THE ASYMPTOTIC EXPANSION FOR THE TRACE OF THE HEAT KERNEL ON A GENERALIZED SURFACE OF REVOLUTION

A Dissertation presented

by

Ping-Charng Lue

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August, 1979

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

PING CHARNG LUE

We, the dissertation committee for the above candidate for the Ph.D degree, hereby recommend acceptance of the dissertation.

David Ebin, Chairman

Jeff Cheeger, First Reader

Ronald G. Douglas

Sei Sujishi, Outside Member

The dissertation is accepted by the Graduate School.

Original Signed By
JACOB BIGELEISEN, DEAN

Jacob Bigeleisen, Dean
Graduate School

Abstract of the Dissertation

THE ASYMPTOTIC EXPANSION FOR THE TRACE OF

THE HEAT KERNEL ON A GENERALIZED SURFACE OF REVOLUTION

by

Ping-Charng Lue

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1979

In this dissertation we construct a parametrix for the heat kernel (or the fundamental solution of the heat equation) on a generalized surface of revolution. Using the parametrix we study the relation between the coefficients of the asymptotic expansion for the trace of the heat kernel on the generalized surface of revolution and the corresponding coefficients on the base of the surface of revolution. One important result is that each of the former coefficients is a linear combination of the latter coefficients We also point out why this result is not trivial if we start with a parametrix in the standard form.

TABLE OF CONTENTS

Acknowledgements

I would like to thank my thesis advisor Professor Jeff Cheeger for his guidance, advice and encouragement.

I would like to thank Professor David Ebin for some helpful conversations.

I would like to thank Professor Detlef Gromoll and Professor Wolfgang Meyer for helping with the beginnings of my study in Differential Geometry.

I would like to thank Ms. Susan Addington for correcting my English.

I thank my wife Ching for supporting me especially the last year of my graduate study.

INTRODUCTION

Given N, an n-dimensional, smooth compact Riemannian manifold without boundary, let Δ be the Laplacian associated with metric tensor $g = (g_{ij})$ on N:

$$(0.1) \qquad \Delta = \sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} g^{ij} \sqrt{\det g} \frac{\partial}{\partial x_j}$$

where $g^{-1} = (g^{ij})$. The heat operator $\Delta + \frac{\partial}{\partial t}$ is defined for all functions in $C^{\infty}(N \times R^{+})$. It has the following physical interpretation: Let an initial temperature distribution $u(x,o)$ on N be given. Here x denotes position and t denotes time. As the heat flows, $u = u(x,t)$ changes in such a way as to satisfy the heat equation $\Delta u + \frac{\partial u}{\partial t} = 0$. The heat kernel $E(x,y,t)$ on N (the fundamental solution of the heat equation on N) also has a physical interpretation: if we are given at a point x an instantaneious unit heat source when $t = 0$, $E(x,y,t)$ will represent the temperature at y at time t. Let $0 = \lambda_0 \le \lambda_1 \le \ldots \le$ etc.$\uparrow \infty$ be the spectrum of Δ, (the set of all the eigenvalues of Δ). If $\{\phi_i\}$* are the eigenfunctions of Δ corresponding to $\{\lambda_i\}$, we have

$$(0.2) \qquad E(x,y,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y).$$

The trace (that is $E(x,x,t)$) has an asymptotic expansion given by Minakshisundran-Pleijel [1]

* $\{\phi_i(x)\}$ are chosen to be an orthonormal basis for $C^{\infty}(N)$.

$$(0.3)\qquad \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^2(x) \underset{t\to 0_+}{\sim} (4\pi t)^{-\frac{n}{2}}(U_0(x,x) + U_1(x,x)t + U_2(x,x)t^2 + \dots).$$

The $\{U_i\}$ are Riemannian invariants, in fact, they are functions of the curvature tensor and its covariant derivatives. Thus they provide us with a measurements of the local derivation from flatness for N.

If we integrate (0.3) with respect to dx, the volume form of N, we will obtain an asymptotic expansion:

$$(0.4)\qquad \sum_{i=0}^{\infty} e^{-\lambda_i t} \underset{t\to 0_+}{\sim} (4\pi t)^{-\frac{n}{2}}(a_0 + a_1 t + a_2 t^2 + \dots).$$

These coefficients $\{a_i\}$ have geometric meaning; for example $a_0 = \mathrm{Vol}(N,g)$ for arbitrary n, $a_1 = \frac{\pi}{3}\chi(M)$ when $n = 2$ ($\chi(M)$ is the Euler characteristic of M). However, it is very difficult to calculate the coefficients. Only the first few of them have been calculated for a general manifold N [1]. For some special manifolds, such as compact symmetric spaces of rank one, explicit formulas for the coefficients have been found by Cahn and Wolf [2].

More generally, we can extend the definition of the heat operator to forms. Consider the space Λ^j of j-forms on N; we can define the operator ${}_j\Delta = d_{j-1}\delta_j + \delta_{j+1}d_j : \Lambda^j \to \Lambda^j$, where

d_j is the exterior derivative and δ_j is the formal adjoint of d_j with respect to the metric g. Moreover, denote the eigenvalues and eigenforms of ${}_j\Delta$ on N by $\{{}_j\lambda_i\}$, $\{{}_j\phi_i\}$ respectively. Let ${}_jE(x,y,t)$ be the heat kernel of j-forms on N. Then we have [8]

$$(0.5) \qquad {}_jE(x,x,t) = \sum_{i=0}^{\infty} e^{-{}_j\lambda_i t}\, {}_j\phi_i(x) \otimes {}_j\phi_i(x)$$

$$\sim \sum_{i=0}^{\infty} B_i\,(x,{}_j\Delta)t^{-\frac{n}{2}+\frac{i}{2}}$$

If we denote index $(\Delta) = \sum_{j=0}^{n}(-1)^j \ker {}_j\Delta = \chi(M)$ we will get an **index theorem** [8] of Δ:

$$(0.6) \qquad \chi(M) = \int_M B_n(x,\Delta)dvol$$

where $B_n(x,\Delta) = \sum_{j=0}^{n}(-1)^j B_n(x,{}_j\Delta)$.

All these show that it is important to study the coefficients. For further discussion of the heat equation proof of the index theorem see [8]. From now on we concentrate on the individual coefficients U_j.

As in mathematical induction, if we want to study properties of manifolds of arbitrary finite dimension, we should study the properties in the lowest dimension where these properties make sense. Then we study how these properties change between manifolds of successive dimensions. The heat kernel on an interval I is well-known, so we shall try to get information about the heat kernel of a manifold M x I in terms of the heat kernel of the manifold M.

In order to make the setting simpler, let us suppose that M is an n-dimensional manifold without boundary. Let the higher dimensional manifold be M x I, where I is an open interval on R. If we adopt the product metric on M x I we will come to the trivial result: the asymptotic expansion for the trace of heat kernel on M x I is equal to the product of the corresponding asymptotic expansions on M and I respectively [1]. Therefore it is natural to choose a new metric on M x I other than the product one. The easiest metric which is suitable for our purpose is $r^2 g + dr \otimes dr$ where g is the metric on M and r is the parameter on I. A more general one is $h^2(r)g + dr \otimes dr$ where $h(r) : I \to R^+$ is a smooth positive function. In [4] Cheeger derived the heat kernel on M x I with metric $r^2 g + dr \otimes dr$ in the following form:

$$(0.7) \qquad E(r_1, r_2, y; t) = \sum_{i=0}^{\infty} (r_1 r_2)^{\frac{1-n}{2}} \frac{1}{2t} e^{-\frac{r_1^2 + r_2^2}{4t}} I_{\nu_i}\left(\frac{r_1 r_2}{2t}\right) \phi(x) \phi_i(y)$$

where $I_{\nu_i}(z)$ is the modified Bessel function of order $\nu_i, \nu_i = \sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda_i}$. As on page 1, $\{\lambda_i\}$ denote eigenvalues of the Laplacian on M. Since $I_{\nu_i}(z) \underset{z\to\infty}{\sim} \frac{e^z}{\sqrt{2\pi z}} \sum_{s=0}^{\infty} \frac{(\nu_i s)}{(-2s)^s}$, the possibility of summing up these asymptotic expansions to give an asymptotic expansion of the heat kernel E suggests itself. If we substitute $\frac{r_1 r_2}{2t}$ for z in the above asymptotic expansion we are led to a factor $\sqrt{\frac{t}{\pi}}\, e^{\frac{r_1 r_2}{2t}}$ in the right hand side of the expansion. It is encouraging for us to find that if we multiply $\sqrt{\frac{t}{\pi}}\, e^{\frac{r_1 r_2}{2t}}$ with the term $\frac{1}{2t}\, e^{-\frac{r_1^2+r_2^2}{4t}}$ in (0.4) we get $\frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(r_1-r_2)^2}{4t}}$, which is the heat kernel on R. Moreover, $\sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$ represents the heat kernel on M (see [1]) and ν_i is a function of λ_i. Both facts suggest that it is reasonable for us to expect the following: the summation of terms containing parts of the asymptotic expansion of $I_{\nu_i}(\frac{r_1 r_2}{2t})$ multiplied by $\phi_i(x)\phi_i(y)$ gives something related to the heat kernel on the section M. Therefore somehow (0.4) gives a relation between the heat kernels on M x I, I and M respectively. We will see that the asymptotic expansion for the trace E will show us how the coefficients of the asymptotic expansion for the trace of the heat kernel on M occur in each of the corresponding coefficients on M x I. Motivated by the above analysis, we wish to obtain a similar asymptotic expansion for M x I with metric $h^2(r)g + dr \otimes dr$. Using the method of separation of variables we write out the heat kernel on M x I as:

$$(0.8) \qquad E(r_1,r_2,x,y;t) = \sum_{i=0}^{\infty} f_i(r_1,r_2,t)\phi_i(x)\phi_i(y).$$

We then find the asymptotic expansion of $f_i(r_1,r_2,t)$ for each i. Here we follow a method of Gelfand and Dikii. But one problem arises: how can the non-uniform sequence of asymptotic expansions be added to form a new asymptotic expansion of the required type? The asymptotic expansion of each f_i has dominant term starting from a term contains $t^{-\frac{1}{2}}$. But the sum has an asymptotic expansion starting with $t^{-(n+1)/2}$ where $n = \dim M$. We deal with this by constructing a parametrix for E and relating the trace of the parametrix to the sum of the traces of f_i. The construction was motivated in part by a certain formal procedure for "adding" the asymptotic expansions of f_i in such a way to obtain the correct power of t.

This work is divided into six sections and an appendix. The first section contains a calculation of a formula for the Laplacian on $M \times I$ in terms of Laplacian on M and derivatives w.r.t. r, the parameter on I. The second section gives the expression for the heat kernel on the metric cone, the case when $h(r) = r$. In section three we adopt some results we need from the paper of Gelfand and Dikii. In section four we construct a parametrix of the heat kernel of $M \times I$ which in turn is exploited in deriving an asymptotic expansion in section five. The last section contains an explanation of the significance of our approach in this work. The appendix is a calculation deriving a relation between the determinants of the exponential maps on $M \times I$ and M. This relation illustrates the difficulties in attempting to carry out a more naive approach to our problem.

$$= \frac{h'(r)}{h(r)} E_i + \frac{1}{h(r)} [E_i, \frac{\partial}{\partial r}]$$

$$= \frac{h'(r)}{h(r)} E_i$$

(1.3) $$[F_i, F_j] = \frac{1}{h^2(r)} [E_i, E_j]$$

(1.4) $$\langle \nabla_{F_i} F_j, F_k \rangle = F_i \langle F_j, F_k \rangle + F_j \langle F_i, F_k \rangle - F_k \langle F_i, F_j \rangle$$

$$+ \langle [F_i, F_j], F_k \rangle - \langle [F_i, F_k], F_j \rangle$$

$$- \langle [F_j, F_k], F_i \rangle$$

$$= \frac{1}{h(r)} \{ \langle [E_i, E_j], E_k \rangle_g - \langle [E_i, E_k], E_j \rangle_g$$

$$- \langle [E_j, E_k], E_i \rangle_g \}$$

$$= \frac{2}{h(r)} \langle \overline{\nabla}_{E_i} E_j, E_k \rangle_g$$

where $\langle \ \rangle$, $\langle \ \rangle_g$ denote the metrics on $M \times I$ and M respectively and ∇, $\overline{\nabla}$ denote Levi-Civita connections on $M \times I$ and M respectively. Hence

(1.5) $$\langle \nabla_{F_i} F_j, F_k \rangle = \frac{1}{h(r)} \langle \overline{\nabla}_{E_i} E_j, E_k \rangle_g$$

Similarly:

(1.6) $$\langle \nabla_{F_i} F_j, \frac{\partial}{\partial r} \rangle = - \frac{h'(r)}{h(r)} \delta_{ij}$$

$$\langle \nabla_{F_i} \frac{\partial}{\partial r}, F_j \rangle = \frac{h'(r)}{h(r)} \delta_{ij}$$

where δ_{ij} is the Kronecker delta. Moreover:

I.2. Asymptotic expansion.

