UNIFORMIZATION OF HYPERELLIPTIC SURFACES

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Abstract of the Dissertation

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Let $S$ be a closed Riemann surface of genus $g \geq 2$. Suppose $S$ is hyperelliptic and $J : S \to S$ is the hyperelliptic involution. Let $S$ be represented as $D/\Gamma$ where $D$ is the unit disc and $\Gamma$ is a fixed point free Fuchsian group of the first kind. Moreover, let $j(z) = -z$ be a lift of the involution to $D$.

We show that $\Gamma$ has a fundamental polygon (called a hyperelliptic polygon) $P \subset D$ with the following properties: $P$ is a $4g$ sided, simple, convex hyperbolic polygon which is $j$-invariant. The Weierstrass points of $S$ correspond to the midpoints of the sides of $P$, the vertices of $P$ and the origin.

Let $T(\Gamma)$ be the Teichmüller space of $\Gamma$. It is known that
$T(\Gamma)$ is a complex manifold of dimension $3g-3$. For each $[\mu] \in T(\Gamma)$ there exists a well defined Jordan domain $D_\mu \subset \mathbb{C} \cup \{\infty\}$ and a quasi-Fuchsian group $\Gamma^\mu$ acting on $D_\mu$ such that $D_\mu/\Gamma^\mu \equiv S_\mu$ is a Riemann surface of genus $g$. The map $j$ acts in a natural manner on $T(\Gamma)$ and the fixed point set $T(\Gamma)^j$ corresponds to a component of the hyperelliptic locus in $T(\Gamma)$.

The Bers fiber space $F(\Gamma) = \{([\mu], z) \in T(\Gamma) \times \mathbb{C} \cup \{\infty\}; [\mu] \in T(\Gamma)$ and $z \in D_\mu\}$ is also a complex manifold of dimension $3g-2$. For each $[\mu] \in T(\Gamma)^j$ we show the existence of a canonical fundamental polygon $P_\mu^j \subset D_\mu$ for $\Gamma^\mu$. The vertices of $P_\mu^j$ lie on holomorphic sections of $F(\Gamma)$ over $T(\Gamma)^j$.

Using Teichmüller theory we compute the polynomial of the surface corresponding to a specific hyperelliptic polygon.

We then study topological properties of dissections of hyperelliptic surfaces of the type given by hyperelliptic polygons. We consider the possibility of prescribing for a surface of genus 2 (necessarily, hyperelliptic) the periods of holomorphic differentials on half the loops of such a dissection.
To my parents.
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I. **INTRODUCTION**

Canonical polygons for finitely generated Fuchsian groups were first studied by Fricke (1920's) and later by Keen (1966). In this thesis we study a different type of polygon which is canonical for Fuchsian groups whose orbit space is a hyperelliptic Riemann surface. The involution of the surface is represented by a symmetry of the polygon and the Weierstrass points are displayed symmetrically. We call such a polygon a **hyperelliptic polygon**.

It turns out that hyperelliptic polygons were first considered by Whittaker [18] in 1899. Whittaker's proofs however, are incomplete and his treatment relies mainly on specific examples.

By the use of quasiconformal maps, in particular the Ahlfors-Bers solution to the mapping problem for a given dilatation (and the continuity of that solution when the dilatation depends linearly on a real parameter $t$), we are able to prove the existence of hyperelliptic polygons for all hyperelliptic surfaces.

In Chapter II, after establishing the preliminaries, we count real parameters for hyperelliptic polygons. Their number coincides with the dimension of the Teichmüller space of hyperelliptic surfaces.

In II.4 we prove the existence of a simple, convex hyperelliptic polygon for any given hyperelliptic surface. This
polygon gives a specific set of generators for the Fuchsian group which uniformizes the hyperelliptic surface.

In Chapter III we look at uniformizations of hyperelliptic surfaces by quasi-Fuchsian groups and find similar fundamental polygons. In fact we show that if the quasi-Fuchsian group belongs to the identity component of the hyperelliptic locus in the Teichmüller space, then the vertices of the polygon lie on holomorphic sections of the Bers fiber space over the Teichmüller space. Finally, using a theorem of Kravetz, we compute the polynomial of the surface corresponding to a specific hyperelliptic polygon.

In Chapter IV we consider topological dissections of hyperelliptic surfaces of the type given by hyperelliptic polygons. We compute the intersection matrix of such a dissection and establish a relation among periods of two $C^1$ closed differentials.

In the last two sections we consider the possibility of prescribing for a surface of genus 2 (necessarily, hyperelliptic), the periods of holomorphic differentials on half the loops of such a dissection.
II. UNIFORMIZATION AND HYPERELLIPTIC SURFACES

II.1 Preliminaries. We will work with groups $\Gamma$ whose elements are Mobius transformations $T(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$. Hence the elements of $\Gamma$ are conformal self maps of the extended complex plane $\mathbb{C} \cup \{\infty\}$.

If $z \in \mathbb{C} \cup \{\infty\}$, let $\Gamma_z = \{ \gamma \in \Gamma; \gamma(z) = z \}$. We say $\Gamma$ is discontinuous at $z$ if

i) $\Gamma_z$ is finite, and

ii) there is a neighborhood $U$ of $z$ such that $\gamma(U) = U$ for all $\gamma \in \Gamma_z$, and $\gamma(U) \cap U = \emptyset$ for all $\gamma \in \Gamma - \Gamma_z$.

Let $\Omega = \Omega(\Gamma) = \{ z \in \mathbb{C} \cup \{\infty\}; \Gamma \text{ is discontinuous at } z \}$. We call $\Omega$ the region of discontinuity of $\Gamma$ and say that $\Gamma$ is discontinuous if $\Omega \neq \emptyset$. We call $\Lambda = \Lambda(\Gamma) = \mathbb{C} \cup \{\infty\} - \Omega$ the limit set.

It can be shown (Ford [9]) that card $\Lambda = 0, 1, 2$ or $\infty$.

If card $\Lambda \leq 2$ then $\Gamma$ is called an elementary group. If $\Omega$ is not empty and card $\Lambda > 2$ then $\Gamma$ is called a (non-elementary) Kleinian group.

If there exists a component $\Xi$ of the region of discontinuity of a Kleinian group $\Gamma$ which is invariant (i.e. $\gamma(\Xi) = \Xi$ for all $\gamma \in \Gamma$), then $\Gamma$ is called a function group.

If $\Gamma$ is a Kleinian group the projection $p : \Omega \rightarrow \frac{\Omega}{\Gamma}$ onto the set of orbits of $\Gamma$ in $\Omega$ induces a unique conformal structure on $\frac{\Omega}{\Gamma}$. 

Let $S$ be a Riemann surface and $\Gamma$ a function group with invariant component $\Sigma$. If $\Gamma$ acts freely on $\Sigma$ and $\frac{\Sigma}{\Gamma} = S$ we say that $\Gamma$ uniformizes $S$.

Let $\Delta$ be an open subset of $\mathbb{C} \cup \{\infty\}$ that is invariant under a discontinuous group $\Gamma$ and $\Delta \subset \Omega$. By a fundamental domain $R$ for $\Gamma$ in $\Delta$ we will mean an open subset of $\Delta$ with the following properties:

i) whenever $yz_1 = z_2$ for some $y \in \Gamma$, $z_1, z_2 \in R$ then $y = \text{id}$.

ii) every point $z \in \Delta$ is equivalent to a point of $\overline{R}$ ($\overline{R}$ is the closure of $R$).

iii) the two dimensional Lebesgue measure of $\overline{R} - R$ is zero.

If $\Gamma$ is a Kleinian group and there is a circle $C \subset \mathbb{C} \cup \{\infty\}$ such that the interior of $C$ is fixed by $\Gamma$, then $\Gamma$ is called Fuchsian. In this case $\Lambda \subset C$ and $\Gamma$ is called of the first kind if $\Lambda = C$, otherwise it is of the second kind (see [12]).

We will consider only Fuchsian groups of the first kind which leave the open unit disc $D$ fixed. We note that if $T(z)$ is a Mobius transformation belonging to such a group then $T(z) = \frac{az + b}{bz + a}$ and $|a|^2 - |b|^2 = 1$ (see [9]). We denote by $\text{Aut } D$ the group of all Mobius transformations which fix $D$.

By Koebe's uniformization theorem [17] every compact surface $S$ of genus $g \geq 2$ may be obtained via a fixed point
free Fuchsian group $\Gamma$ as $\frac{D}{\Gamma}$.

We will assume throughout that $g \geq 2$. (For uniformizations of surfaces of genus 0 or 1 see [11].) A Riemann surface $S$ of genus $g$ is hyperelliptic if there exists a conformal self map $J : S \to S$ such that $J^2 = \text{id}$, and $J$ has $2g + 2$ fixed points. It is known that every hyperelliptic surface corresponds to the surface of a polynomial of two variables $z$ and $w$: 
\[ w^2 = \prod_{i=1}^{2g+2} (z - a_i) \]
where the $a_i \in \mathbb{C}$ are distinct (see [17]).

There are various other characterizations of hyperelliptic surfaces (see [17]). The only ones we will use are the existence of an involution with $2g + 2$ fixed points or equivalently the existence of a representation of the surface as a twofold cover of the sphere branched at $2g + 2$ points.

The Poincaré metric on the unit disc is defined by
\[ ds = \frac{2 \lvert \text{d}z \rvert}{1 - \lvert z \rvert^2}. \]
In this metric geodesics are arcs of circles orthogonal to the unit circle (including straight lines through the origin).

Let $a_1, \ldots, a_{n+1}$, $n > 1$, be a set of points such that $a_j \neq a_k$, $1 \leq j, k \leq n$ and $a_{n+1} = a_1$, which are joined together (in order) by Jordan arcs $A_1, \ldots, A_n$. By a polygon $P$ we will mean the closed curve formed by the arcs $A_1, \ldots, A_n$. The points $a_1, \ldots, a_n$ are called the vertices of $P$ and the arcs $A_1, \ldots, A_n$ are called the sides of $P$.

If a polygon $P$ is contained in $D$ and its sides are hyperbolic line segments then we say $P$ is a hyperbolic polygon.
A hyperbolic polygon is **simple** if no two non-adjacent sides intersect and adjacent sides intersect only at a vertex. For a simple hyperbolic polygon \( P \), it is easy to construct a homeomorphism from the unit circle onto \( P \). Thus \( P \) is a Jordan curve and we denote by \( P^0 \) that component of the complement of \( P \) which is contained in \( D \). We say a simple hyperbolic polygon is a **fundamental polygon** for a Fuchsian group \( \Gamma \) if \( P^0 \) is a fundamental domain for \( \Gamma \) in \( D \).

A simple hyperbolic polygon is called **convex** if any two points \( z_1, z_2 \in P^0 \cup P \) may be joined by a geodesic line segment which is contained in \( P^0 \cup P \).

We will work with simple hyperbolic polygons \( P \subset D \) which have \( 4g \) sides and satisfy the following properties:

1) \( P \) is invariant under the transformation \( j(z) = -z \).

2) (We denote the sum of the interior angles of \( P \) by \( \mathcal{P} \).) \( \mathcal{P} = 2\pi \).

3) If we give \( P \) the counterclockwise orientation and label the sides \( A_1, \ldots, A_{4g} \) (in counterclockwise order) then there exist Mobius transformations \( T_i \in \text{Aut} D \), \( i = 1, \ldots, 2g \), such that \( T_i(A_1) = A_{2g+1} \) with the reverse orientation. (Note that by the invariance property \( A_{2g+1} = j(A_i), i = 1, \ldots, 2g \).

We call such a polygon a **hyperelliptic polygon** (Fig. 1).

In what follows we will prove that every hyperelliptic surface may be uniformized by a Fuchsian group \( \Gamma \) which has

---

\( j(A_i) \) refers to the point set with no implied orientation.
a hyperelliptic polygon as a fundamental polygon.

II.2 Poincaré's Theorem. We will need the following theorem of Poincaré to prove that every hyperelliptic polygon is a fundamental polygon for a Fuchsian group which uniformizes a hyperelliptic surface. Since such polygons lie in $D$, we will omit consideration of polygons with vertices on the unit circle.

**Poincaré's Theorem.** Let $P \subset D$ be a simple, hyperbolic polygon which satisfies the following: 2

1) the sides of $P$ are identified in pairs by elements of $\text{Aut } D$ which generate a group $G$.

ii) the vertices of $P$ distribute themselves into sets of $G$-equivalent vertices which we call cycles, and the sum of the angles of a cycle is equal to $2\pi /\nu$, where $\nu$ is a positive integer.

Then $G$ is a Fuchsian group and $P$ is a fundamental polygon for $G$. (For a proof see [15].)

In order to apply Poincaré's theorem to hyperelliptic polygons we need to verify condition ii).

**Lemma II.2.1.** Let $P$ be a hyperelliptic polygon. Then there is one equivalence class of vertices.

**Proof.** Let $A_1$ be the first side of the polygon. We label its vertices $a_1$ and $a_2$ and denote the oriented hyperbolic line joining $a_1$ to $a_2$ by $<a_1,a_2>$. We label the rest of the vertices in counterclockwise order. Thus $A_1 = <a_1,a_{l+1}>$,
$i = 1, \ldots, 4g-1$, and $A_{4g} = \langle a_{4g}, a_1 \rangle$ (see Fig. 1). (Note that $-a_1 = a_{2g+1}, i = 1, \ldots, 2g$.)

![Diagram](image.png)

Fig. 1

Let $T = \prod_{n=0}^{2g-2} \epsilon_n$, where $\epsilon_n = \begin{cases} 1 & n \text{ odd} \\ -1 & n \text{ even} \end{cases}$. By tracing the identifications in Fig. 1, we obtain that $T(a_1) = a_2$.

Thus $a_1$ is equivalent to $a_2$. By relabeling the transformations suitably we obtain that $a_i$ is equivalent to $a_{i+1}$, $i = 1, \ldots, 4g-1$. Thus all vertices are equivalent.

**Corollary II.2.2.** Let $P$ be a hyperelliptic polygon. Then the group $T$ generated by the $T_i$, $i = 1, \ldots, 2g$, is a Fuchsian group and $P$ is a fundamental polygon for $T$.

**Proof.** Lemma II.2.1 and Poincaré's theorem yield the

---

3 When multiplying two Mobius transformations $R$ and $S$ we will use the notation $S \circ R$ to mean $R$ followed by $S$.

Thus $\prod_{n=0}^{m} S_n = S_m \circ \cdots \circ S_1 \circ S_0$. 


desired result.

Let \( P \) be a hyperelliptic polygon. By Corollary II.2.2, the group generated by the \( T_i, \ i = 1, \ldots, 2g \), is Fuchsian. We call \( \Gamma \) the group associated to \( P \).

**Lemma II.2.3.** Let \( P \) be a hyperelliptic polygon with side pairing transformations \( T_i, \ i = 1, \ldots, 2g \). Then

\[
T_i(z) = \frac{\alpha_i z + \beta_i}{\bar{\beta}_i z + \alpha_i}
\]

where \( |\alpha_i|^2 - |\beta_i|^2 = 1 \) and \( \alpha_i \in \mathbb{R} \).

**Proof.** Each \( T_i \) leaves \( D \) fixed and is thus of the form

\[
T_i(z) = \frac{\alpha_i z + \beta_i}{\bar{\beta}_i z + \alpha_i}.
\]

Since \( T_i(A_i) = j(A_i) \) with the reverse orientation, \( T_i(a_i) = -a_{i+1} \) and \( T_i(a_{i+1}) = -a_i \), where \( a_i \) and \( a_{i+1} \) are the vertices of \( A_i = \langle a_i, a_{i+1} \rangle \) and \( a_i \neq a_{i+1} \).

