FREDHOLM AND INVERTIBLE TUPLES

OF BOUNDED LINEAR OPERATORS

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RAUL ENRIQUE CURTO

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Abstract of the Dissertation

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We consider the sets I(H), D(H) and N(H) of commuting invertible tuples, doubly commuting invertible tuples and normal invertible tuples, respectively, of bounded linear operators on a Hilbert space H, where invertible is to be understood in the sense of J. L. Taylor: A joint spectrum for several commuting operators, J. Funct. Anal. 6,2(1970). We prove that D(H) and N(H) are arcwise connected, regardless of the dimension of H, and that the same is true for I(H) when H is finite dimensional. Along the way we develop a number of techniques which generalize nicely those of the "one variable" situation. In particular, Sp(T,H)=Sp(T,L(H)) carries over to n-tuples.

We define $F(H)$ to be the set of almost commuting (=commuting modulo the compact operators) tuples of operators on H which are invertible in the Calkin algebra. We obtain an integer-valued index, which is continuous, invariant under compact perturbations and onto Z. A natural question is to determine whether index is the only invariant for the arcwise components of $F(H)$. This is the deformation problem. We solve it in several special cases, while delineating a general approach to its solution. For instance, we prove that (z_1,z_2) and (z_1^*,z_2^*) on $H^2(S^1\times S^1)$ lie in the same path-component. At the same time, we give a comprehensive account of all basic facts of this "several variables" theory, in complete harmony with the classical knowledge. We prove that an essentially normal n-tuple with all commutators in trace class has necessarily index zero ($n\geq 2$), and that a natural generalization of Atkinson's theorem holds in $F(H)$.

Dedication

To my wife, Ines. To my daughter, Carina.

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CHAPTER 0: INTRODUCTION

Given an algebra B with identity 1 and an element a of B, one says that a is invertible if there exists $b \in B$ such that $ab=ba=1$. For two or more commuting elements a_i $(i=1,\dots,n)$ of B, there is a classical notion of joint nonsingularity for the case B commutative, which requires the existence of elements $b_i \in B$ satisfying the equation $a_1b_1+\dots+a_nb_n=1$. The spectrum thus obtained retains most of the properties of the "one variable" spectrum. In the general case, one can either replace B by some commutative subalgebra containing the a_i's or ask for a solution (in B) of the preceding equation. In one case the spectrum will depend strongly on the choice of the subalgebra; in the other, some of the usual properties will not hold.

When B is L(X), the algebra of bounded operators on a Banach space X, the open mapping theorem implies at once that an operator T is invertible iff $\ker T=(0)$ and $\operatorname{ran} T=X$. Thus, invertibility can be defined in terms of the action of T, rather than the existence of an operator $S \in L(X)$ with $TS=ST=I$. Unfortunately, the classical notion of nonsingularity for a commuting tuple of operators on X, as explained in the first paragraph, fails to reflect the actions of the operators. J. L. Taylor discovered a new definition, independent of the subalgebra, and that does reflect those actions. He used the Koszul complex for the tuple and defined invertibility as exactness of the complex. It turned out that all standard results of the classical theory carry over when one uses this new notion, which we take for our work.

The topological structure of the set of invertible operators on a Hilbert space H has been of great importance in Operator Theory. In this thesis we study the related question for tuples of operators. We prove that the sets D(H) and N(H) of doubly commuting invertible tuples and commuting tuples of normal operators, respectively, are arcwise connected, and that the same holds for I(H) (=commuting invertible tuples) when the dimension of H is finite. Along the way we develop a number of techniques which generalize nicely those of the "one variable" situation. In particular, Sp(T,H)= Sp(T,L(H)) carries over to n-tuples.

An operator T is said to be Fredholm if its image in the Calkin algebra A(H) (=L(H) modulo the compacts) is invertible. A well known theorem of Atkinson states that T is Fredholm iff ran T is closed and both ker T and ker T^* are finite dimensional. One then defines the index of T as dim ker T-dim ker T^* and shows that it is continuous, invariant under compact perturbations and onto Z. In connection with these ideas, we consider almost commuting (=commuting modulo the compacts) tuples of operators on H and define a notion of Fredholmness (as in [4]), in complete harmony with the preceding. We obtain an index, which is continuous, invariant under compact perturbations and onto Z. We also get a natural generalization of Atkinson's theorem. At the same time, we give a complete account of all basic facts of this "several variables" theory.

Associated with index, the deformation problem arises naturally. We solve it in many particular cases, while delineating a general approach to its solution. For instance, we prove that (z_1, z_2) and (z_1^*, z_2^*) on $H^2(S^1 \times S^1)$ lie in the same path-component.

The organization of the paper, intended to be expository on the subject, is as follows:

Chapter I is devoted to studying the joint spectrum, summarizing the main results in [12] and [13] as needed for our purposes. At the same time we prove some additional facts on the Koszul complex and obtain a matrix representation for a tuple.

We then consider, in Chapter II, questions of Fredholmness and invertibility of tuples. In this direction we prove Proposition 3.1, a key step for the subsequent results. In particular, invertibility turns out to be equivalent to the usual notion of invertibility for the associated matrix representation. Consequently, $Sp(T,H)=Sp(T,L(H))$ remains true when we consider tuples. Many fundamental facts on invertible tuples, including special manipulations of the coordinates, which will prove to be useful in dealing with the deformation problem, and a natural generalization of Atkinson's theorem, complete the chapter.

Index is the main subject studied in III. In addition to the properties mentioned at the beginning, we obtain an "Euler characteristic formula" version. We then compute the indices of $(z_1,\ldots,z_n)$ acting on both $H^2(S^1\times\ldots\times S^1)$ and $H^2(S^{2n-1})$, and study the way index behaves under algebraic perturbations of the coordinates. Interesting enough is the fact that an essentially normal n-tuple with all commutators in trace class has index zero ($n\geq 2$). Finally, using index arguments we show that $Sp(z_1,\ldots,z_n)$ is $\prod_{i=1}^{n} D_i$ or B^{2n}, according to the space we consider.

The deformation problem is studied at length in Chapter IV. We first solve it in many easy cases and then give a complete exposition of the essentially normal situation, proved in [4]. By showing an explicit path from (z_1,z_2) on the bidisc to (z_1,z_2) on the sphere, we conclude that they lie in the same path-component. We then show how, in case our tuples have first coordinate almost doubly commuting with the rest, and closed range, the problem reduces to tuples with at least one partial isometry V. Next, we solve the deformation problem when V is semi-Fredholm, reducing the situation to our knowledge of $H^2(S^1 \times S^1)$.

In Chapter V we only consider commuting tuples. After a series of algebraic lemmas (interesting in their own right) involving invertibility and some manipulations with the Koszul boundary maps, we get to the connectedness of invertible tuples in finite dimensional spaces. We show this by using a simultaneous upper triangular form for commuting matrices. A transfinite induction argument then plays a central role in our proof of the connectedness of doubly commuting invertible tuples, along with an exhaustive analysis of the elementary cases. The connectedness of normal invertible tuples complete the chapter.

We now establish our notation.

Notation

We denote by H a (complex) Hilbert space, L(H) is the algebra of bounded linear operators on H, K(H) is the ideal of compact operators and A(H) the Calkin algebra $L(H)/_{K(H)}$. The canonical projection is $\pi : L(H) \longrightarrow A(H)$, sometimes called the Calkin map. Whenever L(H) and A(H) are in the context, we shall agree to denote the elements in L(H) by capital letters and reserve small letters for those in A(H). Thus, if T and t are used, we mean: $T\in L(H)$, $t\in A(H)$ and $\pi(T)=t$.

We shall consider $L^2(S^{2n-1})$, $n=1,2,\ldots$, the space of square integrable functions with respect to surface area measure. $H^2(S^{2n-1})$ is the subspace of functions which are boundary values of analytic functions in B^{2n}. For $k\in Z^n$, $z\in S^{2n-1}$ we define $|k|=\sum_{i=1}^{n} k_i$, $k!=\prod_{i=1}^{n} k_i!$, $z^k=\prod_{i=1}^{n} z_i^{k_i}$. There is a natural orthonormal basis for $H^2(S^{2n-1})$, namely: $e_k=c_k z^k$, where $c_k=\frac{1}{\sqrt{2\pi^n}}\sqrt{\frac{(n+|k|-1)!}{k!}}$ (cf. [2]).

We shall write $L^2(S^1\times\ldots\times S^1)$ (n factors) for the space of square integrable functions on $S^1\times\ldots\times S^1$ with respect to the product measure induced by taking the Haar measure on each circle. Then $f_k=z^k$ defines an orthonormal basis for $L^2(S^1\times\ldots\times S^1)$ and $H^2(S^1\times\ldots\times S^1)$ is then defined as the subspace of functions with $(f,f_k)=0$ whenever $k\notin(Z^+)^n$. Under the identification $L^2(S^1\times\ldots\times S^1)$ $=L^2(S^1)\otimes\ldots\otimes L^2(S^1)$, $H^2(S^1\times\ldots\times S^1)$ becomes $H^2(S^1)\otimes\ldots\otimes H^2(S^1)$. The functions in $H^2(S^1\times\ldots\times S^1)$ can be thought of as boundary values of $L^2(\prod_{i=1}^{n} D_i)$-functions which are analytic on the interior of the polydisc.

For $\varphi\in L^{\infty}(S^{2n-1})$ (respectively $L^{\infty}(S^1\times\ldots\times S^1)$), we define the Toeplitz operator $T_\varphi\in L(H^2(S^{2n-1}))$ (respectively $W_\varphi\in L(H^2(S^1\times\ldots\times S^1))$) by $T_\varphi f=P(\varphi f)$ (resp. $W_\varphi f=P(\varphi f)$), where $f\in H^2$ and P is the orthogonal projection in $L(L^2)$ onto H^2. Under the above identifications, it follows that, if $\varphi_i\in L^{\infty}(S^1)$ $(i=1,\ldots,n)$ and φ is defined on $S^1\times\ldots\times S^1$ by $\varphi(z_1,\ldots,z_n)=\varphi_1(z_1)\ldots\varphi_n(z_n)$, then $\varphi\in L^{\infty}(S^1\times\ldots\times S^1)$ and W_φ is $T_{\varphi_1}\otimes\ldots\otimes T_{\varphi_n}$. In particular, if $\varphi(z_1,\ldots,z_n)=z_i$, then W_φ is $I\otimes\ldots\otimes T_z\otimes\ldots\otimes I$, where T_z is the unilateral shift on $H^2(S^1)$.

Preliminary Results

We now state and prove some standard facts that will be needed in our work.

PR 1: An operator $T\in L(H)$ is

left invertible iff T^*T is invertible

right invertible iff TT^* is invertible

invertible iff T^*T and TT^* are invertible.

More generally, the same holds for elements of any C^*-algebra.