Let $\{\Psi_i\}$ be a sequence of functions which are defined in a neighborhood of x_0. $\{\Psi_i\}$ is called an asymptotic sequence for $x \to x_0$ in this neighborhood if for each i $\Psi_{i+1} = o(\Psi_i)$ as $x \to x_0$. This definition is due to Poincare. Let f(x) be a function which is also defined in this neighborhood. Then we say that f(x) has an asymptotic expansion in terms of $\{\Psi_i\}$ at x_0 and write

$$(1.10) \qquad f(x) \underset{x \to x_0}{\sim} \sum_{i=0}^{\infty} G_i \Psi_i(x)$$

if $f(x) = \sum_{i=0}^{N} G_i \Psi_i(x) + o(\Psi_N)$ as $x \to x_0$ holds for any N. It is easy to see that for given $\{\Psi_i\}$, the asymptotic expansion of f(x) at x_0 is unique. We will need the following standard result.

I.2.1. Theorem: If f(z) is analytic in $-\alpha < \theta < \alpha$, $0 < r < b$ $z = re^{i\theta}$ and if $f(z) \underset{z \to 0}{\sim} \sum_{k=0}^{\infty} z^k$ uniformly in θ then

$$f'(z) \underset{z \to 0}{\sim} \sum_{k=1}^{\infty} k\, G_{k-1} z^{k-1} \text{ uniformly in any smaller sector}$$

$$-\alpha < \alpha_1 < \theta < \alpha_2 < \alpha$$

Proof: [9].

If $\{\lambda_i\}$ are eigenvalues of the Laplacian Δ on a compact Riemannian manifold, $\sum_{i=0}^{\infty} e^{-\lambda_i t}$ is then analytic in a sector region as above [1].

I.2.2. Corollary: If $\sum_{i=0}^{\infty} e^{-\lambda_i t} \underset{t \to 0}{\sim} t^{-\frac{n}{2}}(a_0 + a_1 t + a_2 t^2 + \ldots)$ then $\sum_{i=0}^{\infty} \lambda_i e^{-\lambda_i t} \underset{t \to 0}{\sim} \frac{n}{2} t^{-\frac{n}{2}-1} a_0 + (\frac{n}{2} - 1) t^{-\frac{n}{2}} a_1$

$$+ \ldots$$

I.2.3. Theorem: Suppose F(s) is the image of f(t) under the Laplace transformation and F(s) has a half-plane of convergence. If:

$$f(t) \underset{t\to 0}{\sim} \sum_{\nu=0}^{\infty} c_\nu t^{\lambda_\nu} \quad (-1<R\lambda_0<R\lambda_1\ldots<\ldots)$$

then

$$F(s) \underset{s\to\infty}{\sim} \sum_{\nu=0}^{\infty} c_\nu \frac{\Gamma(\lambda_\nu+1)}{s^{\lambda_\nu+1}}$$, where $R\lambda_i$ represents the real part of λ_i.

Proof: [6].

I.3. Bessel functions and Hankel transforms.

The Bessel equation:

$$(1.11) \qquad z^2 \frac{d^2\omega}{dz^2} + z\frac{d\omega}{dz} + (z^2 - \nu^2)\omega = 0$$

has a solution as $J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)}$. A second independent solution is $J_{-\nu}$ unless ν is an integer. When ν is an integer, $\nu = n$ (positive), $J_{-n}(z) = (-1)^n J_n(z)$ where

$J_{-n}(z) = \sum_{m=n}^{\infty} (-1)^m \frac{(z/2)^{2m-n}}{m!(m-n)!}$, i.e., J_n and J_{-n} are no longer independent. If the function $Y_\nu(z)$ is defined by $Y_\nu(z)$ $= \frac{(\cos \pi\nu)J_\nu(z) - J_{-\nu}(z)}{\sin \nu\pi}$ then

$$Y_n(z) = \lim_{\nu\to n} Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}}{\partial \nu}\right]_{\nu=n},$$

will be the second independent solution of the Bessel equation of order n. J_ν is called the Bessel equation of the first kind of order ν, Y_ν is called the Bessel equation of the second kind of order ν,

Consider a differential equation which differs from the above one by a sign:

$$(1.12) \qquad z^2 \frac{d^2\omega}{dz} + z\frac{d\omega}{dz} - (z^2 + \nu^2)\omega = 0 .$$

This equation will be transformed into the above one by changing z into iz. Since we are interested in real functions of z we choose a solution of the form $e^{-\frac{1}{2}\nu\pi i}J_\nu(iz)$ and denotes it by $I_\nu(z)$. Thus $I_\nu(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)}$,

$I_\nu(z)$ is called a modified Bessel function of the first kind. It has the following asymptotic expansion as $z \to \infty$

$$(1.13) \qquad I_\nu(z) \underset{z\to\infty}{\sim} \frac{e^z}{\sqrt{2\pi z}} \sum_{s=0}^{\infty} \frac{(\nu,s)}{(-2z)^s} + \frac{e^{-z-(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \frac{(\nu,s)}{(2z)^s} ,$$

where

$$(\nu,s) = \frac{\Gamma(\nu+\frac{1}{2}+s)}{s!\Gamma(\nu+\frac{1}{2}-s)} \qquad -\frac{3}{2}\pi < \arg z < \tfrac{1}{2}\pi$$

or

$$I_\nu(z) \underset{z\to\infty}{\sim} \frac{e^z}{\sqrt{2\pi z}} \sum_{s=0}^{\infty} \frac{(\nu,s)}{(-2z)^s} + \frac{e^{-z}\cos(\nu+\frac{1}{2})\pi}{\sqrt{2\pi z}} \sum_{s=0}^{\infty} \frac{(\nu,s)}{(2z)^s}$$

in the real case [11].

If $f(r)$ is a function which is defined on R^+ and satisfies appropriate conditions (see [12]) then the conventional Hankel transform $F(p)$ of order $\nu \geq -\frac{1}{2}$ of $f(r)$ is defined as:

$$F(p) = \int_0^\infty \sqrt{pr}\, f(r) J_\nu(pr)\,dr, \tag{1.14}$$

then we have $f(r) = \int_0^\infty \sqrt{pr}\, F(p) J_\nu(rp)\,dp$, which is often called Hankel's inversion theorem. For a more detailed description see Zemanian's Generalized Integral Transformations [12].

II The Heat Kernel of a Metric Cone

Basically we follow the presentation of [4] in this section. The fundamental solution of the heat equation $(\Delta + \frac{\partial}{\partial t})\, u = 0$ [1] on a manifold is called the heat kernel* on the manifold. When the manifold is compact we know that the heat kernel can be expressed as $\sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x)\phi_i(y)$ [1], where ϕ_i is the eigenfunction of the Laplacian on the manifold with eigenvalue λ_i. With this motivation, following the method of separation of variables, we try to give an expression for the heat kernel on $M \times I$ with metric $h(r)^2 g + dr \otimes dr$:

$$(2.1) \qquad E(r_1,r_2,x,y;t) = \sum_{i=0}^{\infty} f_i(r_1,r_2,t)\phi_i(x)\phi_i(y)$$

where ϕ_i still represent eigenfunction of $\underline{\Delta}$.

When $h(r) = r$ the relation between the Laplacians Δ and $\underline{\Delta}$ is:

$$(2.2) \qquad \Delta = -\left(\frac{\partial}{\partial r}\right)^2 - \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\underline{\Delta} .$$

If $m = S^n$ this is just the formula for the Laplacian on R^{n+1}, expressed in polar coordinates where $\underline{\Delta}$ represents the intrinsic Laplacian on S^n. Moreover if $I = R^+ \cup \{0\}$, we call $M \times I$ the metric cone with base M. The metric cone has a singularity at $r = 0$. Now consider the heat kernel of the

---

*For definition see Section IV.

metric cone. In this case the function f_i in the above expression can be found explicitly in terms of modified Bessel functions as we are deriving now. In fact f_i should satisfy

$$(2.3)\qquad [-(\frac{\partial}{\partial r_1})^2 - \frac{n}{r}\frac{\partial}{\partial r_1} + \frac{\partial}{\partial t}]f_i(r_1,r_2,t) = 0$$

or

$$(2.4)\qquad [\frac{\partial^2}{\partial r_1^2} - \frac{(\frac{n}{2}(\frac{n}{2}-1)+\lambda_i)}{r_1^2} - \frac{\partial}{\partial t}]r_1^{\frac{n}{2}}\, f_i(r_1,r_2,t) = 0$$

with $r_1^{\frac{n}{2}} f_i(r_1,r_2,t)$ converging to $r_2^{-\frac{n}{2}}\,\delta_{r_1-r_2}$ as $t \to 0$, $r_1^{\frac{n}{2}} f_i(r_1,r_2,t)$ converges to 0 as $t \to \infty$, $r_1^{\frac{n}{2}} f_i(r_1,r_2,t)$ converges to 0 as $r_1 \to \infty$, $r_1^{\frac{n}{2}} f_i(r_1,r_2,t)$ remains finite as $r \to 0$ [12].

Now if we let $N_\mu = r_1^{\mu+\frac{1}{2}}\frac{\partial}{\partial r_1} r_1^{-\mu-\frac{1}{2}}$

$$M_\mu = r_1^{-\mu+\frac{1}{2}}\frac{\partial}{\partial r_1} r_1^{\mu+\frac{1}{2}}$$

then we have

$$(2.5)\qquad M_{\mu_i}N_{\mu_i}r_1^{\frac{n}{2}}\, f_i - \frac{\partial}{\partial t}\, r_1^{\frac{n}{2}}\, f_i = 0$$

where

$$\mu_i = \sqrt{(\frac{n-1}{2})^2 + \lambda_i}\ .$$

If $U(\rho,t) = H_{\mu_i}(r_1^{\frac{n}{2}} f_i)$, where H_{μ_i} denotes the conventional

Hankel transformation, then

$$(2.6) \qquad -\rho^2 U(\rho,t) - \frac{\partial}{\partial t} U(\rho,t) = 0$$

$$U(\rho,t) = A(\rho)^{-\rho^2 t}$$

$$A(\rho) = \int_0^\infty r_2^{-\frac{n}{2}} \delta_{r_1-r_2} \sqrt{r_1\rho}\, J_{\mu_i}(r_1\rho)dr_1$$

therefore

$$A(\rho) = r_2^{-\frac{n}{2}+\frac{1}{2}} \sqrt{\rho}\, J_{\mu_i}(r_2\rho)$$

$$(2.7) \quad U(\rho,t) = r_2^{-\frac{n}{2}+\frac{1}{2}} \sqrt{\rho}\, J_{\mu_i}(r_2\rho)e^{-\rho^2 t}$$

$$r_1^{\frac{n}{2}} f_i(r_1,r_2;t) = \int_0^\infty r_2^{-\frac{n}{2}+\frac{1}{2}} \sqrt{\rho}\, J_{\mu_i}(r_2\rho)e^{-\rho^2 t}\sqrt{r_1\rho}\, J_{\mu_i}(r_1\rho)d\rho$$

$$= r_2^{-\frac{n}{2}+\frac{1}{2}} r_1^{\frac{1}{2}} \cdot \frac{1}{2t} e^{-\frac{r_1^2+r_2^2}{4t}} I_{\mu_i}\left(\frac{r_1 r_2}{2t}\right)$$

or

$$(2.8) \qquad f_i(r_1,r_2;t) = (r_1 r_2)^{\frac{1-n}{2}} \frac{1}{2t} e^{-\frac{r_1^2+r_2^2}{4t}} I_{\mu_i}\left(\frac{r_1 r_2}{2t}\right)$$

Finally,

$$(2.9) \qquad E(r_1,r_2,x,y;t) = \sum_{i=0}^{\infty} (r_1 r_2)^{\frac{1-n}{2}} \frac{1}{2t} e^{-\frac{r^2+r^2}{4t}}$$

$$\cdot\, I_{\mu_i}\left(\frac{r_1 r_2}{2t}\right)\phi_i(x)\phi_i(y)$$

where $\mu_i = \sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda_i}$; $\{\lambda_i\}$ are eigenvalues corresponding to $\{\phi_i\}$.

