(S \cap E_i)}{m(E_i)} = 1$$

for each sequence $\{E_i\}$ shrinking nicely to a point $s_0 \in S \cap L_f$. The points of density for S thus form a subset of S having full measure.

6.2. <u>Proposition</u>: Suppose γ is a one-to-one function in $AC_\Psi(I)$. and μ is arclength measure on $\Gamma = \gamma(I)$. Let S be a μ-measurable subset of Γ. Then for μ - almost all $s \in S$ we have

$$\text{vi)} \quad \lim_{r\to 0} \frac{\mu(S \cap J_r(s))}{\mu(J_r(s))} = 1$$

where, for $r > 0$, $J_r = J_r(s) = \{\gamma(t) \in \Gamma : |\gamma(t) - s| < r\}$.

A point $s \in S$ satisfying vi) will be called a <u>μ-concentration point</u> for S.

<u>Proof</u>: It clearly suffices to obtain the result when $S \subset \gamma(I_0)$, since a μ-concentration point for $S \cap \gamma(I_0)$ is also a μ-concentration point for S. Also we may assume, without loss of generality, that $\mu(S) > 0$ and that $\gamma^{-1}(S) \subset (0,1)$.

Let $S_{(\mu)}$ be the collection of μ-concentration points for S, and suppose that for each $s_0 \in S$ we can find an open subarc J of Γ containing s_0 such that $\mu((S - S_{(\mu)}) \cap J) = 0$. It follows immediately that each compact subset of $S - S_{(\mu)}$ has μ-measure zero, and hence $\mu(S \setminus S_{(\mu)}) = 0$ by regularity of μ. Therefore select $s_0 = \gamma(t_0) \in S$, and fix $\alpha \in (0, \pi/4)$. We shall show that $\gamma(I_\alpha)$ is a suitable choice for J in the above. (I_α is defined by (i)).

For some $R > 0$ we may establish each coordinates for α, R at s_0. Write S^α for $S \cap \gamma(I_\alpha)$ and let $\tilde{L}$ be the collection of points of density (with respect to m) for $\tilde{S}^\alpha = \operatorname{Re}(S^\alpha)$; also set $L = \tilde{\gamma}(\tilde{L})$. Since $m(\tilde{S}^\alpha \setminus \tilde{L}) = 0$, it follows by i) that

$$\text{vii)} \quad \mu(S^\alpha \setminus L) = \int_{\tilde{S}^\alpha \setminus L} \sec \tilde{\theta}(x)dx = 0.$$

Now fix $s' = \gamma(t') \in L$; we claim s' is a μ-concentration point of S^α. Take a sequence of open discs $D_{r_i} = D_{r_i}(s')$, $i = 1,2,\ldots$, with radii r_i decreasing to zero. For r_i sufficiently small we have $\Gamma \cap D_{r_i} \subset I_\alpha$; we may assume the numbering of the $D_{r_i} = D_{r_i}(s')$ begins at this point. As we have seen, the sets $J_{r_i} = J_{r_i}(s') = \Gamma \cap D_{r_i}$ are open subarcs of I_α. We claim the sequence $E_i = \operatorname{Re}(J_{r_i})$, $i = 1,2,\ldots$, shrinks nicely to $x' = \operatorname{Re}(s')$, with respect to m.

For one thing, we have $E_i \subset I_{r_i}$ for each I, where $I_{r_i}(x') = (x' - r_i, x' + r_i)$. Secondly, the estimate $\mu(J_{r_i}) \geq 2r_i$ is immediate, since J_{r_i} passes through the center s' of each $D_{r_i}(s')$. Consequently $m(\mathrm{Re}(J_{r_i})) \geq 2r_i \cos \alpha$, $i = 1,2,\ldots$, and the claim is established.

Now set $\overline{\theta} = \theta(t')$. Given arbitrary $\epsilon > 0$, one can find (using 6.1 c)) a positive integer i_0 such that, for $i \geq i_0$,

viii) $(1 - \epsilon) \sec \overline{\theta} \leq \sec \theta(t) \leq (1 + \epsilon) \sec \overline{\theta} \qquad t \in \gamma^{-1}(J_{r_i})$.

By i) we have

$$\frac{\mu(S^\alpha \cap J_{r_i})}{\mu(J_{r_i})} = \frac{\int_{\tilde{S}^\alpha \cap E_i} \sec \tilde{\theta}(x)dx}{\int_{E_i} \sec \tilde{\theta}(x)dx},$$

and therefore, since $\tilde{\theta}(x) = \tilde{\theta}(x(t)) = \theta(t)$,

$$\left(\frac{1 - \epsilon}{1 - \epsilon}\right) \frac{m(\tilde{S}^\alpha \cap E_i)}{m(E_i)} \leq \frac{\mu(S^\alpha \cap J_{r_i})}{\mu(J_{r_i})} \leq \left(\frac{1 + \epsilon}{1 - \epsilon}\right) \frac{m(\tilde{S}^\alpha \cap E_i)}{m(E_i)} .$$

Now the E_i's shrink nicely to x' which is a (Lebesgue) point of density for $\tilde{S}^\alpha$. Passage to the limit as $i \to \infty$ in the above therefore yields

$$\frac{1 - \epsilon}{1 + \epsilon} \leq \lim_{i \to \infty} \frac{\mu(S^\alpha \cap J_{r_i})}{\mu(J_{r_i})} \leq \frac{1 + \epsilon}{1 - \epsilon},$$

and since $\epsilon > 0$ was arbitrary,

$$\lim_{i \to \infty} \frac{\mu(S^\alpha \cap J_{r_i})}{\mu(J_{r_i})} = 1.$$

Thus s' is a μ-concentration point for S^α, which means $L \subset S_{(\mu)}$. Therefore we have by (vii)

$$\mu((S \setminus S_{(\mu)}) \cap I_\alpha) = \mu\ (S^\alpha \setminus S_{(\mu)}) \leq \mu(S^\alpha \setminus L) = 0$$

which completes the proof.

Observe that, in the statement of 6.2, our attention was confined to a particular type of shrinking sequence of sets, namely intersections of Γ with concentric shrinking discs. By employing a more involved estimation procedure we could eliminate this restriction, and obtain 6.2 for sequences of sets $\{J_i\}$ which merely <u>shrink nicely</u> with respect to μ to some point of Γ. (It is clear from the definition for Lebesgue measure what we mean by this). Also, in the special case $\gamma \in C^1$ the proof is greatly simplified. But we choose not to develop these points here, as the present form of 6.2 is adequate to our needs.

We continue with more local analysis. S will be a fixed measurable subset of Γ with $\mu(S) > 0$, and $s_0 = \gamma(t_0)$ will be a μ-concentration point for $S \cap \gamma(I_0)$. Let us set $\alpha = .01$ radians As we have seen, $\exists\ \tilde{R} > 0 \ni J_{\tilde{R}} = D_{\tilde{R}}(s_0) \cap \Gamma$ is an open subarc of I_α for $0 < R \leq \tilde{R}$. Since s_0 is a μ-concentration point for S we can find, for each $\epsilon > 0$, a number $R_0 \in (0,\tilde{R}]$ such that

ix) $\mu(J_{R_0} \setminus S) < \epsilon\, \mu(J_{R_0})$.

It will turn out that certain key estimates are independent of this choice of R_0; we shall therefore feel free to postpone the precise selection of R_0 until a later time (for those who are interested, a value corresponding to $\epsilon = \sin^2\alpha/100$ will suffice). We shall be working exclusively in local coordinates for α, R_0 at $\gamma(t_0)$. For $\frac{1}{2}R_0 \leq R \leq R_0$ let us write

$$D'_R = D_R(iR) = \{\zeta \in \mathbb{C} : |\zeta - iR| < R\}$$

$$C'_R = \partial D'_R = \{\zeta \in \mathbb{C} : |\zeta - iR| = R\}$$

$$J'_R = \Gamma \cap D'_R = \{\zeta \in \Gamma : |\zeta - iR| < R\} .$$

Also let us define the lines

$$\Lambda_1 = \Lambda_1(\alpha) = \{\gamma(t_0) + re^{i(\theta+\alpha)} | -\infty < r < \infty\}$$

$$\Lambda_2 = \Lambda_2(\alpha) = \{\gamma(t_0) + re^{i(\theta-\alpha)} | -\infty < r < \infty\}.$$

Each Λ_K has two rays Λ_K^+ and Λ_K^- originating at s_0, corresponding to non-negative and non-positive values, respectively, of the parameter r.

For $R_0 \geq R > \frac{R_0}{2}$ $\gamma(t_0) = iR_0$ lies in the open disc D'_R; for those R we let $\lambda_K^{\pm}(R)$ be the point of intersection of $\Lambda_K^{\pm}$ and C'_R. The (open) minor arcs $\widehat{\lambda_2^+(R), \lambda_1^+(R)}$ and $\widehat{\lambda_2^-(R), \lambda_1^-(R)}$ shall be denoted by $A^+(R)$ and $A^-(R)$ respectively.

Also let us define

$$X_\alpha = \{\gamma(t_0) + re^{i(\theta+\epsilon)} : -\infty < r < \infty,\ |\epsilon| < \alpha\}$$

with X_α^+, X_α^- the halves of X_α corresponding to non-negative and non-positive values of r, respectively. We obviously have $\partial X_\alpha = \Lambda_1 \cup \Lambda_2$; also $X_\alpha^\pm \cap C_R^! = A^\pm(R)$ for $R \in (\frac{1}{2} R_0, R_0]$.