Proof: It suffices to show the first statement, by virtue of the existence of faithful representations and spectral permanence. Assume T is left invertible; then T^* is right invertible, i.e., T^* is onto. Since ran $T^*=\mathrm{ran}(T^*T)^{\frac{1}{2}}$

by polar decomposition, we see that T^*T is onto, or T^*T is invertible, being self-adjoint. Conversely, if T^*T is invertible, then $(T^*T)^{-1}T^*$ is a left inverse for T. The other assertions follow in the same way.

PR 2: Let B be a C^*-algebra and d be a square matrix over B. Let d^* be the matrix adjoint of d in the sense that the (i,j)-entry of d^* is the adjoint of the (j,i)-entry of d. Consider d acting on B. Then $\ker d=\ker d^*d$. Consequently, $\ker d^*\cap \operatorname{ran} d=(0)$.

Proof: Let k be the order of d and assume that $d^*da=0$, where $a=\begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}$ and $a_i\in B$ $(i=1,\dots,k)$. Let b be the $k\times k$-matrix over B whose first column is a and the rest are zero. Then $d^*db=0$. Therefore, $\|db\|^2=\|b^*d^*db\|=0$, so that $db=0$, or $da=0$. (The norm we use here is the natural norm that makes $M_k(B)$ into a C^*-algebra.) We have thus proved that $\ker d^*d\subset \ker d$. The other inclusion is obvious.

PR 3: Let B be a C^*-algebra. We know that there exists a $*$-isometric isomorphism ϕ of B into $L(H)$ for some Hilbert space H. Therefore, every element a of B has a polar decomposition, when seen in $L(H)$, of the form QW, where Q is the image of $(aa^*)^{\frac{1}{2}}$. When B is a W^*-algebra, W is also in $\operatorname{im}\phi$. Similarly, we can get decompositions of the form VP.

PR 4: Let T be an operator in $L(H)$ and $T=VP=QW$ its polar decompositions. Then $\operatorname{ran} T$ is closed iff $\operatorname{ran} P$ is closed iff ran T^* is closed iff ran Q is closed.

Proof: If $Px_n \to y$, then $VPx_n \to Vy$. If ran T is closed, $Vy=Tz$ for some z. Then $V^*Vy=V^*Tz=Pz$. But y is in $(\ker P)^\perp=(\ker V)^\perp$, so that $V^*Vy=y$. Therefore, ran P is closed. Next, ran P=ran T^*, so that ran P closed $\Longrightarrow$ ran T^* closed. By the first argument, if ran T^* is closed, so is ran Q. Finally, ran Q=ran T, so that ran Q closed $\Longrightarrow$ ran T closed.

PR 5: Let P be a positive operator. Then ran P is closed iff ran P^2 is closed.

Proof: "Only if". Assume that $P^2x_n \to y$. Then $y=Pz$ for some z, being in the closure of ran P. Moreover, z can be chosen in $(\ker P)^\perp$=ran P, that is, $z=Px$ for some x. Thus, $y=P^2x$.
"If". $\overline{\text{ran } P}=(\ker P)^\perp=(\ker P^2)^\perp$ (by PR 2)=ran $P^2 \subset$ ran P, so that ran P is closed.

PR 6: Let M, N be subspaces of H and $P=P_M$. Then

$$M\cap(PN)^\perp=M\cap N^\perp.$$

Proof: Let $x\in M\cap(PN)^\perp$, $y\in N$. Then $(x,y)=(Px,y)=(x,Py)=0$, showing one inclusion. Conversely, let $x\in M\cap N^\perp$ and $y\in N$. then $(x,Py)=(Px,y)=(x,y)=0$.

We conclude this chapter studying the T_{z_i}'s and W_i's (see also [2]).

We shall first compute the polar decompositions of the

T_{z_i}'s. Let $T_{z_i}=S_iP_i$. We know that $P_i^2=T_{z_i}^*T_{z_i}$. A simple calculation shows that

$$T_{z_i}^* = \begin{cases} 0 & k_i=0 \\ \dfrac{c_{{}^i k}}{c_k}\, e_{{}^i k} & k_i\geq 1 \end{cases},$$

where ${}^i k=(k_1,\ldots,k_i-1,\ldots,k_n)$.

Then

$$P_i e_k = \frac{c_k}{c_{k^{(i)}}}\, e_k .$$

Therefore,

$$S_i e_k = e_{k^{(i)}} \qquad (k^{(i)}=(k_1,\ldots,k_i+1,\ldots,k_n).$$

We now observe that

$$\Big(\sum_{i=1}^{n} T_{z_i}^*T_{z_i}\Big)e_k = \sum_{i=1}^{n} P_i^2 e_k = \sum_{i=1}^{n} \frac{c_k^2}{c_{k^{(i)}}^2}\, e_k = e_k ,$$

that is,

$$T_{z_1}^*T_{z_1}+\ldots+T_{z_n}^*T_{z_n}=I .$$

By explicit computation, one can show that $\left[T_{z_i}^*, T_{z_i}\right]$ is compact ($i=1,\ldots,n$), so that the T_{z_i}'s form a commuting collection of essentially normal operators.

We now consider $H^2(S^1\times\ldots\times S^1)$ (n times).

<u>PR 7</u>: Let $f\in H^2(S^1\times\ldots\times S^1)$ and assume that $f(z_1,\ldots,\lambda_i,\ldots,z_n)=0$ for some λ_i of modulus less than 1 and all z_j: $|z_j|<1$ ($i\neq j$). Then there exists $g\in H^2(S^1\times\ldots\times S^1)$ such that $f=(W_i-\lambda_i)g$.

<u>Proof</u>: Since f is analytic in the interior of the polydisc and $f(z_1,\ldots,\lambda_i,\ldots,z_n)=0$, there exists g analytic such that $f(z)=(z_i-\lambda_i)g(z)$. Using the power series representation of f in $|z_i-\lambda_i|<\varepsilon$, $|z_j|<1$ ($j\neq i$), it is easy to verify that g is actually in $H^2(S^1\times\ldots\times S^1)$.

CHAPTER I: THE JOINT SPECTRUM

Given a commutative Banach algebra B with identity and an n-tuple $a=(a_1,\ldots,a_n)$ of elements of B, one can say that a is nonsingular provided that there exists an n-tuple $b=(b_1,\ldots,b_n)$ of b_i's in B such that $ab=a_1b_1+\ldots+a_nb_n=1$; equivalently, if a is not contained in any proper maximal ideal of B. The spectrum $Sp_B(a)$ is the set $\{\lambda\epsilon C^n;\ a-\lambda=(a_1-\lambda_1,\ldots,a_n-\lambda_n)$ is singular$\}$. It is then possible to define nonsingularity for a commuting tuple $a=(a_1,\ldots,a_n)$ of bounded linear operators acting on a Banach space X by considering a certain commutative Banach subalgebra of L(X) containing the a_i's. It is unfortunate that the spectrum gotten in this way depends very strongly on the algebra considered, rather than on the actions of the a_i's on X.

In [12] J. L. Taylor introduced a new notion of spectrum that does not involve any algebra and does reflect those actions. We shall spend the rest of the chapter studying this notion. In section 1 we look at the algebraic machinery, the Koszul complex, which is the key ingredient in Taylor's definition. We also obtain a recursive method to get the boundary maps and associate a matrix to any commuting tuple of elements of an algebra with involution. Although we do not need that for section 3, it will be important for our work in the subsequent chapters. In section 3, a summary of the results on the spectrum we shall need later is given, along with the appropriate references for the reader not familiar with them.

1. The Koszul complex. Consider the complex exterior algebra E^n on n generators, that is, E^n is the complex algebra with identity e generated by indeterminates $e_1,\dots,e_n$ such that $e_i\wedge e_j=-e_j\wedge e_i$, where $\wedge$ denotes multiplication. E^n is graded, $E^n=\bigoplus_{p=0}^{\infty}E_p^n$, with $E_p^n\wedge E_q^n\subset E_{p+q}^n$. The elements $e_{j_1}\wedge\dots\wedge e_{j_p}$ with $1\leq j_1<\dots<j_p\leq n$ form a basis for E_p^n $(p>0)$, while $E_0^n=\mathbb{C}e$. It follows easily that $E_n^n=\mathbb{C}(e_1\wedge\dots\wedge e_n)$ and $E_p^n=0$ for $p>n$. Moreover, $\dim E_p^n=\binom{n}{p}$, so that, as a vector space over $\mathbb{C}$, E_p^n is isomorphic to $\mathbb{C}^{\binom{n}{p}}$. We also define $E_p^n=0$ for $p<0$.

If B is a complex algebra and X is a left B-module, we set $E_p^n(X)=X\otimes_{\mathbb{C}}E_p^n$ and consider $E_p^n(X)$ as a left B-module. Given an n-tuple $a=(a_1,\dots,a_n)$ of elements in the center of B, we define a boundary map $d_p:E_p^n(X)\to E_{p-1}^n(X)$ by $d_p(x\otimes e_{j_1}\wedge\dots\wedge e_{j_p})=\sum_{i=1}^{p}(-1)^{i+1}a_{j_i}x\otimes e_{j_1}\wedge\dots\wedge\hat{e}_{j_i}\wedge\dots\wedge e_{j_p}$ when $p>0$ and $d_p=0$ for $p\leq 0$. ($\hat{}$ means deletion.)

It is easy to see that $d_pd_{p+1}=0$ for all p, so that ran $d_{p+1} \subset \ker d_p$ (all p). In other words, $\{E_p^n(X), d_p\}$ is a chain complex, called the Koszul complex for a and denoted $E(X,a)$. As a vector space, $E_p^n(X)$ is $X^{\binom{n}{p}}$, where X^k denotes direct sum of k copies of X. If we split the basis of E_p^n $(n>1,\ p\geqslant 1)$ into $\{e_{j_1}\wedge\ldots\wedge e_{j_p} : 1\leq j_1<\ldots<j_p\leq n-1\}$ and $\{e_{j_1}\wedge\ldots e_{j_{p-1}}\wedge e_n : 1\leq j_1<\ldots<j_{p-1}\leq n-1\}$ we get a corresponding direct sum decomposition $E_p^n(X)=E_p^{n-1}(X)\oplus E_{p-1}^{n-1}(X)$ (where we have made obvious identifications), so that $d_p^{(n)}=d_p(E_p^n(X))$ can be written as a two by two matrix (observe that $d_p^{(n)}: E_p^{n-1}(X)\oplus E_{p-1}^{n-1}(X)\longrightarrow E_{p-1}^{n-1}(X)\oplus E_{p-2}^{n-1}(X))$. The action of $d_p^{(n)}$ on E_p^{n-1} is that of $d_p^{(n-1)}=d_p(E_p^{n-1}(X),(a_1,\ldots,a_{n-1}))$ and

$$d_p^{(n)}(x\otimes e_{j_1}\wedge\ldots\wedge e_{j_{p-1}}\wedge e_n)=\sum_{i=1}^{p-1}(-1)^{i+1}a_{j_i}x\otimes e_{j_1}\wedge\ldots\wedge \hat{e}_{j_i}\wedge\ldots\wedge e_{j_{p-1}}\wedge e_n+(-1)^{p+1}a_n x\otimes e_{j_1}\wedge\ldots\wedge e_{j_{p-1}}.$$

It then follows that

$$(1)\qquad d_p^{(n)}=\begin{pmatrix} d_p^{(n-1)} & (-1)^{p+1}\operatorname{diag}(a_n)\\ 0 & d_{p-1}^{(n-1)}\end{pmatrix}\qquad (n>1,\ p\geqslant 1)$$

This gives a recursive method to obtain the d_p's for $(a_1,\ldots a_n)$ knowing those for $(a_1,\ldots,a_{n-1})$ $(n>1)$.