Starting from the above expression for the heat kernel, Cheeger finds an asymptotic expansion of the trace of E by calculating the residue of the Mellin transform of

$$\sum_{j=0}^{\infty} e^{-\frac{r_1 r_2}{2t}} I_{\mu_j}\left(\frac{r_1 r_2}{2t}\right), \ [4].$$

A more naive approach is the following: first of all consider the asymptotic expansion of each $I_{\mu_j}(\frac{r_1 r_2}{2t})$ when $t \to 0$; we then attempt to add these up to get an asymptotic expansion for E. The fundamental difficulty which arises in carrying out this procedure is, as mentioned in the introduction, the legitimacy of the addition. In fact, although each of the summands has dominant term with power $t^{-\frac{1}{2}}$ in its asymptotic expansion at $t = 0$, we know that we should have dominant term with $t^{-\frac{n+1}{2}}$ in the expansion of the sum. We do not solve these problems directly for $h(r) = r$ but solve the more general case when $h(r)$ is a smooth positive function.

III Some Results from a Paper by Gel'fand and Dikii

In a joint paper [7], Gel'fand and Dikii studied the kernel of the resolvent of a Sturm-Liouville equation which includes as a particular case the Green's function for any fixed self-adjoint boundary conditions. In particular, they derived an asymptotic expansion for the trace of the kernel.

III.1. Definition: A resolvent for a Sturm-Liouville equation $-\varphi'' + [u(x) + \zeta]\varphi = 0$ is defined to be a function $R(x,y;\zeta)$ such that:

a) R is continuous in x and y;

b) R is symmetric in x and y;

c) R satisfies the differential equation as of either x or y when the other is kept fixed and $x \neq y$;

d) $\lim_{x \to y}(R_x - R_y) = 1$; where $R_x = \frac{\partial}{\partial x}R$ $R_y = \frac{\partial}{\partial y}R$

e) $R(x,y;\zeta)$ converges to zero exponentially as $\zeta \to \infty$, $x \neq y$.

$R(x,y;\zeta)$ converges to zero exponentially as $\zeta \to \infty$ means that $R(x,y;\zeta)$ converges to zero faster than any power of ζ^{-1} as $\zeta \to \infty$.

Now let us cite some results from [7] that we will need later.

III.2. Theorem:

(3.1) $$R(x,x;\zeta) \underset{s\to\infty}{\sim} \sum_{\ell=0}^{\infty} \frac{R_\ell[u]}{\zeta^{\ell+\frac{1}{2}}}$$

where

$$R_\ell[u] = \sum_{N=\ell}^{\infty} \sum_{k_1+\dots k_N=2\ell-2N} M_{k_1\dots k_N}\, u^{(k_1)}\cdots u^{(k_N)}$$

$$M_{k_1\dots k_N} = \frac{1}{k_1!\dots k_N!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\{|\eta_1|+|\eta_1-\eta_2|+\dots+|\eta_N|\}} \cdot \eta_1^{k_1}\cdots\eta_N^{k_N}\, d\eta_1\dots d\eta_N$$

Proof: Expand the operator $(-\frac{d^2}{dx^2} + u + \zeta)^{-1}$ into a power series in $(-\frac{d^2}{dx^2} + \zeta)^{-1}$:

(3.2) $$-(-\frac{d^2}{dx^2} + u + \zeta)^{-1} = (-\frac{d^2}{dx^2} + \zeta)^{-1} - (-\frac{d^2}{dx^2} + \zeta)^{-1}\, u(-\frac{d^2}{dx^2} + \zeta)^{-1} + \dots .$$

Find the kernel of the resolvent by means of the series

(3.3) $$R = R_0 - R_0 \circ u \circ R_0 + R_0 \circ u \circ R_0 \circ u \circ R_0 - \dots$$

where R_0 is the kernel of the resolvent of $(-\frac{d^2}{dx^2} + \zeta)^{-1}$.

Here we let $R_0 = \frac{e^{-\sqrt{\zeta}\,|x-1|}}{2\sqrt{\zeta}}$ and we let $R_0 \circ u \circ R_0 \circ u \circ \dots\dots$ denotes

(3.4) $\int \ldots \int R_0(x,x_1;\zeta)\, u(x_1) R_0(x_1,x_2,\zeta)\, u(x_2) \ldots dx_1 \ldots dx_N$

represent $u(x_1)$ as a Taylor series:

$$u(x_1) = \sum_{k_1=0}^{\infty} \frac{(x_1 - x)^{k_1}}{k_1!} u^{(k_1)}(x) \tag{3.5}$$

and substitute into (3.3). For more details see [7].

Remark: There is another approach to calculating the R_ℓ. It was proposed in [6] by Dikii and was developed in full generality for arbitrary elliptic pseudo-differential operator in [10] by Seeley.

III.3. Proposition: If we denote $R(x,x,\zeta)$ by R

$$- RR'' + (R')^2 + 4(u + \zeta)R^2 = 1 + c(\zeta) \tag{3.6}$$

where $c(\zeta)$ is exponentially small as $\zeta \to \infty$.

Proof: The straightforward derivation makes use of the properties b), c), e), of the kernel of the resolvent and the fact that the Wronskian of a Sturm-Liouville equation is constant.

Differentiating the above equation gives

III.4. Corollary:

$$- R''' + 4(u(x) + \zeta)R' + 2u'(x) = 0 \tag{3.7}$$

III.5. Proposition:

(3.8)
$$R_0 = \tfrac{1}{2},\ R_1 = -\frac{1}{4}u$$

$$4R_{\ell+1} = 2\sum_{k=0}^{\ell-1} R_k R''_{\ell-k} - \sum_{k=1}^{\ell-1} R'_k R'_{\ell-k}$$

$$- 4u\sum_{k=0}^{\ell} R_k R_{\ell-k} - 4\sum_{k=1}^{\ell} R_k R_{\ell-k+1}$$

$$R'_{\ell+1} = \frac{1}{4}R'''_{\ell} - uR'_{\ell} - \tfrac{1}{2}uR_{\ell}\ .$$

Proof: Substitute (3.1) into (3.6), (3.7).

III.6. Proposition:

(3.9)
$$\frac{\partial}{\partial u} R_{\ell}[u] = -(\ell - \tfrac{1}{2})R_{\ell-1}[u]$$

Where we view u, u', u'' as independent variables and $\frac{\partial}{\partial u} R_{\ell}$ means taking partial derivative of R_{ℓ} with respect to u, for example $\frac{\partial}{\partial u} 3uu'' = 3u''$.

Proof: By induction on making use of (3.8).

IV The Construction of a Parametrix for the Heat Kernel on the Generalized Surfaces of Revolution

Let us recall the definition of a fundamental solution of the heat equation on a Riemannian manifold.

IV.1. Definition: A fundamental solution of the heat equation on a smooth Riemannian manifold N is defined as a function F on $N \times N \times R_+^*$ which satisfies the following:

(i) F is C^0 in the three variables, C^2 in the second variable, C^1 in the third variable.

(ii) $(\Delta_2 + \frac{\partial}{\partial t})F = 0$. Here Δ_2 is the Laplacian with respect to the second variable.

(iii) $\lim_{t \to 0} F(x, \cdot, t) = \delta_x$ for any $x \in N$. **

Remark: When N is compact without boundary, or when more generally N is a complete noncompact manifold with so called bounded geometry, the fundamental solution is unique and can be constructed by the parametrix method; see [1] [3]. In this case we will call the unique fundamental solution the heat kernel. As a consequence of the construction, the properties of the heat kernel at the diagonal are determined by those of a sufficiently good parametrix. The parametrix in turn, is determined by the local geometry at the diagonal. Thus for example, by studying a parametrix on an open set U with

**For R_+^*, δ_x see Definition IV,2.

some Riemannian metric, we obtain the behaviorer of the pointwise trace $E(x,x,t)$ $(x \in U)$ if the fundamental solution for any compact Riemannian manifold N which contains U as an open submanifold.

If M and N are two compact Riemannian manifolds and $M \times N$ is the product manifold with the product metric, then the heat kernel of $M \times N$ is just the product of the kernels of M and N. Moreover if E^g denotes the heat kernel of M with metric g and a is a constant, we have $E^{ag}(x,y,t) = a^{-\frac{n}{2}} E^g(x,y,a^{-1}t)$ [1]. Now consider $M \times I$ with metric $h^2(r)g + dr \otimes dr$, for the time being letting h = const. The heat kernel in this case is:

$$(4.1) \qquad \frac{1}{\sqrt{4\pi t}} e^{-\frac{(r_1-r_2)^2}{4t}} h^{-n} \sum_{i=0}^{\infty} e^{-\frac{\lambda_i t}{h^2}} \phi_i(x)\phi_i(y)$$

where $\phi_i(x)$, λ_i still represent the eigenfunction and eigenvalue of the Laplacian on (M,g). Comparing this with expression (2.1), we are led to write:

$$(4.2) \qquad \frac{1}{2\sqrt{\pi}} \sum_{i=0}^{\infty} h^{-\frac{n}{2}}(r_1)h^{-\frac{n}{2}}(r_2)e^{-\frac{(r_1-r_2)^2}{4t}}$$

$$\cdot \sum_{j=0}^{\infty} a_j(r_1,r_2,\lambda_i)\phi_i(x)\phi_i(y)t^{j-\frac{1}{2}}$$

as a plausible form for a parametrix of the heat kernel on the generalized surface of revolution $M \times I$ with metric $h^2(r)g + dr \otimes dr$. Before we show that this does represent an appropriate form for a parametrix we would like to make a remark and recall the definition of a parametrix for the heat operator.

Remark: From the standard point of view a parametrix of the heat kernel on the generalized surface of revolution should have the following form:

$$(4.3) \qquad \left(\frac{1}{\sqrt{4\pi t}}\right)^{n+1} e^{-\frac{\rho^2}{4t}} (u_0 + u_1 t + u_2 t + \dots)$$

where the phase function $\rho = \operatorname{dist}((r_1, x); (r_2, y))$. In our case, the phase function is equal to the distance function only when $h = \text{const}$. We will explain why the standard form (4.3) is not useful for our calculation [Section VI].

IV.2. Definition: We call H a parametrix of $\square = \Delta + \frac{\partial}{\partial t}$ if it satisfies:

(i) $H \in C^\infty(N \times N \times R_+^*)$,

(ii) $\square H$ can be extended to become a function in $C^0(N \times N \times R_+)$,

(iii) $\lim\limits_{t \to 0} H(x, \cdot, t) = \delta_x$ for all $x \in N$.

In the definition, R_+ denotes the non-negative real numbers, R_+^* denotes the positive reals and δ_x denotes the Dirac function at the point x.

We now apply the heat operator $\Box_2 = \Delta_2 + \frac{\partial}{\partial t}$ to the expression (4.2) with respect to the second variable (r_2, y). Collect those terms with the same power of t and set the resulting expressions equal to 0. In particular, by setting the coefficient of $t^{j-\frac{1}{2}}$ equal to zero, we get:

$$(4.4)\quad \left(u_1(r_2) - \frac{\lambda_1}{h(r_1)h(r_2)}\right)G_{j-1} - \left(\frac{\partial}{\partial r_2}\right)^2 G_{j-1} - (r_1 - r_2)\frac{\lambda_1 h'(r_2)}{h(r_1)h^2(r_2)}G_{j-1}$$

$$+ jG_j - (r_1 - r_2)\frac{\partial}{\partial r_2}G_j - \frac{\lambda_1(h(r_2)h''(r_2) - 2(h'(r_2)^2)}{h(r_1)h^3(r_2)}G_{j-2}$$

$$- 2\frac{\lambda_1 h'(r_2)}{h(r_1)h^2(r_1)}\frac{\partial}{\partial r_2}G_{j-2} - \left(\frac{\lambda_1 h'(r_2)}{h(r_1)h(r_2)}\right)^2 G_{j-3} = 0,$$

where $j \geq 0$ $G_{-3} = G_{-2} = G_{-1} = 0$.

$$(4.5)\qquad u_1(r) = \frac{n}{2} - \left(\frac{n}{2} - 1\right)\frac{h'(r)^2}{h^2(r)} + \frac{n}{2}\frac{h''(r)}{h(r)} + \frac{\lambda_1}{h^2(r)}\ .$$

We can then solve for G_j successively: G_0 = const. Here, in order to satisfy the third condition of definition IV. 2., we should choose $G_0 = 1$.

$$(4.6)\qquad G_1 = \frac{1}{r_1 - r_2}\int_{r_1}^{r_2} u_1(r)dr + \frac{\lambda_1}{h(r_1)h(r_2)}\ .$$

The lower limit is so chosen because we want to avoid the singularities which arise when $r_1 = r_2$, if we had chosen any other lower limit. Moreover G_1 is unique when we choose the fixed lower limit.