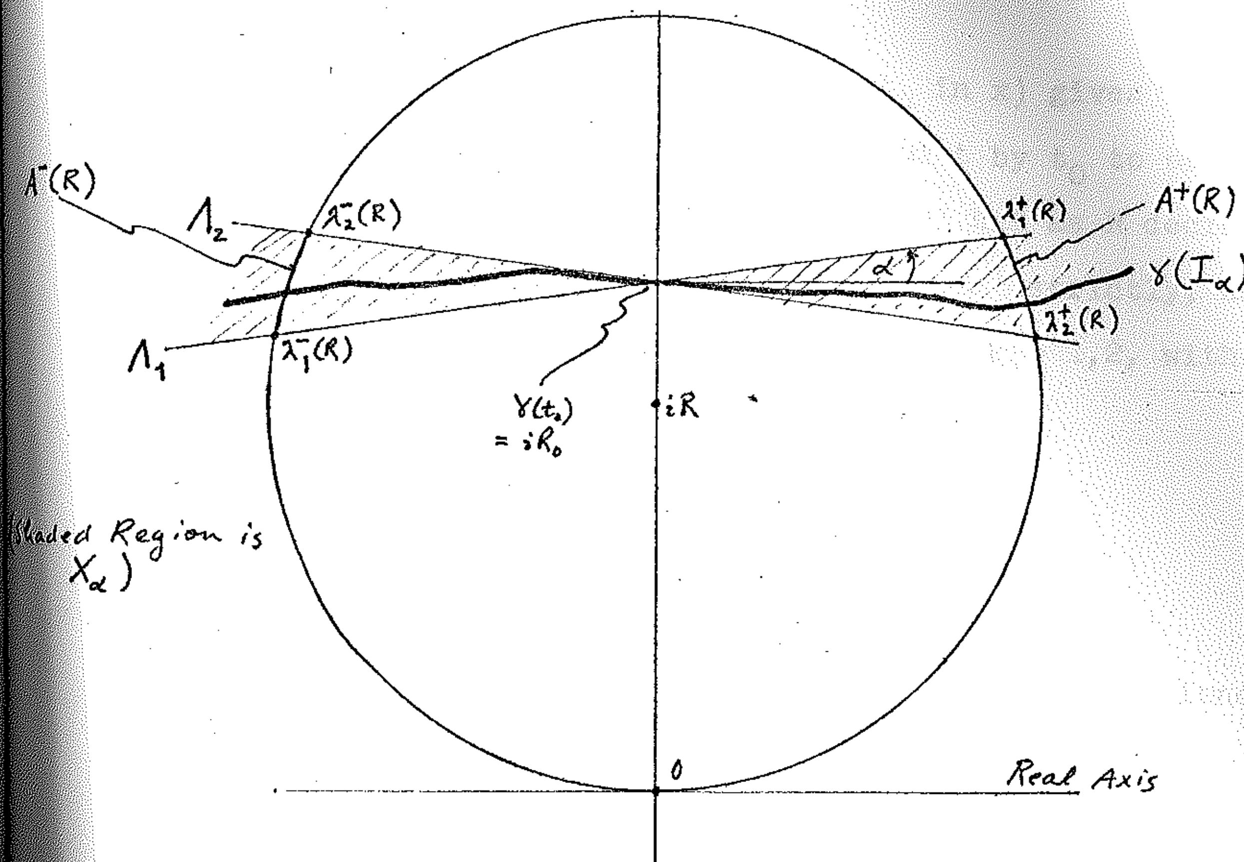


Diagram 2: The circle $C_R^!$, $R \in (\frac{R_0}{2}, R_0]$

Recall we had written $I_\alpha = (\tau_1, \tau_2)$; we now set $I_\alpha^+ = (t_0, \tau_2)$, $I_\alpha^- = (\tau_1, t_0)$. For $t_1 \in I_\alpha$, $t \in (t_1, \tau_2)$ we have

$$|\mathscr{I}m(\gamma(t) - \gamma(t_1))| = \int_{t_1}^{t} \mathscr{I}m\ \gamma'(t)dt = \int_{t_1}^{t} |\gamma'(t)| \sin\theta(t)dt$$

$$< (\sin\alpha)\mu(\gamma(t_1), \gamma(t)),$$

and likewise

$\mathrm{Re}(\gamma(t) - \gamma(t_1)) > (\cos\alpha)\mu(\gamma(t_1), \gamma(t_1))$. Therefore,

$$x)\quad 0 \le \frac{|\mathscr{I}m(\gamma(t) - \gamma(t_1))|}{\mathrm{Re}(\gamma(t) - \gamma(t_1))} < \tan\alpha$$

for $t_1, \in I_\alpha, t \in (t, \tau_2)$. Setting $t = t_0$ in x) gives $I_\alpha^+ \subset X_\alpha^+$; moreover, if $\bar{t} \in I_\alpha^-$, replace t_1 by $\bar{t}$ and t by t_0 to obtain $I_\alpha^- \subset X_\alpha^-$. Now it is clear (see Diagram 2) hat as R decreases on $(0, {}^R/_2]$ the functions arg $\lambda_k^+(R)$ increase strictly, and the functions arg $\lambda_k^-(R)$ decrease strictly, for $k = 1,2$, with $^\pi/2$ the limit for all four as $R \to 0$. Since

$$\arg(\lambda_1^+(R_0)) = \tfrac{1}{2}(\tfrac{\pi}{2} - \alpha)$$
$$\arg(\lambda_2^+(R_0)) = \tfrac{1}{2}(\tfrac{\pi}{2} + \alpha)$$

(see Diagram 3a)) we must have unique numerical solutions R_1, R_2 in $(\frac{1}{2} R_0, R_0)$ of the equations

$$\text{xi)}\quad \arg\lambda_2^+(R_1) = {}^\pi/4, \qquad \arg\lambda_1^+(R_2) = \frac{\pi}{2} - 2\alpha.$$

These can be solved using elementary geometric methods (particularly the law of sines). For the first equation this gives

$$\frac{R_0 - R_1}{\sin\alpha} = \frac{R_1}{\sin(\frac{\pi}{2} - \alpha)}$$

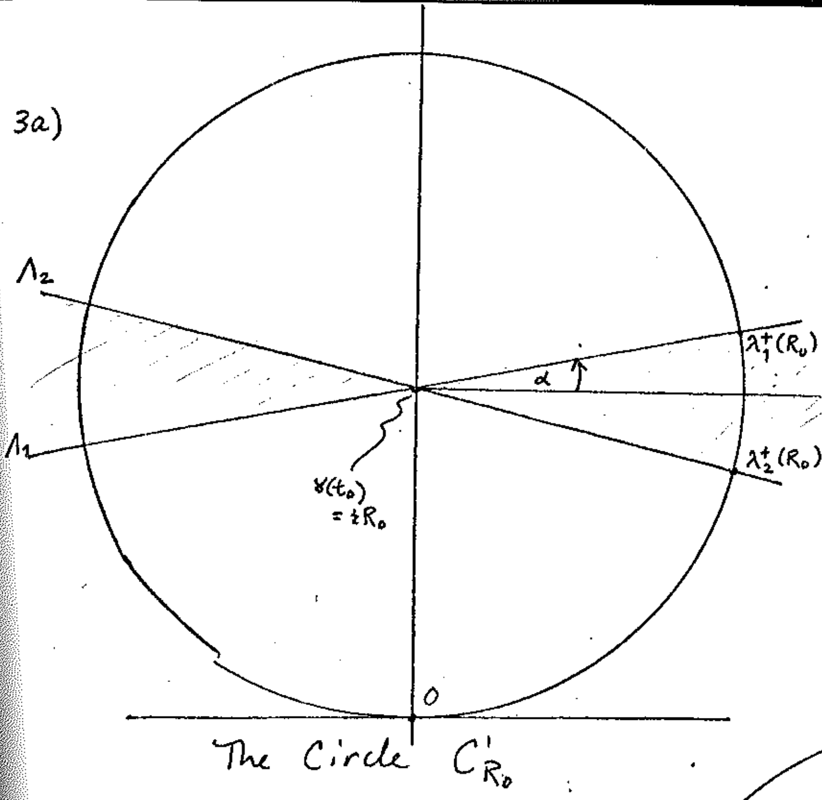
(see Diagram 3b) and for the second we obtain