Therefore

\[
T_i(a_i) = \frac{\alpha_i a_i + \beta_i}{\bar{\beta}_i a_i + \alpha_i} = -a_{i+1}
\]

and

\[
T_i(a_{i+1}) = \frac{\alpha_i a_{i+1} + \beta_i}{\bar{\beta}_i a_{i+1} + \alpha_i} = -a_i
\]

and we obtain the following equations

\[
(II.2.1) \quad -a_i (\bar{\beta}_i a_{i+1} + \alpha_i) = \alpha_i a_{i+1} + \beta_i
\]

\[
(II.2.2) \quad -a_{i+1} (\bar{\beta}_i a_i + \alpha_i) = \alpha_i a_i + \beta_i
\]

Subtracting \((II.2.1)\) from \((II.2.2)\) we get

\[
(a_i - a_{i+1}) \bar{\alpha}_i = \alpha_i (a_i - a_{i+1}). \quad \text{Since} \ a_i - a_{i+1} \neq 0, \ \bar{\alpha}_i = \alpha_i \text{ and } \alpha_i \in \mathbb{R}.
\]

**Lemma II.2.4.** Under the same hypothesis of Lemma II.2.3, \( j \) conjugates each generator \( T_i \) to its inverse. Moreover,
\[
T_i \cdot o_j = \begin{cases} \epsilon_{i_1} \epsilon_{i_2} \cdots \epsilon_{i_n} & \text{if } l \leq i_k \leq 2g, k \in \mathbb{N}^+ \text{ and } \epsilon_{i_k} = \mp 1.
\end{cases}
\]

**Proof.** We use induction on \( n \). If \( \epsilon_{i_1} = 1 \) then by Lemma II.2.3

\[
T_{i_1}(z) = \frac{\alpha_{i_1} z + \beta_{i_1}}{\beta_{i_1} z + \alpha_{i_1}} \quad \text{where } \alpha_{i_1} \in \mathbb{R}.
\]

Thus

\[
T_{i_1} o_j(z) = T_{i_1}(-z) = \frac{- \alpha_{i_1} z + \beta_{i_1}}{- \beta_{i_1} z + \alpha_{i_1}} = \frac{- (\alpha_{i_1} z + \beta_{i_1})}{\beta_{i_1} z - \alpha_{i_1}}
\]

\[
= - T_{i_1}^{-1}(z) = j o T_{i_1}^{-1}(z)
\]

and \( j \) conjugates \( T_{i_1} \) to its inverse.

Assume \( T_i \cdot o_j = j o T_i \cdot o_j \). Let

\[
T = T_{i_1}^{-1} \cdots T_{i_n}^{-1} \quad \text{and } T^* = T_{i_1}^{-1} \cdots T_{i_n}^{-1}.
\]

If \( \epsilon_{i_1} = 1 \) then again using Lemma II.2.3 we obtain

\[
(\text{II.2.3}) \quad T_i (j o T^*(z)) = T_i (-T^*(z)) = - T_i^{-1} (T^*(z)) = j o T_i^{-1} o T^*(z).
\]

Since \( T o j = j o T^* \), \( T_i (j o T^*(z)) = T_i \cdot o_j (z) = T_i \cdot o_{i_1} \cdots o_{i_n} (z) \).

Now from (II.2.3) we obtain
\[ T_{n}^{\ldots} T_{1}^{\ldots} o j(z) = j o T_{n}^{\ldots} T_{1}^{\ldots} o j(z) = j o T_{n}^{\ldots} T_{1}^{\ldots} o j(z) \]

which is the desired result. If \( \varepsilon_{i} = -1 \) a similar computation

yields \[ T_{n}^{\ldots} T_{1}^{\ldots} o j(z) = j o T_{n}^{\ldots} T_{1}^{\ldots} o j(z). \]

Corollary II.2.5. Let \( P \) be a hyperelliptic polygon with associated Fuchsian group \( \Gamma \). Then \( \frac{D}{\Gamma} \) has an involution \( J \).

Proof. Define \( J : \frac{D}{\Gamma} \to \frac{D}{\Gamma} \) by \( J([z]) = [j(z)] \). To show that \( J \) is well defined suppose \( [z_{0}] = [z_{1}] \). Then there exists \( a \in \Gamma \) such that \( T(z_{0}) = z_{1} \) and by Lemma II.2.4 \( j o T(z_{0}) = T^{*} o j(z_{0}) \) where \( T^{*} \in \Gamma \). Thus \( J([z_{0}]) = J([z_{1}]) \).

In order to prove that \( \frac{D}{\Gamma} \) is hyperelliptic we will show that the involution \( J \) of Corollary II.2.5 has \( 2g + 2 \) fixed points. Trivially \( [o] \) is a fixed point. Since the vertices \( a_{i}, i = 1, \ldots, 4g \), are equivalent and their orbit \( \{a_{i}\} \) contains the negative of each vertex, \( \{a_{i}\} \) is a fixed point of \( J \). We will show that the hyperbolic midpoints \( c_{i} \) of the sides \( A_{i}, i = 1, \ldots, 2g \), are also fixed points.

Lemma II.2.6. Let \( A \) be a Mobius transformation of the form \( A(z) = \frac{az + \beta}{\beta \bar{z} + \alpha} \) where \( |\alpha|^{2} - |\beta|^{2} = 1, \beta \neq 0 \) and \( \alpha \in \mathbb{R} \). Then \( A(z) = -z \) has a unique solution \( z_{0} \in D \).

Proof. The solutions to \( \frac{az + \beta}{\beta \bar{z} + \alpha} = -z \) are \( \frac{-\alpha \bar{z} + 1}{\beta} \).

\[ \]
\[ \frac{-\alpha + 1}{\beta} = \sqrt{\frac{\alpha^2 - 2\alpha + 1}{\alpha^2 - 1}} \text{ (resp. } \frac{-\alpha - 1}{\beta} = \sqrt{\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 - 1}} \text{) and since} \]
\[ \alpha > 1, \ \alpha^2 - 1 > \alpha^2 - 2\alpha + 1 \text{ (resp. } \alpha^2 + 2\alpha + 1 > \alpha^2 - 1). \text{ Thus} \]
\[ 1 > \frac{\alpha^2 - 2\alpha + 1}{\alpha^2 - 1} \text{ (resp. } \frac{\alpha^2 + 2\alpha + 1}{\alpha^2 - 1} > 1 \text{) and } 1 > \frac{-\alpha + 1}{\beta} \text{ (resp.} \]
\[ 1 < \frac{-\alpha - 1}{\beta} \).} \]

The generating transformations \( T_i, \ i = 1, \ldots, 2g, \) of a group \( \Gamma \) associated to a hyperelliptic polygon are of the form
\[ T_i(z) = \frac{a_i z + b_i}{c_i z + d_i}, \text{ where } a_i \in \mathbb{R}. \text{ By Lemma II.2.6, } T_i(z) = -z \text{ has a unique solution } c_i \in \mathbb{D}. \{c_i\} \text{ is thus a fixed point of } J. \text{ To show that the } \{c_i\}, \ i = 1, \ldots, 2g, \text{ are distinct we need to know that they are not congruent modulo } \Gamma. \]

**Lemma II.2.7.** Let \( T_i \) map \( A_i = \langle a_i, a_{i+1} \rangle \) onto \( \langle -a_i, -a_{i+1} \rangle \) and let \( c_i \) be the hyperbolic midpoint of \( \langle a_i, a_{i+1} \rangle \). Then \( T_i(c_i) = -c_i \).

**Proof.** Since \( c_i \) is the hyperbolic midpoint of \( A_i \) (see Fig. 1)
\[ \int_{a_i}^{c_i} \frac{|dz|}{1-|z|^2} = \int_{c_i}^{a_{i+1}} \frac{|dz|}{1-|z|^2} \text{ where the line integral is along the unique geodesic joining the limits of integration. But the Poincaré metric is invariant under the transformations } j \text{ and } T_i. \text{ Hence} \]
\[ \int_{-c_i}^{c_i} \frac{|dz|}{1-|z|^2} = \int_{a_i}^{a_i+1} \frac{|dz|}{1-|z|^2} = \int_{T_i(a_i+1)}^T \frac{|dz|}{1-|z|^2}. \]

Thus \[ \int_{-c_i}^{c_i} \frac{|dz|}{1-|z|^2} = \int_{T_i(a_i+1)}^T \frac{|dz|}{1-|z|^2} = \int_{-a_i}^{a_i} \frac{|dz|}{1-|z|^2} \]

and \( T_i(c_i) = -c_i. \)

We have shown that the \( c_i, 1 = 1, \ldots, 2g, \) lie on a fundamental polygon for \( \Gamma. \) They are distinct since the sides \( A_i \) intersect only at vertices and none of the \( c_i \) is a vertex.

Since \( P \) is a fundamental polygon and identifications occur in pairs it is obvious that the \( c_i \) are not congruent. Similarly \( O \) is not congruent to any of the vertices or the hyperbolic midpoints and none of the hyperbolic midpoints is congruent to a vertex.

Thus \( \{0\}, \{a_i\}, \{c_i\}, \ldots, \{c_{2g}\} \) are distinct fixed points of \( J. \) Since it is well known that a conformal automorphism of a compact surface of genus \( g \) has at most \( 2g+2 \) fixed points, these are all the fixed points of \( J. \)

We have proven the following:

**Lemma II.2.8.** Let \( P \) be a hyperelliptic polygon with associated Fuchsian group \( \Gamma. \) Then \( \overline{D / \Gamma} \) is hyperelliptic and the fixed points of the hyperelliptic involution are \( \{0\}, \{a_1\} \) where \( a_1 \) is the first vertex, and \( \{c_i\}, 1 = 1, \ldots, 2g, \) where \( c_i \) is the hyperbolic midpoint of the side \( A_i. \)
II.3. **Parameters.** Before proving that any hyperelliptic surface of genus $g$ may be uniformized by a fixed point free Fuchsian group which has a hyperelliptic fundamental polygon, one would like to know that the number of parameters which determine hyperelliptic polygons agrees with the dimension of the Teichmüller space of hyperelliptic surfaces (see Chapter III).

In this section we informally construct a space of hyperelliptic polygons which is disconnected and not well defined. Our construction will show that this space depends on $4g-2$ real parameters. On the other hand, the Teichmüller space of hyperelliptic surfaces is a complex manifold of dimension $2g-1$.

Let $R_1, \ldots, R_{2g}$ be rays through the origin such that

\[ \sum_{i=1}^{2g-1} \alpha_i < \pi, \text{ where } \alpha_i > 0, \quad i = 1, \ldots, 2g-1, \]

is the angle between $R_i$ and $R_{i+1}$. Clearly there are $2g-1$ real parameters which determine this construction. We let $R_1 = R$, the real axis (Fig. 2).

We now fix a choice of rays $R_1, \ldots, R_{2g}$ and construct a $2g-1$ real parameter family of polygons with vertices on the given rays satisfying properties i) and ii) of II.1, page 6.

Let $t_i$ and $-t_i$, $i = 1, \ldots, 2g$, be the points of intersection of $R_i$ with $|z| = 1$. Let $L_1$ be the arc of a circle orthogonal to $|z| = 1$ which joins the points $-t_{2g}$ and $t_2$ and lies inside $D$. (We call $L_1$ the hyperbolic line joining $-t_{2g}$ and $t_2$ (Fig. 3).)
\( L_1 \) intersects \( R_1 \) at a point \( r_1, 0 < |r_1| < 1. \)

Choose \( a_1 \in R_1 \) such that \( r_1 < a_1 < 1 \) and construct the polygon \( P_1 = \langle a_1, t_2, t_3, \ldots, t_{2g}, -a_1, -t_2, \ldots, -t_{2g}, a_1 \rangle \) (Fig. 4).

\((z_1, z_2, \ldots, z_n, z_1) \) denotes the hyperbolic polygon formed by joining the points \( z_i, i = 1, \ldots, n, \) in their given order by hyperbolic lines.) Each angle at the vertices of \( P_1 \) is 0 except for the angle \( \beta_1 \) at the vertices \( a_1 \) and \( -a_1 \). Since \( r_1 < a_1 < 1, \beta_1 < \pi \) and \( P_1 \) is less than \( 2\pi. \)

Let \( L_2 \) be the hyperbolic line joining \( a_1 \) to \( t_3 \) and let \( t'_2 \) be the point of intersection of \( L_2 \) with \( R_2. \) Let

\[ P'_1 = \langle a_1, t'_2, t_3, \ldots, t_{2g}, -a_1, -t'_2, -t_3, \ldots, -t_{2g}, a_1 \rangle \] (Fig. 5).

\( \hat{A}_1 > 2\pi \) since the angle \( \beta_2 \) at the vertices \( t'_2 \) and \( -t'_2 \) is equal to \( \pi. \)

For each choice of \( r \in R_2 \) such that \( |t'_2| < |r| < |t_2| \)

we obtain a polygon

\[ P^r_1 = \langle a_1, r, t_3, \ldots, t_{2g}, -a_1, -r, -t_3, \ldots, -t_{2g}, a_1 \rangle \] and \( P^{t_2}_1 = P_1, \)

\[ P^t_1 = P'_1 \] (Fig. 6).

As \( r \) varies from \( |t'_2| \) to \( |t_2| \) along \( R_2, \hat{A}_1 \) is strictly decreasing from \( k_1 > 2\pi \) to \( 2\beta_1 < 2\pi. \) (This last fact may be shown by isolating the triangles \( \langle a_1, r, t_3, a_1 \rangle \). As \( r \) varies the sum of the interior angles of \( \langle a_1, r, t_3, a_1 \rangle \) is strictly decreasing since the hyperbolic area of \( \langle a_1, r, t_3, a_1 \rangle \) is strictly increasing (Fig. 7).)

---

The continuity of \( \hat{A}_1 \) will be shown in Lemma II.4.15.
Thus for some \( r_2 \in R_2 \), \( |t_2| < |r_2| < |t_2| \), \( P_1^\Omega = 2\pi \) and \( P_1 < 2\pi \) for all \( r \in R_2 \) such that \( |r_2| < |r| < |t_2| \).

Choose \( a_2 \in R_2 \) such that \( |r_2| < |a_2| < |t_2| \).

We now repeat the same process to find an interval from which to choose \( a_3 \in R_3 \).
Let $L_3$ be the hyperbolic line joining $a_2$ to $t_4$ and let $t_3^1$ be the point of intersection of $L_3$ with $R_3$. Construct the polygons $P_2^1 = \langle a_1, a_2, t_3^1, t_4, \ldots, t_2^{g'}, -a_1, -a_2, -t_3^1, \ldots, -t_2^{g'}, a_1 \rangle$ and $P_2 = \langle a_1, a_2, t_3, t_4, \ldots, t_2^{g'}, -a_1, -a_2, -t_3, -t_4, \ldots, -t_2^{g'}, a_1 \rangle$.

For each $r \in R_3$ such that $|t_3^1| < |r| < |t_3|$ let $P_2^r = \langle a_1, a_2, r, t_4, \ldots, t_2^{g'}, -a_1, -a_2, -r, -t_4, \ldots, -t_2^{g'}, a_1 \rangle$.

As in the previous case there is an $r_3 \in R_3$, $|t_3^1| < |r_3| < |t_3|$, such that $P_2^{r_3} = 2\pi$; and for all $r \in R_3$ such that $|r_3| < |r| < |t_3|$, $P_2^r < 2\pi$ (Fig. 8).

We continue this process and choose points $a_i \in R_1$, $i = 1, \ldots, 2g-1$, such that $P_2^{a_2^{g-1}} < 2\pi$ (Fig. 9). Let $L_2^{g}$ be the
Fig. 8

Fig. 9
hyperbolic line joining \(a_{2g-1}\) to \(-a_1\) and let \(t_{2g}\) be the intersection of \(L_{2g}\) with \(R_g\). As before there exists a uniquely determined point \(a_{2g} \in R_{2g}\) such that \(P_{2g-1} = 2\pi\).

Thus we have constructed a \(4g-2\) real parameter family of polygons satisfying properties i) and ii).

As it stands we need to know the existence of Mobius transformations pairing opposite sides of each polygon (property iii)) before we can claim they are hyperelliptic polygons.

The following Lemma shows that they all satisfy property iii).

**Lemma II.3.1.** Let \(a, b \in D - \{0\}, a \neq b\). Then there exists a unique Mobius transformation \(A\) which fixes \(D\) such that \(A(a) = -b\) and \(A(b) = -a\).

**Proof.** Since both determinants

\[
D_1 = \begin{vmatrix} \frac{1-|b|^2}{1-|a|^2} \end{vmatrix} > 0 \quad D_2 = \begin{vmatrix} \frac{1-|b|^2}{1-|a|^2} & \frac{1-b\bar{a}}{1-\bar{a}b} \\ \frac{1-b\bar{a}}{1-a\bar{b}} & \frac{1-|a|^2}{1-|b|^2} \end{vmatrix} = 0
\]

are \(\geq 0\), we have sufficient conditions for the existence of an analytic map \(A\) with the desired properties (Ahlfors [1]).

By Pick's lemma and the equality \(\frac{|A(a)-A(b)|}{|1-A(a)A(b)|} = \frac{|a-b|}{|1-ab|}\), \(A\) is fractional linear (Ahlfors [1]).
To prove uniqueness let B be another transformation with the same properties. By Lemma II.2.7 both A and B map the hyperbolic midpoint of the line joining a to b onto the hyperbolic midpoint of the line joining -a to -b. Thus $B^{-1}A$ fixes the line joining a to b and $B^{-1}A = \text{id}$.