$$(4.7) \quad G_2(r_1,r_2,\lambda_1) = \frac{1}{2(r_1-r_2)^2}\left(\int_{r_1}^{r_2} u_1(r)dr\right)^2$$

$$+\frac{1}{r_1-r_2}\left(\int_{r_1}^{r_2} u_1(r)dr\right)\frac{\lambda_1}{h(r_1)h(r_2)}$$

$$-\frac{2}{(r_1-r_2)^3}\int_{r_1}^{r_2} u_1(r)dr - \frac{1}{(r_1-r_2)^2}(u_1(r_k) + u_1(r_2))$$

$$+\frac{1}{2}\frac{\lambda_1}{h(r_1)h(r_2)}$$

$$(4.8) \quad G_3(r_1,r_2,\lambda) = \frac{1}{3!}\left(\frac{1}{r_1-r_2}\int_{r_1}^{r_2} u_1(r)dr + \frac{\lambda_1}{h(r_1)h(r_2)}\right)^3$$

$$+\frac{12}{(r_1-r_2)^5}\int_{r_1}^{r_2} u_1(r)dr - \frac{3}{(r_1-r_2)^4}\left(\int_{r_1}^{r_2} u_1(r)dr\right)^2$$

$$+\frac{6}{(r_1-r_2)^4}(u_1(r_1) + u_1(r_2))$$

$$-\frac{u_1(r_1) + u_1(r_2)}{(r_1-r_2)^3}\int_{r_1}^{r_2} u_1(r)dr - \frac{2}{(r_1-r_2)^3}\frac{\lambda_1}{h(r_1)h(r_2)}$$

$$\cdot \int_{r_1}^{r_2} u_1(r)dr$$

$$-\frac{1}{(r_1-r_2)^3}\int_{r_1}^{r_2} u_1^2(r)dr + \frac{1}{(r_1-r_2)^3}\int_{r_1}^{r_2} u''(r)dr$$

$$-\frac{1}{(r_1-r_2)^2}\frac{\lambda_1}{h(r_1)h(r_2}[u_1(r_1) + u_1(r_2)].$$

In general we can express the G_j successively as:

$$(4.9)\quad G_{j+1} = \frac{1}{(r_1-r_2)^{j+1}} \int_{r_1}^{r_2} \{-(r_1 - r)^j \left(\frac{\lambda_i h'(r_2)}{h(r_1)h(r_2)}\right)^2 G_{j-2}$$

$$- 2(r_1 - r)^j \frac{\lambda_k h'(r_2)}{h(r_1)h^2(r_2)}$$

$$\cdot \frac{\partial}{\partial r_2} G_{j-1} - (r_1 - r)^j \frac{\lambda_i(h(r_2)h''(r_2)-2(h'(r_2))^2)}{h(r_1)h^3(r_2)} G_{j-1}$$

$$- (r_1 - r)^j \frac{\partial^2}{\partial r_2^2} G_j + (r_1 - r)^j \left(u_i(r) - \frac{\lambda_i}{h(r_1)h(r_2)}\right) G_j$$

$$- (r_1 - r)^{j+1} \frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)} G_j\} dr.$$

The G_{j+1} is also unique.

IV.3. Proposition: $G_{j+1}(r_1, r_2, \lambda_i)$ is C^∞ in the variables

r_1, r_2.

Proof: By induction, write $G_{j+1} = \frac{1}{(r_1-r_2)^{j+1}} \int_{r_1}^{r_2} (r_1 - r)^j$ $A_{j+1}(r_1, r, \lambda) dr$. From the induction hypothesis, A_{j+1} is C^∞ in r_1, r_2. Since it is sufficient to check that $\left(\frac{\partial}{\partial r_1}\right)^k \left(\frac{\partial}{\partial r_2}\right)^\ell G_{j+1}$ exist for $r_1 = r_2$ we express $A_{j+1}(r_1, r, \lambda_i)$ as a Taylor series in r with remainder contains $(r - r_1)^{k+\ell}$ and then integrate the result term by term. Taking the limit as $r_2 \to r_1$ will show that the partial derivative exists at $r_1 = r_2$.

Remark: Notice that (4.9) shows $\deg_{\lambda_i} G_{j+1} \le j + 1$.

IV.4. Proposition: $\lim_{t\to 0} H_k(r_1,x,.,t) = \delta_{(r_1,x)}$. Here by H_k we mean expression (4.2) with the second summation replaced by the finite sum $\sum_{j=0}^{k}$.

Proof: Let $f(r_2,y)$ be a smooth function with compact support which contains the point (r_1,x). Then

$$(4.10)\quad \lim_{t\to 0} \int_{M\times I} H_k(r_1,x,r_2,y;t)\, f(r_2,y)dV$$

$$= \lim_{t\to 0} \int_{M\times I} H_k(r_1,x,r_2,y;t)\, f(r_2,y)h^n(r_2)dr_2dy$$

$$= \lim_{t\to 0} \int_I \{\frac{1}{2\sqrt{\pi}}\, h^{-\frac{n}{2}}(r_1)\, h^{-\frac{n}{2}}(r_2)e^{-\frac{(r_1-r_2)^2}{4t}}\, t^{-\frac{1}{2}}$$

$$\sum_{j=0}^{k} [\int_M \sum_{i=0}^{\infty} e^{-\frac{\lambda_i t}{h(r_1)h(r_2)}} \phi_i(x)\phi_i(y)f(r_2,y)$$

$$\cdot\, G_j(r_1,r_2,\lambda_i)t^j dy]h^n(r_2)\}dr_2$$

$$= \lim_{t\to 0} \sum_{j=0}^{k} \int_M \sum_{i=0}^{\infty} e^{-\frac{\lambda_i t}{h^2(r_1)}} \phi_i(x)\phi_i(y)f(r_1,y)$$

$$\cdot\, G_j(r_1,r_1,\lambda_i)t^j dy$$

$$= \lim_{t\to 0} \int_M \sum_{i=0}^{\infty} e^{-\frac{\lambda_i t}{h^2(r_1)}} \phi_i(x)\phi_i(y)f(r_1,y)dy = f(r_1,x).$$

The second from the last equality holds since $\deg_{\lambda_i} G_j(r_1,r_2,\lambda_i)$ is finite. Formally,

$$\lim_{t\to 0} \int_M \sum_{i=0}^{\infty} e^{-\frac{\lambda_i t}{h^2(r_1)}} \lambda_i^s \phi_i(x)\phi_i(y) f(r_1,y)dy = \lim_{t\to 0}$$

$$\cdot \int_M e^{-\Delta^t} \Delta^s f(r_1,y) = \Delta^s f(r_1,x).$$ s is a positive integer.

Q.E.D.

In order to prove that H_k does represent a parametrix it remains to check the second condition of definition IV. 2. For this we need the following proposition:

IV.5. Proposition: For given $T > 0$,

$$|\Box H_k| \le \text{const.}\ t^{[\frac{k}{3}]-\frac{n}{2}-\frac{3}{2}} \quad t < T .$$

The constant in this inequality depends on $h(r)$. Before we prove the proposition we need some inequalities about the degree of λ_i in $G_\ell(r_1,r_2,\lambda_i)$. The significance of knowing the degree of λ_i in G_ℓ is that each increase of the degree of λ_i in the expression $\sum_{i=0}^{\infty} \lambda_i^s e^{-\lambda_i t}$ will lower the degree of t by 1 in the t power series asymptotic expansion at 0 of this expression. In fact, we can apply theorem I.2.1. to the expression and the property mentioned above will follow easily.

IV.6. Claim: (i) $\deg_{\lambda_1} G_\ell(r,r,\lambda_1) \le [\frac{2\ell}{3}]$

(ii) $\deg_{\lambda_1}((\frac{\partial}{\partial r_2})^k G_\ell)(r,r,\lambda_1) \le [\frac{2\ell+k-1}{3}],\ 0 < k \le \ell$

(iii) $\deg_{\lambda_1}((\frac{\partial}{\partial r_2})^k G_\ell)(r,r,\lambda_1) \le \ell \qquad k \ge \ell$

Proof: By induction on ℓ, suppose the above inequalities hold for all $\ell' < \ell$ then:

$$(4.11) \qquad \deg_{\lambda_1} G_\ell(r,r,\lambda_1) \le \max\{[\frac{2\cdot(\ell-1)}{3}],\ 1 + [\frac{2(\ell-2)}{3}],\ 1 + [\frac{2(\ell-2)+1-1}{3}],\ 2 + [\frac{2(\ell-3)}{3}]\} = [\frac{2\ell}{3}]$$

which proves the first inequality.

Consider

$$(4.12) \qquad [(\frac{\partial}{\partial r_2})^k(\ell G_\ell - (r_1 - r_2)\frac{\partial}{\partial r_2} G_\ell)]_{r_1=r_2}$$
$$= [\ell(\frac{\partial}{\partial r_2})^k G_\ell + \binom{k}{1}(\frac{\partial}{\partial r_2})^k G_\ell - (r_1 - r_2)^{k+1} G_\ell]_{r_1=r_2}$$
$$= ((\ell + k)(\frac{\partial}{\partial r_2})^k G_\ell)(r,r,\lambda_1)$$

and

$$(4.13) \qquad \ell G_\ell - (r_1 - r_2)\frac{\partial}{\partial r_2} G_\ell = -(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)})G_{\ell-1}$$
$$+ (\frac{\partial}{\partial r_2})^2 G_{\ell-1} + (r_1 - r_2)\frac{\lambda_i h'(r_2)}{h(r_1)h(r_2)} G_{\ell-1}$$
$$+ \frac{\lambda_i(h(r_2)h''(r_2) - 2(h'r_2))^2)}{h(r_1)h^3(r_2)} G_{\ell-2}$$
$$+ 2\frac{\lambda_i h'(r_2)}{h(r_1)h(r_2)^2}\frac{\partial}{\partial r_2} G_{\ell-2} + (\frac{\lambda_i h'(r_2)}{h(r_1)h(r_2)})^2 G_{\ell-3}$$

if we want to know the degree of λ_1 in $((\frac{\partial}{\partial r_2})^k G_\ell)(r,r,\lambda_1)$ we need only know the degree of the k-th derivation w.r.t. r_2 of the r.h.s. of the above expression. But:

$$(4.14)\quad (\frac{\partial}{\partial r_2})^k[(u_1(r_2) - \frac{\lambda_1}{h(r_1)h(r_2)})G_{\ell-1}] = (u_1(r_2) - \frac{\lambda_1}{h(r_1)h(r_2)})$$

$$\cdot (\frac{\partial}{\partial r_2})^k G_{\ell-1} + \binom{k}{1}\frac{\partial}{\partial r_2}(u_1(r_2) - \frac{\lambda_1}{h(r_1)h(r_2)})(\frac{\partial}{\partial r_2})^{k-1} G_{\ell-1}$$

$$\cdot +\ldots+ \binom{k}{m}(\frac{\partial}{\partial r_2})^m(u_1(r_2) - \frac{\lambda_1}{h(r_1)h(r_2)}(\frac{\partial}{\partial r_2})^{k-m} G_{\ell-1}$$

$$\cdot +\ldots+ (\frac{\partial}{\partial r_2})^k(u_1(r_2) - \frac{\lambda_1}{h(r_1)h(r_2)})G_{\ell-1}.$$

Therefore:

$$(4.15)\quad \deg_{\lambda_1}(\frac{\partial}{\partial r_2})^k[(u_1(r_2) - \frac{\lambda_1}{h(r_1)h(r_2)})G_{\ell-1}]_{r_1=r_2}$$

$$\leq \max\{1 + [\frac{2(\ell-1) + k-1+1}{3}],\ldots 1 + [\frac{2(\ell-1) + k-m-1}{3}]$$

$$\cdot ,\ldots 1 + [\frac{2(\ell-1)}{3}]\} = [\frac{2\ell + k - 1}{3}] .$$

Similarly we have:

$$(4.16)\quad \deg_{\lambda_1}((\frac{\partial}{\partial r_2})^k(\frac{\partial}{\partial r_2})G_{\ell-1})_{r_1=r_2} \leq [\frac{2\ell + k - 1}{3}]$$

$$\deg_{\lambda_1}\{(\frac{\partial}{\partial r_2})^k[(r_1 - r_2)\frac{\lambda_1 h'(r_2)}{h(r_1)h(r_2)}G_{\ell-1}]\}_{r_1=r_2} \leq [\frac{2\ell + k - 1}{3}]$$

$$\deg_{\lambda_1}\{(\frac{\partial}{\partial r_2})^k[\frac{\lambda_1(h(r_1)h''(r_2) - 2(h'(r_2))^2)}{h(r_1)h^3(r_2)}G_{\ell-2}]\}_{r_1=r_2}$$

$$\leq [\frac{2\ell + k - 1}{3}]$$

$$\deg_{\lambda_i}\{(\frac{\partial}{\partial r_2})^k(\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}\frac{\partial}{\partial r_2}G_{\ell-2})\}_{r_1=r_2} \le [\frac{2\ell + k - 1}{3}]$$

$$\deg_{\lambda_i}\{(\frac{\partial}{\partial r_2})^k((\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)})^2 G_{\ell-3})\}_{r_1=r_2} \le [\frac{2\ell + k - 1}{3}] .$$

Combining the above inequalities we get:

$$(4.17) \qquad \deg_{\lambda_i}((\frac{\partial}{\partial r_2})^k G_\ell)(r,r,\lambda_i) \le [\frac{2\ell + k - 1}{3}]$$

so we have proved (ii). (iii) is trivial since $\deg_{\lambda_i} G_\ell(r_1,r_2,\lambda_i)$ is never greater than ℓ.