It is important to associate a matrix to every n-tuple $a=(a_1,\ldots,a_n)$; this will become apparent in Chapter II. The way we proceed is the following: given the tuple a, we consider the boundary maps d_p of the Koszul complex for a and construct a matrix $\hat{a}$ by setting:

$$\hat{a} = \begin{pmatrix} d_1 & & & \\ d_2^* & d_3 & & \\ & d_4^* & \ddots & \end{pmatrix}, \tag{2}$$

where d_i is regarded as an $\binom{n}{i-1}$ by $\binom{n}{i}$ matrix, and d_i^* is the adjoint of d_i as a matrix, i.e., the (j,k)-entry of d_i^* is the adjoint of the (k,j)-entry of d_i (as we said in the introductory paragraph, we are doing this only in case B has an involution).

Examples: (i) If $a=(a_1,a_2)$, then

$$\hat{a} = \begin{pmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{pmatrix}$$

(ii) If $a=(a_1,a_2,a_3)$, then

$$\hat{a} = \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ -a_2^* & a_1^* & 0 & a_3 \\ -a_3^* & 0 & a_1^* & -a_2 \\ 0 & -a_3^* & a_2^* & a_1 \end{pmatrix}$$

It is clear from the above definition that a is a square matrix of order 2^{n-1}, since $\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k}$ and $\sum_k \binom{n}{k} = 2^n$. By looking closely at example (ii), we see that $\hat{a}$ is also

$$\begin{pmatrix} a_{12} & \tilde{a}_3 \\ -\tilde{a}_3^* & a_{12}^* \end{pmatrix},$$

where $a_{12}=\widehat{(a_1,a_2)}$ and $\tilde{a}_3=\mathrm{diag}(a_3)$. Thus

$$(\widehat{a_1,a_2,a_3}) = (\widehat{(\widehat{a_1,a_2}),\tilde{a}_3}).$$

In the general case we have:

$$(3) \qquad (\widehat{a_1,\ldots,a_n}) \cong (\widehat{(\widehat{a_1,\ldots,a_{n-1}}),\tilde{a}_n}) ,$$

where $\tilde{a}_n = \mathrm{diag}(a_n)$, of order 2^{n-2}, and $\cong$ means that some permutations of rows and columns are needed to get equality.

In [14], Vasilescu gives another way of assigning a matrix to a commuting tuple of operators on a Hilbert space H which turns out to be self-adjoint, acting on $H \otimes C^{2^n}$. For our purposes, however, our construction will be more advantageous, especially in studying the index of an almost commuting tuple of bounded linear operators on H, which will be defined in terms of the index of the corresponding $\wedge$.

We conclude this section with the following

<u>Lemma 1.1</u>. Let B be a C^*-algebra, $a=(a_1,\ldots a_n)$ be a commuting tuple of elements of B and $\hat{a}$ be associated as before. Then $\hat{a}$ is normal if and only if a_1 is normal.

<u>Proof:</u> A straightforward computation shows that $\hat{a}^*\hat{a}-\hat{a}\hat{a}^*$ is a block diagonal matrix whose diagonal entries are either $a_1^*a_1-a_1a_1^*$ or $a_1a_1^*-a_1^*a_1$. The result then follows easily from this.

2. Taylor's spectrum

Definition 2.1. Let B be a Banach algebra, X be a Banach space which is a left B-module, $a_1,\dots,a_n$ be elements in the center of B and E(X,a) be the Koszul complex for a. Then, a is said to be nonsingular if E(X,a) is exact. We write Sp(a,X) for the set of tuples $\lambda\in C^n$ such that $a-\lambda=(a_1-\lambda_1,\dots,a_n-\lambda_n)$ is singular. The particular choice of B does not affect Sp(a,X), that depends only on the actions of the a_i's (cf. [12]).

Examples:

(i) If $T\in L(X)$, the Koszul complex for T is $0\to X\xrightarrow{T}X\to 0$, so that E(X,T) is exact iff T is invertible, and $Sp(T,X)=\sigma(T)$. That is, this notion of nonsingularity for a single operator is identical with invertibility. Although we shall use both words to indicate a tuple satisfies the above definition, the preceding example seems to show that the latter is the right one.

(ii) If $T_1,T_2\in L(X)$ and they commute, the Koszul complex for $T=(T_1,T_2)$ is $0\to X\xrightarrow{d_2}X\oplus X\xrightarrow{d_1}X\to 0$, where $d_2x=-T_2x\oplus T_1x$ and $d_1(x\oplus y)=T_1x+T_2y$. Thus, E(X,T) is exact iff d_2 is one-to-one, ran $d_2=\ker d_1$ and d_1 is onto; or, $\ker T_1\cap\ker T_2=0$, ran T_1 + ran T_2 =X and, if $T_1x+T_2y=0$, there is a $z\in X$ such that $x=-T_2z$, $y=T_1z$. For instance, it suffices (but it is not necessary) to have T_1 (or T_2)

invertible. Thus $Sp(T,X) \subset \sigma(T_1) \times \sigma(T_2)$. Much more is true in general.

Proposition 2.2. (Lemma 3.1 and Theorem 3.2 in [12]) Let $a_1,\dots,a_n,a_{n+1}$ be in the center of B and set $a=(a_1,\dots,a_n)$ and $a'=(a_1,\dots,a_n,a_{n+1})$. If $P: C^{n+1} \to C^n$ is the projection on the first n coordinates, then $P(Sp(a',X))=Sp(a,X)$. More generally, if $s: \{1,\dots,k\} \to \{1,\dots,n\}$ is an injection, $s^*a=(a_{s(1)},\dots,a_{s(k)})$ and $s^*z=(z_{s(1)},\dots,z_{s(k)})$, then $s^*Sp(a,X)=Sp(s^*a,X)$. In particular, if p is a permutation, then a is nonsingular iff p^*a is nonsingular.

Corollary 2.3. (Corollary to Theorem 3.2 in [12]) If X is a nonzero Banach space, then $Sp(a,X) \neq \emptyset$ for each tuple $a=(a_1,\dots,a_n)$ of elements of the center of B.

Proposition 2.4. (Theorem 3.1 in [12]) With a, B, X as before and $X \neq (0)$, $Sp(a,X)$ is a nonempty compact subset of the closed polydisc $D_{r(a)}$ of multiradius $r(a)=(r(a_1),\dots,r(a_n))$, where $r(a_i)=\lim_n \|a_i^n\|^{1/n}$ is the spectral norm of a_i.

There is an important connection between the ideal theoretical notion of joint spectrum and Taylor's.

Proposition 2.5. (Lemma 1.1 in [12]) Let $a_1,\dots,a_n$ be elements of the center of some complex algebra B and X be a left B-module. If $a_1b_1+\dots+a_nb_n=1$ for some $b_1,\dots,b_n\in B$, then $a=(a_1,\dots,a_n)$ is nonsingular. Consequently, $Sp(a,X)\subset Sp_B(a)$.

We shall illustrate the proposition by considering a pair $a=(a_1,a_2)$. Assume that there exist b_1, $b_2\in B$ such that $a_1b_1+a_2b_2=1$ and that the b_i's commute with the a_j's. If $d_2x=0$, then $a_1x=a_2x=0$. Multiplying on the left by b_1 and b_2, respectively, we get $b_1a_1x=0$ and $b_2a_2x=0$, so that $x=(b_1a_1+b_2a_2)x=0$, or $\ker d_2=0$. If $d_1(x\oplus y)=0$, then $a_1x+a_2y=0$, so that $b_1a_1x+b_1a_2y=0$, or $(1-a_2b_2)x+a_2b_1y=0$, or $x=-a_2(b_1y-b_2x)$. Set $z=b_1y-b_2x$. Then $a_1z=a_1b_1y-a_1b_2x=y-a_2b_2y-a_1b_2x=$ $y-b_2(a_2y+a_1x)=y$, as desired. Finally, if $z\in X$, let $x=b_1z$, $y=b_2z$. Then $a_1x+a_2y=z$, showing that d_1 is onto.

The inclusion $Sp(a,X)\subset Sp_B(a)$ can actually be proper. In his paper, Taylor showed, using a 5-tuple $a=(a_1,\dots,a_5)$ that $Sp(a,X)\neq Sp_{(a)'}(a)$, where $(a)'$ is the commutant of the set $a_1,\dots,a_5$. (In general, $Sp(a,X)\subset$ $Sp_{(a)'}(a)\subset Sp_{(a)''}(a)\subset Sp_{(a)}(a)$ –(a) being the Banach algebra generated by the a_i's–, so that the cut was made at the precise point.) There was the possibility, however, of having equality for shorter tuples (the given example vanished for $n<5$). In a letter to R. G. Douglas, Taylor mentioned the fact that (W_1,W_2) on $H^2(S^1\times S^1)$ (W_i being multiplication by the coordinate z_i), which we shall see produces a commuting invertible pair in the Calkin algebra $A(H^2(S^1\times S^1))$, is an example where proper containment also holds. We shall give a proof of that in next chapter, when we study that pair.

In case X=B and B is regarded as a left B-module under the left regular representation, $Sp(a,B)=Sp_B(a)$. There are also geometric conditions on $Sp(a,X)$ which actually force equality. For instance, if $Sp(a,X)$ is polynomially convex, then $Sp(a,X)=Sp_B(a)$ for any closed subalgebra $B\subset L(X)$ with $a_1,\dots,a_n$ in its center (see [13] for a complete treatment of the subject).

We now proceed to state the functional calculus.

Proposition 2.6. (Theorem 4.8 in [13]) Let $a=(a_1,\dots,a_n)$ be a commuting tuple in $L(X)$, U be a domain containing $Sp(a,X)$ and $f_1,\dots,f_m$ be holomorphic on U. Let $f:U\to C^m$ be defined by $f(z)=(f_1(z),\dots,f_m(z))$ and $f(a)$ be the tuple $(f_1(a),\dots,f_m(a))$. Then $Sp(f(a),X)=f(Sp(a,X))$.