We include a geometric proof of the existence part of Lemma II.3.1.

Proof. We will first construct a transformation $T \in \text{Aut D}$ such that $T(a) = r > 0$ and $T(b) = -r$. Let $c$ be the hyperbolic midpoint of the line segment $<a, b>$ and let $A_1 \in \text{Aut D}$ be such that $A_1(c) = 0$ (see Ford [9]). $A_1$ maps the line segment $<a, b>$ onto a line segment through the origin with zero as midpoint. Let $A_2$ be a rotation about the origin such that

$$A_2 \circ A_1(a) = r > 0 \text{ and } A_2 \circ A_1(b) = -r.$$ Let

$$T = A_2 \circ A_1 \text{ and } A = j_0T^{-1}ojoT.$$ Now $A(a) = j_0T^{-1}ojoT(a) = j_0T^{-1}(-r) = -b$.

Similarly $A(b) = -a$ and trivially $A \in \text{Aut D}$.

II.4. Continuity argument. Let $S$ be a surface of genus $g$ with a hyperelliptic involution $J$ and let $w_1, \ldots, w_{2g+2}$ be the Weierstrass points of $S$. We denote this data by the symbol $(S, J, w_1, \ldots, w_{2g+2})$.

In this section we will construct a hyperelliptic polygon
and, by means of quasiconformal maps, deform the polygon to obtain a new hyperelliptic polygon whose associated Fuchsian group uniformizes $S$.

We begin by defining quasiconformal maps. A homeomorphism $w$ of the open unit disc $D$ onto itself is quasiconformal (or $\mu$-conformal) if it has generalized locally square integrable derivatives which satisfy the equation $\frac{w_z}{z} = \mu(z)w_z$ almost everywhere on $D$, where $\mu$ is a measurable complex valued function on $D$ with $\text{ess. sup.} |\mu(z)| = k < 1$.

We say $\mu$ is a Beltrami coefficient for a Fuchsian group $\Gamma$ if $\mu_0A = \frac{A_1}{A}$ for all $A \in \Gamma$.

The following theorem will be needed.

**Theorem II.4.1 (Ahlfors-Bers [3])**. Let $\mu$ be a measurable complex valued function on $D$ with $\text{ess. sup.} |\mu(z)| = k < 1$. Then there exists a unique quasiconformal self map $w^\mu$ of $D$ satisfying the Beltrami equation $\frac{w^\mu_z}{z} = w^\mu_z\mu(z)$ and normalized by $w^\mu(0) = 0$, $w^\mu(1) = 1$.

We now construct a specific hyperelliptic polygon which is somewhat easier to work with. Subdivide the unit disc $D$ by lines $R_1, \ldots, R_{2g}$ through the origin such that the angle between $R_1$ and $R_{1+1}$ is $\frac{2\pi}{4g}$. Let $R_1$ be on the real axis (Fig. 10).
For each \( 0 < r < 1 \) choose points \( r_1, -r_1 \in R_1 \) such that 
\[ |r_1| = r \] 
and construct the non-euclidean polygon 
\[ P_r = \langle r_1, \ldots, r_{2g}, -r_1, \ldots, -r_{2g}, r_1 \rangle \] (Fig. 11).

The interior angle \( \alpha_r \) at each vertex of \( P_r \) is strictly increasing from 0 to \( \pi - \frac{2\pi}{4g} \) as \( r \) decreases from 1 to 0. We show this fact by isolating the hyperbolic triangle 
\[ A_r = \langle 0, r_1, r_2, 0 \rangle \] (Fig. 12). The hyperbolic area \( |A_r| \) of \( A_r \) is strictly decreasing to 0 as \( r \to 0 \). Thus 
\[ |A_r| = \pi - \frac{2\pi}{4g} - \alpha_r = 0 \] as \( r \to 0 \) and \( \alpha_r \to \pi - \frac{2\pi}{4g} \).

The sum of the interior angles of \( P_r \) is thus strictly increasing from 0 to \( 4g(\pi - \frac{2\pi}{4g}) = \pi(4g-2) > 2\pi \). Hence there exists an \( r_0 \in (0, 1) \) for which \( \alpha_{r_0} = 2\pi \). We let \( P_{r_0} = P^* \) and call it the perfectly symmetric polygon.

If we label the sides of \( P^* \) by \( A^*_i, i = 1, \ldots, 4g \), (in counterclockwise order) then Lemma II.3.2 guarantees the existence of Mobius transformations \( T^*_i, i = 1, \ldots, 2g \), identifying opposite sides so that \( P^* \) is a hyperelliptic polygon.

Let \( \Gamma^* \) be the group associated to \( P^* \) (as defined in II.2) and denote the Weierstrass points of \( \frac{D}{\Gamma^*} \) as follows: \( w^*_1 = \{ c^*_i \} \), \( i = 1, \ldots, 2g \), where \( c^*_i \) is the hyperbolic midpoint of \( A^*_i \), \( w^*_{2g+1} = \{ 0 \} \) and \( w^*_{2g+2} = \{ a^*_j \} \) where \( a^*_j \) is any vertex.

We extend the group \( \Gamma^* \) by the map \( z \mapsto -z \) to obtain the group \( [j, \Gamma^*] \). As is well known, \( \frac{D}{[j, \Gamma^*]} \) is conformally a sphere with \( 2g+2 \) distinguished points \( \{ w^*_1 \}, \ldots, \{ w^*_{2g+2} \} \), each of which

\[^5\] The continuity of \( \alpha_r \) will be shown in Lemma II.4.15.
is the fixed point of an elliptic element of order two in \(\{j, \Gamma^*\}\).

Choose a quasiconformal map \(f: \frac{D}{\{j, \Gamma^*\}} \to \frac{S}{\mathcal{J}}\) with the property that \(f(\{w_i^*\}) = w_i^*, \ i = 1, \ldots, 2g+2\). Such a map exists by Bers' theorem [5]. The map \(f\) defines on \(P^*\) a function \(\mu(z) = \frac{f(\bar{z})}{f(z)}\) and we extend \(\mu\) to all of \(D\) by requiring that it be compatible with \(\{j, \Gamma^*\}\). \(\mu\) is thus a Beltrami coefficient for \(\{j, \Gamma^*\}\).

Throughout the remainder of this section \(P^*, f,\) and \((S, J, w_1, \ldots, w_{2g+2})\) will remain fixed.

**Lemma II.4.2.** Let \(\mu\) be a Beltrami coefficient for the group \(\{j, \Gamma^*\}\) and let \(w^\mu\) be the unique normalized quasiconformal self map of \(D\) such that \(w^\mu = \frac{w^\mu}{f(z)}\). Then \(w^\mu \circ j \circ w^\mu = j\).

**Proof.** By Bers' theorem [5] \(w^\mu \circ j \circ w^\mu_{-1}\) is an elliptic transformation of order two which fixes \(D\) with fixed point \(w^\mu(0) = 0\). Thus \(w^\mu \circ j \circ w^\mu_{-1} = j\).

Let \(a_1^*, \ldots, a_{4g}^*\) be the vertices of \(P^*\) and let \(t \in [0, 1]\). Trivially \(tu\) is also a Beltrami coefficient for the group \(\{j, \Gamma^*\}\). We define \(P_{tu} = \langle w^t \mu(a_1^*), \ldots, w^t \mu(a_{4g}^*), w^t \mu(a_1^*)\rangle\).

For convenience we denote each vertex \(w^t \mu(a_i^*),\)
\(i = 1, \ldots, 4g,\) by \(a_1^*\). Each side \(A_i^* = \langle a_1^*, a_i^* \rangle, \ i = 1, \ldots, 4g-1,\)
is oriented from the lower to the higher index. When \(i = 4g\)
we orient \(A_i^* = \langle a_1^*, a_i^* \rangle\) from \(a_4^*\) to \(a_1^*\).
The angle \( 0 \leq \alpha_i^t < 2\pi \) at the vertex \( a_i^t \) to the left of the line \( A_{i-1}^t, i = 2, \ldots, 2g \), is called the \textit{interior angle} of \( P_{t\mu} \) at \( a_i^t \). When \( i = 1 \), \( 0 \leq \alpha_1^t < \pi \) is the angle to the left of \( A_{2g}^t \). \( A_{t\mu}^* \) is the sum of the interior angles of \( P_{t\mu} \).

\textbf{Lemma II.4.3.} Let \( t \in [0,1] \). Then the following properties hold.

i) \( P_{t\mu} \) is \( j \)-invariant.

ii) The transformation \( T_i^t = w^{t\mu}T_i^c(w^{t\mu})^{-1}, i = 1, \ldots, 2g \), identify opposite sides of \( P_{t\mu} \).

\textbf{Proof.}

i) This is a trivial consequence of Lemma II.4.2 and the fact that \( j \) is an isometry.

ii) Consider the sides \( A_i^t = \langle a_i^t, a_{i+1}^t \rangle \), \( i = 1, \ldots, 2g \).

Then \( T_i^t(a_i^t) = w^{t\mu}T_i^c(w^{t\mu})^{-1}(w^{t\mu}(a_i^t)) = w^{t\mu}T_i^c(a_i^*) \)

\[ = w^{t\mu}(-a_{i+1}^*) = -a_{i+1}^t. \] In the same manner

\( T_i^t(a_{i+1}^t) = -a_i^t. \) By Bers' theorem [5] the elements \( T_i^t \) are Mobius transformations which preserve the unit disc. Since \( A_i^t \) is the geodesic joining \( a_i^t \) with \( a_{i+1}^t \) \( T_i^t(A_i^t) = -A_i^t. \)

Before proceeding we make two observations.

First, the group \( \Gamma^t = \{ T_i^t \}, i = 1, \ldots, 2g \), is a Fuchsian group (see Bers [5]). Second, since the transformations \( T_i^t \)
identify opposite sides of $P_{t_\mu}$, the same argument used in II.2.1 shows that all vertices are equivalent under $T_t$.

In order to prove that $P_{t_\mu}$ is a simple polygon we use a continuity argument. We will show in Lemmas II.4.13 and II.4.14 that $A = \{ t \in [0,1] : P_{t_\mu} \text{ is a simple polygon} \}$ is both open and closed and thus equal to $[0,1]$.

We will need the preliminary Lemmas II.4.4-11.

Lemma II.4.4. Let $G = \{ \gamma_1, \gamma_2, \gamma_3 \}$ be a Fuchsian group generated by three elliptic transformations $\gamma_1, \gamma_2$ and $\gamma_3$ of order two whose fixed points are not equivalent in $G$. Then the fixed points are not colinear.

Proof. Suppose the fixed points $z_1, z_2$ and $z_3$ lie on a hyperbolic line $L$. Since $\gamma_1, \gamma_2$ and $\gamma_3$ are elliptic of order two, $L$ is invariant under $G$. Since $G$ is Fuchsian, the unit circle is also invariant under $G$. Thus the two points of intersection of $L$ with the unit circle is a closed invariant set. Since any closed invariant set consisting of at least two points must contain all the limit points (see Ford [9]), $G$ has at most two limit points. $G$ is thus a finitely generated elementary Fuchsian group. This is a contradiction since the only finitely generated elementary Fuchsian groups are either cyclic or have exactly two elliptic conjugacy classes (see Greenberg [10]). On the other hand, $G$ has three non-conjugate elliptic elements.
Lemma II.4.5. No two distinct sides of $P_{tu}$ are colinear.

Proof. Suppose the two sides $A_1^t = \langle a_1^t, a_{i+1}^t \rangle$ and $A_{i}^t = \langle a_{i}^t, a_{i+1}^t \rangle$, $0 < i$, $1 \leq 4g$, $i \neq 1$ are colinear.

According to Lemma II.4.3 the transformations $T_i^t$ and $T_{i}^t$ identify each side with its corresponding opposite side. Thus the transformations $-T_i^t$ and $-T_{i}^t$ are elliptic of order two with fixed points at the midpoints of the sides $A_i^t$ and $A_{i}^t$ respectively.

Since all vertices of $P_{tu}$ are equivalent there exists a transformation $T_{a_{i}^t} \in \Gamma^t$ such that $T_{a_{i}^t} (a_{i}^t) = -a_{i}^t$. Thus $-T_{a_{i}^t}$ is also elliptic of order two and its fixed point is $a_{i}^t$.

Let $G = \{-T_i^t, -T_{i}^t, -T_{a_{i}^t}\} \subset \{j, \Gamma^t\}$ be the group generated by $-T_i^t$, $-T_{i}^t$ and $-T_{a_{i}^t}$. Note that no two of these elements can be conjugate since the corresponding elements in $\{j, \Gamma^t\}$ are not conjugate (Lemma II.2.8). Moreover, by our original assumption the fixed points of these elements are colinear. This contradicts Lemma II.4.4.

Corollary II.4.6. Let $\alpha_i^t$ be the interior angles of $P_{tu}$, $i = 1, \ldots, 4g$. Then $\alpha_i^t \neq 0$ and $\alpha_i^t \neq \pi$

Proof. It suffices to note that if $\alpha_i^t = 0$ or $\pi$ then the two sides of $P_{tu}$ which determine $\alpha_i^t$ are colinear. This contradicts Lemma II.4.5.
Throughout the remainder of the section we will adopt the following conventions.

Given distinct points $z_1, z_2 \in D$, $L_{z_1}^{z_2}$ denotes the hyperbolic line determined by $z_1$ and $z_2$ which is oriented from $z_1$ to $z_2$.

Given a hyperbolic line segment $<z_1, z_2>$, we say that $<z_1, z_2>$ separates the set $A$ from the set $B$ if $A$ and $B$ lie on distinct sides of $L_{z_1}^{z_2}$.

The notation $U_{z_1}^\varepsilon$ will mean an $\varepsilon$-neighborhood centered at $z_1$.

Lemma II.4.7 and Corollary II.4.8 are obvious and we omit the proofs.

**Lemma II.4.7.** Let $a, b, c \in D$ be distinct points and let $c$ lie to the left of $L_a^b$. Then there exists $\varepsilon > 0$ such that the $\varepsilon$-neighborhoods $U_a^\varepsilon$, $U_b^\varepsilon$ and $U_c^\varepsilon$ have the following properties.

i) $U_a^\varepsilon$, $U_b^\varepsilon$ and $U_c^\varepsilon$ are pairwise disjoint.

ii) Given $a_1 \in U_a^\varepsilon$, $b_1 \in U_b^\varepsilon$ and $c_1 \in U_c^\varepsilon$ then $c_1$ lies to the left of $L_{a_1}^{b_1}$.

**Corollary II.4.8.** Let $a, b, c, d \in D$ be distinct points and let $L_a^b$ separate $c$ from $d$. Then there exists $\varepsilon > 0$ such that the $\varepsilon$-neighborhoods $U_a^\varepsilon$, $U_b^\varepsilon$, $U_c^\varepsilon$ and $U_d^\varepsilon$ have the following properties.

i) $U_a^\varepsilon$, $U_b^\varepsilon$, $U_c^\varepsilon$ and $U_d^\varepsilon$ are pairwise disjoint.

ii) Given $a_1 \in U_a^\varepsilon$, $b_1 \in U_b^\varepsilon$, $c_1 \in U_c^\varepsilon$ and $d_1 \in U_d^\varepsilon$ the line $L_{a_1}^{b_1}$ separates $c_1$ from $d_1$. 
Lemma II.4.9. Let $\alpha_i^t$, $i = 1, \ldots, 4g$, be the interior angles of $P_{\mathbf{t}_\mathbf{u}}$. Then $0 < \alpha_i^t < \pi$.

Proof. We will use a continuity argument. We assume $i$ is fixed and for simplicity $i \neq 1$. By relabeling suitably the same proof may be used when $i = 1$.

Let $B_i = \{ t \in [0,1] \text{ such that } 0 < \alpha_i^t < \pi \}$. We note first that $B_i$ is not empty since each interior angle of $P^*$ lies in the interval $(0, \pi - \frac{2\pi}{4g})$ (see p.23).

To show that $B_i$ is open let $t_o \in B_i$. Thus $a_{i+1}^t$ lies to the left of $A_{i-1}^t = \langle a_{i-1}^t, a_i^t \rangle$. Let $\varepsilon > 0$ be such that the $\varepsilon$-neighborhoods $U^\varepsilon_{t_o}, U^\varepsilon_{t_o},$ and $U^\varepsilon_{t_o}$ satisfy the properties $a_{i-1}^t, a_i^t, a_{i+1}^t$ of Lemma II.4.7. (Fig. 13).