Q.E.D.

Now let us return to the proof of the proposition. Since

$$(4.18) \quad \Box_2 H_k = \sum_{i=0}^{\infty}[(r_1 - r_2)\frac{\partial}{\partial r_2}G_{k+1}(r_1,r_2,\lambda_i)$$
$$- (k+1)G_{k+1}(r_1,r_2,\lambda_i)]G_i t^{k-\frac{1}{2}}$$
$$+ \sum_{i=0}^{\infty}[\frac{-\lambda_i(h(r_2)h''(r_2) - 2(h'(r_2))^2)}{h(r_1)h(r_2)^3}G_k(r_1,r_2\lambda_i)$$
$$- 2\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)}\frac{\partial}{\partial r_2}G_k(r_1,r_2,\lambda_i) - (\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)})^2$$
$$G_{k-1}(r_1,r_2,\lambda_i)]G_i t^{k+\frac{1}{2}} + \sum_{i=0}^{\infty}[-(\frac{\lambda_i h'(r_2)}{h(r_1)h^2(r_2)})^2$$
$$G_k(r_1,r_2,\lambda_i)]G_i t^{k+\frac{3}{2}}$$

where

$$G_i = \frac{1}{2\sqrt{\pi}} h^{-\frac{n}{2}}(r_1)h^{-\frac{n}{2}}(r_2)e^{-\frac{(r_1 - r_2)^2}{4t}} e^{\frac{-\lambda_i t}{h(r_1)h(r_2)}}\phi_i(k)\phi_i(y) .$$

Since we know that

$$(4.19)\qquad \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y) \le \text{const.}\cdot t^{-\frac{n}{2}} \qquad t < T$$

$$(4.20)^{*}\qquad \sum_{i=0}^{\infty} \lambda_i^s e^{-\lambda_i t}\phi_i(x)\phi_i(y) \le \text{const.}\cdot t^{-\frac{n}{2}-s} \qquad t < T$$

and

$$(4.21)\qquad \left|e^{-\frac{(r_1-r_2)^2}{4t}}(r_2 - r_1)^k\right| \le \text{const.}\cdot t^{-\frac{k}{2}} \qquad k \ne 0$$

it is easy to see that:

$$(4.22)\qquad \left|\sum_{i=0}^{\infty} e^{-\frac{(r_1-r_2)^2}{4t}} \frac{(r_1 - r_1)^k}{k!}\left(\frac{\partial^k}{\partial r_2^k} G_\ell(r_1,r_2,\lambda_i)\right)_{r_1=r_2} \cdot e^{-\frac{\lambda_i t}{h(r_1)h(r_2)}}\phi_i(x)\phi_i(y)\right| \le \text{const.}\cdot t^{-\frac{n}{2}+\frac{k}{2}-\left[\frac{2\ell+k-1}{3}\right]}.$$

Now, we expand $G_\ell(r_1,r_2,\lambda_i)$ into a Taylor series at $r_2 = r_1$ with r_2 as variable. If we denote the sum of all terms with power of $(r_2 - r_1)$ greater than ℓ by $\overline{G}_\ell(r_1,r_2,\lambda_i)$, then:

$$(4.23)\qquad G_\ell(r_1,r_2,\lambda_i) = G_\ell(r_1,r_1,\lambda_i) + (r_2 - r_1)\left(\frac{\partial}{\partial r_2} G_\ell(r_1,r_2,\lambda_i)\right)_{r_1=r_2} + \dots + \frac{(r_2 - r_1)^k}{k!}\left[\left(\frac{\partial}{\partial r_2}\right)^k G_\ell(r_1,r_2,\lambda_i)\right]_{r_1=r_2} + \dots + \frac{(r_2 - r_1)^\ell}{\ell!}\overline{G}_\ell(r_1,r_2,\lambda_i).$$

Note that $\deg_{\lambda_i}\overline{G}_\ell(r_1,r_2,\lambda_i) \le \ell$, we have pointed this out in the remark following IV. 3. In order to estimate G_i, we multiply each term of the Taylor series expansion by G_i, and then use the above inequalities to obtain

$$(4.24)\qquad |G_\ell(r_1,r_2,\lambda_i)G_i| \le \text{const.}\cdot t^{-\frac{n}{2}-\left[\frac{2\ell}{3}\right]}.$$

---

*See I.22

Similarly we can estimate $\Box_2 H_k$ since we can estimate each term of it.

$$(4.25) \qquad |\Box_2 H_k| \le \text{const.}\cdot t^{k-\frac{1}{1}-\frac{n}{2}-[\frac{2(k+1)}{3}]}$$
$$+ \text{const.}\cdot t^{k+\frac{1}{2}-\frac{n}{2}-\frac{n}{2}-[\frac{2k}{3}]-1}$$
$$+ \text{const.}\cdot t^{k+\frac{1}{2}-\frac{n}{2}-[\frac{2(k-1)}{3}]-2}$$
$$+ \text{const.}\cdot t^{k+\frac{3}{2}-\frac{n}{2}-[\frac{2k}{3}]-2}$$
$$\le \text{const.}\cdot t^{[\frac{k}{3}]-\frac{n}{2}-\frac{3}{2}} .$$

This completes the proof of the proposition and shows that H_k does represent a parametrix for heat kernel on M x I when $[\frac{k}{3}] > \frac{n}{2} + \frac{3}{2}$. Since the heat kernel is symmetric w.r.t. its variables (r_1,x), (r_2,y), if we represent it as $\sum_{i=0}^{\infty} f_i(r_1,r_2,t)\phi_i(x)\phi_i(y)$ we can see that $f_i(r_1,r_2,t)$ is also symmetric w.r.t. r_1 and r_2. For some special functions $h(r)$, $f_i(r_1,r_2,t)\phi_i(x)\phi_i(y) = \sum_{j=0}^{\infty} G_j(r_1,r_2,\lambda_i)G_i t^{j-\frac{1}{2}}$ is convergent for each i. Since G_i is symmetric w.r.t. r_1 and r_2, in these special cases $G_j(r_1,r_2,\lambda_i)$ is also symmetric w.r.t. r_1 and r_2. But if we look at the construction of G_j more carefully we will find that the form of G_j in terms of $h(r)$ is universal* so that we get the following

* G_j is the summation of $\frac{1}{(r_1-r_2)}$ to some power multiplied by the integral of a rational function of $h(r), h'(r), \ldots$; the coefficients of the denominator and numerator of the rational function are independent of $h(r)$. See (4.8) for $j = 3$.

IV.8. Corollary:

$$(4.26) \quad \text{Trace } E \underset{t\to 0}{\sim} \sum_{j=0}^{\infty} \frac{1}{2\sqrt{\pi}} h^{-n}(r) \sum_{i=0}^{\infty} e^{-\frac{\lambda_i t}{h^2(r)}} a_j(r,r,\lambda_i)\phi_i^2(x) t^{j-\frac{1}{2}}$$

V The Asymptotic Expansion of the Heat Kernel on M x I

We begin with some formal arguments which will show us how to apply the results we cite in section III to the calculation of $G_j(r,r,\lambda_i)$ of Section IV. Recall Cauchy's integral formula in the theory of a complex variable:

$$(5.1) \qquad f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\,d\xi,$$

where $f(z)$ is analytic in a neighborhood of the closed curve C and z is inside C. If D is an unbounded operator we define

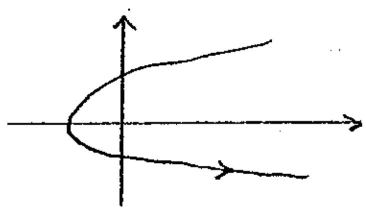
$$(5.2) \qquad f(D) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi I - D}\,d\xi,$$

where C is a curve about spec D and f is analytic and bounded in a neighborhood of spec D [8].

Now let us consider the case when $D = \Delta$ and $f(z) = e^{-zt}$

$$(5.3) \qquad e^{-\Delta t} = \frac{1}{2\pi i}\int_C \frac{e^{-\xi t}}{\xi I - \Delta}\,d\xi,$$

where C is of the form



Since the Laplace transform of $e^{-\xi t}$ is $(\xi + s)^{-1}$:

$$(5.4) \qquad \int_0^\infty e^{-\xi t}e^{-st}dt = (\xi - s)^{-1}$$

therefore

$$(5.5)\qquad \int_0^\infty e^{-\Delta t} e^{-st} dt = (\Delta + s)^{-1} .$$

Although the above derivation is true only in some special circumstances, it does suggest that if we express the heat kernel on $M \times I$ with the metric $h^2(r)g + dr \otimes dr$ as

$$(5.6)\qquad E(r_1, r_2, x, y; t) = \sum_{i=0}^{\infty} f_i(r_1, r_2, t)\phi_i(x)\phi_i(y)$$

then the Laplace transform of E must be the kernel of a resolvent of Δ and

$$(5.7)\qquad (\Delta_2 + s)\sum_{i=0}^{\infty} \hat{f}_i(r_1, r_2, s)\phi_i(x)\phi_i(y) = 0 .$$

Notice that $\hat{f}$ means the Laplace transform of f.

Therefore

$$(5.8)\qquad \left[-\left(\frac{\partial}{\partial r_2}\right)^2 - \frac{nh'(r_2)}{h(r_2)}\frac{\partial}{\partial r_2} + \frac{\lambda_i}{h^2(r_2)} + s\right]\hat{f}_i(r_1, r_2, s) = 0$$

or

$$(5.9)\qquad \left[-\left(\frac{\partial}{\partial r_2}\right)^2 + \left(\frac{n}{2}\left(\frac{n}{2}-1\right)\frac{h'(r_2)^2}{h^2 r_2)} + \frac{n}{2}\frac{h''(r_2)}{h(r_2)} + \frac{\lambda_i}{h^2(r_2)}\right)\right]$$

$$\cdot\; h^{\frac{n}{2}}(r_2)\hat{f}(r_1, r_2, s) = -\,sh^{\frac{n}{2}}(r_2)\hat{f}(r_1, r_2, s).$$

This in turn suggests that $h^{\frac{n}{2}}(r_1)h^{\frac{n}{2}}(r_2)\hat{f}_i(r_1, r_2, s)$ is the kernel of the resolvent of $-d^2 + (u_i(r) + s)$ where, as before,

$$(5.10)\qquad u_i(r) = \frac{n}{2}\left(\frac{n}{2}-1\right)\frac{h'(r)^2}{h^2(r)} + \frac{n}{2}\frac{h''(r)}{h(r)} + \frac{\lambda_i}{h^2(r)} .$$

Apply the results from Section III:

$$(5.11) \qquad h^n(r)\hat{f}_i(r,r,s) \underset{s\to\infty}{\sim} \sum_{\ell=0}^{\infty} \frac{R_\ell[u_i]}{s^{\ell+\frac{1}{2}}}$$

Although the converse of theorem I.2.3 is true, only under special circumstances. The theorem does imply that if the asymptotic expansion exists, the only candidate is

$$(5.12) \qquad h^n(r)f_i(r,r,t) \underset{t\to 0}{\sim} \sum_{\ell=0}^{\infty} \frac{R_\ell[u_i]}{\Gamma(\ell+\frac{1}{2})} t^{\ell-\frac{1}{2}} ,$$

because the asymptotic expansion w.r.t. a asymptotic sequence at some special point is unique.

The following proposition tells us that it is possible to factor $e^{-u_i t}$ out of the above asymptotic expansion.

V.1. Proposition: Summing up those terms in $\sum_{\ell=0}^{\infty} \frac{R_\ell[u_i]}{\Gamma(\ell+\frac{1}{2})} t^{\ell-\frac{1}{2}}$ which contain a fixed monomial of derivatives of u_i of the form $u_i^{(k_1)} u_i^{(k_2)} \ldots u_i^{(k_j)}$ $k_j \neq 0$ gives the sum as

$$A_{k_1\ldots k_j} e^{-u_i t} u_i^{(k_1)} \ldots u_i^{(k_j)} t^{k_{k_1}\ldots k_j}$$

Proof: From the proposition III.6. we have

$$\frac{\partial}{\partial u_i} R_\ell[u_i] = -(\ell - \tfrac{1}{2}) R_{\ell-1}[u_i]$$

which implies

$$(5.13) \qquad \frac{\partial}{\partial u_i} \frac{R_\ell[u_i]}{\Gamma(\ell+\frac{1}{2})} = - \frac{R_{\ell-1}[u_i]}{\Gamma(\ell-1+\frac{1}{2})}$$

Q.E.D.