We conclude this chapter with a definition. If the tuple a is singular, then at least one of the homology modules $H_p(E(X,a)) \overset{def}{=} \ker d_p / \operatorname{ran} d_{p+1}$ is nonzero. Each nonzero element of $H_p(E(X,a))$ represents a singularity of a certain type for a. For instance, if $k\in H_n(E(X,a))$,

$k = x \otimes e_1 \wedge \ldots \wedge e_n$, then $a_i x = 0$ for all $i = 1, \ldots, n$. So H_n can be thought of as $\bigcap_{i=1}^{n} \ker a_i$. Similarly H_0 is $X / \sum_{i=1}^{n} a_i X$, where $a_i X$ is the image of a_i.

CHAPTER II: FREDHOLM AND INVERTIBLE TUPLES

The classical definition of a Fredholm operator $T \in L(H)$ requires the range to be closed and both ker T and ker T^* to be finite dimensional. A fundamental theorem of Atkinson says that this is equivalent to the invertibility of $\pi(T)$ in the Calkin algebra (cf. [3]). Consequently, many authors prefer this algebraic definition, from which invariance of the Fredholm class F under compact perturbations and openness in $L(H)$ follow trivially. But the classical approach serves well to define index(T) as dim ker T - dim ker T^*. It can be proved that index is continuous, invariant under compact perturbations and that the arcwise components of F can be put into a one-to-one correspondence with Z (cf. [3]).

Since we have a notion of invertibility for n-tuples, it seems reasonable to consider the possibility of extending the above to almost commuting tuples of operators on H (see [4] where that idea first appeared). We do this in section 1, in a slightly more general setting. Section 2 is devoted to study the main examples: multiplication by the coordinates z_i on both $H^2(S^{2n-1})$ and $H^2(S^1 \times \ldots \times S^1)$. In section 3, we obtain a necessary and sufficient condition for invertibility (when X is A(H), a

W^*-algebra or H), from which a chain of corollaries is derived, along with Theorem 3.7, that states an n-tuple a is invertible iff $\hat{a}$ is invertible. We then conclude that $Sp(T,H)=Sp(T,L(H))$ for any commuting tuple T, and that, for W^*-algebras A,B such that $A\subset B$, $Sp(a,A)=Sp(a,B)$ for any commuting tuple a of elements of A. In section 4, a natural generalization of Atkinson theorem is obtained, together with a proposition which allows us to multiply in one coordinate without leaving F.

1. Let H be a Hilbert space, $\{n_k\}_{k\in Z}$ be a sequence of nonnegative integers with $n_k=0$ for $k<0$, $H_k=H\otimes C^{n_k}$ and $D_k\in L(H_k,H_{k-1})$ such that D_kD_{k+1} is compact for all k. We consider the system:

$$(D)\quad \dots\xrightarrow{D_{k+1}}H_k\xrightarrow{D_k}H_{k-1}\xrightarrow{D_{k-1}}\dots\xrightarrow{D_2}H_1\xrightarrow{D_1}H_0\longrightarrow 0\quad,$$

and the complex:

$$(d)\quad \dots\xrightarrow{d_{k+1}}A_k\xrightarrow{d_k}A_{k-1}\xrightarrow{d_{k-1}}\dots\xrightarrow{d_2}A_1\xrightarrow{d_1}A_0\longrightarrow 0\quad,$$

where $A_k=A(H)\otimes C^{n_k}$ (n_k copies of the Calkin algebra $A(H)$) and d_k is the matrix associated to D_k in the canonical way (i.e., the entries of d_k are the projections on $A(H)$ of the entries of D_k).

If $A=(A_1,\dots,A_n)$ is an almost commuting tuple of operators on H (i.e., $[A_i,A_j]\in K(H)$, all i,j), the Koszul system (D(A)) is the one we get by taking $n_k=\binom{n}{k}$ and

$$D_k(x\otimes e_{j_1}\wedge\dots\wedge e_{j_k})=\sum_{i=1}^{k}(-1)^{i+1}A_{j_i}x\otimes e_{j_1}\wedge\dots\wedge\hat{e}_{j_i}\wedge\dots\wedge e_{j_k}\quad,$$

as in I.1. Although $D_k D_{k+1}$ need not be zero this time, the compactness of the commutators forces it to be compact (indeed, one can directly show this calculating $D_k D_{k+1}(x \otimes e_{j_1} \wedge \ldots \wedge e_{j_{k+1}})$; a possibly easier way is to draw a proof by induction, using the two by two matrix representation of D_k explained in I.1(1)).

Definition 1.1. A system (D) is said to be Fredholm if the associated complex (d) is exact (that is, ker d_k= ran d_{k+1}, all k).

Definition 1.2. An almost commuting tuple $A=(A_1,\ldots,A_n)$ is Fredholm (in symbols, $A \in F$) if the associated Koszul system is Fredholm, i.e., if $\pi(A)=(\pi(A_1),\ldots,\pi(A_n))$ is nonsingular.

Definition 1.3. The essential spectrum $Sp_e(A)$ of an almost commuting tuple A is $Sp(\pi(A),A(H))$.

Remark: Although we have not made any explicit reference to dimension(H), we shall always understand it infinite in case the word compact is in the context.

2. Examples

(i) Any almost commuting tuple $A=(A_1,\ldots,A_n)$ with one of the A_i's Fredholm. This follows easily from I.Proposition 2.5.

(ii) $W=(W_1,\ldots,W_n)$ on $H^2(S^1\times\ldots\times S^1)$ (n times).
If $n=1$, this is well known. We now give a proof for $n=2$ and then an induction argument based in I.1(1).
In what follows, we denote with a the projection of A into the Calkin algebra A(H).
Since W_1 is an isometry, we clearly have $\ker w_1=0$, and so d_2 is one-to-one.
Assume now that $w_1a+w_2b=0$. Multiplying by w_1^* on the left, we get $a=-w_1^*w_2b$. Since W_1^* also commutes with W_2, $a=-w_2w_1^*b$. Let $c=w_1^*b$. Then $a=-w_2c$ and $b=w_1c$, because $b=-w_1w_2^*a$ (multiply the given equation by w_2^* on the left).
Finally, to show that $\operatorname{ran} w_1+\operatorname{ran} w_2=A(H)$, take $a=w_1^*$ and $b=p_1w_2^*$, where P_i is the orthogonal projection onto the kernel of W_i^*. Then:
$w_1a + w_2b = w_1w_1^* + w_2p_1w_2^*=1-p_1+p_1w_2w_2^*=1-p_1+p_1(1-p_2)= 1-p_1+p_1-p_1p_2=1$ (observe that $p_1p_2=0$, being P_1P_2 a rank one projection).
We now proceed to the case $n>2$. Assume that $(W_1,\ldots,W_{n-1})$ is Fredholm on $H^2(S^1\times\ldots\times S^1)$ ($n-1$ factors). We denote by $D_k^{(m)}$ the kth boundary map in the Koszul system for $(W_1,\ldots,W_m)$. We want to show that $\ker d_k^{(n)}=\operatorname{ran} d_{k+1}^{(n)}$ for all k. If $k=0$, this amounts to showing that $\operatorname{ran} w_1+\ldots+\operatorname{ran} w_n=A(H)$. For this, take $a_1=w_1^*$, $a_2=p_1w_2^*,\ldots,$ $a_n=p_1\ldots p_{n-1}w_n^*$ (as before, P_i is the projection onto

the kernel of W_1^*). Then:

$w_1a_1+\ldots+w_na_n=w_1w_1^*+w_2p_1w_2^*+\ldots+w_np_1\ldots p_{n-1}w_n^*=$
$1-p_1+p_1(1-p_2)+\ldots+p_1\ldots p_{n-1}(1-p_n)=1-p_1+p_1-p_1p_2+\ldots+$
$p_1\ldots p_{n-1}-p_1\ldots p_n=1$, because $p_1\ldots p_n=0$.

When $k>0$, we use the decomposition:

$$d_k^{(n)}=\begin{pmatrix} d_k^{(n-1)} & (-1)^{k+1}\mathrm{diag}(w_n) \\ 0 & d_{k-1}^{(n-1)} \end{pmatrix}. \quad (n>1,\ k\geq 1).$$

If $d_k^{(n)}\binom{a}{b}=0$, then $d_k^{(n-1)}a+(-1)^{k+1}\mathrm{diag}(w_n)b=0$ and $d_{k-1}^{(n-1)}b=0$. We want to deduce that $b=d_k^{(n-1)}c$ for some c. This follows by the induction hypothesis when $k>1$ and by the following argument when $k=1$. Since

$$(w_1\ldots w_{n-1}\ w_n)\binom{a}{b}=0, \text{ we have:}$$

$(w_1\ldots w_{n-1})a+w_nb=0$; that is

$$d_1^{(n-1)}a+w_nb=0.$$

Multiplying by w_n^* on the left

$$w_n^*d_1^{(n-1)}a+b=0.$$

But $w_n^*d_1^{(n-1)}=d_1^{(n-1)}\mathrm{diag}(w_n^*)$, so that

$$b=d_1^{(n-1)}(-\mathrm{diag}(w_n^*)a), \text{ as desired.}$$

We remark that, even when we are assuming that $(w_1,\ldots,w_{n-1})$ is invertible on $A(H^2(S^1\times\ldots\times S^1))$ (n-1 factors) and we are dealing with n factors (to consider $w=(w_1,\ldots,w_n)$,we can still use our induction hypothesis because of the algebraic calculations involved, as exemplified in the cases $n=2$ and $n>2$, $k=0,1$.

We now return to the proof. Since $b=d_k^{(n-1)}c$ for some c, we have:

$$d_k^{(n-1)}a+(-1)^{k+1}\mathrm{diag}(w_n)d_k^{(n-1)}c=0.$$

Now, $\mathrm{diag}(w_n)$ and $d_k^{(n-1)}$ commute, so that

$$d_k^{(n-1)}(a+(-1)^{k+1}\mathrm{diag}(w_n)c)=0.$$

By induction hypothesis, there exists d satisfying

$$a+(-1)^{k+1}\mathrm{diag}(w_n)c=d_{k+1}^{(n-1)}d.$$

Then:

$$d_{k+1}^{(n)}\binom{d}{c}=\begin{pmatrix} d_{k+1}^{(n-1)} & (-1)^{k+2}\mathrm{diag}(w_n) \\ 0 & d_k^{(n-1)} \end{pmatrix}\binom{d}{c}=\binom{a}{b},$$

as desired.