For each $m$, $m = 1-1, 1, 1+1$, the map $t \mapsto w^{t\mathbf{u}}(s_m^x)$ is continuous (see [3]). Thus there exist $0 < \delta_m$ such that $w^{t\mathbf{u}}(s_m^x) \in U^\varepsilon_{t_o}$ for all $|t_o - t| < \delta_m$, $t \in [0,1]$. Let

$$\delta = \min \{ \delta_m \}. \text{ Then for all } |t_o - t| < \delta, \ t \in [0,1],$$

$$w^{t\mathbf{u}}(s_m^x) \in U^\varepsilon_{t_o}. \text{ Thus the vertex } a_{i+1}^t \text{ lies to the left of } a_m^t$$

the geodesic line determined by $A_{i-1}^t$ and $0 < \alpha_i^t < \pi$. Consequently, $B_i$ is open.

In the above we have assumed $0 < t_o < 1$. A similar argument works for $t_o = 0$ and $t_o = 1$. 
To show that $B_1$ is closed let $t_k \to t$ where $t_k \in B_1$.

By Corollary II.4.6 we know that $\sigma^t_1 \neq 0$ or $\pi$. Thus we assume $\sigma^t_1 > \pi$ and $a^t_{i+1}$ lies to the right of $A^t_{i-1}$ (Fig. 14).

Let $\epsilon > 0$ be such that the $\epsilon$-neighborhoods $U^\epsilon_{a^t_{i-1}}, U^\epsilon_{a^t_i}$ and $U^\epsilon_{a^t_{i+1}}$ satisfy the properties of Lemma II.4.7 (with the orientation reversed). Let $t_{k_n} \in B_1$ be such that $w^{t_{k_n}}_{t^{k_n}}(a^*_{m}) \in U^\epsilon_{a^t_m}$, $m = i-1, i, i+1$. Thus the point $a^t_{i+1}$ lies to the right of $A^t_{i-1}$ and $\sigma^t_1 > \pi$. This contradicts $t_{k_n} \in B_1$.

We have shown that $0 < \sigma^t_1 < \pi$ and $B_1$ is closed. Since $B_1$ is both open and closed $B_1 = [0,1]$.

**Corollary II.4.10.** Let $P_{tu}$ $t \in [0,1]$ be simple. Then $P_{tu}$ is convex.

**Proof.** By Lemma II.4.9 $0 < \sigma^t_1 < \pi$, $i = 1, \ldots, 4g$. A simple polygon with this property is convex (see Magnus [14]).

**Lemma II.4.11.** Let $P \subset D$ be a simple, convex, hyperbolic polygon with sides $A^t_i$, $i = 1, \ldots, 4g$. Then there exist convex neighborhoods $H^t_i$ of $A^t_i$ with the following property:

$$H^t_k \cap H^t_l = \emptyset$$

whenever $A^t_k \cap A^t_l = \emptyset$, $1 \leq k, l \leq 4g$.

**Proof.** For each fixed side $A^t_i$, $i = 1, \ldots, 4g$, of $P$, we
denote by $A_1, \ldots, A_{4g-3}$ the sides of $P$ which are disjoint from $A_1$.

Clearly, since $P$ is convex, there exists for each side $A_i$ a line $L_i$ which separates $A_i$ from $\bigcup_{j=1}^{4g-3} A_{ij}$ (Fig. 15). Let $H_i$ (resp. $H_i'$) be the open half plane determined by $L_i$ which contains $A_i$ (resp. $\bigcup_{j=1}^{4g-3} A_{ij}$). Note that $H_i$ and $H_i'$ are disjoint, open, convex sets.

Let $H_i = H_i' \cap \bigcap_{j=1}^{4g-3} H_{ij}$). We note that $H_i$ is open and convex. Moreover, $A_i \subset H_i$ and since $A_{ij}, j = 1, \ldots, 4g-3$, is not adjacent to $A_i$, $A_i \subset H_{ij}$. Thus $A_i \subset H_i$.

---

Fig. 15
Suppose $A_k$ and $A_l$, $1 \leq k, l \leq 4g$, are disjoint sides. If $z \in H_k$, then $z \in H_k'$ and $z \notin H_k''$. But $A_l = A_k''$ for some $1 \leq r \leq 4g - 3$. Thus $z \notin H_k''$ and $z \notin H_k = H_k' \cap \bigcap_{j=1}^{4g-3} H_k''$.

Hence $H_k' \cap H_k = \emptyset$.

**Corollary II.4.12.** Let $P \subset D$ be a simple, convex, hyperbolic polygon with vertices $a_i, i = 1, \ldots, 4g$. Then there exists $\varepsilon > 0$ such that the $\varepsilon$-neighborhoods $U^\varepsilon_{a_i}$ have the following properties.

1) $U^\varepsilon_{a_i} \cap U^\varepsilon_{a_j} = \emptyset$ whenever $a_i \neq a_j, 1 \leq i, j \leq 4g$

2) Given $z_i \in U^\varepsilon_{a_i}, i = 1, \ldots, 4g$, the non-adjacent sides of polygon $<z_1, \ldots, z_{4g}, z_1>$ are disjoint.

**Proof.** Let $H_i, i = 1, \ldots, 4g$, be convex neighborhoods as described in Lemma II.4.11. Thus $a_i \in H_i \cap H_{i-1}, i = 2, \ldots, 4g,$ and $a_i \in H_i \cap H_{4g}$. Choose $\varepsilon > 0$ small enough so that $a_i \in U^\varepsilon_{a_i} \subset H_i \cap H_{i-1}, i = 2, \ldots, 4g,$ and $a_i \in U^\varepsilon_{a_i} \subset H_i \cap H_{4g}$.

Clearly, the $U^\varepsilon_{a_i}, i = 1, \ldots, 4g$, are pairwise disjoint.

We need to verify property 2).

Let $<z_i, z_{i+1}>$ and $<z_j, z_{j+1}>$, $1 \leq i, j < 4g$, be non-adjacent sides of $<z_1, \ldots, z_{4g}, z_1>$.

Clearly, $A_1 = <a_1, a_1 + 1>$ and $A_j = <a_j, a_j + 1>$ are not adjacent. Thus $H_i \cap H_j = \emptyset$. But $U^\varepsilon_{a_i}, U^\varepsilon_{a_j} \subset H_i$ and $U^\varepsilon_{a_i}, U^\varepsilon_{a_j} \subset H_j$, thus $<z_i, z_{i+1}> \subset H_i$ and $<z_j, z_{j+1}> \subset H_j$. Therefore $<z_i, z_{i+1}> \cap <z_j, z_{j+1}> = \emptyset$. The same argument works if one of the sides is of the form $<z_{4g}, z_1>$. 
Therefore \(<z_i, z_{i+1}> \cap <z_j, z_{j+1}> = \emptyset\). The same argument works if one of the sides is of the form \(<z_{i+k}, z_{j}>\).

We are now in a position to prove that the set A previously defined is both open and closed. We note first that A is not empty since \(0 \in A\).

**Lemma II.4.13.** A is open.

**Proof.** Let \(t_0 \in A\). Thus \(P_{t_0}\) is simple and by Corollary II.4.10 it is convex. Let \(\varepsilon > 0\) be such that the \(\varepsilon\)-neighborhoods \(U^\varepsilon_{t_0, i}\), \(i = 1, \ldots, 4g\), satisfy the properties of Corollary II.4.12.

As in the proof of Lemma II.4.9 we choose \(0 < \delta\) such that \(w^t(a^*_1) \in U^\varepsilon_{t_0, i}\) for all \(|t_0 - t| < \delta\), \(t \in [0, 1]\) and \(i = 1, \ldots, 4g\). Thus the non-adjacent sides of \(P_{t_0}\) are disjoint. By Lemma II.4.5 the adjacent sides of \(P_{t_0}\) are not colinear. Thus \(P_{t_0}\) is simple.

In the above we have assumed \(0 < t_0 < 1\). A similar argument works for \(t_0 = 0\) and \(t_0 = 1\).

**Lemma II.4.14.** A is closed.

**Proof.** Let \(t_k \rightarrow t\) where \(t_k \in A\). We need to show that \(P_{t_k}\) is simple. By Lemma II.4.5 no two sides of \(P_{t_k}\) are colinear thus the proof will follow from the definition of a simple
polygon when we prove that no two non-adjacent sides of $P_{t_0}$ intersect at exactly one point.

We assume that there exist sides $A_j^t$ and $A_l^t$, $0 < j, l < 4g$, $j \neq l - 1, l, l + 1$ which intersect in a single point (Figs. 16 and 17). The point of intersection may be a vertex as in Fig. 16 or an interior point of both sides as in Fig. 17. In either case we may assume that the side $A_l^t$ separates $a_j^t$ from $a_{j+1}^t$.

Let $\epsilon > 0$ be such that the $\epsilon$-neighborhoods satisfy the properties of Corollary II.4.8.

Let $t_k \in A$ be such that $w_{k_n}(a^*) \in U_t^c$, $m = j-1, j, l-1, l$. Thus the line $A_l^t$ separates the point $a_j^t$ from $a_{j+1}^t$. This is a contradiction since by Corollary II.4.10 $P_{t_k}$ is convex and no side can separate two vertices.

**Lemma II.4.15.** Let $a_j^t, j = 1, \ldots, 4g$ be the interior angles of $P_{t_0}$. Then $a_j^t$ is continuous function of $t$.

**Proof.** Suppose $z_1, z_2, z_3 \in D$ are distinct points. We first construct a Möbius transformation $T$ which fixes $D$ such that $T(z_2) = 0$ and $T(z_1) \in \mathbb{R}^+$. Let $T(z) = \frac{az - az_2}{-az_2z + a}$ where

$$a = \sqrt{\frac{1}{1 - |z_2|^2}} e^{-i\theta/2} \quad \text{and} \quad \theta = \arg\left(-\frac{z_1 - z_2}{z_2z_1 + 1}\right).$$
We need to verify that $T$ has the required properties.

1. $|a|^2 - |z_2|^2 |z_2|^2 = |a|^2(1-|z_2|^2) = \frac{1-|z_2|^2}{1-|z_2|^2} = 1.$

Thus $T$ fixes $D$.

2. $T(z_2) = az_2 - az_2 = 0$

3. $T(z_1) = \frac{az_1 - a}{-az_2 + \bar{a}} = \frac{a}{\bar{a}} \text{re}^i\theta = r.$

(Note that $r = \frac{|z_1 - z_2|}{-z_2 + \bar{z}_1} > 0.$)

Join $z_1$ to $z_2$ (resp. $z_2$ to $z_3$) by a geodesic $l_1$ (resp. $l_2$) oriented from $z_1$ to $z_2$ (resp. $z_2$ to $z_3$). Since $T$ is a conformal map, the angle formed at $z_2$ to the left of the line $l_1$ is equal to

$$2\pi - \arg T(z_3)$$

(In the above formula we use the principal branch of the argument lying between 0 and $2\pi$.)

Now consider the points $w^{t_1}(a^*_j - 1), w^{t_1}(a^*_j)$ and $w^{t_1}(a^*_j + 1)$. From (II.4.1) we obtain

$$a^t_j = 2\pi - \arg(\frac{a^*_j w^{t_1}(a^*_j + 1) - a^*_j w^{t_1}(a^*_j)}{-a^*_j w^{t_1}(a^*_j) w^{t_1}(a^*_j + 1) + \bar{a}_t})$$

where $a^t_j = \frac{1}{\sqrt{1 - |w^{t_1}(a^*_j)|^2}} e^{-i\theta/2}$ and
\[(\text{II.4.3}) \quad \theta_j^t = \arg\left(\frac{w_j^t(a_{j-1}^*) - w_j^t(a_j^*)}{-w_j^t(a_j^*) w_j^t(a_{j-1}^*) + 1}\right)\]

By Lemma II.4.9 $0 < \alpha_j^t < \pi$. Thus the values of the argument function in formula (II.4.2) lie in the open interval $(\pi, 2\pi)$ and the argument function is therefore continuous. In formula (II.4.3) the argument function is allowed to vary continuously with respect to $t$. All other functions considered are continuous, thus $\alpha_j^t$ is a continuous function of $t$. (Note that we have again assumed that $j \neq 1$. The same proof may be used when $j=1$ by a suitable relabeling of indices.)

We also note that in general if $z_1^t, z_2^t, z_3^t \in \mathcal{D}$ are three distinct points varying continuously with respect to a parameter $t$, then $\alpha^t$ (defined analogously) is also a continuous function of $t$ provided $\alpha^t$ lies in an interval properly contained in $(0, 2\pi)$. In particular, this applies to the construction of the perfectly symmetric polygon in this section and the polygons constructed in section II.3.

**Lemma II.4.16.** Let $t \in [0, 1]$. Then $\hat{P}_{t^*} = 2\pi$.

**Proof.** Since the elements $T_i^*$, $i = 1, \ldots, 2g$, generate the group $\Gamma^*$ the elements $T_i^t$ also generate a Fuchsian group and satisfy the same relation as the $T_i^*$ (Bers [5]). Thus

\[
(T_2^t)^{-1} \circ T_{2g-1}^t \circ (T_2^t)^{-1} \circ T_{2g-2}^t \circ \cdots \circ T_2^t \circ T_{2g-1}^t = 1.
\]

Let $\alpha_i^t$, $i = 1, \ldots, 4g$, be the interior angles of $P_{t^*}$. Since
\[(T_{2g}^t)^{-1} \circ \cdots \circ (T_{2g}^t)^{-1} \circ (T_1^t)^{-1} \circ \cdots \circ T_{2g}^t \circ T_1^t(A_1) = A_1 \text{ the angle} \]
\[\hat{\theta}_{t\mu} = \Sigma c_i^t \text{ fills out a circle and is a multiple of } 2\pi.\]
But \(\hat{\theta}_{t\mu}\) is a continuous function of \(t\) (Lemma II.4.15) and \(\hat{\theta}_{t\mu} = 2\pi.\)

**Corollary II.4.17.** \(P_\mu\) is a hyperelliptic polygon and is a fundamental polygon for \(\Gamma_\mu.\) \(\frac{D}{\Gamma_\mu}\) is conformally equivalent to \(S.\)

**Proof.** Since \(A\) is both open and closed \(A = [0,1]\) and \(P_\mu\) is a simple polygon. From Lemma II.4.4 and II.4.16 we obtain that \(P_\mu\) is \(j\)-invariant, the transformations \(T_i^t,\)
\(i = 1, \ldots, 2g,\) identify opposite sides of \(P_\mu\) and \(\hat{\theta}_\mu = 2\pi.\)
Using Poincaré's theorem \(P_\mu\) is a fundamental polygon for \(\Gamma_\mu.\)

To prove the last assertion we construct the diagram (with commutative squares)

```
\[
\begin{array}{ccc}
S & \xrightarrow{f_{\mu}} & D \\
\downarrow & & \downarrow \\
\frac{D}{\Gamma_{\mu}} & \xrightarrow{w_{\mu}'} & \frac{D}{\Gamma_{\mu}'} \\
\downarrow & & \downarrow \\
\frac{D}{\Sigma} & \xrightarrow{w_{\mu}''} & \frac{D}{\Sigma'} \\
\end{array}
\]
```

where \(h_{\mu_1}^1\) and \(h_{\mu_2}^1\) are the induced maps.
The maps $f$ and $h^\mu_2$ have the same dilation. Thus $h^\mu_2 \circ f^{-1} : \frac{S}{J} \rightarrow \frac{S}{\{j_i, \Gamma^\mu\}}$ is conformal (see Bers [5]). Consequently $S$ and $\frac{D}{\Gamma^\mu}$ are twofold covers of $\frac{S}{J}$ branched over the same $2g+2$ points and are conformally equivalent.

For clarity we restate Corollary II.4.17 as a theorem.

**Theorem II.4.18.** Let $(S,J,w_1,\ldots,w_{2g+2})$ be a hyperelliptic surface. Then there exists a hyperelliptic polygon $P$ with associated Fuchsian group $\Gamma$ such that $S$ is conformally equivalent to $\frac{D}{\Gamma}$.

The conformal equivalence is given by a map $\hat{\alpha}$ with the property that $\hat{\alpha}(w_1) = \{c_i\}$, $i = 1, \ldots, 2g$, $\hat{\alpha}(w_{2g+1}) = \{0\}$ and $\hat{\alpha}(w_{2g+2}) = \{a_j\}$ where $a_j$ is any vertex of $P$.