Since our goal is to find a technique which enables us to compute the asymptotic expansion of the trace of E. Factoring $e^{-u_1 t}$ out of $\sum_{\ell=0}^{\infty} \frac{R_\ell[u_1]}{\Gamma(\ell+\frac{1}{2})} t^{\ell-\frac{1}{2}}$ is a little bit too much, as we see if we compare it with the asymptotic expansion of the trace of E that we got in the end of the last section where we have factor $e^{-\frac{\lambda_1 t}{h^2(r)}}$ in the expression not $e^{-u_1 t}$. Fortunately we can remedy this problem by splitting u_1 into two terms, one containing λ_1 and the other not containing λ_1. The first term is exactly $\frac{\lambda_1}{h^2(r)}$, and the other we denote by $q(r)$.

Clearly,

$$(5.14) \qquad q(r) = \frac{n}{2}\left(\frac{n}{2} - 1\right)\frac{h'(r)^2}{h^2(r)} + \frac{n}{2}\,\frac{h''(r)}{h(r)}$$

Then

$$(5.15) \qquad f_1(r,r,t) \underset{t\to 0}{\sim} \frac{1}{2\sqrt{\pi}}\, h^{-\frac{n}{2}}(r)\, e^{-\frac{\lambda_1 t}{h^2(r)}} \sum_{j=0}^{\infty} \tilde{G}_j(r,\lambda_1) g^{j-\frac{1}{2}} ,$$

where we multiply $e^{-q(r)t}$ back into the expression. Therefore if we denote $R_\ell[u_1]$ more explicitly by $R_\ell[u_1, u_1', u_1'', \ldots]$ then $\tilde{G}_j(r,\lambda_1)$ is nothing but $\frac{R_j(q, u_1', u_1'' \ldots)}{\Gamma(j+\frac{1}{2})} \cdot 2\sqrt{\pi}$; i.e., replace u_1 (but not its derivative) in R_ℓ by q.

Up to now most of the work we have done in this section is just formal. Therefore a theorem such as the following is needed if we want to apply these results to compute the asymptotic expansion of the trace of E.

V.2. Theorem $a_j(r,r,\lambda_i) = \tilde{a}(r,\lambda_i)$

Proof: Consider the expression

$$(5.16)\quad K_l(r_1,r_2,\lambda_i) = u_i(r_2)[a_{\ell-2}\frac{\lambda_i}{h(r_1)h(r_2)} + \frac{1}{2!}a_{\ell-3}$$

$$\cdot\ (\frac{\lambda_i}{h(r_1)h(r_2)})^2 + \ldots + (1-1)^{\ell-1}\frac{1}{\;}\ (\frac{\lambda_i}{h(r_1)h(r_2)})^{\ell-1} a_0]$$

$$+ \ell[a_\ell - a_{\ell-1}\frac{\lambda_i}{h(r_1)h(r_2)} + \ldots + (-1)^\ell \frac{1}{\ell!}(\frac{\lambda_i}{h(r_1)h(r_2)})^\ell a_0]$$

$$+ (r_1 - r_2)\frac{\partial}{\partial r_2}[a_\ell - a_{\ell-1}\frac{\lambda_i}{h(r_1)h(r_2)} + \ldots + (-1)^\ell \frac{1}{\ell!}$$

$$\cdot\ (\frac{\lambda_i}{h(r_1)h(r_2)})^\ell a_0]$$

$$+ \frac{\partial^2}{\partial r_2}[a_{\ell-1} - a_{\ell-2}\frac{\lambda_i}{h(r_1)h(r_2)} + \ldots + (-1)^\ell \frac{1}{(\ell-1)!}$$

$$\cdot\ (\frac{\lambda_i}{h(r_1)h(r_2)})^{\ell-1} a_0]\ .$$

We can express the right hand side of above as the sum of terms of the following form

$$(5.17)\quad (-1)^k\frac{1}{k!}(\frac{\lambda_i}{h(r_1)h(r_2)})^k[(u_i(r_2) - \frac{\lambda_i}{h(r_1)h(r_2)})a_{\ell-k-1}$$

$$- (\frac{\partial}{\partial r_2})^2 a_{\ell-k-1} + (r_1 - r_2)\frac{\lambda_i}{h(r_1)}(\frac{1}{h(r_2)})' + (\ell - k)a_{\ell-k}$$

$$- (r_1 - r_2)\frac{\partial}{\partial r_2}a_{\ell-k} + \frac{\lambda_i}{h(r_1)}(\frac{1}{h(r_2)})'' a_{\ell-k-2}$$

$$+ \frac{\lambda_i}{h(r_1)}(\frac{\lambda_i}{h(r_2)})'\frac{\partial}{\partial r_2}a_{\ell-k-2} - \frac{\lambda_i}{h(r_1)}[(\frac{1}{h(r_2)})']^2 a_{\ell-k-3}]\ .$$

Since G_ℓ satisfies (4.4) of the last section, each term of this form must be zero and so is their sum, i.e. $K_1(r_1,r_2,\lambda_1) = 0$. Similarly, if we denote by $K_2(r_1,r_2,\lambda_1)$ the expression which comes from $K_1(r_1,r_2,\lambda_1)$ by interchanging the roles played by r_1 and r_2, then we also have $K_2(r_1,r_2,\lambda_1) = 0$. Moreover:

$$(5.18)\qquad \begin{aligned} &\frac{\partial}{\partial r_2} K_1(r_1,r_2,\lambda_1) = 0 \\ &3\frac{\partial}{\partial r_1} K_1(r_1,r_2,\lambda_1) = 0 \\ &3\frac{\partial}{\partial r_2} K_2(r_1,r_2,\lambda_1) = 0 \\ &\frac{\partial}{\partial r_1} K_2(r_1,r_2,\lambda_1) = 0 \,. \end{aligned}$$

Combining these identities, and letting $r_1 = r_2$, we get

$$(5.19)\qquad \begin{aligned} &(\ell - \tfrac{1}{2})[G_\ell - \frac{\lambda_1}{h^2}G_{\ell-1} + \tfrac{1}{2}(\frac{\lambda_1}{h^2})^2 G_{\ell-2}+\dots]' \\ &= \frac{1}{4}[G_{\ell-1} - G_{\ell-1}\frac{\lambda_1}{h^2} + \frac{1}{2!}(\frac{\lambda_1}{h^2})^2 G_{\ell-3}+\dots]''' \\ &- \tfrac{1}{2}u_1'[G_{\ell-1} - G_{\ell-2}\cdot\frac{\lambda_1}{h^2}+\dots] \\ &- u_1[G_{\ell-1} - G_{\ell-2}\frac{\lambda_1}{h^2}+\dots]' \,. \end{aligned}$$

Now, since by construction of $\tilde{G}_\ell$ we have

$$(5.20)\qquad [\tilde{G}_\ell - \tilde{G}_{\ell-1}\frac{\lambda_1}{h^2} +\dots+ \frac{1-}{\ell!} G_0(\frac{\lambda_1}{h^2})^\ell] = \frac{R_\ell}{\Gamma(\ell+\frac{1}{2})} 2\sqrt{\pi}$$

and from corollary III.4. we have

$R_\ell' = \frac{1}{4}R_{\ell-1}''' - uR_{\ell-1}' - \frac{1}{2}u'R_{\ell-1}$, then by substitution we will see that G_ℓ satisfies the same differential equation as above. Since both of them satisfy the same differential equation of first order they may differ by a constant. But because this constant is universal,* it is sufficient to check the special case h = const. Then we will get that it must be zero, so that $G_j(r,r,\lambda_i) = \tilde{G}_j(r,\lambda_i)$

Q.E.D.

V.3. Corollary:

$$\text{(5.21)} \quad \text{Trace } E \underset{t\to 0}{\sim} \sum_{j=0}^{\infty} \frac{1}{\sqrt{4\pi}} h^{-n}(r) \sum_{i=0}^{\infty} e^{-\frac{\lambda_i t}{h^2(r)}} \tilde{G}_j(r,\lambda_i)\phi_i^2(x)t^{j-\frac{1}{2}} .$$

This proves the legitimacy of the addition of the infinitely many asymptotic expansions to form the asymptotic expansion we wanted.

Now we are ready to calculate the coefficients.

Now if

$$\text{(5.22)} \quad \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^2(x) \underset{t\to 0}{\sim} \frac{1}{\sqrt{4\pi}^n}\left(C_0 t^{-\frac{n}{2}} + C_1 t^{-\frac{n}{2}+1} + \dots + C_s t^{-\frac{n}{2}+s} + \dots\right)$$

then

$$\text{(5.23)} \quad \sum_{i=0}^{\infty} \lambda_i^\ell e^{-\lambda_i t}\phi_i^2(x) \underset{t\to 0}{\sim} \frac{1}{\sqrt{4\pi}^n}\left[C_0\left(\tfrac{n}{2}\right)\left(\tfrac{n}{2}+1\right)\dots\left(\tfrac{n}{2}+\ell-1\right)t^{-\frac{n}{2}-\ell}\right.$$
$$+ C_1\left(\tfrac{n}{2}-1\right)\tfrac{n}{2}\dots\left(\tfrac{n}{2}+\ell-2\right)t^{-\frac{n}{2}-\ell+1}$$
$$+ \dots$$

---

*From the constructions of both G_ℓ and $\tilde{G}_\ell$ we see that the coefficients involve in them are independent of h(r).

$$+ C_s(\tfrac{n}{2} - s)\dots(\tfrac{n}{2} + \ell - s)t^{-\frac{n}{2}-\ell+s}$$

$$+\dots] \qquad \text{where } \ell \geq 1$$

and

$$(5.24) \quad \sum_{i=0}^{\infty} \lambda_i^{\ell} e^{-\frac{\lambda_i t}{h^2(r)}} \phi_i^2(x) \underset{t\to 0}{\sim} \frac{1}{\sqrt{4\pi}^n} [C_0(\tfrac{n}{2})(\tfrac{n}{2} + 1)\dots(\tfrac{n}{2} + \ell - 1) \cdot (\frac{t}{h^2(r)})^{-\frac{n}{2}-\ell}$$

$$+\dots$$

$$+ C_s(\tfrac{n}{2} - s)\dots(\tfrac{n}{2} + \ell - s) \cdot (\frac{1}{h^2(r)})^{-\frac{n}{2}-\ell+s}$$

$$+\dots]$$

Suppose we denote the coefficients of λ_i^{ℓ} in $\tilde{G}_j(r,\lambda_i)$ by $G_j^{\ell}(r)$,

and then consider those terms containing $t^{-\frac{n+1}{2}+k}$ which come from $\tilde{G}_j(r,\lambda_i)$. They will have the form

$$(5.25) \quad \frac{1}{\sqrt{4\pi}} h^{-n}(r) \frac{1}{\sqrt{4\pi}^n} G_j^{\ell}(r) C_s(\tfrac{n}{2} - s)\dots(\tfrac{n}{2} + \ell - s)(\frac{t}{h^2(r)})^{-\frac{n}{2}+s-\ell} t^{j-\frac{1}{2}}$$

with $-\ell + s + j = k$.

But since $\deg_{\lambda_i} \tilde{G}_j = \deg_{\lambda_i} G_j(r,r,\lambda_i) \leq [\frac{2j}{3}]$ and $s \geq 0$ then $j \leq 3k$. Now let j vary and collect all those terms contain $t^{-\frac{n+1}{2}+k}$. We will have

$$(5.26)\quad \sum_{j=0}^{[\frac{2j}{3}]} \sum_{j=0}^{3k} \frac{1}{\sqrt{4\pi t}^{\,n+1}} \mathfrak{a}_j^{\ell}\, C_{k-j+1}(\tfrac{n}{2}-k+j-\ell)\cdots(\tfrac{n}{2}-k+j-1) \cdot \left(\frac{t}{h^2(r)}\right)^k h^{2j}(r) .$$

If we denote

$$(5.27)\quad d_k = \sum_{\ell=0}^{[\frac{2j}{3}]} \sum_{j=0}^{3k} \mathfrak{a}_j^{\ell}\, C_{k-j+1}(\tfrac{n}{2}-k+j-\ell)\cdots(\tfrac{n}{2}-k+j-1)h^{2j}(r)$$

then

$$(5.28)\quad \text{Trace } E \underset{t\to 0}{\sim} \sum_{k=0}^{\infty} \frac{1}{\sqrt{4\pi t}^{\,n+1}} d_k \left(\frac{t}{h^2(r)}\right)^k$$

Remark: When $\ell = 0$, $(\frac{n}{2}-k+j-\ell)\cdots(\frac{n}{2}-k+j-1)$ will be replaced by 1.