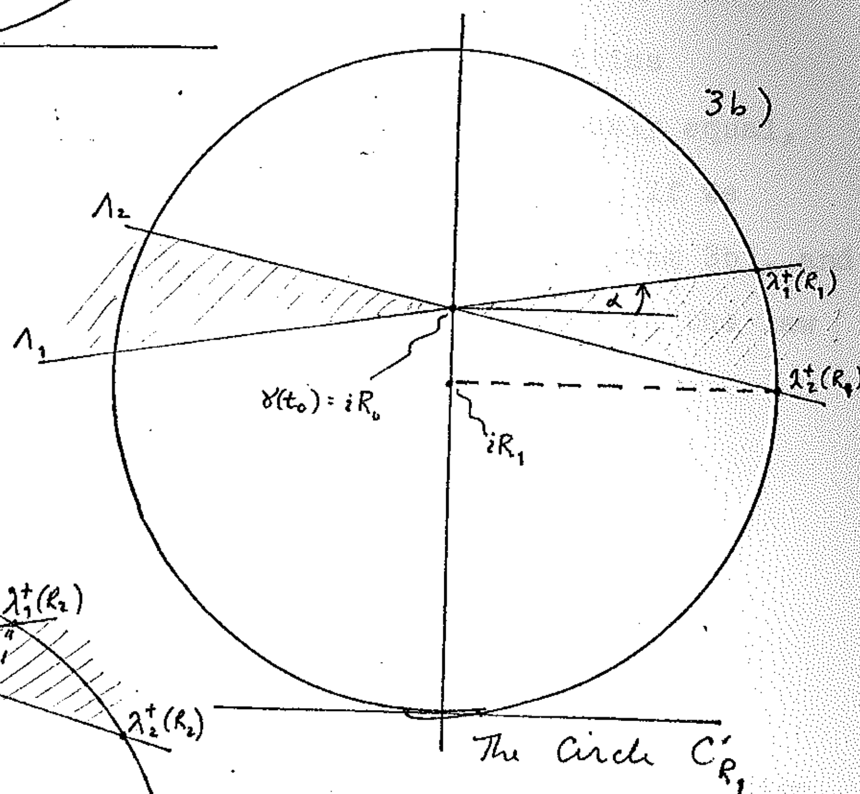
$$\frac{R_0 - R_2}{\sin(\frac{\pi}{2} - 5\alpha)} = \frac{R_2}{\sin(\frac{\pi}{2} + \alpha)}$$

(see Diagram 3c). Hence the desired solutions (in terms of R_0) are

xii) $R_1 = \left(\frac{\cos\ \alpha}{\cos\ \alpha + \sin\ \alpha}\right)R_0$

$$R_2 = \left(\frac{\cos\ \alpha}{\cos\ \alpha + \cos\ 5\alpha}\right)R_0 .$$





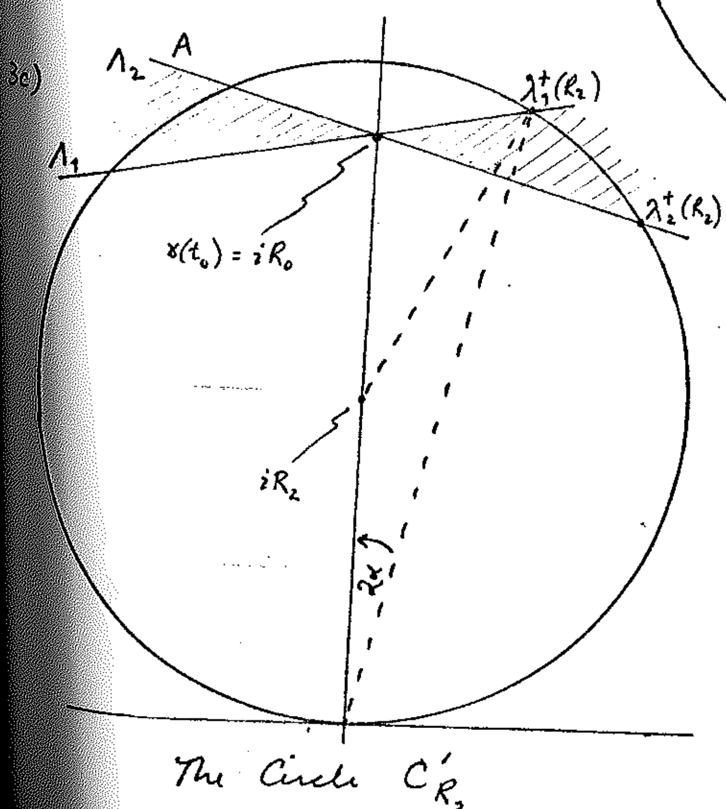


Diagram 3: The circles C'_{R_0}, C'_{R_1}, C'_{R_2}

So far we have only used the fact that $\alpha < \pi/4$; we may thus conclude from xii) that by initially choosing α small we shall have R_1, R_2 as close as we like to R_0, $\frac{1}{2} R_0$, respectively. This degree of closeness, once α is fixed, is independent of our subsequent choice of R_0, so long as $D_{R_0}(s_0) \cap \Gamma \subset I_\alpha$. (For our present choice of α we have $R_1 > (.99)R_0$ and $R_2 < (1.01)\frac{1}{2} R_0$.)

We also need explicit formulas for the quantities $|\lambda_1^+(R_1) - iR_0|$ and $|\lambda_2^+(R_2) - iR_0|$; these are obviously equal, respectively, to $|\lambda_2^-(R_1) - iR_0|$ and $|\lambda_1^-(R_2) - iR_0|$. If we construct the segment joining iR_1 and $\lambda_1^+(R_1)$ in diagram 3b), and the segment joining iR_2 and $\lambda_2^+(R_2)$ in diagram 3c) then the law of sines can be applied. This gives

$$\text{xiii)} \quad |\lambda_1^+(R_1) - iR_0| = \frac{\cos 2\alpha}{\cos \alpha} R_1 = \frac{\cos 2\alpha}{\cos \alpha + \sin \alpha} R_0$$

$$|\lambda_2^+(R_2) - iR_0| = \frac{\sin 6\alpha}{\cos \alpha} R_2 = \frac{\sin 6\alpha}{\cos \alpha + \cos 5\alpha} R_0$$

Note that the remarks following xii) also apply here.

Set $c_R(w) = iR - iRe^{iw}$; then C'_R has the parameterization $C'_R = \{c_R(w) : 0 \le w < 2\pi\}$. Now evidently

$$\text{xiv)} \quad \arg c_R(w) = \tfrac{1}{2}w,$$

also w is the direction angle of the tangent to C'_R at $c_R(w)$. We may write

$$A^+(R) = \{c_R(w) : w_2 < w < w_1\}$$

for some values w_1, w_2; evidently $\lambda_2^+(R) = c_R(w_2)$, $\lambda_1^+(R) = c_R(w_1)$. As we have seen, the relation

$$\frac{\pi}{2} - 2\alpha \geq \arg \lambda_1^+(R) > \arg \lambda_2^+(R) \geq \pi/4$$

holds for $R_2 \leq R \leq R_1$, and therefore

xv) $\pi - 4\alpha \geq w_1 > w_2 \geq \pi/2$.

Now let γ_1, γ_2 be distinct points of I_α^+; by x) we have

$$\left|\frac{\mathcal{I}m(\gamma_2 - \gamma_1)}{\mathrm{Re}(\gamma_2 - \gamma_1)}\right| < \tan \alpha,$$

so that the direction angle of the segment $\overline{\gamma_1\gamma_2}$ is either ϵ or $\epsilon + \pi$ for some angle ϵ with $|\epsilon| < \alpha$. If γ_1 and γ_2 are <u>both</u> points of the same $C_R^!$, then $\gamma_1, \gamma_2 \in A^+(R)$ (since $I_\alpha^! \subset X_\alpha$ and $X_\alpha^+ \cap C_R^! = A^+(R)$); hence by the Mean Value Theorem $\exists$ point $c_R(w_0) \in A^+(R)$ (between γ_1 and γ_0) such that the tangent to $C_R^!$ at $c_R(w_0)$ is parallel to $\overline{\gamma_1\gamma_2}$. Comparison of direction angles shows this to be impossible; hence for $R \in [R_2, R_1]$ the circle $C_R^!$ contains at most one point of I_α^+. Similar reasoning shows $C_R^!$ contains at most one point of I_α^-. But $\gamma(t_0)$ lies in the interior $D_R^!$ of $C_R^!$, so it isn't hard to see that $\Gamma \cap C_R^! = I_\alpha \cap C_R^!$ contains at least two points. Thus $C_R^! \cap I_\alpha^+$ is a single point $\gamma_+(R) = \gamma(t_+)$, and $C_R^! \cap I_\alpha^-$ is a single point $\gamma_-(R) = \gamma(t_-)$; moreover the intersection of I_α and $D_R^!$ is the arc

xvi) $(\gamma_-(R), \gamma_+(R)) = (\gamma(t_-), \gamma(t_+)) = \{\gamma(t) : t_- < t < t_+\}$.

Also, suppose $R_1 \geq R' > R'' \geq R_2$; then we claim $t' > t''$, where t', t'' are defined by

$$\gamma(t') = \gamma_+(R'); \; \gamma(t'') = \gamma_+(R'').$$

The proof is quite simple: If $t'' > t'$ then by x) we have $\gamma(t'') = \gamma(t') + re^{i\epsilon}$ where $|\epsilon| < \alpha$ and $r > 0$. At each $c_{R'}(w)$ of $C'_{R'}$ the direction angles $w + \varphi$ for $\pi < \varphi < 2\pi$ are all exterior for C'_R. By xv) we see then that ϵ is an exterior direction at all points of $A^+(R')$, including $\gamma(t') = \gamma_+(R')$; thus $\gamma(t'')$ is exterior to $C'_{R'}$ contradiction. The case $t'' = t'$ is trivially excluded.

$$\begin{aligned} \text{Now, } c_R(w) &= iR - iRe^{iw} \\ &= iR - iR(\cos w + i \sin w) \\ &= iR(1 - \cos w) + R \sin w \end{aligned}$$

and so, for $R \in [R_2, R_1]$, the derivative

$$\frac{d}{dw} \operatorname{Re} c_R(w) = \cos w$$

is strictly negative on $A^+(R) = \{c_R(w) : w_2 < w < w_1\}$ by xv). It follows that

$$\text{xvii)} \quad \operatorname{Re} \lambda_1^+(R) = \inf \{\operatorname{Re} z : z \in A^+(R)\}$$

$$\operatorname{Re} \lambda_2^+(R) = \sup \{\operatorname{Re} z : z \in A^+(R)\}$$

for $R \in [R_2, R_1]$.