We shall now show that (w_1,w_2) is not invertible in the classical sense, that is, the equation $w_1b_1+w_2b_2=1$ cannot be solved for b_1, b_2 in the commutant of w_1 and w_2. Suppose there is such a pair. Since $(w_1^*,p_1w_2^*)$ is a solution (obviously not in that commutant) and $\ker d_1^{(2)}=\mathrm{ran}\ d_2^{(2)}$, any other solution must be of the form $(w_1^*-w_2c,p_1w_2^*+w_1c)$ for some c. Fix c such that $b_1=w_1^*-w_2c$, $b_2=p_1w_2^*+w_1c$. Since $w_1b_1=b_1w_1$, we have: $w_1w_1^*-w_1w_2c=w_1^*w_1-w_2cw_1$, or $p_1=w_2(cw_1-w_1c)$, or $p_1w_2^*+w_1c=cw_1$. Then $c=w_1^*cw_1$. Moreover $w_1^*-w_2c$ commutes with w_2, so that $cw_2=w_2c$. We have therefore obtained:

$w_1^*cw_1=c \qquad cw_2=w_2c \qquad \text{and } cw_1-w_1c=p_1w_2^*$.

If we write $H^2(S^1\times S^1)$ as $\ker W_1^*\oplus(\ker W_1^{*2}\ominus\ker W_1^*)\oplus\ldots$, we have:

$$C=\begin{pmatrix} C_{00} & C_{01} & C_{02} & \cdots \\ C_{10} & C_{00} & C_{01} & \cdots \\ C_{20} & C_{10} & C_{00} & \\ & & & \ddots \end{pmatrix},$$

by using the first equation; the third equation says that $(C_{02}\ \ C_{03}\ \ C_{04}\ \ \ldots)$ is compact and that $C_{01}-S^*$ is compact (here S is the unilateral shift acting on $\ker W_1^*$, i.e., $S=W_2|\ker W_1^*$). By taking a compact perturbation, if necessary, we can then write C as

$$\begin{pmatrix} C_{00} & C_{01} & 0 & 0 \\ C_{10} & C_{00} & C_{01} & C_{02} \\ C_{20} & C_{10} & C_{00} & C_{01} \\ & & & & \ddots \end{pmatrix}.$$

But CW_2-W_2C is compact, so:

$$\begin{pmatrix} C_{00}S-SC_{00} & C_{01}S-SC_{01} & 0 \\ C_{10}S-SC_{10} & C_{00}S-SC_{00} & C_{01}S-SC_{01} \\ C_{20}S-SC_{20} & C_{10}S-SC_{10} & C_{00}S-SC_{00} \\ & & & \ddots \end{pmatrix}$$

is compact. Thus $C_{01}S=SC_{01}$. Then C_{01} is an analytic Toeplitz operator (identifying $\ker W_1^*$ with $H^2(S^1)$) and a compact perturbation of S^*. But this says that S^* is analytic, which is a contradiction.

(iii) $T_z=(T_{z_1},\ldots,T_{z_n})$ on $H^2(S^{2n-1})$.

We already know that $T_{z_1}^* T_{z_1} + \dots + T_{z_n}^* T_{z_n} = I$, that the T_{z_i}'s are essentially normal and they almost commute. Consequently, if t_i denotes $\pi(T_{z_i})$, then $t = (t_1, \dots, t_n)$ is a commuting tuple of normal elements of $A(H)$, satisfying the equation $t_1^* t_1 + \dots + t_n^* t_n = 1$. Therefore $Sp_{(t)'}(t)$ does not contain 0. Since $Sp(t, A(H)) \subset Sp_{(t)'}(t)$, we conclude that t is invertible.

3. Proposition 3.1. Let B be a W^*-algebra, $A(H)$ or H, $0 \leq n_k \in Z$, $n_k = 0$ for $k < 0$, $B_k = B \otimes C^{n_k}$ and $d_k \in L(B_k, B_{k-1})$ be an n_{k-1} by n_k matrix over B (or $d_k \in L(H_k, H_{k-1})$) with $d_k d_{k+1} = 0$ for all k. Then the complex: $\dots \to B_k \xrightarrow{d_k} B_{k-1} \to \dots$ is exact (at every stage) if and only if $l_k = d_k^* d_k + d_{k+1} d_{k+1}^*$ is invertible (all k). (Here d_k^* is the matrix adjoint of d_k.)

Proof: (only if) Since $B_{-1} = 0$, we have $d_0 = 0$. By exactness, d_1 is onto. Hence $d_1 d_1^*$ is invertible (0.PR 1), or l_0 is invertible. Let us now assume that l_j is invertible for all $j \leq k$ and prove that so is l_{k+1}. We first need a direct sum decomposition of B_{k+1} into $\ker d_{k+1} \dot{+} \operatorname{ran} d_{k+1}^*$. Clearly $\ker d_{k+1} \cap \operatorname{ran} d_{k+1}^* = 0$ (0. PR 2). If $b \in B_{k+1}$ then $d_{k+1} b \in B_k = \operatorname{ran} l_k$, so that there exists $c \in B_k$ such that $d_{k+1} b = l_k c = d_k^* d_k c + d_{k+1} d_{k+1}^* c$. Then $d_{k+1}^* d_{k+1} b = d_{k+1}^* d_k^* d_k c + d_{k+1}^* d_{k+1} d_{k+1}^* c = d_{k+1}^* d_{k+1} d_{k+1}^* c$, because $d_k d_{k+1} = 0$. Thus $b - d_{k+1}^* c$ belongs to $\ker d_{k+1}^* d_{k+1} = \ker d_{k+1}$ (0. PR 2). Therefore, $b \in \ker d_{k+1} + \operatorname{ran} d_{k+1}^*$.

Once we have obtained such a decomposition, we can prove that l_{k+1} is onto (that is, invertible, being self-adjoint). Given $b \in B_{k+1}$, there exist c in ker d_{k+1} and d in ran d_{k+1}: $b=c+d_{k+1}^{*}d$ (notice that since l_{k-1} is invertible, $B_k = \ker d_k \dot{+} \operatorname{ran} d_k^{*}$ and $d_{k+1}^{*}d_k^{*}=0$, so that d can be chosen in ker $d_k = \operatorname{ran} d_{k+1}$).

Since c is in ker d_{k+1}, exactness implies there is e in B_{k+2} such that $c=d_{k+2}e$. Consequently:

(1) $$b=d_{k+2}e+d_{k+1}^{*}d.$$

But $d=d_{k+1}f$ for some f in B_{k+1}. Moreover, by polar decomposition, ran $d_{k+2} \subset \operatorname{ran}(d_{k+2}d_{k+2}^{*})^{\frac{1}{2}}$, so that

(2) $$d_{k+2}e=(d_{k+2}d_{k+2}^{*})^{\frac{1}{2}}g$$

for some g in B_{k+1}. By the direct sum decomposition for B_{k+1}, $g=g_1+d_{k+1}^{*}g_2$, with $g_1 \in \ker d_{k+1}$ and $g_2 \in B_k$. But then there is $h \in B_{k+2}$: $g_1=d_{k+2}h \in \operatorname{ran} d_{k+2} \subset \operatorname{ran}(d_{k+2}d_{k+2}^{*})^{\frac{1}{2}}$, so that $g_1=(d_{k+2}d_{k+2}^{*})^{\frac{1}{2}}k$ for some $k \in B_{k+1}$. Thus:

(3) $$g=(d_{k+2}d_{k+2}^{*})^{\frac{1}{2}}k+d_{k+1}^{*}g_2.$$

Combining (1), (2) and (3) we get:

$$b=d_{k+2}e+d_{k+1}^{*}d=(d_{k+2}d_{k+2}^{*})^{\frac{1}{2}}g+d_{k+1}^{*}d_{k+1}f=$$

$$(d_{k+2}d_{k+2}^{*})k+(d_{k+2}d_{k+2}^{*})^{\frac{1}{2}}d_{k+1}^{*}g_2+d_{k+1}^{*}d_{k+1}f=$$

$d_{k+2}d_{k+2}^{*}k+d_{k+1}^{*}d_{k+1}f$, since $d_{k+2}d_{k+2}^{*}d_{k+1}^{*}=0$ and therefore $(d_{k+2}d_{k+2}^{*})^{\frac{1}{2}}d_{k+1}^{*}=0$ (O. PR 2).

To complete the proof, we observe that k can be chosen in ker d_{k+1} and f in ran d_{k+1}^{*}. Thus:

$l_{k+1}(k+f)=d_{k+1}^{*}d_{k+1}f+d_{k+2}d_{k+2}^{*}k=b$, as desired.

(if) Assume that $d_k b=0$. Then $l_k b=d_{k+1}d_{k+1}^{*}b$. Since l_k is invertible, $b=l_k^{-1}d_{k+1}d_{k+1}^{*}b$ and the conclusion will follow once we prove that l_k and $d_{k+1}d_{k+1}^{*}$ commute. But this is obvious.

<u>Remark</u>: Although the preceding proof made no distinction between a W^{*}-algebra or $A(H)$ and a Hilbert space H, it can actually be simplified in the latter case (for instance, the direct sum decomposition needs no proof and is an orthogonal direct sum).

<u>Corollary 3.2</u>. An almost commuting (respectively commuting) tuple $(A_1,\ldots,A_n)$ is Fredholm (resp. invertible) if and only if $L_k=D_k^{*}D_k+D_{k+1}D_{k+1}^{*}$ is Fredholm (resp. invertible) for all k, where $D_k=D_k(A_1,\ldots,A_n)$.

<u>Proof</u>: $\pi(L_k)=l_k$.

<u>Corollary 3.3</u>. Let $A=(A_1,\ldots,A_n)$ be an almost commuting (resp. commuting) tuple of operators on H. If $A\in F(H)$ (resp. A is invertible), so are $\sum_{i=1}^{n}A_i^{*}A_i$ and $\sum_{i=1}^{n}A_iA_i^{*}$.

<u>Proof</u>: $\sum_{i=1}^{n}A_i^{*}A_i=D_n^{*}D_n$ and $\sum_{i=1}^{n}A_iA_i^{*}=D_1D_1^{*}$. But $L_n=D_n^{*}D_n$ and $L_0=D_1D_1^{*}$.

The statement in parenthesis has been proved by Vasilescu in [14].

Corollary 3.4. An almost doubly commuting (resp. doubly commuting) tuple $A=(A_1,\dots,A_n)$ (i.e., $[A_i,A_j^*]$ also compact (resp. zero) for all $i\neq j$) is Fredholm (resp. invertible) if and only if $\sum_{i=1}^{n} {}^{f}A_i$ is Fredholm (respectively invertible) for every $f:\{1,\dots,n\}\to\{0,1\}$, where

$$ {}^{f}A_i=\begin{cases} A_i^*A_i & f(i)=0 \\ A_iA_i^* & f(i)=1 \end{cases} .$$

Proof: A direct calculation shows that in this case $l_k=d_k^*d_k+d_{k+1}d_{k+1}^*$ is a block diagonal matrix of order $\binom{n}{k}$ whose diagonal entries are precisely the $\binom{n}{k}$ different combinations $\sum_{i=1}^{n} {}^{f}A_i$, for $f:\{1,\dots,n\}\to\{0,1\}$ with $\#\{i:\ f(i)=0\}=k$.