V.4. Proposition: In the case of metric cone, $\tilde{\mathfrak{a}}_j(r)$ satisfies the recursive formula

$$(5.29)\quad \tilde{\mathfrak{a}}_{\ell+1} = \frac{1}{\ell+1}\left(\frac{\lambda_i^2\tilde{\mathfrak{a}}_{\ell-2}}{r^6} - \frac{2\ell\lambda_i\tilde{\mathfrak{a}}_{\ell-1}}{r^4} + \frac{\ell(\ell+1)\tilde{\mathfrak{a}}_{\ell}}{r^2} - \frac{\frac{n}{2}(\frac{n}{2}-1)\tilde{\mathfrak{a}}_{\ell}}{r^2}\right)$$

Actually we can prove a more precise formula:

$$(5.30)\quad \mathfrak{a}_{\ell+1} = \frac{1}{\ell+1}\left(\frac{\lambda_i^2\mathfrak{a}_{\ell-2}}{r_1^3 r_2^3} - \frac{2\ell\lambda_i\mathfrak{a}_{\ell-1}}{r_1 r_2} - \frac{\frac{n}{2}(\frac{n}{2}-1)\mathfrak{a}_{\ell}}{r_1 r_2}\right)$$

and $\mathfrak{a}_0 = 1$.

Making use of the fact that, for all ℓ, $\mathfrak{a}_\ell = \dfrac{e_\ell(\lambda_i)}{(r_1 r_2)^\ell}$,

where $e_\ell(\lambda_i)$ are independent of r_1 and r_2, substitution of the above formula into the differential equation will show that it does work.

If we drop $\frac{1}{r^{2\ell}}$ from $\tilde{\alpha}_\ell$ then $e_\ell(\lambda_1)$ will satisfy

$$(5.31)\quad e_{\ell+1} = \frac{1}{\ell+1}(\lambda_1^2 e_{\ell-2} - 2\ell\, e_{\ell-1} + \ell(\ell+1)e_\ell - \frac{n}{2}(\frac{n}{2}-1)e_\ell).$$

Moreover we have the corollary

V.5. Corollary:

$$(5.32)\quad e^{\ell\,*}_{j+1} = \frac{1}{j+1}(e^{\ell-2}_{j-2} - 2j\, e^{\ell-1}_{j-1} + j(j+1)e^{\ell}_{j} - \frac{n}{2}(\frac{n}{2}-1)e^{\ell}_{j}).$$

Therefore in the case of the metric cone we have

$$(5.33)\quad d_k = \sum_{\ell=0}^{[\frac{2j}{3}]} \sum_{j=0}^{3k} e^{\ell}_{j}\, c_{k_{j+1}} (\frac{n}{2} + k-j-\ell)\ldots(\frac{n}{2}+k-j-1).$$

*e^{ℓ}_{j+1} is the coefficient of λ_1^{ℓ} in $e_{j+1}(\lambda_1)$.

VI The Significance of the Approach and Applications of the Result

In the end of the last chapter when $h(r) = r$ we got a formula for d_k in terms of C_j and e_j^{ℓ} where e_j^{ℓ} is given by a recursive formula. From the form of the recursive formula we have reason to believe that this is the most explicit expression we can find. For more general cases since G_j^{ℓ} is the coefficient of λ_j^{ℓ} in G_j, we should know G_j. Although G_j can be constructed by the recusive formula in section IV, a generating function for them will be more helpful. Gel'fand and Dikii have derived a generating function for R_{ℓ} which can be exploited to construct G_{ℓ}. They proceeded by first viewing R_{ℓ} as a polynomial in $u, u', u'', \ldots$ and then translating R_{ℓ} into the so called symbolic polynomial, which in turn has a generating function. Judging from the sophisticated way they derived the function, we expect that a generating function or even a more directly recursive formula for d_k, than the one we got, is very unlikely.

It is reasonable to ask if it is possible to find a more direct way to arrive at our results. We will make some remarks concerning this point. Let us consider first the case of the metric cone. Since the first coefficient of the asymptotic expansion of the heat kernel is the reciprocal of the square root of the determinant of the exponential map, we should find the relation between the determinant of the exponential map on the cone and the determinant of the exponential map on the base*. In the appendix we will show that

$$(6.1) \qquad \theta((p,\tau),(Q,\varkappa)) = \underline{\theta}(P,Q)\left(\frac{\ell}{\sin\ell}\right)^{n-1}$$

where $\theta((p,\tau), (Q,\varkappa))$ denotes the determinant of the exponential map from tangent space $M \times I_{(p,\tau)}$ to $M \times I$ evaluated at $(\exp_{(p,\tau)})^{-1}(Q,\varkappa)$. $\underline{\theta}(P,Q)$ denotes the corresponding notion on M, and ℓ denotes the distance between P and Q on M. In order to find the coefficients of the asymptotic expansion of the heat kernel we should apply Δ to $\theta^{-\frac{1}{2}}$ [1,p.208]. From the relation $\Delta = \frac{\underline{\Delta}}{r^2} - \frac{n}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2}$ and the fact that the right hand side of the above relation is independent of r we have

$$\Delta^k\theta^{-\frac{1}{2}} = \frac{\underline{\Delta}}{r^{2k}}(\underline{\Delta} + 2n-2\cdot3)(\underline{\Delta} + 4n - 4\cdot5)\ldots(\underline{\Delta} + (2k-2)n - (2k-2)(2k-1))\,\underline{\theta}^{-\frac{1}{2}}\left(\frac{\ell}{\sin\ell}\right)^{-\frac{n-1}{2}} .$$

This formula shows the complication of a direct computation. There is the difficulty not only of writing out explicitly the operator in the right hand side into a polynomial of $\underline{\Delta}$ but also of applying each term to a product of two functions. Moreover the expression does not lead immediately to the formula of correct type, i.e. coefficients on M x I are linear combination of coefficients on base M. That means our result

*More general, the k-th coefficients are something like $\Delta^k\theta^{-\frac{1}{2}}$ on the trace (i.e. $r_1 = r_2$, $x = y$) [1, p.208].

implied are interesting cancellation. In the case of more general $h(r)$ it becomes even more hopeless since the relation between the determinant of the exponential maps must involve r. This involvement of r will make the application of Δ to the determinant even more complicated and it will be unlikely there is any expression as the above one.

For the first application, if we set $d_k = 0$ in formula (5.33) this will give us a recursive formula for C_j which is just the coefficient of the asymptotic expansion of the trace of the heat kernel on S^n. The formula so obtained is no more complicated than the corresponding formula obtained by Cahn and Wolf [2]. In their formula they have two cases, when n is even or odd. We have a single formula for all n. However, their approach is more general and they apply it to the cases of compact symmetric spaces of rank one.

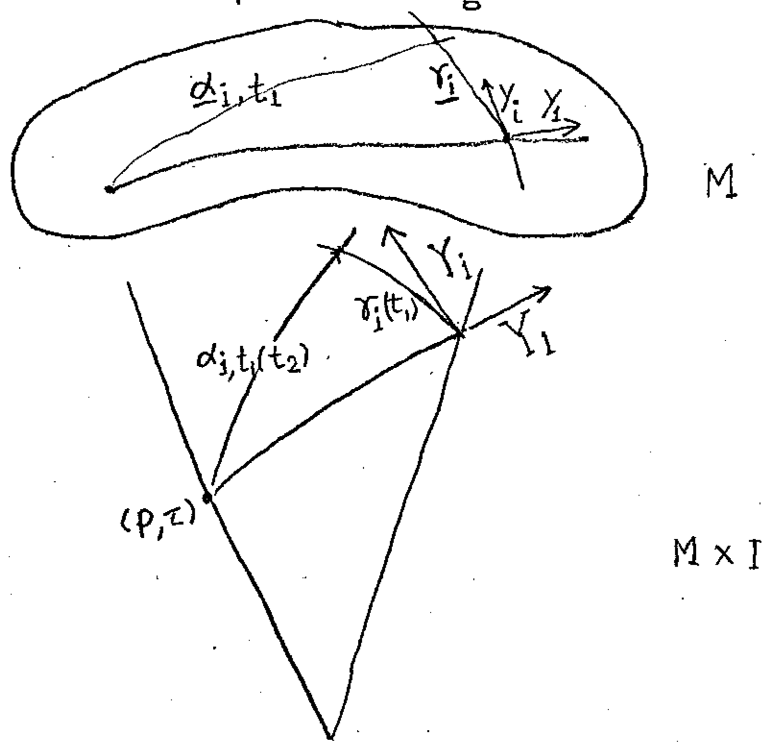
The other application is that if $h(0) = 0$, $h'(0) \neq 0$ we have a singularity ar $r = 0$. Consider $M \times I$ with the following metrics: $h^2(r)g + dr \otimes dr$ and $(h'(0)r)^2 g + dr \otimes dr$. We denote by d_k and $\tilde{d}_k$ the coefficients of asymptotic expansion of the trace of the heat kernels of $M \times I$ with the above metrics respectively. Then a direct calculation by first writing out d_k and $\tilde{d}_k$ according to the formula (5.27) then with the help of L'Hopital's Rule will give

$\lim_{r \to 0} \frac{d_k}{\tilde{d}_k} = 1$ when both d_k and $\tilde{d}_k$ are not zero. This gives us some information about the behavior of d_k and $\tilde{d}_k$ at the cone-like singularity.

Appendix

In this appendix we will give the calculation of the formula (6.1). Let c(t) be the geodesic on M x I with

$c(0) = (p,\tau)$ $\dot{c}(0) = \left(\frac{\sqrt{1-\lambda^2}}{\tau} E, \lambda\right)$ $|E|_g = 1$



then

$c(t) = (\underline{c}(\cos^{-1}(\frac{\tau+t\lambda}{\varkappa})), \varkappa)$ where $\underline{c}(s)$ is the geodesic on M with $\underline{c}(0) = p$, $\dot{\underline{c}}(0) = E$

$$\varkappa = \sqrt{t^2 + \tau^2 + 2\tau t\lambda}$$

$$\dot{c}(t) = \left(\dot{\underline{c}}(\cos^{-1}(\frac{\tau+t\lambda}{\varkappa})) \cdot \frac{\tau(1-\lambda^2)^{\frac{1}{2}}}{\varkappa^2}, \frac{t+\tau\lambda}{\varkappa}\right)$$

$$\dot{c}(0) = \left(\dot{\underline{c}}(0) \cdot \frac{(1-\lambda^2)^{\frac{1}{2}}}{\tau}, \lambda\right)$$

Let $Y_1 = \dot{c}(t)$

$$y_1 = \dot{\underline{c}}(\cos^{-1}(\frac{\tau+t\lambda}{\kappa})$$

We can find $y_2, \ldots y_n$ κ so that $y_1, y_2 \ldots y_n$ form an orthonormal base of $T_{\underline{c}(\cos^{-1}\frac{\tau+t\lambda}{\kappa})}M$ then $Y_1, Y_i = (y_i \frac{1}{\kappa}, 0)$ $i = 2\ldots n$ and $Y_{n+1} = (\dot{\underline{c}}(\cos^{-1}(\frac{\tau+t\lambda}{\kappa}))\cdot\frac{t+\tau\lambda}{\kappa^2}, \frac{\tau(1-\lambda^2)^{\frac{1}{2}}}{\kappa})$ form an orthonormal base of $T_{c(t)}(M \times I)$. Let γ_i be the geodesics on M with

$$\underline{\gamma_i}(0) = \mathring{\underline{c}}(\cos^{-1}(\frac{\tau+t\lambda}{\kappa}))$$

$$\dot{\underline{\gamma_i}}(0) = y_i \qquad i = 2\ldots n$$

then the geodesics with $\gamma_i(0) = c(t)$, $\dot{\gamma}_i(0) = Y_i$, $i = 2,\ldots n$ will be $\gamma_i(t_1) = (\underline{\gamma_i}(\cos^{-1}(\frac{\kappa}{\kappa_i}), \kappa_i)$ where $\kappa_i = \sqrt{t_1^2 + \kappa^2}$. Moreover, if γ_{n+1} is the geodesic with $\gamma_{n+1}(0) = c(t)$ $\mathring{\gamma}_{n+1}(0) = Y_{n+1}$, then

$$\gamma_{n+1}(t_1) = (\underline{\gamma}_{n+1}(\cos^{-1}(\kappa+t_1\circ\frac{\tau(1-\lambda^2)^{\frac{1}{2}}}{\kappa_{n+1}})), \kappa_{n+1})$$

where

$$\underline{\gamma}_{n+1}(0) = \underline{c}(\cos^{-1}\frac{\tau+t\lambda}{\kappa})$$

$$\dot{\underline{\gamma}}_{n+1}(0) = \dot{\underline{c}}(\cos^{-1}\frac{\tau+t\lambda}{\kappa})$$

$$\kappa_{n+1} = [t_1^2 + \kappa^2 + 2\tau t_1(1-\lambda^2)^{\frac{1}{2}}]^{\frac{1}{2}} .$$