**Corollary II.4.19.** Let $(S,J,w_1,\ldots,w_{2g+2})$ be a hyperelliptic surface. Then $S$ may be uniformized by a Fuchsian group generated by elements $T_i$, $i = 1, \ldots, 2g$, where $T_i(z) = \frac{a_i z + b_i}{E_i z + a_1}$, $a_1 \in \mathbb{R}$, and

\[(II.4.4) \quad T_2^{-1} \circ \cdots \circ T_{2g-1} \circ T_{2g}^{-1} \circ \cdots \circ T_{2g-2} \circ T_{2g-1} \circ T_{2g} \circ T_1 = id.
\]

Moreover any uniformization of $S$ by a Fuchsian group (with no elliptic elements) has a set of generators satisfying (II.4.4).

**Proof.** Let $P$ be a hyperelliptic polygon whose associated group $\Gamma$ uniformizes $S$. The pairing transformations $T_i$, $i = 1, \ldots, 2g$, generate $\Gamma$ and $T_i(z) = \frac{a_i z + b_i}{E_i z + a_1}$, where $a_1 \in \mathbb{R}$.
by Lemma II.2.3. If $\Gamma'$ is another uniformization of $S$ (without elliptic elements) then it is well known that $\Gamma' = T_0 \Gamma T^{-1}$ for some motion $T$. The elements $T'_i = T_0 T_i T^{-1}$ generate $\Gamma'$ and satisfy (II.4.4).
III. MODULI OF HYPERELLIPTIC-surfaces

IIIi Teichmüller spaces. In order to study the moduli of
hyperelliptic surfaces we need to establish some facts about
Teichmüller spaces. All the material in this section is
expository.

We first define quasi-Fuchsian groups and then give
brief descriptions of the Teichmüller space and the Bers
fiber space of a Fuchsian group. (For a more detailed
description see [4].)

Let $\Gamma$ be a discontinuous group of Mobius transformations
and let $C$ be an oriented Jordan curve such that the domains
$D'$ and $D''$, interior and exterior to $C$ respectively, are in-
vvariant under $\Gamma$. $\Gamma$ is called a quasi-Fuchsian group and the
domains $D'$ and $D''$ are called the invariant components of $\Gamma$.
In this case $\Lambda(\Gamma)$ is contained in $C$ and $\Gamma$ is said to be of
the first kind if $\Lambda(\Gamma) = C$, otherwise it is of the second
kind (see [12]).

As in Chapter II, $D$ will denote the open unit disc. We
denote by $\text{Aut } D$ the group of all Mobius transformations which
fix $D$.

Let $\Gamma$ be a quasi-Fuchsian group with an invariant com-
ponent $D'$ and let $\pi : D \to D'$ be a Riemann map. Define the
group

$$G = \{ \gamma \in \text{Aut } D : \gamma = \pi^{-1} \circ T \pi \text{ for some } T \in \Gamma \}.$$ 

$G$ is a Fuchsian group (see Kra [12]) and is called the Fuchsian
equivalent of $\Gamma$. Clearly $\frac{D}{G} = \frac{D'}{T}$.

A polygon will be called a smooth Jordan polygon if it is a simple polygon whose sides are analytic arcs.

Let $\Gamma$ be a Fuchsian group with no parabolic elements. $L^\infty(\Gamma)$ is the space of Beltrami differentials for $\Gamma$ (II.4 ) with norm $\|\mu\|_\infty = \text{ess. sup.}\{|\mu(z)|; z \in D\}$.

The open unit ball $M(\Gamma)$ of $L(\Gamma)$ is the space of Beltrami coefficients for $\Gamma$.

Let $\mu \in M(\Gamma)$ and extend $\mu$ to all of $\mathbb{C} \cup \{\infty\}$ by requiring that $\mu|D^C = 0$. We denote by $w_\mu$ the unique quasiconformal self map of $\mathbb{C} \cup \{\infty\}$ which fixes 1, -1 and 1, and satisfies the Beltrami equation $\left(\frac{\mu}{\overline{\mu}\mu} \right) = \mu(z)(\frac{w_\mu}{\overline{w_\mu}})z$ (see [3]).

We say that $\mu, \nu \in M(\Gamma)$ are equivalent (and write $\mu \sim \nu$) if and only if $w_\mu = w_\nu$ on the unit circle. $T(\Gamma)$ is the space of equivalence classes in $M(\Gamma)$. $\bar{\pi}$ is the projection of $M(\Gamma)$ onto $T(\Gamma)$. The domain $w_\mu(D)$ depends only on the equivalence class $\bar{\pi}(\mu)$ of $\mu$.

The Bers fiber space is defined by $F(\Gamma) = \{(\bar{\pi}(\mu), z) \in T(\Gamma) \times \mathbb{S} \cup \{\infty\}; \mu \in M(\Gamma)$ and $z \in w_\mu(D)\}$.

Both $T(\Gamma)$ and $F(\Gamma)$ are complex manifolds (see [4]).

The group $\Gamma$ acts discontinuously on $F(\Gamma)$ as a group of biholomorphic self maps by the rule

$$(\bar{\pi}(\mu), z)\gamma = (\bar{\pi}(\mu), \gamma^\mu(z)),$$

where $\mu \in M(\Gamma)$, $z \in w_\mu(D)$ and $\gamma^\mu = w_\gamma \circ \gamma(w_\mu)^{-1}$ (see Bers [4]).
\[ V(\Gamma) = \frac{F(\Gamma)}{\Gamma} \] is also a complex manifold and the projection \( (\delta(\mu), z) \mapsto \delta(\mu) \) is a holomorphic map from \( V(\Gamma) \) onto \( T(\Gamma) \). The inverse image of \( \delta(\mu) \) is the closed surface \( \frac{w(\mu)}{w(\mu, \Gamma \circ w(\mu))^{-1}} \). Note that \( \Gamma^\mu = w(\mu, \Gamma \circ w(\mu))^{-1} \) is a quasi-Fuchsian group with invariant component \( w(\mu, D) \).

Let \( \Gamma \) be a fixed point free Fuchsian group such that \( D/\Gamma \) is a closed surface of genus \( g \), and let \( H \) be a nontrivial group of automorphisms of \( D/\Gamma \). \( p : D \to D/\Gamma \) is the natural projection map.

Let \( \Gamma' = \{ \gamma \in \text{Aut } D; \rho \circ \gamma = \rho \circ h \text{ for some } h \in H \} \). It is well known that \( \Gamma' \) is a Fuchsian group, \( \Gamma \) is a normal subgroup of \( \Gamma' \) and \( \Gamma'/\Gamma \cong H \).

We will now define actions of \( \Gamma'/\Gamma \) on \( T(\Gamma) \) and of \( \Gamma' \) on \( F(\Gamma) \).

\( \Gamma' \) acts on the space \( M(\Gamma) \) of Beltrami coefficients for \( \Gamma \) by

\[(III.1.1) \quad \mu \cdot g = (\mu \circ g)\delta_{g'}, \text{ for all } \mu \in M(\Gamma), \ g \in \Gamma'. \]

The subgroup \( \Gamma \) acts trivially on \( M(\Gamma) \) and the action of \( g \in \Gamma' \) depends only on its equivalence class modulo \( \Gamma \).

Therefore \( (III.1.1) \) induces an action of \( H \) on \( T(\Gamma) \) by the rule

\[ \delta(\mu) \cdot \alpha(g) = \delta(\mu \cdot g) \text{ for all } \mu \in M(\Gamma), \ g \in \Gamma', \]

where \( \alpha : \Gamma' \to H \) is the quotient map.

We now define the action of \( \Gamma' \) on \( F(\Gamma) \) by the rule
\((\Phi(\mu), z) \cdot g = (\Phi(\mu \cdot g), g^\mu (z)),\)

where \(\mu \in M(\Gamma), z \in w_\mu(D), g \in \Gamma',\) and \(g^\mu \circ w_\mu \cdot g = w_\mu \cdot g.\)

**III.2 The hyperelliptic locus.** In this section we consider the subspace of Teichmüller space which consists of hyperelliptic surfaces. In order to do so we will first define the Teichmüller modular group.

Using the material from Chapter II, we then obtain new results on uniformizations of hyperelliptic surfaces by quasi-Fuchsian groups.

Let \(\theta\) be an automorphism of the Fuchsian group \(\Gamma\) with the property that \(\theta(\gamma) = w_\gamma \circ w^{-1}\) for all \(\gamma \in \Gamma\), where \(w\) is a quasiconformal self map of \(D\). \(\theta\) is called a geometric automorphism.

The map \(w\) induces a biholomorphic self map of \(M(\Gamma)\) by sending \(\mu \in M(\Gamma)\) into the Beltrami coefficient of \(w_\mu \circ w^{-1}\). It is easy to check that this mapping preserves equivalence classes and hence induces a biholomorphic self map \(\theta^* : T(\Gamma) \to T(\Gamma)\) which depends only on the conjugacy class of \(\theta\) modulo the group of inner automorphisms of \(\Gamma\).

Thus the (Teichmüller) modular group \(\text{Mod } \Gamma\), which is defined to be the quotient of the group of geometric automorphisms by the normal subgroup of inner automorphisms, acts on \(T(\Gamma)\) as a group of biholomorphic self maps.

Let \(H = \{\Phi(\mu) \in T(\Gamma); \frac{w_\mu(D)}{\mu} \text{ is hyperelliptic}\}\). Thus if
\( \forall (u) \in H \text{ then } \frac{w_u(D)}{r_u} \) has an involution \( J_u \). The map \( w_u : D \rightarrow \frac{w_u(D)}{r_u} \) induces a quasiconformal map \( f_u : \frac{D}{r_u} \rightarrow \frac{w_u(D)}{r_u} \).

Hence \( f^{-1}J_u \) of \( J_u \) is a quasiconformal self map of \( \frac{D}{r_u} \) and we may lift it to a geometric automorphism of \( \Gamma \). In this manner \( J_u \) induces a self map \( J^*_u \) of \( T(\Gamma) \).

Let \( T(\Gamma) \) be the fixed point set of the map \( J^*_u \). \( J^*_u \) and \( J^*_v \) are called equivalent if \( T(\Gamma) \) is a component of \( H \) and that when \( g > 2 \) there are infinitely many inequivalent \( J^*_u \). It is not hard to show that each component of \( H \) contains a copy from each conformal equivalence class of hyperelliptic surfaces. When \( g = 2 \) it is well known that \( T(\Gamma) = T(\Gamma) \).

Proposition III.2.1. Let \( \Gamma \) be a fixed point free quasi-Fuchsian group and \( D' \) an invariant component of \( \Gamma \). Let \( \frac{D'}{r} \) be hyperelliptic with involution \( J \) and let \( \{ z_0 \} \) be a Weierstrass point of \( \frac{D'}{r} \). Let \( E : D' \rightarrow D' \) be a lift of the involution such that \( E^2 = \text{id} \), \( D' \) and \( E(z_0) = z_0 \). Then there exists a fundamental region \( R \subset D' \) which is bounded by a smooth Jordan polygon \( P' \) invariant under \( E \). Moreover, if we label the sides of \( P' \) by \( A'_1, A'_2, \ldots, A'_{4g} \) (in order), there exist generating transformations \( T'_1, \ldots, T'_{2g} \) for \( \Gamma \) such that \( T'_i(A'_1) = E(A'_1), \) \( i = 1, \ldots, 2g \).
Proof. Let \( \pi : D \rightarrow D' \) be a Riemann map such that \( \pi(0) = z_0 \). Then \( \pi^{-1} \) is a conformal self map of \( D \) with a fixed point at zero, thus \( \pi^{-1} \in \text{Hom} = j|D \) (see [9]). Let \( G \) be the Fuchsian equivalent of \( \Gamma \). Thus \( \frac{D}{G} \) is hyperelliptic and by Theorem II.4.16, \( G \) has a fundamental polygon \( P \subset D \) with sides \( A_1, \ldots, A_{2g} \) and generating transformations \( T_i \in G, i = 1, \ldots, 2g \), such that \( T_i(A_1) = -A_1 \). Let \( R = \pi(P^0) \). It is easy to verify that \( R \) is a fundamental domain for \( \Gamma \) bounded by \( \pi(P) = P' \) a smooth Jordan polygon with sides \( \pi(A_1) = A_1' \). The transformations \( T'_i = \pi \circ T_i \circ \pi^{-1} \) generate \( \Gamma \) and \( T'_i(A_1') = E(A_1') \).

**Lemma III.2.2.** Let \( \xi(\mu) \in H \subset T(\Gamma) \) and let \( J_\mu \) be the involution of \( \frac{w(\mu)}{\Gamma(\mu)} \). Let \( \{z_0\} \) be a Weierstrass point of \( \frac{w(D)}{\Gamma(\mu)} \). Then there exists a lift \( E : \frac{w(\mu)}{\Gamma(\mu)} \rightarrow \frac{w(D)}{\Gamma(\mu)} \) of the involution \( J_\mu \) such that \( E^2 = \text{id} \cdot \frac{w(D)}{\Gamma(\mu)} \) and \( E(z_0) = z_0 \).

**Proof.** Let \( E' \) be a lift of \( J_\mu \) to \( \frac{w(D)}{\Gamma(\mu)} \). Since \( \{z_0\} \) is a Weierstrass point \( E'(z_0) = T(z_0) \). But \( T^{-1}E' \) is also a lift of \( J_\mu \) and \( T^{-1}E'(z_0) = z_0 \). Let \( E = T^{-1}E' \), then \( E^2 = S \cdot \frac{w(D)}{\Gamma(\mu)} \) where \( S \in \Gamma(\mu) \). Since \( E^2 \) has a fixed point \( S = \text{id} \).

**Corollary III.2.3.** Let \( \xi(\mu) \in H \subset T(\Gamma) \). Then there exists a fundamental region \( R \subset \frac{w(D)}{\Gamma(\mu)} \) as described in Proposition III.2.1.
Proof. Lemma III.2.2 and Proposition III.2.1 yield the desired result.

Let \( G \) be a Fuchsian group such that \( \frac{D}{G} \) is hyperelliptic. It is well known that \( G \) is conjugate to a Fuchsian group \( \Gamma \) which is \( j \)-invariant. We call \( \Gamma \) a **normalized** hyperelliptic group.

If \( \Gamma \) is a normalized hyperelliptic Fuchsian group then \( j(z) = -z \) is a lift of the involution \( J \) of \( \frac{D}{\Gamma} \). Let \( \langle J \rangle \) be the group of conformal self maps of \( \frac{D}{\Gamma} \) generated by \( J \). Let \( \Gamma' \) be the group of all lifts of elements of \( \langle J \rangle \) to \( D \). It is easy to verify that \( \{j, \Gamma\} = \Gamma' \) and \( \frac{\Gamma'}{\Gamma} = \langle J \rangle \).

We will assume throughout the rest of this section that \( \Gamma \) is a normalized hyperelliptic Fuchsian group and \( \Gamma' = \{j, \Gamma\} \).

We now turn our attention to the fundamental regions in the fibers over \( T(\Gamma)^{j} \subset H \subset T(\Gamma) \). We call \( T(\Gamma)^{j} \) the **identity component** of \( H \).

**Proposition III.2.4.** Let \( \mu \in M(\Gamma) \) be an even coefficient (i.e. \( \mu = \mu_{oj} \)). Then

1) \( w_{\mu}(D) / T^{\mu} \) is hyperelliptic and \( j^{\mu} = w_{\mu}oj(w_{\mu})^{-1} \) is an elliptic Möbius transformation of order 2 that fixes \( w_{\mu}(D) \). Moreover, \( j^{\mu} | w_{\mu}(D) \) is a lift of the hyperelliptic involution of \( w_{\mu}(D) / T^{\mu} \).

2) the quasi-Fuchsian group \( T^{\mu} \) has a fundamental region \( R \subset w_{\mu}(D) \) whose boundary is a smooth \( j^{\mu} \)-invariant Jordan polygon with sides \( A_{1}^{i}, \ldots, A_{2g}^{i} \). Moreover, \( T^{\mu} \) is generated by transformations \( T_{i}^{j} \), \( i = 1, \ldots, 2g \), such that \( T_{i}^{j}(A_{i}^{1}) = j^{\mu}(A_{i}^{1}) \).
Proof.

1) Since $\mu$ is even, the mapping $\gamma \rightarrow \gamma^\mu$ (where $\gamma \in \{j, \Gamma\}$) maps elliptic elements onto elliptic elements of the same order (see [4]). Since the group $\{j^\mu, \Gamma^\mu\}$ is quasi-Fuchsian, it follows trivially that $j^\mu \mid w_\mu(D)$ induces an involution of $w_\mu(D)/\Gamma^\mu$ with $2g + 2$ fixed points.

ii) The proof follows from part i of this lemma and Proposition III.2.1.