By symmetry we also see that

$$\text{xvii)}' \quad \operatorname{Re} \lambda_2^-(R) = -\operatorname{Re} \lambda_1^+(R) = \sup \{\operatorname{Re} z : z \in A^-(R)\}$$

$$\operatorname{Re} \lambda_1^-(R) = -\operatorname{Re} \lambda_2^+(R) = \inf \{\operatorname{Re} z : z \in A^-(R)\}.$$

Now xvii), together with iii) implies that

$$\begin{aligned}
\mu(\gamma(t_0), \gamma_+(R_1)) &\geq m(\operatorname{Re}(\gamma(t_0), \gamma_+(R_1))) \\
&= \operatorname{Re} \gamma_+(R_1) \\
&\geq \operatorname{Re} \lambda_1^+(R_1) \\
&= |\lambda_1^+(R_1) - iR_0| \cos \alpha
\end{aligned}$$

$$\begin{aligned}
\mu(\gamma(t_0), \gamma_+(R_2)) &\leq m(\operatorname{Re}(\gamma(t_0), \gamma_+(R_2))) \sec \alpha \\
&= \operatorname{Re} \gamma_+(R_2) \cdot \sec \alpha \\
&\leq \operatorname{Re} \lambda_2^+(R_2) \cdot \sec \alpha \\
&= |\lambda_2^+(R_2) - iR_0|.
\end{aligned}$$

Similarly, by (xvii)',

$$\begin{aligned}
\mu(\gamma_-(R_1), \gamma(t_0)) &\geq |\lambda_2^-(R_1) - iR_0| \cos \alpha \\
&= |\lambda_1^+(R_1) - iR_0| \cos \alpha
\end{aligned}$$

$$\mu(\gamma_-(R_2), \gamma(t_0)) \geq |\lambda_1^-(R_2) - iR_0| = |\lambda_2^+(R_2) - iR_0|.$$

Therefore, using xiii), we obtain

$$\begin{aligned}
\mu(J'_{R_1}) &= \mu(\gamma_-(R_1), \gamma_+(R_1)) \\
&= \mu(\gamma_-(R_1), \gamma(t_0)) + \mu(\gamma(t_0), \gamma_+(R_1))
\end{aligned}$$

(continued)

$$= 2|\lambda_1^+(R) - iR_0| \cos\alpha = \frac{2\cos 2\alpha \cos\alpha}{\cos\alpha + \sin\alpha} R_0;$$

and $\mu(J'_{R_2}) = \mu(\gamma_-(R_2), \gamma(t_0)) + \mu(\gamma(t_0), \gamma_+(R_2))$

$$\leq 2|\lambda_2^+(R_2) - iR_0|$$

$$= \frac{2\sin 6\alpha}{\cos\alpha + \cos 5\alpha} R_0 .$$

Let us now define

$$K = K_{\alpha,R_0} = J'_{R_1} \setminus J'_{R_2} .$$

Using the above estimates for $\mu(J'_{R_1})$, $\mu(J'_{R_2})$ we see that

$$\mu(K) \geq \left[\left(\frac{2\cos 2\alpha\cos\alpha}{\cos\alpha + \sin\alpha}\right) - \left(\frac{2\sin 6\alpha}{\cos\alpha + \cos 5\alpha}\right)\right] R_0 .$$

Since by ii) we have $\mu(J'_{R_0}) \leq 2R_0 \sec\alpha$, it follows that

xviii) $$\frac{\mu(K)}{\mu(J'_{R_0})} \geq \frac{1}{2}\left[\left(\frac{2\cos 2\alpha\cos\alpha}{\cos\alpha + \sin\alpha}\right) - \left(\frac{2\sin 6\alpha}{\cos\alpha + \cos 5\alpha}\right)\right]\cos\alpha .$$

The quantity on the right of xviii) will be denoted by h_α. Observe that h_α is independent of R_0. (For our present α, $h_\alpha > .97$).

Now we may write

$$K = (D'_{R_1} \setminus D'_{R_2}) \cap \Gamma = \left(\bigcup_{R_1 > R > R_2} C'_R\right) \cap \Gamma = \bigcup_{R_1 > R > R_2} (C'_R \cap \Gamma)$$

$$= \bigcup_{R_1 > R > R_2} \{\gamma_-(R), \gamma_+(R)\} = K^+ \cup K^-,$$

where $K^\pm = \{\gamma_\pm(R) \mid R_1 > R > R_2\}$. It is clear that

xix) $K \cap I_\alpha^+ = K^+, \qquad K \cap I_\alpha^- = K^-.$

On the other hand, if $t_1^-, t_2^-, t_2^+, t_1^+ \in I_\alpha$ are such that

$$\gamma(t_K^\pm) = \gamma\pm(R_K), \qquad k = 1,2,$$

then we have $t_1^- < t_2^- < t_0 < t_2^+ < t_1^+$ (see paragraph following xvi), so that

$$K = (\gamma(t_1^-), \gamma(t_1^+)) \setminus (\gamma(t_2^-), \gamma(t_2^+))$$

$$= (\gamma(t_1^-), \gamma(t_2^-)) \cup (\gamma(t_2^+), \gamma(t_1^+)).$$

From xix) we see that

$$K^+ = (\gamma(t_2^+), \gamma(t_1^+))$$

$$K^- = (\gamma(t_1^-), \gamma(t_2^-)) .$$

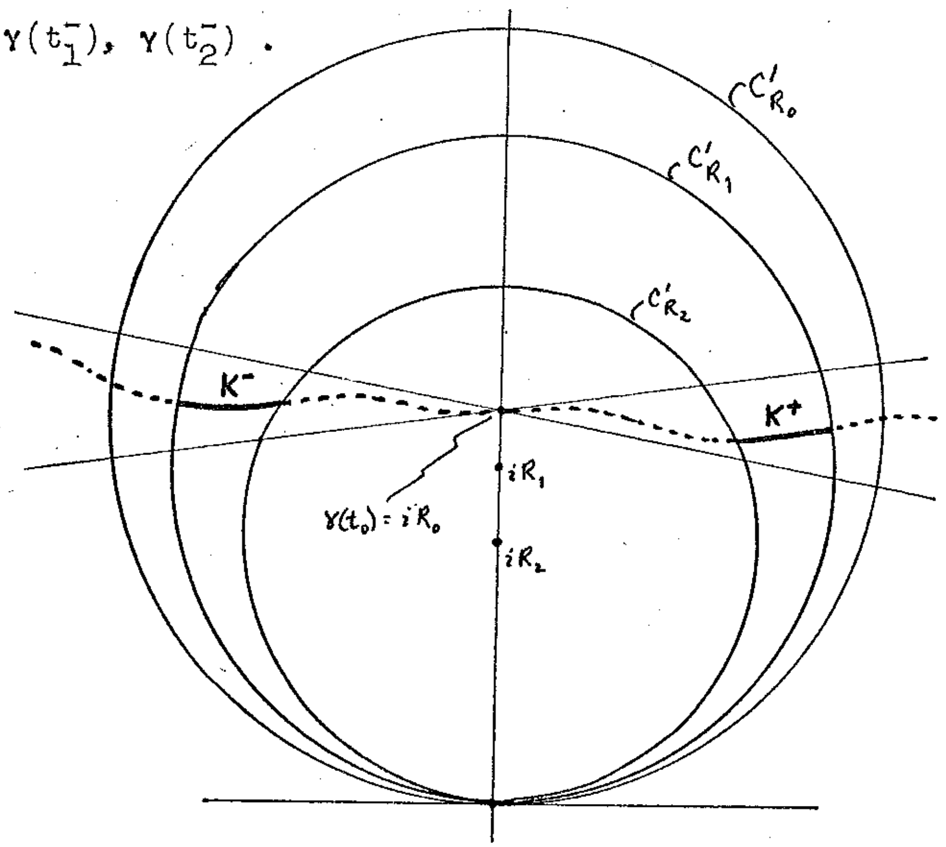


Diagram 4: The set $K = K^+ \cup K^-$

It has been shown, then, that each $\gamma(t) \in K^+$ is $\gamma_+(R)$ for some $R \in (R_2, R_1)$; this R is obviously unique. Also we know that no C_R' contains more than a single point of I_α^+, hence of K^+, for $R_2 < R < R_1$. Thus the pairing $(R, \gamma_\pm(R))$ defines a bijection of Y and $K^\pm$. More concretely, we have bijections

$$r_\pm : K^\pm \to Y = (R_2, R_1)$$

defined by the equations

$$r_\pm(\gamma_\pm(R)) = R.$$

Let us now digress briefly to discuss some measure theory. We define the mearure $\tilde{\mu}$ on I by $\tilde{\mu}(E) = \int_E |\gamma'(t)| dt$ for $E \in \mathcal{B}_I$. Then $\tilde{\mu}(E) = \mu(\gamma(E))$, $E \in \mathcal{B}_I$, and so

$$\gamma(I, \tilde{\mu}) \to (\Gamma, \mu)$$

is a measure space equivalence. Therefore

$$\text{xx)} \quad \int_E f \, d\tilde{\mu} = \int_{\gamma(E)} (f \circ \gamma^{-1}) d\mu$$

for Borel function f on I, $E \in \mathcal{B}_I$. Also, if $\frac{df}{d\tilde{\mu}}(t)$ exists, then so does $\frac{d(f \circ \gamma^{-1})}{d\mu}(\gamma(t))$ and

$$\text{xxi)} \quad \frac{df}{d\tilde{\mu}}(t) = \frac{d(f \circ \gamma^{-1})}{d\mu}(\gamma(t))$$

Now since $|\gamma'(t)| > 0$ a.e. (m) on I, the measures $\tilde{\mu}$ and m are mutually absolutely continuous. In particular $\frac{dm}{d\tilde{\mu}}$ exists, and

$$\frac{dm}{d\tilde{\mu}}(t) = 1\Big/\frac{d\tilde{\mu}}{dm}(t) = 1\Big/|\gamma'(t)|$$

whenever $\gamma'(t)$ exists.