Now let $\alpha_{i,t_1}(t_2)$ be the geodesic connecting (p,τ) and $\gamma_i(t_1)$ then

$$\alpha_{i,t_1}(t_2) = (\underline{\alpha}_{i,t_1}(\sin^{-1}(\frac{t_2\cdot\sqrt{t_1^2+k^2}\sin\ell_i(t_1)}{d_i(t_1)\rho_{i,t_1}(t_2)})),\ \rho_{i,t_1}(t_2))$$

where $\underline{\alpha}_{i,t_1}$ is the geodesic connecting p and $\underline{\gamma}_i(\cos^{-1}(\frac{\kappa}{\sqrt{t_1^2+k^2}}))$ on M, and $\ell_i(t_1)$ is the distance between them.

$$\rho_{i,t_1}(t_2) = [(1 - \frac{t_2}{d_i(t_1)})^2 \cdot \tau^2 + (\frac{t_2}{d_i(t_1)})^2(t_1^2+\kappa^2)$$

$$+ 2\frac{t_2}{d_i(t_1)}(1 - \frac{t_2}{d_i(t_1)})\ \tau\sqrt{t_1^2+\kappa^2}\ \cos\ell_i(t_1)]^{\frac{1}{2}}$$

$$d_i(t_1) = [\tau^2 + t_1^2 + \kappa^2 - 2\tau\sqrt{t_1^2+\kappa^2}\ \cos\ell_i(t_1)]^{\frac{1}{2}}$$

$$\dot{\alpha}_{i,t_1}(0) = (\dot{\underline{\alpha}}_{i,t_1}(0)\cdot\frac{\sqrt{t_1^2+\kappa^2}\ \sin\ell_i(t_1)}{d_i(t_1)\tau},$$

$$\frac{1}{d_i(t_1)}(-\tau+\sqrt{t_1^2+\kappa^2}\cdot(\kappa\ell_i)t_1))$$

$$\frac{d}{dt_1}d_i(t_1)\dot{\underline{\alpha}}_{i,t_1}(0)\Big|_{t_1=0}$$

$$= (\frac{d}{dt_1}\dot{\underline{\alpha}}_{i,t_1}(0)\Big|_{t_1=0}\cdot\frac{\kappa\sin(\cos^{-1}\frac{\tau+t\lambda}{\kappa})}{\tau},\ 0)$$

$$= (\frac{d}{dt_1}\dot{\underline{\alpha}}_{i,t_1}(0)\Big|_{t_1=0}\cdot\frac{t\sqrt{1-\lambda^2}}{\tau},\ 0)$$

since

$$\frac{d}{dt_1}\ell_i(t_1)\dot{\underline{\alpha}}_{i,t_1}(0)\Big|_{t_1=0} = \cos^{-1}(\frac{\tau+t\lambda}{\kappa})\frac{d}{dt_1}\dot{\underline{\alpha}}_{i,t_1}(0)\Big|_{t_1=0}$$

$$= \frac{d}{d\tilde{t}_1} \ell_1(t_1)\dot{\underline{\alpha}}_{1,t_1}(0)\Big|_{\tilde{t}_1=0} \cdot \frac{d\tilde{t}_1}{dt_1}\Big|_{t_1=0} = \frac{d}{d\tilde{t}_1} \ell_1(t_1)\dot{\underline{\alpha}}_{1,t_1}(0) \cdot \frac{1}{\kappa}$$

where $\tilde{t}_1 = \cos^{-1}(\frac{\kappa}{\sqrt{t_1^2+\kappa^2}})$

$$\therefore \frac{d}{dt_1} \dot{\underline{\alpha}}_{1t_1}(0)\Big|_{t_1=0} = \frac{1}{\cos^{-1}(\frac{\tau+t\lambda}{\kappa})} \circ \frac{1}{\kappa} \frac{d}{d\tilde{t}_1} \ell_1(t_1)\dot{\underline{\alpha}}_{1,t_1}(0) .$$

Therefore

$$\frac{d}{dt_1} d_1(t_1)\dot{\alpha}_{1,t_1}(0)\Big|_{t_1=0}$$

$$= (\frac{d}{d\tilde{t}_1} \ell_1(t_1)\dot{\underline{\alpha}}_{1,t_1}(0) \quad \frac{t\sqrt{1-\lambda^2}}{\kappa(\cos^{-1}(\frac{\tau+t\lambda}{\kappa}))\tau} , 0).$$

Now consider γ_{n-1}. Let $\alpha_{n-1,t_1}(t_2)$ be the geodesic connecting

(p,τ) and $(\underline{\gamma}_{n+1}(\cos^{-1}(\frac{\kappa+t_1 \cdot \frac{\tau(1-\lambda^2)^{\frac{1}{2}}}{\kappa}}{\kappa_{n+1}})), \kappa_{n+1})$

$$\alpha_{n+1,t_1}(t_2) = (\underline{\alpha}_{n+1,t1}(\sin^{-1} \frac{t_2 \circ \kappa_{n+1} \sin \ell_{n+1}(t_1)}{d_{n+1}(t_1) \ \rho_{n+1,t_1}(t_2)}), \ \rho_{n+1,t_1}(t_2))$$

where $\underline{\alpha}_{n+1,t_1}$ is the geodesic connecting p and

$\underline{\gamma}_{n+1}(\cos^{-1} \frac{\kappa+t_1 \cdot \tau(\frac{1-\lambda^2}{\kappa})^{\frac{1}{2}}}{\kappa_{n+1}})$ on M, and $\ell_{n+1}(t_1)$ is the distance

between them.

$$\rho_{n+1,t_1}(t_2) = [(1 - \frac{t_2}{d_{n+1}(t_1)})^2 \tau^2 + (\frac{t_2}{d_{n+1}(t_1)})^2 \kappa_{n+1}^2$$

$$+ 2 \frac{t_2}{d_{n+1}(t_1)} (1 - \frac{t_2}{d_{n+1}(t_1)}) \tau \kappa_{n+1} \cos \ell_{n+1}(t_1)]^{\frac{1}{2}}$$

$$d_{n+1}(t_1) = [\tau^2 + \kappa_{n+1}^2 - 2\tau \kappa_{n+1} \cos \ell_{n+1}(t_1)]^{\frac{1}{2}}$$

$$\dot{\alpha}_{n+1,t_1}(0)$$

$$= (\underline{\dot{\alpha}}_{n+1,t_1}(0) \cdot \frac{\kappa_{n+1} \sin \ell_{n+1}(t_1)}{\tau\, d_{n+1}(t_1)}, \frac{1}{d_{n+1}(t_1)}(-\tau + \kappa_{n+1} \cos \ell_{n+1}(t_1)))$$

$$d_{n+1}(t_1)\dot{\alpha}_{n+1,t_1}(0) = (\underline{\dot{\alpha}}_{n+1,t_1}(0) \frac{\kappa_{n+1} \sin \ell_{n-1}(t_1)}{\tau},$$

$$(-\tau + \kappa_{n+1} \cos \ell_{n+1}(t_1)))$$

$$\frac{d}{dt_1}(d_{n+1}(t_1)\dot{\alpha}_{n+1,t_1}(0))|_{t_1=0}$$

$$= (\underline{\dot{\alpha}}_{n+1,t_1}(0)|_{t_1=0} \frac{d}{dt_1} \frac{\kappa_{n+1} \sin \ell_{n+1}(t_1)}{\tau}|_{t_1=0},$$

$$\frac{d}{dt_1}(\kappa_{n+1} \cos \ell_{n+1}(t_1)))$$

$$= (\dot{\underline{c}}(0)(\frac{[\frac{1}{\varkappa}\tau(1-\lambda^2)^{\frac{1}{2}}\cdot\sin(\cos^{-1}\frac{\tau+t\lambda}{\varkappa})+\varkappa\cos(\cos^{-1}\frac{\tau+t\lambda}{\varkappa})}{\tau}$$

$$\frac{\cdot\sqrt{\frac{\varkappa^2-\tau^2(1-\lambda^2)}{\varkappa^2}}]}{}\quad, \frac{1}{\varkappa}\tau(1-\lambda^2)^{\frac{1}{2}}\cos\cdot\cos^{-1}(\frac{\tau t\lambda}{\varkappa})$$

$$-\varkappa\cdot\sin\cos^{-1}(\frac{\tau+t\lambda}{\varkappa})\cdot\sqrt{\frac{\varkappa^2-\tau^2(1-\lambda^2)}{\varkappa^2}})$$

$$= (\dot{\underline{c}}(0)\ \frac{\lambda(t^2+\tau^2)+2t\tau}{\varkappa^2\tau}\ ,\ \frac{(1-\lambda^2)^{\frac{1}{2}}(\tau^2-t^2)}{\varkappa^2})$$

Let $L_i = \left|\frac{d}{dt_1} d_i(t_1)\dot{\alpha}_{i,t_1}(0)\,|_{t_1=0}\right|$ $\qquad i = 2,\ldots n+1$

$$\underline{L}_i = \left|\frac{d}{dt_1} \ell_i(t_1)\dot{\underline{\alpha}}_{i,t_1}(0)\,|_{t_1=0}\right| \qquad i = 2\ldots n$$

then

$$L_i = \underline{L}_i\ \frac{t\sqrt{1-\lambda^2}}{\varkappa(\cos^{-1}(\frac{\tau+t\lambda}{\varkappa}))}$$

$$L_{n+1} = \frac{\sqrt{\tau^2(\lambda(t^2+\tau^2)+2t\tau)^2+(1-\lambda^2)(\tau^2-t^2)^2}}{\varkappa^2}$$

$$\prod_{i=2}^{n+1} L_i = (\prod_{i=2}^{n}\underline{L}_i)\cdot(\frac{t\sqrt{1-\lambda^2}}{\varkappa\cos^{-1}(\frac{\tau+t\lambda}{\varkappa})})^{n-1}\cdot L_{n+1}$$

$$\theta^{-1}((p,\tau),c(t)) = \underline{\theta}^{-1}\left(p,\underline{c}\left(\cos^{-1}\left(\frac{\tau+t\lambda}{\kappa}\right)\right)\circ\left(\frac{t\sqrt{1-\lambda^2}}{\kappa\cos^{-1}\left(\frac{\tau+t\lambda}{\kappa}\right)}\right)^{n-1}\right)$$

Recall that $\kappa = \sqrt{t^2+\tau^2+2t\lambda}$

$\lambda = \langle\mathring{c}(0),\frac{\partial}{\partial r}\rangle$ $\quad\langle\ \rangle$ represents the metric on M x I.

$\theta((p,\tau),(Q\kappa))$

$$= \underline{\theta}(p,Q)\left(\frac{\kappa\cos^{-1}\frac{\tau+t\lambda}{\kappa}}{t\sqrt{1-\lambda^2}}\right)^{n-1}$$

$$= \underline{\theta}(p,Q)\left(\frac{\ell}{\sin\ell}\right)^{n-1}$$ $\quad \ell = d(p,Q)$ on M.

since $\ell = \cos^{-1}\frac{\tau+t\lambda}{\kappa}$,

$$t\sqrt{1-\lambda^2} = \kappa\sin\ell \,.$$

Bibliography

(1) Berger, Gauduchon, Mazet: Le Spectre d'une Variété Riemannienne, Lecture Note in Mathematics 194. Barline-Heidelberg-New York: Spring-Verlag 1971.

(2) Cahn, Wolf: Zeta Functions and Their Asymptotic Expansions for Compact Symmetric Space of Rank One [preprint].

(3) Cheeger, Yau: A Lower Bound for the Heat Kernel [preprint].

(4) Cheeger: Spectral Geometry of Space with Cone-like Singularities [to appear].

(5) Doetsch: Introduction to the theory and application of the Laplace transformation. Spring-Verlag 1974.

(6) Dikii: Trace formula for Strum-Liouville Operator, Uspekhi Mat. Nauk 13:3 (1958).

(7) Gel'fand, Dikii: Asymptotic Behavior of The Resolvent of Strum-Liouville Equations and The Algebra of The Korteweg Vaies Equations. Russian Mathematical Surveys, vol. 30 No. 5, 1975.

(8) Gilkey: The Index Theorem and The Heat Equation.

(9) Lauwerier: Asymptotic analysis, Mathematical center tracts, Amsterdam 1977.

(10) Seeley: The index of elliptic systems of singular integral operator, J. Math. Anal. Appl. 7 (1963).

(11) Watson: A treatise on the theory of Bessel functions Cambridge University Press 1966.

(12) Zemanian: Generalized Integral Transformations, New York, Interscience Publishers 1968.