**Corollary III.2.5.** Let $\mu \in M(\Gamma)$ such that $\hat{\xi}(\mu) \in T(\Gamma)^J$. Then there exists a $\mu' \in \hat{\xi}(\mu)$ such that $j^{\mu'}$ is an elliptic Mobius transformation of order 2, $w_\mu(D)$ is $j^{\mu'}$-invariant, and the group $\Gamma^\mu$ has a $j^{\mu'}$-invariant fundamental region as defined in Proposition III.2.4.

**Proof.** Let $\hat{\xi}(\mu) \in T(\Gamma)^J$, thus $\mu \sim \mu o j$. In order to apply Proposition III.2.4 we need to show that there exists a $\mu' \in \hat{\xi}(\mu)$ such that $\mu' = \mu' o j$.

We first show that in general if $\nu_1 \sim \nu_2$ then $\nu_1 o j \sim \nu_2 o j$. Let $w_{\nu_1}$ and $w_{\nu_2}$ be the normalized solutions to the mapping problem for $\nu_1$ and $\nu_2$ respectively. Then $w_{\nu_1 o j}$ and $w_{\nu_2 o j}$ are solutions for $\nu_1 o j$ and $\nu_2 o j$. Since $w_{\nu_1}(z) = w_{\nu_2}(z)$ on the unit circle we have that $w_{\nu_1 o j} = w_{\nu_2 o j}$ on the unit circle. Thus for some $0 < \theta < 2\pi$, $e^{i\theta}w_{\nu_1 o j}$ and $e^{i\theta}w_{\nu_2 o j}$ are normalized solutions for $\nu_1 o j$ and $\nu_2 o j$. Hence $\nu_1 o j \sim \nu_2 o j$. 

Now let $\mu' \in \mathcal{E}(\mu)$ be the unique Teichmüller differential with minimal maximal dilation (see [13]). By the above argument $\mu' \circ \jmath \sim \mu'$. But the maximal dilation of $\mu'$ is the same as that of $\mu' \circ \jmath$, hence $\mu' \circ \jmath = \mu'$. The proof now proceeds as that of Proposition III.2.4 using the coefficient $\mu'$.

The fundamental regions for the groups $\Gamma^j$, where $\mathcal{E}(\mu) \in T(\Gamma)^j$ may be described more precisely.

Let $z_0 \in \mathcal{D}$ such that $\{z_0\}$ is a Weierstrass point of $\mathcal{D}/\Gamma$. As in the proof of Lemma III.2.2, there exists an elliptic element of order two, $T_{z_0} \in \Gamma'$ such that $T_{z_0}(z_0) = z_0$.

Since $j$ and $T_{z_0}$ are equivalent modulo $\Gamma$, $T(\Gamma)^j = T(\Gamma)^0$.

As in the proof of Corollary III.2.5, if $\mathcal{E}(\mu) \in T(\Gamma)^j$, there exists a $\mu' \in \mathcal{E}(\mu)$ such that $\mu' = \mu' \circ \jmath$. Thus, we define the section $s_{z_0} : T(\Gamma)^j \to F(\Gamma)$ by $s_{z_0}(\mathcal{E}(\mu))$

$$= (\mathcal{E}(\mu), \nu_{\mu'}(z_0)).$$

**Lemma III.2.6.** The map $s_{z_0} : T(\Gamma)^j \to F(\Gamma)$ is a holomorphic section.

**Proof.** It follows from a theorem of Rauch (see [6]) that the natural map $f : T(\Gamma') \to T(\Gamma)^j \subset T(\Gamma)$ is a complex analytic embedding. Thus the map $f^* : F(\Gamma') \to F(\Gamma)$ given by $f^*(\mathcal{E}(\mu),z) = (f(\mathcal{E}(\mu)),z)$ is also a complex analytic embedding.

It is well known that $s_{z_0}^* : T(\Gamma') \to F(\Gamma')$ defined by
\( s^*_D(\bar{\varphi}(\mu)) = (\varphi(\mu), w_{\mu}(z_0)) \) is a holomorphic section (see [7]).

Now \( s^*_D = f^*s^*_D \) of \( f^{-1} \).

(Note that when \( g = 2 \), \( s^*_D \) is a global section, i.e. \( s^*_D : T(\Gamma) \to F(\Gamma) \)).

When \( z_0 = 0 \) we call \( s^*_D \) the zero section. Thus Corollary III.2.5 states that every group \( \Gamma^1 \) such that \( \varphi(\mu) \in T(\Gamma)^j \) has a fundamental region symmetric about a point in the zero section. We will show that the vertices of these regions may also be chosen to lie on holomorphic sections.

Before continuing we need to define the Poincaré metric \( \lambda_D(\mu) \) for \( w_{\mu}(D) \). \( \lambda_D(\mu) \) is defined as follows:

Let \( \lambda_D = \frac{1}{1-|z|^2} \) be the Poincaré metric on the unit disc.

Choose \( \pi : D \to w_{\mu}(D) \), a Riemann map and define \( \lambda_D(\mu) \) by

\[
\lambda_D(\mu)(\pi(z)) |\pi'(z)| = \lambda_D(z) \text{ for } z \in D.
\]

\( \lambda_D(\mu) \) is well defined (see Kra [12]). In fact geodesics in \( w_{\mu}(D) \) are the images under \( \pi \) of geodesics in \( D \).

By Theorem II.4.18, \( \Gamma \) has a hyperelliptic fundamental polygon with vertices \( a_1, \ldots, a_{4g} \). Let \( T_{a_1} \) be a lift of the involution \( J \) on \( \mathbb{D}_1 \) such that \( T_{a_1}^j(a_i) = a_{i+1} \text{ if } i = 1, \ldots, 4g \). Thus \( s^*_{a_1}(T(\Gamma)^j) = \{ (\bar{\varphi}(\mu), w_{\mu}(a_1)) : \varphi(\mu) \in T(\Gamma)^j \} \) is a holomor-
phic section.

Let $P_{\mu}', \subset w_{\mu}(D)$ be the polygon formed by joining the points $w_{\mu}(a_1), i = 1, \ldots, 4g$, by geodesics in $\lambda_D(\mu)$.

**Theorem III.2.7.** Let $\phi(\mu) \in T(\Gamma)^{\mu}$. Then $P_{\mu}', \subset w_{\mu}(D)$ is a simple $j^{\mu'}$-invariant polygon and $P_{\mu}', \text{ (the interior of)}$ $P_{\mu}'$, is a fundamental region for $\Gamma^\mu$.

**Proof.** Let $\tau : D \to w_{\mu}(D)$ be a Riemann map such that $\tau(0) = w_{\mu}(0)$ and $\tau(1) = 1$ (see [2]). As in the proof of Proposition III.2.1, $\tau^{-1}j^{\mu'} \circ \tau = j$. $w = \tau^{-1}ow_{\mu}$, is a quasiconformal self map of $D$ (see [2]) which is $j$-invariant and satisfies $w(0) = 0$. $w$ extends to a self map $\hat{w}$ of $G \cup \{\infty\}$ with the following properties: $\hat{w}(0) = 0$, $\hat{w}(1) = 1$ and $\hat{w}(\frac{1}{z}) = \frac{1}{\hat{w}(z)}$ (see [2]).

Thus $\hat{w}ow_{\mu}\hat{w}^{-1}$ is a Fuchsian group and by Theorem III.4.8 $P_{\mu}^w = \langle w(a_1), \ldots, w(a_{4g}), w(a_1) \rangle$ is a hyperelliptic polygon. Now $\tau(P_{\mu}^w) = P_{\mu}'$, and it is easy to verify that $P_{\mu}'$, has the required properties.

In order to simplify our results for the identity component $T(\Gamma)^{\mu}$, we construct a fiber space over $T(\Gamma)^{\mu}$ in which the fibers are $j$-invariant.

Let $\mu \in M(\Gamma')$ and $\phi_\mu$ be the unique quasiconformal automorphism of $G \cup \{\infty\}$ which fixes 0, 1 and $\infty$, and satisfies the
Beltrami equation \((\phi_\mu)' = \mu(z)(\phi_\mu)_z\) where \(\mu|_{\partial^C} = 0\) (see [3]).

**Lemma III.2.8.** Let \(\mu, \nu \in \mathcal{M}(\Gamma')\). Then \(\mu \sim \nu\) if and only if \(\phi_\mu = \phi_\nu\) on the unit circle.

**Proof.** Since \(w_\mu\) and \(\phi_\mu\) (resp. \(w_\nu\) and \(\phi_\nu\)) are solutions of the same Beltrami equation, there exists a Möbius transformation \(\alpha_\mu\) (resp. \(\alpha_\nu\)) such that \(\alpha_\mu w_\mu = \phi_\mu\) (resp. \(\alpha_\nu w_\nu = \phi_\nu\)) (see [3]).

If \(\phi_\mu = \phi_\nu\) on the unit circle, then \((\alpha_\nu)^{-1} \circ \alpha_\mu \circ w_\mu = w_\nu\) on the unit circle. Since \(w_\nu (w_\mu)^{-1}\) fixes 1, −1 and i, we obtain that \((\alpha_\nu)^{-1} \circ \alpha_\mu\) = id. and \(w_\mu = w_\nu\) on the unit circle.

Suppose conversely that \(w_\mu = w_\nu\) on the unit circle. Then

\[(\text{II.2.1}) \quad \alpha_\nu \circ (\alpha_\mu)^{-1} \circ \phi_\mu = \phi_\nu\]

on the unit circle. Since the transformations \(\phi_\mu \circ \phi_\nu (\phi_\mu)^{-1}\) and \(\phi_\nu \circ \phi_\mu (\phi_\nu)^{-1}\) both are of order 2 and fix 0 and \(\infty\), they are equal. We obtain hence from (II.2.1)

\[(\text{II.2.2}) \quad \phi_\mu \circ \phi_\nu (\phi_\mu)^{-1} = \phi_\nu \circ \phi_\mu (\phi_\nu)^{-1}\]

\[= \alpha_\nu \circ (\alpha_\mu)^{-1} \circ \phi_\mu \circ \phi_\nu (\phi_\mu)^{-1} \circ \alpha_\mu \circ (\alpha_\nu)^{-1}\]

on the image of the unit circle under \(\phi_\nu\).

Since the transformations of (II.2.2) are Möbius, (III.2.2) holds for all \(z \in \mathcal{S} \cup \{\infty\}\). Thus \(\phi_\mu \circ \phi_\nu (\phi_\mu)^{-1}\) and \(\alpha_\nu \circ (\alpha_\mu)^{-1}\) commute and must have the same fixed points (see [9]), or
since they are of order 2, one interchanges the fixed points of the other. We assume the latter possibility does not occur. By definition, \( \alpha_\nu \) and \( \alpha_\mu \) fix 1. Thus \( \alpha_\nu \circ (\alpha_\mu)^{-1} = \text{id} \). The desired result follows from (II.2.1).

We must verify that \( \alpha_\nu \circ (\alpha_\mu)^{-1} \) cannot interchange the two points 0, \( \infty \). Since \( w_\mu \) and \( w_\nu \) agree on the unit circle,

\[
w_\mu \circ \gamma \circ (w_\mu)^{-1} = w_\nu \circ \gamma \circ (w_\nu)^{-1} \quad \text{for all } \gamma \in \Gamma!
\]

In particular \( w = (w_\nu)^{-1} \circ w_\mu \) commutes with each \( \gamma \in \Gamma \). Let \( \gamma = j \). Thus \( w \) either fixes 0 and \( \infty \) or permutes them. It is well known that a quasiconformal map that commutes with \( \Gamma \) must map each component of \( \Gamma \) onto itself (see [16]). In particular \( w_\mu(0) = w_\nu(0) \) and \( w_\nu(\infty) = w_\nu(\infty) \).

The relations \( \alpha_\mu \circ w_\mu = \phi_\mu \) (and the similar one for \( \nu \)) show that \( \alpha_\mu(w_\mu(0)) = \alpha_\nu(w_\nu(0)) \) and \( \alpha_\mu(w_\mu(\infty)) = \alpha_\nu(w_\nu(\infty)) \). Thus \( \alpha_\mu \) and \( \alpha_\nu \) agree on the three points \( w_\mu(0) \), \( w_\mu(1) = 1 \) and \( w_\mu(\infty) \). Hence they are the same Mobius transformation.

We now define \( \mathcal{F}(\Gamma') = \{(\phi(\mu), z) \in T(\Gamma') \times \mathcal{C}; \mu \in M(\Gamma') \} \) and \( z \in \phi_\mu(D) \). Note that by Lemma III.2.8 \( \mathcal{F}(\Gamma') \) is well defined.

**Lemma III.2.9.** \( \mathcal{F}(\Gamma') \) is a manifold isomorphic to \( F(\Gamma') \) (i.e. there exists a biholomorphic map \( f : F(\Gamma') \to \mathcal{F}(\Gamma') \)) such that the following diagram commutes.
where \( p \) and \( \widetilde{p} \) are the natural projection maps.

**Proof.** We first construct a Mobius transformation \( \alpha_\mu \) such that \( \alpha_\mu(1) = 1 \), \( \alpha_\mu(w_\mu(0)) = 0 \) and \( \alpha_\mu(w_\mu(\infty)) = \infty \). We consider three cases.

**Case 1.** \( w_\mu(0) \neq \infty \) and \( w_\mu(\infty) \neq \infty \). Then

\[
\alpha_\mu(z) = \frac{a_\mu z + b_\mu}{c_\mu z + d_\mu}
\]

where

\[
\begin{align*}
a_\mu &= w_\mu(\infty) - 1, \\
b_\mu &= (1 - w_\mu(\infty))w_\mu(0), \\
c_\mu &= (w_\mu(0) - 1) \\
d_\mu &= w_\mu(\infty)(1 - w_\mu(0))
\end{align*}
\]

We need to verify that \( \alpha_\mu \) has the required properties.

1) \( \alpha_\mu(1) = \frac{(w_\mu(\infty) - 1) + (1 - w_\mu(\infty))w_\mu(0)}{(w_\mu(0) - 1) + (w_\mu(\infty)(1 - w_\mu(0)))} = 1 \)

(Note that neither the numerator nor the denominator can be 0 since \( w_\mu(0) \neq 1 \) and \( w_\mu(\infty) \neq 1 \).)
2) \( a_{\mu}(w_{\mu}(0)) = \frac{(w_{\mu}(\infty) - 1)w_{\mu}(0) + (1 - w_{\mu}(\infty))w_{\mu}(0)}{w_{\mu}(0) - 1}w_{\mu}(0) + w_{\mu}(\infty) - w_{\mu}(0)) = 0 \)

(Note that since \( w_{\mu}(\infty) \neq 1 \) and \( w_{\mu}(0) \neq w_{\mu}(\infty) \) the numerator can never be 0.)

Case 2. \( w_{\mu}(0) = \infty \). Then \( a_{\mu}(z) = \frac{1 - w_{\mu}(\infty)}{1 - w_{\mu}(\infty)} \)

and

1) \( a_{\mu}(1) = \frac{1 - w_{\mu}(\infty)}{1 - w_{\mu}(\infty)} = 1 \)

2) \( a_{\mu}(w_{\mu}(0)) = \frac{1 - w_{\mu}(\infty)}{w_{\mu}(0) - w_{\mu}(\infty)} = 0 \)

3) \( a_{\mu}(w_{\mu}(\infty)) = \frac{1 - w_{\mu}(\infty)}{w_{\mu}(\infty) - w_{\mu}(\infty)} = \infty \)

Case 3. \( w_{\mu}(\infty) = \infty \). Then \( a_{\mu}(z) = \frac{z - w_{\mu}(0)}{1 - w_{\mu}(0)} \)

and

1) \( a_{\mu}(1) = \frac{1 - w_{\mu}(0)}{1 - w_{\mu}(0)} = 1 \)

2) \( a_{\mu}(w_{\mu}(0)) = \frac{w_{\mu}(0) - w_{\mu}(0)}{1 - w_{\mu}(0)} = 0 \)
3) \( \alpha_{\mu}(w_\mu(\infty)) = \frac{w_\mu(\infty) - w_\mu(0)}{1 - w_\mu(0)} = \infty \)

In all cases \( \alpha_{\mu} \) has the required properties. Since \( \phi_{\mu} \) is unique, we thus have \( \alpha_{\mu} \circ \phi_{\mu} = \phi_{\mu} \).

Let \( f : F(T') \to \tilde{F}(\Gamma') \) be defined by \( f(\phi(\mu), z) = (\phi(\mu), \alpha_{\mu}(z)) \). Thus \( f^{-1} : \tilde{F}(\Gamma') \to F(T') \) is given by \( f^{-1}(\phi(\mu), z) = (\phi(\mu), (\alpha_{\mu})^{-1}(z)) \).