Set $h = \frac{dm}{d\tilde{\mu}} \circ \gamma^{-1}$ on Γ, and define a measure $\tilde{m}$ on $\mathcal{B}_\Gamma$ by

$$\tilde{m}(B) = \int_B h \, d\mu \qquad B \in \mathcal{B}_\Gamma .$$

Then by xx)

$$\tilde{m}(\gamma(E)) = \int_{\gamma(E)} (\frac{dm}{d\tilde{\mu}} \circ \gamma^{-1}) d\mu = \int_E \frac{dm}{d\tilde{\mu}} d\tilde{\mu}$$

$$= m(E) \qquad \text{for } E \in \mathcal{B}_I .$$

Thus also the measure spaces (I,m) and $(\Gamma,\tilde{m})$ are equivalent via γ, and we have

$$\text{xx})' \quad \int_E (g\circ\gamma)dm = \int_{\gamma(E)} g \, d\tilde{m} ,$$

together with the corresponding formula

$$\text{xxi})' \quad \frac{d(g\circ\gamma)}{dm} = \frac{dg}{d\tilde{m}} ,$$

for Borel functions g on Γ, and sets $E \subset \mathcal{B}_I$. Note also that $\frac{d\tilde{m}}{d\mu}$ exists a.e. (μ), and

$$\text{xxii}) \quad \frac{d\tilde{m}}{d\mu}(\gamma(t)) = h(\gamma(t)) = \frac{1}{|\gamma'(t)|} .$$

Now suppose we are given some γ_0 in $K = K^+ \cup K^-$; we know then that γ_0 equals $\gamma_+(R)$ or $\gamma_-(R)$ for some $R \in Y = (R_2,R_1)$.

Since $\gamma_0 \in C'_R$ we must have

$$|\gamma_0 - iR|^2 = R^2$$

which in fact may be solved explicitly to yield

$$R = \frac{|\gamma_0|^2}{2\,\mathcal{J}m\,\gamma}\,.$$

Thus r_+ and r_- are the restrictions to K^+ and K^- respectively of the function

$$R(\zeta) = |\zeta|^2/2\,\mathcal{J}m\,\zeta.$$

$R(\zeta)$ is very nicely behaved in a neighborhood of K. If we set

$$\tilde{r}_\pm(t) = r_\pm(\gamma(t)) \qquad t \in \gamma^{-1}(K^\pm)$$

Then evidently the $\tilde{r}_\pm$ are differential almost everywhere, with

xxiii) $$\frac{d\tilde{r}_\pm}{dm}(t) = \frac{d}{dt}\left(\frac{|\gamma(t)|^2}{2\,\mathcal{J}m\,\gamma(t)}\right)$$

$$= \frac{2|\gamma(t)|\,\frac{d|\gamma(t)|}{dt}\,\mathcal{J}m\,\gamma(t) - \mathcal{J}m\,\gamma'(t)\,|\gamma(t)|^2}{2\,\mathcal{J}m^2\gamma(t)}$$

$$= \frac{2|\gamma(t)|\;|\gamma'(t)|\,\cos\beta(t)\,\mathcal{J}m\,\gamma(t) - \mathcal{J}m\,\gamma'(t)\,|\gamma(t)|^2}{2\,\mathcal{J}m^2\gamma(t)}\,.$$

The last equality because

$$\frac{d|\gamma(t)|}{dt} = \frac{(\gamma'(t),\,\gamma(t))}{|\gamma(t)|} = |\gamma'(t)|\,\cos\beta(t)$$

when we consider the (real) inner product of γ, γ' as vector functions in $\mathbb{R}^2$, and $\beta(t)$ is the angle between $\gamma(t)$ and $\gamma'(t)$. Also, from xxi)' and xxii)

$$\frac{dr_\pm}{d\mu}(\gamma(t)) = \frac{dr_\pm}{d\tilde{m}}(\gamma(t))\ \frac{d\tilde{m}}{d\mu}(\gamma(t)) = \frac{1}{|\gamma'(t)|}\ \frac{d\tilde{r}_\pm}{dm}(t)$$

which gives, together with xxiii) above,

$$\text{xxiv)}\quad \frac{dr_\pm}{d\mu}(\gamma(t) = \frac{|\gamma(t)|\ \cos\ \beta(t)}{\mathcal{I}m\ \gamma(t)} - \frac{\mathcal{I}m\ \gamma'(t)\,|\gamma(t)|^2}{2|\gamma'(t)|\ \mathcal{I}m^2\ \gamma(t)}\ .$$

We shall show that $\frac{dr_+}{d\mu}$ is positive on K^+ and $\frac{dr_-}{d\mu}$ negative on K^-; also that both are bounded away from zero with a bound that is independent of R_0.

We recall the parameterization $c_R(w)$ for C_R', $R \in [R_2, R_1]$. Using xiv) and xv) we see that

$$\arg \lambda_1^+(R) = \sup\ \{\arg z\ :\ z \in A^+(R)\}$$

$$\arg \lambda_2^+(R) = \inf\ \{\arg z\ :\ z \in A^+(R)\},$$

and since $\arg \lambda_1^+(R)$, $\arg \lambda_2^+(R)$ increase with decreasing $R \in [R_2, R_1]$ it follows by xi) that

$$\text{xxv)}\quad \frac{\pi}{2} - 2\alpha = \arg \lambda_1^+(R_2) \geq \arg \gamma(t) \geq \arg \lambda_2^+(R_1) \geq \frac{\pi}{4}$$

for $t \in \gamma^{-1}(K^+)$. Similarly

$$\text{xxv)'}\quad \frac{\pi}{2} + 2\alpha \leq \arg \gamma(t) \leq \frac{3\pi}{4}, \qquad t \in \gamma^{-1}(K^-).$$

We shall also employ the identities

$$\mathcal{I}m\ \gamma(t) = |\gamma(t)| \sin(\arg \gamma(t))$$

$$\mathcal{I}m\ \gamma'(t) = |\gamma'(t)| \sin \theta(t)$$

in conjunction with the above, to obtain the desired estimates. Let us begin with $\frac{dr_+}{d\mu}$. If $\theta(t) \geq 0$, $t \in \gamma^{-1}(K^+)$, then $\beta(t) = [\arg \gamma(t)] - \theta(t)$ satisfies

$$\frac{\pi}{2} - 2\alpha \geq \beta(t) \geq \frac{\pi}{4} - \alpha$$

by xxv). Therefore $\cos \beta(t) \geq \cos(\frac{\pi}{2} - 2\alpha) = \sin 2\alpha$ and from xxii) we obtain

$$\frac{dr_+}{d\mu}(\gamma(t)) = \frac{|\gamma(t)|}{\mathcal{I}m\ \gamma(t)} \cos \beta(t) - \frac{1}{2} \frac{\mathcal{I}m\ \gamma'(t)}{|\gamma'(t)|} \left(\frac{|\gamma(t)|}{\mathcal{I}m\ \gamma(t)}\right)^2$$

$$= \frac{\cos \beta(t)}{\sin(\arg \gamma(t))} - \frac{1}{2} \frac{\sin \theta(t)}{\sin^2(\arg \gamma(t))}$$

$$\geq \frac{\sin 2\alpha}{\sin(\frac{\pi}{2} - 2\alpha)} - \frac{1}{2} \frac{\sin \alpha}{\sin^2(\frac{\pi}{4})}$$

$$= \tan 2\alpha - \sin \alpha > \sin \alpha, \text{ for } t \in \gamma^{-1}(K^+)$$

On the other hand, if $\theta(t) < 0$ for some $t \in \gamma^{-1}(K^+)$, then $\frac{\pi}{2} - \alpha \geq \beta(t) \geq \frac{\pi}{4}$, and

$$\frac{dr_+}{d\mu}(\gamma(t)) \geq \frac{|\gamma(t)|}{\mathcal{I}m\ \gamma(t)} \cos \beta(t) \geq \frac{\cos(\frac{\pi}{2} - \alpha)}{\sin(\frac{\pi}{2} - 2\alpha)}$$

$$= \frac{\sin \alpha}{\cos 2\alpha} > \sin \alpha.$$

This proves

$$\text{xxvi)} \quad \frac{dr_+}{d\mu}(\gamma(t)) > \sin\alpha, \qquad t \in \gamma^{-1}(K^+).$$

And an antisymmetric argument using xxv)' gives

$$\text{xxvi)'} \quad \frac{dr_-}{d\mu} < -\sin\alpha, \qquad t \in \gamma^{-1}(K^-) .$$

Thus a lower bound of $\sin\alpha$ is established for $|\frac{dr_\pm}{d\mu}|$ on $K^\pm$. The upper bound $|\frac{dr_\pm}{d\mu}| \le 4$ on $K^\pm$, is a triviality from xxiii).