Corollary 3.5. If the A_i's are essentially normal (resp. normal) and they almost commute (resp. commute), then $A=(A_1,\dots,A_n)$ is Fredholm (resp. invertible) if and only if $\sum_{i=1}^{n} A_i^*A_i$ is Fredholm (resp. invertible).

Proof: Use Fuglede's theorem to conclude that A is an almost doubly commuting (resp. doubly commuting) tuple and then apply the preceding corollary, along with the stability of the Fredholm class under compact perturbations.

Observations: Corollary 3.4 gives an easy proof of the Fredholmness of W (see Example 2(ii)). Corollary 3.5 says that for a commuting tuple of normal elements of $L(H)$ or $A(H)$, the Koszul complex is exact iff it is exact at any stage, a natural generalization of a well known "one variable" fact.

Corollary 3.6. Let $A=(A_1,\dots,A_n)$ be an essentially normal tuple (resp. normal) and M be the maximal ideal space of the C^*-algebra generated by $\pi(A_1),\dots,\pi(A_n)$ (resp. $A_1,\dots,A_n$). Then $Sp_e(A)=M$ (resp. $Sp(A)=M$), when M is regarded as a subset of C^n under the homeomorphism $\varphi\mapsto(\varphi(\pi(A_1)),\dots,\varphi(\pi(A_n)))$ (resp. $\varphi\mapsto(\varphi(A_1),\dots,\varphi(A_n))$).

Proof: By the preceding corollary, A is Fredholm iff $\sum_{i=1}^{n} A_i^*A_i$ is Fredholm. Let B be the C^*-algebra generated by $\pi(A_1),\dots,\pi(A_n)$. Then $B\cong C(M)$. Therefore,

$\lambda\notin Sp_e(A)$ iff $A-\lambda\in F$ iff $\sum_{i=1}^{n}(A_i-\lambda_i)^*(A_i-\lambda_i)\in F$ iff
$\sum_{i=1}^{n}(\pi(A_i)^*-\bar{\lambda}_i)(\pi(A_i)-\lambda_i)$ is invertible iff
$\varphi(\sum_{i=1}^{n}(\pi(A_i)^*-\bar{\lambda}_i)(\pi(A_i)-\lambda_i))\neq 0$ for all $\varphi\in M$ iff
$\sum_{i=1}^{n}|z_i-\lambda_i|^2>0$ for all $z\in M$ iff $\lambda\notin M$.

The statement in parentheses follows in the same way.

The following theorem gives a precise relation between invertibility for a tuple a and for its associated $\hat{a}$ (see I.1(2)).

Theorem 3.7. Let $a=(a_1,\dots,a_n)$ be a commuting tuple of elements in a W^*-algebra B (or $A(H)$) acting on H or B (or on $A(H)$). Then a is invertible if and only if $\hat{a}$ is invertible.

Proof: By 0. PR1, $\hat{a}$ is invertible iff so are $\hat{a}^*\hat{a}$ and $\hat{a}\hat{a}^*$. An easy computation shows that $\hat{a}^*\hat{a}$ is a block diagonal matrix whose entries are the l_k's ($l_k=d_k^*d_k+d_{k+1}d_{k+1}^*$) for odd k's. Similarly, $\hat{a}\hat{a}^*$ contains those l_k's with even k. The theorem now follows by an application of Proposition 3.1.

We immediately get:

Corollary 3.8. An almost commuting (resp. commuting) tuple $A=(A_1,\dots,A_n)$ of operators on H is Fredholm (resp. invertible) iff $\hat{A}\in L(H\otimes C^{2^{n-1}})$ is Fredholm (resp. invertible).

Proof: Obvious.

Corollary 3.9. Let A be a commuting tuple of operators on H. Then $Sp(A,H)=Sp(A,L(H))$.

Proof: This corollary states that these two notions of invertibility for A (when the A_i's act on H and when they multiply on L(H)) are actually the same. It follows easily from Theorem 3.7 and the fact that it is known for singletons.

Corollary 3.10. Let $A=(A_1,\dots,A_n)\in F$ (resp. invertible), $\varphi:\{1,\dots,n\}\to\{1,*\}$ and $\varphi(A_i)=A_i^{\varphi(i)}$. Assume that $[\varphi(A_i),\varphi(A_j)]$ is compact (resp. zero) for all $i\neq j$. Then $\varphi(A)=(\varphi(A_1),\dots,\varphi(A_n))\in F$ (resp. invertible).

Proof: It suffices to prove it when $\varphi(1)=*$, $\varphi(2)=1,\dots,$

$\varphi(n)=1$, because any other φ is a composition of this particular one and transpositions, which are permissible by I.Proposition 2.2.

We now observe that $\widehat{(a_1,a_2)}^*=\widehat{(a_1^*,-a_2)}$. Define f recursively by the conditions: $f(1)=a_1$, $f(k+1)=\widehat{(f(k),\widehat{a}_{k+1})}$. It is almost obvious that $\widehat{a}$ is, up to some permutations of rows and columns, equal to $f(n)$.

Let g be defined by the conditions: $g(1)=a_1^*$, $g(k+1)=\widehat{(g(k),-\widehat{a}_{k+1})}$. Since $f(1)^*=g(1)$, $f(2)^*=g(2)$, it follows at once that $f(n)^*=g(n)$. In other words, $\widehat{a}^*$ is, up to some permutations of rows and columns, equal to $\widehat{(a_1^*,-a_2,\dots,-a_n)}$.

Now, if a is invertible, so is $f(n)$. Then $g(n)$ is invertible, so that $(a_1^*,-a_2,\dots,-a_n)$ is invertible.

It is clear that multiplication by -1 cannot alter invertibility (in fact, we shall give a much more general result in Section 4). Therefore, $\varphi(a)=(a_1^*,a_2,\dots,a_n)$ is invertible, or $\varphi(A)\in F$. For the statement in parenthesis, replace a by A everywhere in the preceding reasoning.

Corollary 3.11. If $A=(A_1,A_2)$ is a doubly commuting invertible pair, then $\ker A_1 \perp \ker A_2$.

Proof: Assume $A_1x=0$. Then $A_1x+A_2^*0=0$. By the preceding corollary, (A_1,A_2^*) is invertible, so that there exists y: $x=-A_2^*y$ and $0=A_1y$. In particular, x belongs to ran $A_2^*=$

$(\ker A_2)^{\perp}$, as needed.

Corollary 3.12. Let B be a C^*-subalgebra of L(H) (resp. A(H)) and $a=(a_1,\ldots,a_n)$ be a commuting tuple of elements of B. Then $Sp(a,B)\subset Sp(a,L(H)$ (resp. $Sp(a,B)\subset Sp(a,A(H))$). Moreover, if B is a W^*-algebra, then $Sp(a,B)=Sp(a,L(H))$.

Proof: Assume that $\lambda\notin Sp(a,L(H))$, i.e., $a-\lambda$ is invertible (acting on L(H)). By Proposition 3.1, $l_k=d_k^*d_k+d_{k+1}d_{k+1}^*$ is invertible (in $M_{\binom{n}{k}}(L(H))$) for all k. By spectral permanence, l_k is invertible in $M_{\binom{n}{k}}(B)$ for all k. A look at the "if" part of the proof of Proposition 3.1 shows that $E(B,a-\lambda)$ is exact, or $\lambda\notin Sp(a,B)$. The statement in parenthesis follows in the same way. The last statement follows immediately from Proposition 3.1.

Remarks: 1) As noticed in I.1(3), given a tuple $A=(A_1,\ldots,A_n)$ $(n>1)$, we can consider the (n-1) first coordinates, form a tuple $'A=(A_1,\ldots,A_{n-1})$, define $\tilde{A}_n=\mathrm{diag}(A_n)\in L(H\otimes C^{2^{n-2}})$ and then have:

$$\hat{A}\cong\widehat{('\hat{A},\tilde{A}_n)}\ ,$$

where $\cong$ means that some permutations of rows and columns are needed to get equality. Since those elementary operations on the matrices will not affect singularity, we conclude (using Corollary 3.8) that, as long as questions of Fredholmness (resp. invertibility) are the context, and we are dealing with almost doubly commuting (resp. doubly commuting) tuples, attention can be restricted to pairs.

2) New proofs of the Fredholmness of examples (ii) and (iii) of Section 2 can now be given as a direct consequence of Corollary 3.8, using results of Coburn [2] and Douglas and Howe [5] for the matrix case. Precisely, as for (ii) we conclude that $W=(W_1,\ldots,W_n)$ is Fredholm iff $\hat{W}\in F$. By the Corollary to Theorem 4 in [5],

$$\widehat{(W_1,W_2)}=\begin{pmatrix} W_1 & W_2 \\ -W_2^* & W_1^* \end{pmatrix}\in F \text{ iff } \begin{pmatrix} T_z & z_2 \\ -\bar{z}_2 & T_{\bar{z}} \end{pmatrix} \text{ and } \begin{pmatrix} z_1 & T_z \\ -T_{\bar{z}} & \bar{z}_1 \end{pmatrix} \text{ are}$$

invertible for all $(z_1,z_2)\in S^1\times S^1$, where T_z is the unilateral shift on $H^2(S^1)$. But this is equivalent to (T_z,z_2) and (z_1,T_z) invertible for all $(z_1,z_2)\in S^1\times S^1$, which is certainly true, by I.2(ii). We use an inductive proof for $n>2$.

As for (iii), we know that T_z is a Toeplitz operator-valued matrix, whose symbol is $(z_1,\ldots,z_n)$. It is easy to show (by induction , for example) that

$$\det \widehat{(z_1,\ldots,z_n)} = \Big(\sum_{i=1}^{n} |z_i|^2\Big)^{n-1} \quad (n>1).$$

We now apply the Corollary to Theorem 1 in [2] to conclude that T_z is Fredholm.

3) If $A=(A_1,\ldots,A_n)\in F$ and $K=(K_1,\ldots,K_n)\in K(H)\otimes C^n$, then

$A+K=(A_1+K_1,\ldots,A_n+K_n)\in F$. This follows from the definition of a Fredholm tuple.

Proposition 3.13. (i) F is an open subset of the set of almost commuting tuples.

(ii) The set I of invertible tuples is an open subset of the set of commuting tuples.

Proof: The map $(A_1,\ldots,A_n)\mapsto\widehat{(A_1,\ldots,A_n)}$ is continuous.

Remark: Using the preceding proposition we can give a different proof that Sp(a,X) is a compact subset of the polydisc of multiradius r(a) (see I.Proposition 2.4), when X is a W^*-algebra, A(H) or H, totally independent of Taylor's paper. We cannot conclude, however, that Sp(a,X) is nonempty. This needs either sheaf theory [12] or the construction of an R-analytic function (the resolvent) in C^n-Sp(a,X), as done in [14] for X a Hilbert space.