Since \( w_{\mu}(0) \) and \( w_{\mu}(\infty) \) are the fixed points of the elliptic transformation \( j^\mu \), they depend (homomorphically) on \( \phi_{\mu}(\mu) \in T(\Gamma') \) (see [3]). Thus, from our previous computations in Cases 1, 2 and 3 of this lemma, we obtain that the coefficients of \( \alpha_{\mu} \) depend (holomorphically) on \( \phi_{\mu}(\mu) \in T(\Gamma') \).

It follows that \( \tilde{F}(\Gamma') \) is a manifold (see [4]) and \( f \) is a biholomorphic map. The commutativity of the above diagram is trivial. The proof is complete.

We define the space

\( \mathcal{F}(T(T)^J) = \{(\phi(\mu), z) \in T(T)^J \times \mathbb{C}; \mu \in M(\Gamma) \text{ and } z \in \phi_{\mu}(D)\} \).

(\( \mathcal{F}(T(T)^J) \)) is the pullback of \( \tilde{F}(\Gamma') \) via the natural map \( g : T(T)^J \to T(\Gamma') \) where \( g(\phi(\mu)) = \phi(\mu') \).

**Proposition III.2.10.** Let \( \phi(\mu) \in T(T)^J \). Then the
following hold:

1) $\phi_{\mu}(D)$ is a $j$-invariant domain.

ii) Let $\Gamma^\mu = \phi_{\mu} \circ \Gamma_0 (\phi_{\mu})^{-1}$. $\Gamma^\mu$ has a fundamental region $R \subset \phi_{\mu}(D)$ whose boundary is a smooth $j$-invariant Jordan polygon with sides $A_1', \ldots, A_{4g}'$. Moreover, $\Gamma^\mu$ is generated by transformations $T_i'$, $i = 1, \ldots, 2g$, such that $T_i'(A_1') = j(A_i')$.

iii) Let $P \subset D$ be a hyperelliptic fundamental polygon for $\Gamma$ and let $a_1, \ldots, a_{4g}$ be the vertices of $P$. Let $\tilde{s}_{a_1} : T(\Gamma)^j \to T(T(\Gamma)^j)$ be defined by

$$
\tilde{s}_{a_1}(\tau(\mu)) = (\tau(\mu), \phi_{\mu}(a_1)), \quad i = 1, \ldots, 4g.
$$

The maps $\tilde{s}_{a_1}$ are holomorphic sections.

iv) Let $P_{\phi_{\mu}} \subset \phi_{\mu}(D)$ be the polygon formed by joining the points $\phi_{\mu}(a_1), \ldots, \phi_{\mu}(a_{4g})$ (in order) with geodesics in $\phi_{\mu}(D)$. $P_{\phi_{\mu}}$ is a $j$-invariant fundamental polygon for $\Gamma^\mu$.

Proof. We note first that $\phi_{\mu} \circ \Gamma_0 (\phi_{\mu})^{-1} = j$, $\phi_{\mu}(D) = \phi_{\mu}(w_{\mu}(D))$ and $\Gamma^\mu = \phi_{\mu} \circ \Gamma_0 (\phi_{\mu})^{-1}$. The assertions now follow from Lemma III.2.9, Corollary III.2.5 and Theorem III.2.7.

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6 The Poincaré metric for $\phi_{\mu}(D)$ is defined in the same manner as that for $w_{\mu}(D)$. 
III.3 The polynomial corresponding to the perfectly symmetric polygon. The perfectly symmetric polygon $P^*$ with sides $A_1^*, \ldots, A_{4g}^*$ was defined in II.4. We note that

\[ A_2^* = e^{\frac{2\pi}{4g}} A_1^*, \quad A_3^* = e^{\frac{4\pi}{4g}} A_1^*, \quad \ldots, A_j^* = e^{\frac{(j-1)2\pi}{4g}} A_1^*, \quad j = 1, \ldots, 4g. \]

Let $\Gamma^*$ be the group generated by the transformations $T_i^* \in \text{Aut } D$ such that $T_i^*(A_1^*) = A_i^* + 2g$ with opposite orientation, $i = 1, \ldots, 2g$. Let $\gamma^* : D \to D$ be the map given by

\[ \gamma^*(z) = (e^{\frac{2\pi}{4g}})z, \text{ thus } \]

\[ T_j^* = (\gamma^*)^{-1} \circ T_i^* \circ \gamma^* = \Gamma^* \quad \text{and} \quad \gamma^* \]

induces an automorphism $\gamma$ of $\frac{D}{\Gamma^*}$ (where $\gamma([z]) = [\gamma^*(z)]$) such that the following diagram commutes (here $p^*$ is the projection $p^* : D \to D/\Gamma^*$).

\[
\begin{array}{ccc}
D & \overset{\gamma^*}{\longrightarrow} & D \\
\downarrow p^* & & \downarrow p^* \\
D/\Gamma^* & \overset{\gamma}{\longrightarrow} & D/\Gamma^*
\end{array}
\]

From the definition of $\gamma^*$ it is obvious that $\gamma^* = \text{id}$. Each map $\gamma^k$, $1 \leq k \leq 4g$, $k \neq 2g$, fixes the points $[0]$ and $[a_j^*]$, where $a_j^*$ is any vertex, and no other points. When $k = 2g$, $\gamma^k$ is the hyperelliptic involution $J$ and its fixed points are $[0]$, $[a_j^*]$ and $[c_j^*]$, $i = 1, \ldots, 2g$, where $c_j^*$ is the hyperbolic midpoint of $A_j^*$.

\[ \text{Here } e^{18}A_j^* = \{e^{18}z; \quad z \in A_j^*\} \]
Consider the twofold cover $D/T^* \xrightarrow{\pi} D/T^*/\langle J \rangle$ where $\langle J \rangle$ is the group generated by $J$ and $J$ is the natural projection.

From (III.3.1) and Lemma 11.2.4 we obtain that $\gamma^{-1} \circ (j, T^*) \circ \gamma = \langle j, T^* \rangle$. Thus $\gamma$ induces an automorphism $\hat{\gamma}$ of $D/T^*/\langle H \rangle = C \cup \{\infty\}$ (where $\hat{\gamma}(\{0\}) = \{\gamma(\{0\})\}$) such that the following diagram commutes:

$$
\begin{array}{ccc}
D/T^* & \xrightarrow{\gamma} & D/T^* \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
C \cup \{\infty\} & \longrightarrow & C \cup \{\infty\}
\end{array}
$$

Since $\gamma^2 = J$, it follows that $\hat{\gamma}^2 = \text{id}$. Hence $\hat{\gamma}$ is elliptic of order $2g$ and its fixed points are $\tilde{J}(\{0\})$ and $J(\{a_1\})$. Moreover,

(III.3.2) $\tilde{J}(\{c_{n+1}\}) = \tilde{J}(\gamma(\{c_n\})) = \hat{\gamma}(\tilde{J}(\{c_n\})), \quad n = 1, \ldots, 2g-1$

and

(III.3.3) $\tilde{J}(\{c_1\}) = \tilde{J}(\gamma(\{c_{2g}\})) = \hat{\gamma}(\tilde{J}(\{c_{2g}\}))$.

By normalizing, we may assume that $\tilde{J}(\{0\}) = 0, \tilde{J}(\{a_1\}) = \infty, \tilde{J}(\{c_1\}) = 1$ and $\hat{\gamma}$ is a rotation about the origin by $\frac{\pi}{2g}$. From (III.3.2) and (III.3.3) we obtain that

$$
\tilde{J}(\{c_{n+1}\}) = e^{i \frac{\pi n}{2g}}, \quad n = 0, \ldots, 2g-1.
$$

The surface corresponding to the polynomial

$$
W^2 = \prod_{n=0}^{2g-1} (z - e^{i \frac{\pi n}{2g}})
$$

is branched over $0, \infty$ and the $2g$ points $e^{i \frac{\pi n}{2g}}, \quad n = 0, \ldots, 2g-1$. Since the conformal equivalence class
of a hyperelliptic surface is uniquely determined by the two-fold cover, we have that the polynomial corresponding to $D/\Gamma^*$ is

$$w^2 = z \prod_{n=0}^{2g-1} (z - e^{i \frac{2\pi n}{2g-1}}), \quad n = 0, \ldots, 2g-1.$$ 

The same result may be obtained by means of Teichmüller theory in the following manner:

Let $A = \langle \gamma \rangle$ be the group generated by $\gamma$ and consider the covering $D/\Gamma^* \xrightarrow{\tilde{\gamma}} (D/\Gamma^*)/A$. $\tilde{\gamma}$ is branched at the points $\{0\}, \{a_j\}$ and $\{c_i\}$ (these are the only fixed points of the elements of $A$). We now compute the branching orders.

Since $\{0\}$ and $\{a_j\}$ are fixed by every element of $A$, the branching order of $\tilde{\gamma}$ at each of these points is $4g-1$. The only element of $A$ which fixes the $\{c_i\}$ is the involution $\gamma^2$. Thus the branching order at each $\{c_i\}$ is one.

We compute the genus $g'$ of $(D/\Gamma^*)/A$ by the Riemann-Hurwitz relation (see [17]). Thus $2g - 2 = 2n(g' - 1) + B$ where $n = 4g$ and $B$ is the sum of the branching orders. Since $B = 10g - 2$ we obtain $g' = 0$.

Let $T(\Gamma^*)$ be the Teichmüller space of $\Gamma^*$ and let $\Gamma'$ be the group of lifts to $D$ of the elements of $A$. Thus

$$\Gamma' = \{ g \in \text{Aut } D; p^* h \circ g \text{ for some } h \in A \}$$

where $p^*: D \to D/\Gamma^*$ is the natural projection. The group $\Gamma'/\Gamma^* \cong A$ acts on $T(\Gamma^*)$ in the manner defined in III.1. $T(\Gamma^*)^A$ (the fixed point set of $A$ in $T(\Gamma^*)$) corresponds to surfaces admitting a group isomorphic to $A$ as a group of conformal automorphisms.
Theorem III.3.7 (Kravetz [13]). The set of points in $T(\Gamma)$ left fixed by every member of $T^*/\Gamma$ is precisely $T(\Gamma^*)$.

(Here $T^*$ is the group of lifts of a group of automorphism of $D^*/\Gamma^*$.)

Using Theorem III.3.7 and our previous computation $T(\Gamma^*)^A = T(0, 3)$ (here $T(0, 3)$ is the Teichmüller space of a three times punctured sphere) and there is only one conformal equivalence class of surfaces in $T(\Gamma^*)^A$.

Let $S^*$ be the surface corresponding to the polynomial

$$w^2 = z \prod_{n=0}^{2g-1} (z-e^{\frac{2\pi i}{g}})$$

$S^*$ admits the automorphism

$$(w, z) \xrightarrow{\alpha} (e^{\frac{2\pi i}{2g}}w, e^{\frac{2\pi i}{2g}}z)$$

which is of order $4g$.

Trivially $\alpha^k, 1 \leq k \leq 4g, k \neq 2g$, fixes $(0, 0), (\infty, \infty)$ and no other points. Since $\alpha^{2g}(w, z) = (-w, z)$, $\alpha^{2g}$ is the hyper-elliptic involution.

Clearly there exists a $\mu \in M(\Gamma^*)$ such that $\varphi(\mu) \in T(\Gamma^*)^A$ and $w_\mu(D)/_{\Gamma^*_\mu} \cong S^*$. Since there is only one conformal equivalence class of surfaces represented in $T(\Gamma^*)^A$, $S^* = D^*/\Gamma^*$. 
IV. TOPOLOGICAL PROPERTIES

IV.1 A topological dissection. In Chapter II, by means of uniformization theory, we were able to find hyperelliptic polygons for hyperelliptic surfaces. In this chapter we will find dissections of hyperelliptic surfaces into polygons of the same type without using uniformizing groups. To be more precise, given a hyperelliptic surface $S$ with involution $J$, we will exhibit a set of $J$-invariant loops $\gamma_1, \ldots, \gamma_{2g}$ on $S$ such that by cutting along these loops one obtains a $4g$ sided $J$-invariant polygon.

Suppose $S$ is the hyperelliptic surface corresponding to the polynomial $w^2 = z \prod_{i=1}^{2g+2} (z-a_i)$. Let $\gamma \subset \mathbb{C} \cup \{ \infty \}$ be an oriented Jordan curve such that the points $a_1, \ldots, a_{2g+2}$ lie on $\gamma$ and are ordered according to the orientation of $\gamma$ (Fig. 18).

Let $L_i$, $i = 1, \ldots, g+1$, be the arc of $\gamma$ which joins $a_{2i-1}$ to $a_{2i}$. Let $M_i$, $i = 1, \ldots, 2g$, be an arc (whose interior is contained in the interior component of $\gamma$) joining $a_2$ to $a_{2+1}$. We assume the arcs $M_i$ intersect only at $a_2$ (Fig. 18).

We make cuts along the arcs $L_i$ labeling one side of each cut $+$ and the other $-$ as in Fig. 19. It is well known that if we take another sphere with the same cuts (we label the points on the second sphere $a_1', \ldots, a_{2g+2}'$, the cuts $L'_1, \ldots, L'_{g+1}$ and the arcs $M'_1, \ldots, M'_{2g}$) and attach the two
spheres along corresponding cuts (identifying + with -), we obtain a conformal representation of $S$. Note that the involution $J$ is the interchange of spheres (Fig. 20).

On the surface $S$ the loops $\gamma_1 = M_1^+ M_1^-$ ($M_1$ followed by $M_1^-$) are simple and $J$-invariant (Fig. 20). We will dissect $S$ along these loops.

We cut each sphere along the arcs $M_1$ and $M_1^-$ respectively and then make the proper identifications to obtain the dissection of $S$.

On the first sphere we start at the point $a_2$ and cut in the direction of $a_{i+2}$, labeling the right hand side of the cut $M_1^+$ and the left hand side $M_1^-$ (Fig. 21). We make similar conventions on the second sphere.
We now attach the two spheres along the cuts $L_1$ and $L'_1$ to obtain the dissection of $S$ given by Fig. 22. (For clarity, in Figs. 22 and 23 we have drawn all arcs as straight lines.)

The sides of Fig. 22 are relabeled as follows:

$$
\gamma_1 = M_1^+M_1^-, \quad \gamma_2 = M_2^+M_2^-, \ldots, \quad \gamma_1 = \begin{cases} 
M_1^+M_1^- & \text{i odd} \\
M_1^-M_1^+ & \text{i even}
\end{cases}
$$

$$
\gamma_{2g+1} = M_1^+M_1^-, \quad \gamma_{2g+2} = M_2^+M_2^-, \ldots, \quad \gamma_{2g+1} = \begin{cases} 
M_1^+M_1^- & \text{i odd} \\
M_1^-M_1^+ & \text{i even}
\end{cases}
$$

The desired $4g$ sided polygon is thus obtained in Fig. 23.

We note that on this polygon the involution $J$ corresponds to a conformal self map which identifies the side $\gamma_1$ with $\gamma_{i+2g}$ and has a fixed point at $a_1$. The cuts $L_1, i = 1, \ldots, g+1$, which appear in Fig. 22 correspond to loops around the handles of $S$.

**IV.2 A relation among periods of closed $C^1$ differentials.**

**Proposition IV.2.1.** Let $\omega$ and $\eta$ be closed $C^1$ differentials on a compact Riemann surface $S$ of genus $g$ and let

$$
\int_{\gamma_1} \omega = A_1, \quad \int_{\gamma_1} \eta = A'_1, \text{ where } \gamma_1, i = 1, \ldots, 2g,
$$

are the loops corresponding to the dissection in IV.1.

Then

$$
\int_S \omega \wedge \eta = \sum_{i,j} \epsilon_{ij} A_i A'_j, \text{ where } 1 \leq i, j \leq 2g,
$$

and

$$
\epsilon_{ij} = \begin{cases} 
1 & j < i \\
0 & j = i \\
-1 & j > i
\end{cases}
$$
Fig. 22
Proof. We dissect $S$ along the loops $\gamma_1$ to obtain the polygon $P$ of IV.1 (Fig. 24). Here we have labeled the vertices $\alpha_1, \ldots, \alpha_{4g}$.