Define the measures $m_\pm$ on $K^\pm$ by integrating $|\frac{dr_\pm}{d\mu}|$; that is, $m_\pm(F) = \int_F |\frac{dr_\pm}{d\mu}| d\mu$ for $F \subset \mathcal{B}_{K^\pm}$. We have

$$|\frac{dr_\pm}{d\mu}| = |\frac{d\tilde{r}_\pm}{d\mu}| \circ \gamma^{-1} \qquad \text{by xxi)}$$

and therefore

$$m_\pm(F) = \int_F (|\frac{d\tilde{r}_\pm}{d\tilde{\mu}}| \circ \gamma^{-1}) d\mu = \int_{\gamma^{-1}(F)} |\frac{d\tilde{r}_\pm}{d\tilde{\mu}}| d\tilde{\mu}$$

$$= \int_{\gamma^{-1}(F)} |\frac{d\tilde{r}_\pm}{dm} \frac{dm}{d\tilde{\mu}}| d\tilde{\mu} = \int_{\gamma^{-1}(F)} |\frac{d\tilde{r}_\pm}{dm}| dm.$$

(The last equality holds because $\frac{dm}{d\tilde{\mu}} = |\gamma'| > 0$.) Now $\tilde{r}_\pm$ is absolutely continuous on the arc $\gamma^{-1}(K^\pm) \subset I$; therefore the graph $\tilde{r}_\pm$ is rectifiable, with arclength measure

$$\lambda_\pm(\tilde{r}_\pm(E)) = \int_E |\frac{dr_\pm}{dm}| dm$$

for $E \in \mathcal{B}_I$ (compare with the above). But since $\tilde{r}_\pm$ is real-valued and strictly monotone on $\gamma^{-1}(K^{\pm})$ it follows that $\lambda\pm$ coincides with Lebesque measure on the graph of $\tilde{r}_{\pm}$, that is on $Y = (R_2, R_1)$. (This is easily verified by recalling the definition of arclength measure, or one may refer to the much more general result in Rudin [ ], Theorem 8.26). We have thus shown that, for $F \in \mathcal{B}_\Gamma$

$$m_{\pm}(F) = \lambda_{\pm}(\tilde{r}_{\pm}(\gamma^{-1}(F)))$$

$$= m(\tilde{r}_{\pm}(\gamma^{-1}(F)))$$

$$= m(r_{\pm}(F)) .$$

We shall denote the bijections $r_+^{-1}r_-$ and $r_-^{-1}r_+$ by q_+, q_- respectively; obviously $q_- = (q_+)^{-1}$ and vice versa. From the above, for $F \subset K^{\pm}$

$$\int_F \left|\frac{dr_{\pm}}{d\mu}\right| d\mu = m_{\pm}(F)$$

$$= m(r_{\pm}(F))$$

$$= m(r_{\mp}(q_{\mp}(F))$$

$$= m_{\mp}(q_{\mp}(F))$$

$$= \int_{q_{\pm}(F)} \left|\frac{dr_{\mp}}{d\mu}\right| d\mu$$

which is to say, if $F_+ \subset K^+$ with $F_- = q_-(F) \subset K^-$, then

xxvii) $\int_{F_+} \left|\frac{dr_+}{d\mu}\right| du = \int_{F_-} \left|\frac{dr_-}{d\mu}\right| d\mu = m(r_{\pm}(F_{\pm}))$.

Using the fact that

$$\text{xxviii)}\quad 4 > \left|\frac{dr_\pm}{d\mu}\right| > \sin\alpha$$

on $K^\pm$, this becomes

$$\text{xxix)}\quad 4\mu(F_\pm) > \mu(F_\mp)\sin\alpha.$$

Now let us suppose that R_0 was chosen so that

$$\mu(J'_{R_0} \setminus S) < h_\alpha \frac{\sin^2\alpha}{100}\,\mu(J'_{R_0})$$

where h_α is given by xvii). Since xxix) holds with $F_\pm = K_\pm$ we have

$$\frac{\mu(K^+)}{\mu(K)} = \frac{\mu(K^+)}{\mu(K^+) + \mu(K^-)} \geq \frac{\sin\alpha}{4 + \sin\alpha}$$

And so, recalling xviii), we see that

$$\frac{\mu(K^+)}{\mu(J'_{R_0})} = \frac{\mu(K^+)}{\mu(K)}\,\frac{\mu(K)}{\mu(J_{R_0})} \geq \frac{h_\alpha \sin\alpha}{4 + \sin\alpha}.$$

This means that

$$\text{xxx)}\quad \frac{\mu(J_{R_0}\setminus S)}{\mu(K^+)} \leq \frac{\left(\frac{\sin^2\alpha\, h_\alpha}{100}\right)\mu(J_{R_0})}{\left(\frac{h_\alpha \sin\alpha}{4 + \sin\alpha}\right)\mu(J_{R_0})} = \frac{\sin\alpha(4 + \sin\alpha)}{100} \leq \frac{8\sin\alpha}{100}$$

And consequently

$$\frac{m(r_+(K^+ \cap S))}{m(r_+(K^+))} = 1 - \frac{m(r_+(K^+\setminus S))}{m(r_+(K^+))}$$

$$= 1 - \frac{\int_{K^+\setminus S}\left|\frac{dr_+}{d\mu}\right|d\mu}{\int_{K^+}\left|\frac{dr_+}{d\mu}\right|d\mu}$$

(continued)

$$\geq 1 - \frac{4\mu(K^+ \setminus S)}{(\sin\alpha)\mu(K^+)}$$

$$\geq 1 - \frac{4\mu(J'_{R_0} \setminus S)}{(\sin\alpha)\mu(K^+)}$$

$$\geq 1 - \left(\frac{4}{\sin\alpha} \cdot \frac{8 \sin\alpha}{100}\right) = \frac{17}{25} .$$

Therefore

$$m(r_+(K^+ \cap S)) \geq \frac{17}{25}\, m(r_+(K^+)) = \frac{17}{25}(R_1 - R_2);$$

and the same argument goes through with + and - interchanged, so that also

$$m(r_-(K^- \cap S)) \geq \frac{17}{25}\, m(r_+(K^+)) = \frac{17}{25}(R_1 - R_2).$$

An easy argument now shows that

xxxi) $m(r_+(K^+ \cap S) \cap r_-(K^- \cap S)) \geq \frac{9}{25}(R_1 - R_2)$.

Write E_0 for $r_+(K^+ \cap S) \cap r_-(K^- \cap S) \subset Y$. Then $R \in E_0$ implies

xxxii) $C'_R \cap \Gamma = \{\gamma_-(R), \gamma_+(R)\} = \{r_-^{-1}(R), r_+^{-1}(R)\} \subset S$

Let ψ be any fractional linear transformation (FLT) taking the origin (in the α, R_0 coordinates) to infinity. We may assume that the local coordinates are the actual ones, since the coordinate transformation itself is an F.L.T. If Λ_I is the imaginary axis; then $\psi(\Lambda_I) = \Lambda$ is a line, since $0 \in \Lambda_I$.

Moreover, $\tilde{\Lambda}_R = \psi(C_R')$ is a straight line for each $R \in Y$, also because $0 \in C_R'$. The circles C_R' are all perpendicular to Λ_I; therefore (see Ahlfors [2] for the elementary properties of F.L.T.'s being employed here) the $\tilde{\Lambda}_R = \psi(C_R')$ form a family of lines perpendicular to Λ (and also parallel to each other). Let $\pi_\Lambda : \psi(\Gamma) \to \Lambda$ denote orthogonal projection and let λ_R be the (finite) point of intersection of Λ and $\tilde{\Lambda}_R$. Then $\pi_\Lambda(z) = \lambda_R$ for any $z \in \tilde{\Lambda}_R \cap \psi(\Gamma)$; in fact,

xxxiii) $\pi_\Lambda^{-1}(\lambda_R) = \tilde{\Lambda}_R \cap \psi(\Gamma)$.

Set $2iE_0 = \{2iR : R \in E_0\}$. For $R \in Y(=(R_2, R_1))$ we have

$$\begin{aligned}\{\psi(2iR)\} &= \psi(C_R' \cap \Lambda_I \setminus \{0\}) \\ &= \psi(C_R') \cap \psi(\Lambda_I) \setminus \{\psi(0)\} \\ &= \tilde{\Lambda}_R \cap \Lambda \setminus \{\infty\} \\ &= \{\lambda_R\} .\end{aligned}$$

Thus

xxxiv) $\psi(2iE_0) = \{\lambda_R : R \in E_0\}$.