We already know that an almost commuting (resp. commuting) tuple is Fredholm (resp. invertible) if one of the coordinates is. In that case, the remaining coordinates are immaterial; they could, for instance, be zero. Similarly, if any k coordinates form a Fredholm (invertible) k-tuple, then the n-tuple is Fredholm (invertible) regardless of what the other coordinates are (use, for example, I.Proposition 2.2 in the Calkin algebra(in H)). One converse to all of this is the following

Proposition 3.14. If $A=(A_1,\dots,A_n)\in F$ and $A_{i_1},\dots,A_{i_k}$ are compact ($k<n$), then the $(n-k)$-tuple formed with the remaining coordinates is Fredholm. An analogous result holds in case F is replaced by I (=invertible tuples).

Proof: Let $s:\{1,\dots,k\}\longrightarrow\{1,\dots,n\}$ be defined by $s(j)=i_j$. Then $s^*Sp(a,A(H))=Sp(s^*a,A(H))$ by I.Proposition 2.2. Since $s^*a=(a_{s(1)},\dots,a_{s(k)})=(a_{i_1},\dots,a_{i_k})=(0,\dots,0)$, we have: $s^*Sp(a,A(H))=0$. Let b be the $(n-k)$-tuple formed with the remaining coordinates, and P be the projection on C^n onto those coordinates. It is apparent that $Sp(a,A(H))\subset$ ran P and so, that b is invertible.

We finish this section with a remark that will be needed in Chapter III.

Remark: If $A\in F$, $p\in S_n$ is a permutation, then $p^*A=(A_{p(1)},\dots A_{p(n)})\in F$ (see I.Proposition 2.2).

4. Given a system (D), there is a natural way of getting a complex, without leaving the space H where (D) acts. In fact, if P_k is the orthogonal projection in L(H) onto ker D_k, and $\tilde{D}_k=P_{k-1}D_k$ (all k), then $(\tilde{D})$ is a complex. One is tempted to believe that since D_kD_{k+1} is compact (all k), then D_k and $\tilde{D}_k$ can differ by only a compact operator. The easiest available counterexample is:

(D) $\quad 0\longrightarrow H\xrightarrow{I}H\xrightarrow{K}H\longrightarrow 0$, K compact, ker K=0.

Here $\tilde{D}_1=D_1$, $\tilde{D}_2=0$, so that $D_2-\tilde{D}_2$ is not compact.

Of course, the (D) shown is not Fredholm, so that one might hope that the statement holds in that case. Moreover, if $n_k=0$ for $k\geq 3$, it does hold, because ran D_1 is closed and therefore there exists $S_1\in L(H_0,H_1)$ such that $S_1D_1=P_1^{\perp}$, and then $D_2-\tilde{D}_2=D_2-P_1D_2=P_1^{\perp}D_2=S_1D_1D_2$, which is compact. Any attempt to extend this proof to the case $n_k\neq 0$ ($k=0, 1, 2, 3$) will fail. Consider:

(D) $\quad 0\to H\xrightarrow{I}H\xrightarrow{K}H\xrightarrow{I}H\to 0$, ker $K=0$, K compact.

(D) is perfectly Fredholm, while $D_3-\tilde{D}_3=I$.

In the general case, a sufficient condition is that all ran D_k be closed.

<u>Proposition 4.1</u>. Let (D) $\quad \ldots\to H_k\xrightarrow{D_k}H_{k-1}\to\ldots$ be a system and $(\tilde{D})$ be its associated complex. Assume that ran D_k is closed (all k). Then $D_k-\tilde{D}_k$ is compact (all k). In particular, $(D)\in F$ iff $(\tilde{D})\in F$.

<u>Proof</u>: Since ran D_k is closed, we can use the open mapping theorem to get $S_k:H_{k-1}\to H_k$ such that $D_kS_k=P_{\text{ran } D_k}$ and $S_kD_k=I-P_{\ker D_k}=P_k^{\perp}$.

Then $D_{k+1}-\tilde{D}_{k+1}=D_{k+1}-P_kD_{k+1}=P_k^{\perp}D_{k+1}=S_kD_kD_{k+1}$.

Since D_kD_{k+1} is compact, the result follows. The rest needs no proof.

The next result resembles Atkinson's theorem.

Theorem 4.2. Let (D): $\cdots \to H_k \xrightarrow{D_k} H_{k-1} \to \cdots$ be a system such that $D_k - \tilde{D}_k$ is compact, all k. The following conditions are equivalent:

(i) $(D) \in F$

(ii) $(\tilde{D}) \in F$

(iii) ran $\tilde{D}_k$ is closed and ker $\tilde{D}_k / \text{ran } \tilde{D}_{k+1}$ is finite dimensional (all k)

(iv) ran D_k is closed and ker $D_k \cap (\text{ran } D_{k+1})^{\perp}$ is finite dimensional (all k)

(v) there exist $S_k \in L(H_{k-1}, H_k)$ $(k \in \mathbb{Z})$ such that

$$S_k D_k + D_{k+1} S_{k+1} - I \text{ is compact (all k).}$$

Remarks: In case $(D)=(D(A))$ for a commuting tuple $A=(A_1, \ldots, A_n)$, (i)$\Longrightarrow$(iv) appears stated (without proof) in the already mentioned letter of Taylor to Douglas. Condition (v) is given as a definition of a Fredholm system in [11].

Proof of the Theorem: (i)$\Longrightarrow$(ii) Clear.

(ii)$\Longrightarrow$(iii) By Proposition 3.1, $\tilde{L}_k = \tilde{D}_k^* \tilde{D}_k + \tilde{D}_{k+1} \tilde{D}_{k+1}^*$ is Fredholm (all k). Being ran $\tilde{D}_{k+1} \subset \ker \tilde{D}_k$, it follows that

$$\text{ran } \tilde{L}_k = \text{ran } \tilde{D}_k^* \tilde{D}_k \oplus \text{ran } \tilde{D}_{k+1} \tilde{D}_{k+1}^* .$$

Since ran $\tilde{L}_k$ is closed, so is ran $\tilde{D}_k^* \tilde{D}_k$. By O. PR 5, the same is true of $\text{ran}(\tilde{D}_k^* \tilde{D}_k)^{\frac{1}{2}} = \text{ran } \tilde{D}_k^*$. O. PR 4 now asserts that ran $\tilde{D}_k$ is closed. Furthermore, $\ker \tilde{L}_k = \ker \tilde{D}_k \cap \ker \tilde{D}_{k+1}^*$.

Since $\widetilde{L}_k$ is Fredholm, we obtain that

$\dim(\ker \widetilde{D}_k / \operatorname{ran} \widetilde{D}_{k+1}) = \dim(\ker \widetilde{D}_k \cap \ker \widetilde{D}_{k+1}^*) = \dim \ker \widetilde{L}_k$, which is finite.

(iii)$\Longrightarrow$(iv) We observe that $\widetilde{D}_k|(\operatorname{ran} \widetilde{D}_{k+1})^{\perp} : (\operatorname{ran}\widetilde{D}_{k+1})^{\perp} \longrightarrow H_{k-1}$ is left semi-Fredholm (closed range and finite dimensional kernel). Since $D_k - \widetilde{D}_k$ is compact, we conclude that $D_k|(\operatorname{ran} \widetilde{D}_{k+1})^{\perp} : (\operatorname{ran} \widetilde{D}_{k+1})^{\perp} \longrightarrow H_{k-1}$ is left semi-Fredholm, too. Then, $\operatorname{ran} D_k = D_k(\operatorname{ran} \widetilde{D}_{k+1})^{\perp}$ is closed (we here use the fact that $\operatorname{ran} \widetilde{D}_{k+1} \subset \ker D_k$) and $\ker D_k \cap (\operatorname{ran} \widetilde{D}_{k+1})^{\perp}$ is finite dimensional. We finish observing that

$\ker D_k \cap (\operatorname{ran} D_{k+1})^{\perp} = \ker D_k \cap (\operatorname{ran} \widetilde{D}_{k+1})^{\perp}$ (0.PR 6).

(iv)$\Longrightarrow$(iii) $D_k|(\operatorname{ran} \widetilde{D}_{k+1})^{\perp} : (\operatorname{ran} \widetilde{D}_{k+1})^{\perp} \longrightarrow H_{k-1}$ is left semi-Fredholm. Therefore, $\widetilde{D}_k|(\operatorname{ran} \widetilde{D}_{k+1})^{\perp} : (\operatorname{ran} \widetilde{D}_{k+1})^{\perp} \longrightarrow H_{k-1}$ has closed range and finite dimensional kernel. But $\widetilde{D}_k(\operatorname{ran} \widetilde{D}_{k+1})^{\perp} = \operatorname{ran} \widetilde{D}_k$ and $\ker \widetilde{D}_k|(\operatorname{ran} \widetilde{D}_{k+1})^{\perp} = \ker\widetilde{D}_k \cap (\operatorname{ran}\widetilde{D}_{k+1})^{\perp}$, as desired.

(iii)$\Longrightarrow$(v) We know that $\widetilde{D}_k$ has closed range. By the Open Mapping Theorem, we can find $S_k \in L(H_{k-1}, H_k)$ such that $S_k\widetilde{D}_k = P_{(\ker \widetilde{D}_k)^{\perp}}$ and $\widetilde{D}_k S_k = P_{\operatorname{ran} \widetilde{D}_k}$ and $\ker S_k = (\operatorname{ran} \widetilde{D}_k)^{\perp}$. Thus $S_k\widetilde{D}_k + \widetilde{D}_{k+1}S_{k+1} = S_k\widetilde{D}_k$ on $(\operatorname{ran} \widetilde{D}_{k+1})^{\perp}$, and $\widetilde{D}_{k+1}S_{k+1}$ on $\operatorname{ran} \widetilde{D}_{k+1}$ (where we use the fact that $\widetilde{D}_k\widetilde{D}_{k+1} = 0$). Since $\ker\widetilde{D}_k/\operatorname{ran}\widetilde{D}_{k+1}$ is finite dimensional, we see that $S_k\widetilde{D}_k + \widetilde{D}_{k+1}S_{k+1} - I$ is compact. But $D_k - \widetilde{D}_k \in K(H_k, H_{k-1})$ (all k), so that $S_kD_k + D_{k+1}S_{k+1} - I$ is compact (all k).

(v)$\Longrightarrow$(i) Passing to the Calkin algebra, we have:

$$s_k d_k + d_{k+1} s_{k+1} = 1 \in M_{n_k}(A(H)),$$

where $s_k = \pi(S_k)$.

If $d_k a=0$, then $d_{k+1}s_{k+1}a=a$, so that $a\in\operatorname{ran} d_{k+1}$, showing that (d) is exact, that is, $(D)\in F$.