(Fig. 24)

Since the interior of $P$ is simply connected, $w = df$ in the interior of $P$. Thus, by Stokes' theorem

$$\int_S w \wedge \eta = \int_P f\eta = \sum_{i=1}^{2g} \left( \int_{\gamma_i} f\eta + \int_{\gamma_i^{-1}} f\eta \right).$$

(IV.2.1)

Let $z_0$ lie in the interior of $P$ and let $z$ and $z'$ be equivalent points on $\gamma_1$ and $\gamma_1^{-1}$ respectively. Join $z$ to $z'$ by a curve $zz'$ passing through $z_0$ and lying in the interior
of $P$ (Fig. 24). Then $f(z) = \int_{z_0}^{z} w$ where the integral is taken along the segment of $\overline{zz'}$ from $z_0$ to $z$. Thus

\[(IV.2.2) \quad \int_{\gamma_1} f\eta + \int_{\gamma_1^{-1}} f\eta = \int_{\gamma_1} ((\int_{z_0}^{z} w) - (\int_{z_0}^{z'} w))\eta = -\int_{\gamma_1} (\int_{\overline{zz'}} w)\eta.\]

Suppose $z \in \gamma_1$. Then $\overline{zz'}$ is homologous to $\overline{\alpha_2} + \gamma_2 + \ldots + \gamma_{2g} + \overline{\alpha_2 z'}$. Thus, since $\int_{\overline{\alpha_2}} w = -\int_{\overline{\alpha_2 z'}} w$, we obtain

\[\int_{\overline{zz'}} w = \int_{\overline{zz'}} df = f(z') - f(z) = \int_{\gamma_2} w + \ldots + \int_{\gamma_{2g}} w.\]

When $z \in \gamma_1$ a similar computation yields

\[(IV.2.3) \quad f(z') - f(z) = \sum_{1 < j} \int_{\gamma_1} w - \sum_{j < 1} \int_{\gamma_j} w\]

Now (IV.2.1), (IV.2.2) and (IV.2.3) yield the desired result.

IV.3. The intersection matrix $P$. We orient the polygon $P$ constructed in IV.1 in a counterclockwise direction. Thus each side $\gamma_i$, $i = 1, \ldots, 2g$, corresponds to an oriented loop in $S$. The positive direction of $\gamma_1$ is from $\alpha_2$ to $\alpha_3$ on the first sphere and from $\alpha_1$ to $\alpha_2$ on the second sphere (Fig. 25).

In general the orientation of $\gamma_1$ will be positive from $\alpha_2$ to $\alpha_{2+1}$ on the first sphere, and positive from $\alpha_{2+1}$ to $\alpha_2$ on the second sphere when $i$ is odd. When $i$ is even, the orientation will be positive from $\alpha_2$ to $\alpha_{2+1}$ on the second sphere, and positive from $\alpha_{2+1}$ to $\alpha_2$ on the first sphere (Fig. 25).
In order to compute the intersection matrix of the loops $\gamma_1$, we need to give a precise definition of the intersection number of two loops. We state the following lemmas without proof (a proof may be found in [8]).

**Lemma IV.2.1.** Let $c$ be a simple closed curve on the Riemann surface $S$. Then there exists a real closed differential $\eta_c$ with the property $\int_c \alpha = (\alpha, \ast \eta_c)$.  

**Lemma IV.2.2.** Let $a$ and $b$ be two homology cycles on $S$. The intersection number of $a$ and $b$ is $a \cdot b = \int_S \eta_a \wedge \eta_b$ and the following properties hold

$$a \cdot b = -b \cdot a$$

$$(a+b) \cdot c = a \cdot c + b \cdot c \quad \text{and} \quad a \cdot b \in \mathbb{Z}.$$  

If we isolate the intersection point $a_2$ of the loops $\gamma_1$ and $\gamma_2$ (Fig. 26) we note that $\gamma_1 \cdot \gamma_2 = 1$ (a proof may be found in [8]).

Similarly, we obtain $\gamma_1 \cdot \gamma_3 = -1$ (Fig. 26).

---

Here $\ast \eta_c$ is the harmonic conjugate of $\eta_c$ and

$$(\alpha, \ast \eta_c) = \int_D \alpha \wedge -\eta_c$$
In general the intersection numbers of the loops \( \gamma_i, i = 1, \ldots, 2g \), are the following (see Fig. 26):

\[
\gamma_1 \cdot \gamma_j = \begin{cases} 
0 & j = 1 \\
-1^j & j > 1 
\end{cases}
\]

\[
\gamma_2 \cdot \gamma_j = \begin{cases} 
-1^j & j < 2 \\
0 & j = 2 \\
-1^{j+1} & j > 2 
\end{cases}
\]

\[
\gamma_3 \cdot \gamma_j = \begin{cases} 
-1^{j+1} & j < 3 \\
0 & j = 3 \\
-1^j & j > 3 
\end{cases}
\]

\[
\vdots
\]

\[
\gamma_1 \cdot \gamma_j = \begin{cases} 
-1^{j+n_1} & j < 1 \\
0 & j = 1 \\
-1^{j+n_2} & j > 1 
\end{cases}
\]

where \( n_1 = \begin{cases} 
1 & i \text{ odd} \\
0 & i \text{ even} 
\end{cases} \)

and \( n_2 = \begin{cases} 
0 & i \text{ odd} \\
1 & i \text{ even} 
\end{cases} \).
This information may be collected in a matrix which we call the intersection matrix $P_0 = (\gamma_i \cdot \gamma_j)$. Note that by Lemma IV.3, $P_0$ is skew-symmetric i.e. $\gamma_i \cdot \gamma_j = -\gamma_j \cdot \gamma_i$. We write the matrix $P_0$ below, its entries are $\gamma_i \cdot \gamma_j$, $1 \leq i, j \leq 2g$.

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\gamma_3$</th>
<th>$\gamma_4$</th>
<th>...</th>
<th>$\gamma_j$</th>
<th>...</th>
<th>$\gamma_{2g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>$\gamma_3$</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$\gamma_4$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>-1</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma_j$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\gamma_i \cdot \gamma_j$</td>
</tr>
<tr>
<td>$\gamma_{2g}$</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

IV.4. A canonical homology basis. To every compact Riemann surface of genus $g \geq 2$, one may associate a set of homology generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ with the intersection matrix

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]
where $0$ is the $g \times g$ zero matrix and $I$ is the $g \times g$ identity matrix. Such a set is called a **canonical homology basis**.

A canonical homology basis appears in Fig. 27.

In IV.1 we associated to every hyperelliptic surface of genus $g \geq 2$ a set of loops $\gamma_1, \ldots, \gamma_{2g}$ (Fig. 25) based at a Weierstrass point $a_2$. (In fact, our construction yields a similar set of loops for any compact surface of genus $g \geq 2$, since topologically such a surface may be obtained in the same manner.)

We will now express the homology classes determined by the loops $\gamma_1, \ldots, \gamma_{2g}$ in terms of the canonical homology loops of Fig. 27 in the following manner:

We first replace each canonical homology loop $\alpha_i$ (resp. $\beta_i$), $i = 1, \ldots, g$, by a homologous loop $\alpha'_i$ (resp. $\beta'_i$) based at $a_2$ (Fig. 28). (A set of loops such as $\alpha'_1, \ldots, \alpha'_g$, $\beta'_1, \ldots, \beta'_g$ is traditionally called a **canonical dissection** of $S$ (see [11]).)

We then write each loop $\gamma_i$, $i = 1, \ldots, 2g$, in terms of the loops $\alpha'_1, \ldots, \alpha'_g$, $\beta'_1, \ldots, \beta'_g$.

We will take products of loops as elements of $\pi_1(S, a_2)$ Since we are only interested in homology, we will write products additively and take equivalence classes of elements in $\pi_1(S, a_2)$ modulo the commutator subgroup of $\pi_1(S, a_2)$. 
Lemma IV.4.1. \( \gamma_1 = -c_i \)

**Proof.** The assertion is obvious by inspection (Figs. 25 and 28).

Lemma IV.4.2. \( \gamma_1 = (-\gamma_{i-1} - a_i) \), whenever \( i < 2g \) is odd.

**Proof.** The loops \(-\gamma_{i-1}\) and \(-a_i\) appear in Fig. 29. Their sum appears in Fig. 30.

Lemma IV.4.3. \( \gamma_1 = (\sum_{j=1}^{1/2} a_j) - \beta_1 + \frac{\beta_{i+2}}{2} \), whenever \( i < 2g \) is even.

**Proof.** The loops \( a_j \) and \( a_j'' \) (Fig. 31) are homologous since they are both homologous to \( a_j \). Thus we obtain that \( \sum_{j=1}^{1/2} a_j'' = \sum_{j=1}^{1/2} a_j \) (Figs. 32 and 33). \( (\sum_{j=1}^{1/2} a_j'') - \beta_1 \) is computed in Fig. 39. The final sum appears in Fig. 35.

Lemma IV.4.4. \( \gamma_{2g} = -\gamma_{2g-1} - \beta_g \)

**Proof.** The loops \( \gamma_{2g-1} \) and \( \beta_g \) appear in Fig. 36. Their sum appears in Fig. 37.
Fig. 29(a) - $\gamma_{i-1}$ (l odd)

Fig. 29(b) - $\alpha'_{i-1}$

Fig. 30(a) - $\gamma'_{i-1} - \alpha'_{i-1}$

Fig. 30(b) - $\gamma'_{i-1} - \alpha'_{i-1}$
Fig. 31(a) $\alpha_j^i$

Fig. 31(b) $\alpha_j^{ii}$

Fig. 32(a) $-\frac{1}{2} \sum_{i=1}^{J-2} \alpha_j^i$ (i even)

Fig. 32(b) $-\frac{1}{2} \sum_{i=1}^{J-2} \alpha_j^{ii}$ (i even)

Fig. 33(a) $-\frac{1}{2} \sum_{i=1}^{J-2} \alpha_j^{ii}$

Fig. 33(b) $-\frac{1}{2} \sum_{i=1}^{J-2} \alpha_j^{ii}$
Fig. 34(a) $\beta_1^i$

Fig. 34(b) $-\sum_{j=1}^{\frac{\nu}{2}} \alpha_j^u + \beta_1^i$ $

Fig. 35(a) - \beta_{i+2}^l$

Fig. 35(b) $-\sum_{j=1}^{\frac{\nu}{2}} \alpha_j^u + \beta_1^i - \beta_{i+2}^l$

Fig. 35(c) $-\sum_{j=1}^{\frac{\nu}{2}} \alpha_j^u + \beta_1^i - \beta_{i+2}^l$
IV.5 Prescribing periods for \( g=2 \). It is well known that to every canonical homology basis one may assign a unique basis of holomorphic differentials \( \eta_1, \ldots, \eta_g \) with the property \( \int_{\alpha_j} \eta_k = \delta_{jk} \). Moreover for this basis, the matrix \( \tau = (B_{jk}) \) with \( B_{jk} = \int_{\beta_j} \eta_k \) is symmetric with positive definite imaginary part. We say such a set of differentials is normalized.

Using the relationship we have just established between the set of loops \( \gamma_1, \ldots, \gamma_{2g} \) and a canonical homology basis \( \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \), one may compute the periods of a normalized set of holomorphic differentials for this basis over the loops \( \gamma_1, \ldots, \gamma_{2g} \).

We will compute these periods for \( g=2 \).

In this case

\[
\int_{\alpha_1} \eta_1 = 1, \quad \int_{\alpha_2} \eta_1 = 0, \quad \int_{\alpha_2} \eta_2 = 0 \quad \text{and} \quad \int_{\alpha_2} \eta_2 = 1
\]

We also have from IV.4

\[
\gamma_1 = -\alpha_1
\]

\[
\gamma_2 = \alpha_1 - \beta_1 + \beta_2
\]

\[
\gamma_3 = -\alpha_1 - \alpha_2 + \beta_1 - \beta_2
\]

\[
\gamma_4 = \alpha_1 + \alpha_2 - \beta_1
\]
Thus

\[ \int_{\gamma_1} \eta_1 = \int_{-\alpha_1} \eta_1 = -1 \]
\[ \int_{\gamma_2} \eta_1 = \int_{\alpha_1} \eta_1 + \int_{-\beta_1} \eta_1 + \int_{\beta_2} \eta_1 = 1 - B_{1,1} + B_{2,1} \]
\[ \int_{\gamma_3} \eta_1 = \int_{-\alpha_1} \eta_1 + \int_{\beta_1} \eta_1 + \int_{-\beta_2} \eta_1 + \int_{-\alpha_2} \eta_1 = -1 + B_{1,1} - B_{2,1} \]
\[ \int_{\gamma_4} \eta_1 = \int_{\alpha_1} \eta_1 + \int_{\alpha_2} \eta_1 + \int_{-\beta_1} \eta_1 = 1 - B_{1,1} \]
\[ \int_{\gamma_1} \eta_2 = 0, \quad \int_{\gamma_2} \eta_2 = -B_{1,2} + B_{2,2} \]
\[ \int_{\gamma_3} \eta_2 = B_{1,2} - B_{2,2} - 1 \text{ and } \int_{\gamma_4} \eta_4 = 1 - B_{1,2}. \]

We collect this information in a matrix \( \tau_0 \).

<table>
<thead>
<tr>
<th></th>
<th>( \gamma_1 )</th>
<th>( \gamma_2 )</th>
<th>( \gamma_3 )</th>
<th>( \gamma_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_1 )</td>
<td>-1</td>
<td>1 - B_{1,1} + B_{2,2}</td>
<td>-1 + B_{1,1} - B_{2,2}</td>
<td>1 - B_{1,1}</td>
</tr>
<tr>
<td>( \eta_2 )</td>
<td>0</td>
<td>-B_{1,2} + B_{2,2}</td>
<td>B_{1,2} - B_{2,2} - 1</td>
<td>1 - B_{1,2}</td>
</tr>
</tbody>
</table>
Given a pair \((\gamma_i, \gamma_j)\), \(i, j = 1, \ldots, 4\), \(i \neq j\) we say that periods are prescribable over \((\gamma_i, \gamma_j)\) if there exists a basis \(w_1, w_2\) of holomorphic differentials such that the matrix of periods over \((\gamma_i, \gamma_j)\) has the following form

\[
\begin{array}{ccc}
\gamma_i & \gamma_j \\
w_1 & 1 & 0 \\
w_2 & 0 & 1
\end{array}
\]

It is obvious that such a basis will exist if the matrix

\[
\begin{array}{ccc}
\gamma_i & \gamma_j \\
\eta_1 & s_{\gamma_i} \eta_1 & s_{\gamma_j} \eta_1 \\
\eta_2 & s_{\gamma_i} \eta_2 & s_{\gamma_j} \eta_2
\end{array}
\]

is invertible. In that case one may solve for \(w_1, w_2\) in terms of \(\eta_1, \eta_2\).

We will show that given any compact surface of genus \(g=2\) and a topological dissection given by curves \(\gamma_1, \ldots, \gamma_4\) as already defined, there always exists a pair \((\gamma_i, \gamma_j)\) on which periods are prescribable. In fact, we can make a stronger statement.

**Lemma IV.5.1.** Let \(S\) be a compact surface of genus \(g=2\). Then if periods are not prescribable over \((\gamma_1, \gamma_2)\), they are...
prescribable over the remaining pairs \((y_1, y_3), (y_1, y_4), (y_2, y_3)\) and \((y_2, y_4)\).

Proof. We examine the possible cases. Suppose \(\text{Det } I_{1,2} = B_{1,2} - B_{2,2} = 0\).

Then i) \(\text{Det } I_{1,3} = 1 + B_{2,2} - B_{1,2} = 1\),

ii) \(\text{Det } I_{1,4} = -1 + B_{1,2} \neq 0\).

Since \(\text{Im } \tau \geq 0\), \(\text{Im } B_{2,2} \neq 0\). By our original assumption \(\text{Im } B_{1,2} = \text{Im } B_{2,2}\)' thus \(\text{Im } B_{1,2} \neq 0\) and \(\text{Det } I_{1,4} \neq 0\).

iii) \(\text{Det } I_{2,3} = -1 + B_{1,1} - B_{2,1} \neq 0\).

If \(-1 + B_{1,1} - B_{2,1} = 0\) then \(\text{Im } B_{1,1} = \text{Im } B_{2,1}\). Since \(\text{Im } B_{1,2} = \text{Im } B_{2,2}\)' we obtain \(\text{Im } B_{1,1} = \text{Im } B_{2,1} = \text{Im } B_{1,2} = \text{Im } B_{2,2}\). This is a contradiction since \(\text{Im } \tau\) is invertible.

iv) \(\text{Det } I_{2,4} = (1 - B_{1,2})(1 - B_{1,1} + B_{2,1})\)

\[ = (\text{Det } I_{1,4}) (\text{Det } I_{2,3}) \neq 0.\]

In all cases the determinant is not zero, thus the corresponding periods are prescribable.
References