Now $\psi(2iE_0) \subset \psi(\Lambda_I) \subset \Lambda$, and the same argument used before (Rudin [15] theorem 8.26) shows that

$$m(\psi(2iE_0)) = \int_{2iE_0} |\psi'| dm > 0$$

(the integral is positive because $m(E_0) > 0$, and the derivative of an F.L.T. is nowhere vanishing on $\mathbb{C}$). Using xxxiv), xxxiii), and xxxii) we see that

$$\pi_\Lambda^{-1}(\psi(2iE_0)) = \bigcup_{R\in E_0} \pi_\Lambda^{-1}(\lambda_R)$$
$$= \bigcup_{R\in E_0} \tilde{\Lambda}_R \cap \psi(\Gamma)$$
$$= \bigcup_{R\in E_0} \psi(C_R' \cap \Gamma) \subset S,$$

so that $E = \psi(2iE_0) \subset \Lambda$ satisfies $m(E) > 0$ and $\pi_\Lambda^{-1}(E) \subset \psi(S)$. If $\eta \in E$ then $\eta = \lambda_R$ for some $R \in E_0$, by xxxiv) above. This gives

$$\pi^{-1}(\eta) = \tilde{\Lambda}_R \cap \psi(\Gamma) = \psi(C_R' \cap \Gamma)$$

and

$$(\pi_\Lambda\psi)^{-1}(\eta) = C_R' \cap \Gamma = \{\gamma_-(R), \gamma_+(R)\}.$$

Thus, starting with an arbitrary subset S of Γ with $\mu(S) > 0$, we have constructed a map ψ satisfying the hypotheses of Proposition 4.6. We have thereby proven

6.3. <u>Theorem</u>: Suppose γ is an injective function in $AC_\Psi[0,1]$, $\Gamma = \gamma[0,1]$, and μ is arclength measure on Γ. Then for each $S \subset \Gamma$ with $\mu(S) > 0$ one can find a line $\Lambda \subset \mathbb{C}$, a subset E of Λ with $m(E) > 0$, and a map ψ holomorphic in a neighborhood of Γ such that $\pi_\Lambda^{-1}(E) \subset \psi(S)$. Moreover $(\pi_\Lambda\psi)^{-1}(\eta)$ has precisely two points of Γ for each $\eta \in E$.

6.4. <u>Corollary</u>: Let (Γ,μ) be as in Theorem 6.3. Then the K-R result holds for (Γ,μ).

§7. A Local Property

7.1. It is a remarkable feature of the problem, and of its treatment in the preceding section, that virtually all of the relevant analysis takes place inside a single, arbitrarily small disc in the complex plane. Points of concentration, pseudocontinuity, and the analysis employing the family $\{C'_R\}$ - all of these involve only local considerations of (Γ,μ). The exception to this is the boundedness of Γ which is required for the Riesz Functional Calculus, which in turn was used to demonstrate (Lemma 4.4) that holomorphic functions in a neighborhood of Γ preserve trace class perturbations.

But we notice that the full strength of Lemma 4.4 is not required, as it was possible to obtain Theorem 6.3 using only fractional linear transformations ψ of Γ. In fact, ψ was confined to a special subclass even within the F.L.T.'s. Recall that we first established local coordinates for α, R_0 at $z_0 = \gamma(t_0)$ by applying the map $z \to e^{-i\theta(t_0)}(z - \zeta_0) = \zeta$, where $\zeta_0 = z_0 - iR_0 e^{i\theta(t_0)}$. The transformation ψ constructed in §6 was required only to take the (new) origin to infinity; we may, for instance choose $\psi(\zeta) = \frac{1}{e^{i\theta(t_0)}\zeta}$, so that composing the two transformations gives

$$1).\ \psi(z) = \psi(\zeta(z)) = \frac{1}{z - \zeta_0}$$

The map ψ of Theorem 6.3 may therefore be assumed to satisfy i) for some $\zeta_0 \notin \Gamma$. Such a map ψ will still preserve trace class perturbations, even on unbounded Γ, for we have the Hilbert Resolvent Identity (also used in proof of Lemma 4.4):

$$\psi(N_2) - \psi(N_1) = (N_2 - \zeta_0)^{-1} - (N_1 - \zeta_0)^{-1}$$
$$= (N_2 - \zeta_0)^{-1}(N_1 - N_2)(N_1 - \zeta_0)^{-1}.$$

We should also note with respect to the preceding that given $S \subset \Gamma$, $\mu(S) > 0$, the point $z_0 = \gamma(t_0)$ may be chosen arbitrarily from a subset of S having full measure, specifically, z_0 must be a point of μ-concentration for S which lies in $\gamma(I_0)$ (see definition 6.1 for I_0). In addition, the parameter R_0 may be chosen arbitrarily small, so that the singularity ζ_0 of ψ lies as close as desired to $\gamma(t_0)$ (these remarks may be verified by reviewing the construction in §6).

Finally, let us recall a property concerning the line Λ and subset E of Λ we constructed for the proof of Theorem 6.3, namely, if $\tilde{\Lambda}_R$ is a line perpendicular to Λ at a point of E, then $\psi^{-1}(\tilde{\Lambda}_R) = C'_R$ for some circle C'_R contained in the closed disc $\overline{D}_{R_0} = \overline{D}_{R_0}(z_0)$. Therefore if we consider orthogonal projection $p_\Lambda : \mathbb{C} \to \Lambda$ (π_Λ is p_Λ restricted to $\psi(S)$), we still have

ii) $(p_\Lambda \psi)^{-1}(E) \subset \overline{D}_{R_0}$.

These properties may be assumed to hold for the objects ψ, Λ, E given by Theorem 6.3. We can now substantiate our assertion that the K-R result involves only local properties of (Γ, μ).

7.2. Theorem: Suppose that for ν - almost all $z \in \Gamma$ we can find a neighborhood Γ_z of z in the $\mathbb{C}$ subspace topology on Γ, such that

a) $\Gamma_z = \gamma(I)$ for some injective map $\gamma \in AC_\psi(I)$

b) Arclength measure on $\gamma(I)$ is equivalent to the restriction of ν.

Then the K-R result holds for (Γ, ν).

Proof: Let Γ' be the collection of $z \in \Gamma$ for which one can find a neighborhood Γ_z as above. Subspaces of $\mathbb{C}$ are Lindelöf; therefore one can find a countable collection $z_1, z_2, \ldots \in \Gamma'$ such that $\Gamma' \subset \bigcup_{k=1}^{\infty} \Gamma_{z_k}$. Suppose now that $S \subset \mathcal{B}_\Gamma$ with $\nu(S) > 0$ for some k. Also let γ_k be an injective function in $AC_\psi(I)$ such that $\Gamma_{z_k} = \gamma_k(I)$, and let μ_k be arclength measure defined with respect to γ_k on $\gamma_k(I)$. Then $\mu_k(S_k) > 0$ by b) and we may apply the results of §6 to the space (Γ_k, μ_k). If I_0 is defined for γ_k as in Definition 6.1, then there is a μ_k-concentration point $z_0 = \gamma_k(t_0)$ for S_k lying simultaneously in $\gamma_k(I_0)$ and the interior of Γ_{z_k}. We choose a disc $D_{R_0} = D_{R_0}(z_0)$ such that $\Gamma \cap \overline{D}_{R_0} \subset \Gamma_{z_k}$ and apply Theorem 6.3 to (Γ, μ). This gives a map ψ holomorphic in a neighborhood of Γ_k, together with a line Λ and subset E of Λ such that

1) $m(E) > 0$

2) $\pi_\Lambda^{-1}(E) \subset \psi(S_k)$, and

3) $(\pi_\Lambda\psi)^{-1}(\eta)$ consists of two points of Γ_{z_k} for each $\eta \in E$.

($\pi_\Lambda : \psi(\Gamma_{z_k}) \to \Lambda$ is perpendicular projection).

Now by the discussion preceding this theorem we may assume $\psi(z) = \frac{1}{z - \zeta_0}$, where ζ_0 lies in $D_{R_0} \setminus \Gamma_{z_k}$. Then ψ is also holomorphic in a neighborhood of Γ. Moreover, if $\tilde{\pi}_\Lambda : \psi(\Gamma) \to \Lambda$ is perpendicular projection defined on $\psi(\Gamma)$, then $\tilde{\pi}_\Lambda = p_\Lambda|_{\tilde{\psi}(\Gamma)}$, and by ii) above we have for each $\eta \in E$,

$$(\tilde{\pi}_\Lambda\psi)^{-1}(\eta) \subset (p_\Lambda\psi)^{-1}(\eta) \subset \bar{D}_{R_0}.$$

It follows that $(\tilde{\pi}_\Lambda\psi)^{-1}(\eta) \subset \Gamma_{z_k}$ and hence that $(\tilde{\pi}_\Lambda\psi)^{-1}(\eta) = (\pi_\Lambda\psi)^{-1}(\eta)$, $\eta \in E$. In particular $(\tilde{\pi}_\Lambda\psi)^{-1}(\eta)$ is finite for each $\eta \in E$, and thus we have shown that the ψ, Λ, E chosen for (Γ_k, μ_k) also satisfy the hypotheses of Proposition 4.6 for (Γ, μ). This completes the proof. ■

7.3. <u>C^1 Curves</u>: As a concrete example of what can be done with this result, let us consider the following situation: Assume that $\Gamma = \gamma(I)$, where γ is a complex valued C^1 function on $I = [0,1]$ and that $\mu(\overline{F}) = 0$, where F is the collection of $z \in \Gamma$ such that $\gamma^{-1}(z)$ contains more than one point (μ is arclength measure on Γ). We claim than that (Γ,μ) satisfies the hypotheses of Theorem 7.2.

Let us set

$$X_0 = \{t \in I : \gamma'(t) = 0\}$$

$$X_1 = \gamma^{-1}(\overline{F}) \cup X_0 \cup \{0,1\}$$

$$U = I \setminus X_1;$$

then evidently U is open in $\mathbb{R}$, and therefore U is the countable union of open intervals U_k. Also we see that

$$\mu(\gamma(X_1)) \leq \mu(\overline{F}) + \mu(\gamma(X_0)) + \mu(\gamma\{0,1\}) = 0$$

and therefore $\mu(\Gamma \setminus U) = 0$.

The restrictions $\gamma_k = \gamma|_{U_k}$ are homeomorphic, for $k = 1,2,\ldots$. To prove this it evidently suffices to show that $\gamma(G)$ is open, where G is any open subset of U_k. First, $I \setminus G$ is compact and therefore so is $\gamma(I \setminus G)$; thus $\Gamma \setminus \gamma(I \setminus G)$ is open in Γ. But because $G \subset U$ we have $\gamma(G) \cap F = \emptyset$ and therefore $\gamma^{-1}(\gamma(G)) = G$. It follows that $\gamma(I \setminus G) = \Gamma \setminus \gamma(G)$ and so $\gamma(G) = \Gamma \setminus \gamma(I \setminus G)$ is open as desired.

Now choose $z \in \gamma(U)$; we claim there is a neighborhood Γ_z of z satisfying the hypotheses of Theorem 7.2. Certainly there is a unique k, and $t \in U_k$, with $\gamma(t) = z$. Let I_z be an open interval containing z with $\overline{I}_z = [s_1, s_2] \subset U_k$. Then $\Gamma_z = \gamma(\overline{I}_z)$ is a neighborhood of z in the $\mathbb{C}$-subspace topology by the above. Also, if we let $t = t(x) = (s_2 - s_1)x + s_1$, then $\tilde{\gamma}(x) = \gamma(t(x))$ is in $AC_\psi(I)$, with $\Gamma_z = \tilde{\gamma}(I)$. Finally, we have

$$\int_{\tilde{\gamma}^{-1}(S)} |\tilde{\gamma}'(x)|dx = \int_{\gamma^{-1}(S)} |\gamma'(t)|dt$$

for measurable $S \subset \Gamma_z$ by change of variables; therefore arc-length measure $\tilde{\mu}$ on Γ_z induced by $\tilde{\gamma}$ is equivalent to μ. (But this is obvious anyway, since all we've done is make a change of parameter). Thus Γ_z has all the desired properties; since z was taken arbitrarily from $\gamma(U)$, and $\mu(\Gamma \backslash \gamma(U)) = 0$, the K-R result for (Γ, μ) follows by Theorem 7.2.

7.3. Kuroda Hypothesis: It is often mentioned in discussions of the Kato-Rosenblum Theorem 1.1 that the hypothesis $A_2 - A_1 = T \in \mathcal{J}_1$ may be replaced by a weaker requirement; namely, that $(A_2 - \zeta)^{-1} - (A_1 - \zeta)^{-1}$ be trace class for some (and hence all) $\zeta \in \rho(A_1) \cap \rho(A_2)$. Thus is the so-called Kuroda Hypothesis for the perturbation problem. We would like to know whether an analogous modification can be made in the statement of 7.1.

More specifically, given (Γ, μ) as in 7.1, and normal operators N_1, N_2 with $\sigma(N_1) \cup \sigma(N_2) \subset \Gamma$, such that

$$(N_2 - \zeta_0)^{-1} - (N_1 - \zeta)^{-1} \in \mathcal{J}_1$$

for some $\zeta_0 \in \rho(N_1) \cup \rho(N_2)$, may we assert that the spectral multiplicity functions δ_{N_1}, δ_{N_2} are equal a.e. (μ) in Γ?

Let us suppose that $\delta_{N_1} > \delta_{N_2}$ on some subset $S \subset \Gamma$, $\mu(S) > 0$. As we saw in the proof of 7.1, there is a point $z_0 \notin \Gamma$, a line Λ, and a subset E of Λ, $m(E) > 0$, such that $\pi_\Lambda^{-1}(E) \subset \psi(S)$ where $\psi(z) = \frac{1}{z - z_0}$, and $\pi_\Lambda : \psi(\Gamma) \to \Lambda$ is orthogonal projection; also $(\pi_\Lambda \psi)^{-1}(\eta)$ is a finite set for each $\eta \in E$.

Now $w = z_0 - \zeta_0$ lies in the resolvent set of $N - \zeta_0$, $i = 1,2$, and therefore $\frac{1}{w}$ lies in the resolvent set of $(N_i - \zeta_0)^{-1}$. It follows that the function

$$\tilde{\psi}(\zeta) = \frac{\zeta}{1 - w\zeta}$$

is holomorphic in a neighborhood of $\sigma((N_1 - \zeta_0)^{-1}) \cup \sigma((N_2 - \zeta_0)^{-1})$, and by Lemma 4.4

$$\tilde{\psi}((N_2 - \zeta_0)^{-1}) - \tilde{\psi}((N_1 - \zeta_0)^{-1})$$

is trace class. But, for $K = 1,2$,

$$\begin{aligned}\tilde{\psi}((N_K - \zeta_0)^{-1}) &= (N_K - \zeta_0)^{-1}(1 - w(N_K - \zeta_0)^{-1})^{-1} \\ &= (N_K - \zeta_0)^{-1}[(N_K - \zeta_0)^{-1}(N_K - z_0)]^{-1} \\ &= (N_K - z_0)^{-1} = \psi(N_K)\end{aligned}$$

and so

$\psi(N_2) - \psi(N_1)$ is trace class, even though nothing about $N_2 - N_1$ is given. The remainder of the proof proceeds along well established lines: the multiplicity functions $\delta_{\pi_\Lambda(\psi(N_1))}$, $\delta_{\pi_\Lambda(\psi(N_2))}$ must be equal a.e. (m) in Λ while finiteness of $(\pi_1\psi)^{-1}(\eta)$ for $\eta \in E$ gives

$$\delta_{\pi_\Lambda(\psi(N_1))} > \delta_{\pi_\Lambda(\psi(N_2))} \text{ on } E$$

just as in the proof of 4.6. The answer to our question, then, is an affirmative one.

[1] Abrahamse and Kreite: "The Spectral Multiplicity of a Multiplication Operator", Indiana University Mathematics Journal; Vol. 22, No. 9 (1973).

[2] Ahlfors, L.V.: Complex Analysis, 2nd Ed. (1966); McGraw Hill.

[3] Berg, I.D.: "An Extension of the Weyl-von Neumann Theorem to Normal Operators", Trans. Amer. Math. Soc. 160, 365-371 (1971).

[4] Birman, M.S.: "Existence Conditions for Wave Operators", Translations Amer. Math. Soc. 54, 91-118 (1966).

[5] Dixmier, J.: Les algebres d'operateurs dans l'espace hilbertien, Paris, Ganthier Villars, 1957.

[6] Foias, C. and Sz-Nagy, B.: Harmonic Analysis of Operators on Hilbert Space (1970) A Kademiai Kiado - Budapest.

[7] Hille, E. and Phillips, R.S.: Functional Analysis and Semi-groups, Revised Ed; Providence, Amer. Math. Soc. 1957.

[8] Hoffman, K.: Banach Spaces of Analytic Functions, Prentice Hall 1962.

[9] Kato, T.: Perturbation Theory for Linear Operators, 2nd Ed. 1976, Springer-Verlag, Berlin Heidelberg.

[10] Krasnosel'skii and Rutickii: Convex Functions and Orlicz Spaces, Groningen, P. Noordhoff, 1961.

[11] Kuroda, S.T.: "On a Theorem of Weyl-von Neumann", Proc. Japan Acad. 34 11-15 (1958).

[12] Privalov, I.I.: Randeigenschaften Analytischer Functionen, Deutscher Verlag der Wissenschaften, Berlin 1956.

[13] Reed, M. and Simon, B.: Methods of Modern Mathematical Physics, Vol. I; Academic Press 1972.

[14] Rudin, W.: Functional Analysis; 1973, McGraw-Hill, Inc.

[15] Rudin, W.: Real and Complex Analysis; 2nd Ed. 1974, McGraw-Hill, Inc.

[16] Voiculescu, D.: "Some Results on Norm-Ideal Perturbations of Hilbert Space Operators"; Preprint Series in Mathematics, No. 34/1978.

[17] Voight, J.: "Störung für Kommutative m-Tupel von selbstadjungierten Operatoren". Universität Munchen 1974.

[18] von Neumann, J.: "Charakterisierung des Spectrums eines Integraloperators". Actualitěs Sci. Ind. 229, 1-20 (1935).

[19] Weyl, H.: "Uber Beschrankte quadratische Formen, deren Differenz Vollstetig inst". Rend. Circ. Mat. Palermo 27, 373-392 (1909)

[20] Birman, M.S. and Solomyak: "Stieltjes Double Integral Operators", from Topics in Mathematical Physics; vol. I; M.S. Birman, Ed. , Consultants Bureau, New York (1967)

[21] Sikonia, W: "The von-Neumann converse of Weyl's theorem". University of Colorado, Department of Math. 1971