Remark: Notice that (i)$\Leftrightarrow$(v) can be extended to:

Let B, n_k, d_k be as in Proposition 3.1. Then the complex $\ldots\to B_k \xrightarrow{d_k} B_{k-1}\to\ldots$ is exact iff there exist $\{s_k : B_{k-1}\to B_k\}_{k\in Z}$ satisfying $s_k d_k + d_{k+1}s_{k+1}=1$.

The "if" part is trivial. For the "only if", use the decomposition $B_k = \ker d_k \dot{+} \operatorname{ran} d_{k+1}^*$.

Corollary 4.3. Let (D): $\ldots\to H_k \xrightarrow{D_k} H_{k-1}\to\ldots$ be a complex. Then $(D)\in F$ iff $\ker D_k / \operatorname{ran} D_{k+1}$ is finite dimensional (all k).

Corollary 4.4. Let (D): $0\to H_2 \xrightarrow{D_2} H_1 \xrightarrow{D_1} H_0 \to 0$ be a system ($n_k=0$ for $k\geq 3$). Then $(D)\in F$ iff $\operatorname{ran} D_1$, $\operatorname{ran} D_2$ are closed and $\ker D_2$, $\ker D_1 \cap (\operatorname{ran} D_2)^{\perp}$ and $(\operatorname{ran} D_1)^{\perp}$ are finite dimensional.

Proof: If $(D)\in F$, then $D_2-\tilde{D}_2$ is compact and (i)$\Rightarrow$(iv) can be used. Conversely, if $\operatorname{ran} D_1$ is closed then $D_2-\tilde{D}_2$ is compact and (iv)$\Rightarrow$(i) applies.

The next proposition will prove to be useful in dealing with questions on connectedness of tuples.

<u>Proposition 4.5</u>. Let B be a Banach algebra, X be a B Banach space which is a left B-module, $a_1,\dots,a_n$ be commuting elements of B and $v\in B$ be an invertible element that commutes with $a_2,\dots,a_n$. The following conditions are equivalent:

(i) $a=(a_1,\dots,a_n)$ is invertible

(ii) $va=(va_1,a_2,\dots,a_n)$ is invertible

(iii) $av=(a_1v,a_2,\dots,a_n)$ is invertible.

<u>Proof</u>: We shall prove by induction that the Koszul complexes $E(X,a)$ and $E(X,va)$ are isomorphic, thus establishing (i)$\Leftrightarrow$(ii). The equivalence of (i) and (iii) will then follow in the same way.

We first observe that va is a commuting tuple. Assume n=2 (for n=1, it is obvious that a is invertible iff so is va); we have

$$E(X,a):\quad 0\longrightarrow X\xrightarrow{d_2}X\oplus X\xrightarrow{d_1}X\longrightarrow 0\ ,$$

and

$$E(X,va):\quad 0\longrightarrow X\xrightarrow{\check{d}_2}X\oplus X\xrightarrow{\check{d}_1}X\longrightarrow 0\ ,$$

where $d_1=(a_1\ \ a_2)$, $\check{d}_1=(va_1\ \ a_2)$, $d_2=\begin{pmatrix}-a_2\\ a_1\end{pmatrix}$ and $\check{d}_2=\begin{pmatrix}-a_2\\ va_1\end{pmatrix}$. Define $T_0^{(2)}: X\to X$, $T_1^{(2)}: X\oplus X\to X\oplus X$ and $T_2^{(2)}: X\to X$ by $x\mapsto vx$, $x\oplus y\mapsto x\oplus vy$ and $x\mapsto x$, respectively. Then:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{d_2} & X\oplus X & \xrightarrow{d_1} & X & \longrightarrow & 0\\
 & & \Big\downarrow T_2^{(2)} & & \Big\downarrow T_1^{(2)} & & \Big\downarrow T_0^{(2)} & & \\
0 & \longrightarrow & X & \xrightarrow{\check{d}_2} & X\oplus X & \xrightarrow{\check{d}_1} & X & \longrightarrow & 0
\end{array}$$

is commutative. Moreover, $T^{(2)}_k$ is an isomorphism $(k=0,1,2)$. Therefore, $E(X,a)$ and $E(X,va)$ are isomorphic.
We now define $T^{(m)}_k : X^{\binom{m}{k}} \longrightarrow X^{\binom{m}{k}}$ by

$$T^{(m)}_k = \begin{pmatrix} T^{(m-1)}_k & 0 \\ 0 & T^{(m-1)}_{k-1} \end{pmatrix}$$

with respect to the decomposition $X^{\binom{m}{k}} = X^{\binom{m-1}{k}} \oplus X^{\binom{m-1}{k-1}}$, as we did in I.1(1).
Assume that $E(X,(a_1,\dots,a_{n-1}))$ and $E(X,(va_1,a_2,\dots,a_{n-1}))$ are isomorphic with the isomorphism given by the $T^{(n-1)}_k$'s.
Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & X^{\binom{n}{n}} \arrow[r, "d^{(n)}_n"] \arrow[d, "T^{(n)}_n"] & \dots X^{\binom{n}{k+1}} \arrow[r, "d^{(n)}_{k+1}"] \arrow[d, "T^{(n)}_{k+1}"] & X^{\binom{n}{k}} \arrow[r, "d^{(n)}_k"] \arrow[d, "T^{(n)}_k"] & \dots X^{\binom{n}{0}} \arrow[r] \arrow[d, "T^{(n)}_0"] & 0 \\
0 \arrow[r] & X^{\binom{n}{n}} \arrow[r, "\tilde{d}^{(n)}_n"] & \dots X^{\binom{n}{k+1}} \arrow[r, "\tilde{d}^{(n)}_{k+1}"] & X^{\binom{n}{k}} \arrow[r, "\tilde{d}^{(n)}_k"] & \dots X^{\binom{n}{0}} \arrow[r] & 0
\end{tikzcd}$$

Since the $T^{(n)}_k$'s are clearly isomorphisms (by the way they were constructed), we need only to prove that in the previous diagram all squares commute.
Now, I.1(1).

$$d^{(n)}_k = \begin{pmatrix} d^{(n-1)}_k & (-1)^{k+1}\operatorname{diag}(a_n) \\ 0 & d^{(n-1)}_{k-1} \end{pmatrix} \quad (n>1,\ k\geq 1).$$

Therefore:

$$T^{(n)}_k d^{(n)}_{k+1} = \begin{pmatrix} T^{(n-1)}_k & 0 \\ 0 & T^{(n-1)}_{k-1} \end{pmatrix} \begin{pmatrix} d^{(n-1)}_{k+1} & (-1)^k \operatorname{diag}(a_n) \\ 0 & d^{(n-1)}_k \end{pmatrix}$$

$$=\begin{pmatrix} T_k^{(n-1)}d_{k+1}^{(n-1)} & (-1)^k T_k^{(n-1)}\operatorname{diag}(a_n) \\ 0 & T_{k-1}^{(n-1)}d_k^{(n-1)} \end{pmatrix}.$$

Since $T_k^{(n-1)}$ is block diagonal and v commutes with a_n, $T_k^{(n-1)}\operatorname{diag}(a_n)=\operatorname{diag}(a_n)T_k^{(n-1)}$. Furthermore,

$$T_k^{(n-1)}d_{k+1}^{(n-1)}=\check{d}_{k+1}^{(n-1)}T_{k+1}^{(n-1)},$$

by induction hypothesis, and also

$$T_{k-1}^{(n-1)}d_k^{(n-1)}=\check{d}_k^{(n-1)}T_k^{(n-1)}.$$

Thus:

$$T_k^{(n)}d_{k+1}^{(n)}=\begin{pmatrix} \check{d}_{k+1}^{(n-1)}T_{k+1}^{(n-1)} & (-1)^k\operatorname{diag}(a_n)T_k^{(n-1)} \\ 0 & \check{d}_k^{(n-1)}T_k^{(n-1)} \end{pmatrix}$$

$$=\begin{pmatrix} \check{d}_{k+1}^{(n-1)} & (-1)^k\operatorname{diag}(a_n) \\ 0 & \check{d}_k^{(n-1)} \end{pmatrix}\begin{pmatrix} T_{k+1}^{(n-1)} & 0 \\ 0 & T_k^{(n-1)} \end{pmatrix}$$

$$=\check{d}_{k+1}^{(n)}T_{k+1}^{(n)}.$$

<u>Proposition 4.6</u>. Let B, X, $a_1,\dots,a_n$ be as before and v be an invertible element of B (not necessarily commuting with $a_2,\dots,a_n$). Then $a=(a_1,\dots,a_n)$ is invertible iff so is $a_v=(va_1v^{-1},\dots,va_nv^{-1})$.

<u>Proof</u>: It is easy to verify that $v_k\colon X^{\binom{n}{k}}\longrightarrow X^{\binom{n}{k}}$ given by $v_k=v\otimes\dots\otimes v$ (k times), $k=0,1,\dots,n$, establish an isomorphism between $E(X,a)$ and $E(X,a_v)$.

<u>Corollary 4.7.</u> Let $A=(A_1,\dots,A_n)\in F$ and V be a Fredholm operator.

(i) If V almost commutes with $A_2,\dots,A_n$, then $VA=(VA_1,A_2,\dots,A_n)$ and $AV=(A_1V,A_2,\dots,A_n)$ are Fredholm.

(ii) If $\tilde{V}$ denotes any "almost inverse" of V, i.e., $\pi(\tilde{V})=\pi(V)^{-1}$, then $A_V=(VA_1\tilde{V},\dots,VA_n\tilde{V})$ is Fredholm.

<u>Remark:</u> It is not hard to see that the positive part P_i in the polar decomposition $T_{z_i}=S_iP_i$ almost doubly commutes with T_{z_j} and S_j (all $j=1,\dots,n$). Using Proposition 3.13(i) and Corollary 4.7, we can show that $(S_1,\dots,S_n)\in F$. For, there exists $\mathcal{E}>0$ such that $(T_{z_1}+\lambda S_1,T_{z_2},\dots,T_{z_n})\in F$ $(|\lambda|\le\mathcal{E})$. Since $T_{z_1}+\mathcal{E}S_1=S_1(P_1+\mathcal{E})$ and $P_1+\mathcal{E}$ is invertible, it follows that $(S_1,T_{z_2},\dots,T_{z_n})$ is Fredholm. We can repeat the argument in each coordinate, thus obtaining the Fredholmness of $(S_1,\dots,S_n)$. Now, since the S_i's are unitarily equivalent to the W_i's (under the same isometric isomorphism), we can conclude (using Proposition 4.6, for instance) that $(W_1,\dots,W_n)$ is Fredholm. Although this proof is notoriously simpler than the one we gave in Section 2, we think that does not reveal the way the W_i's interact to produce exactness in the Calkin algebra. Furthermore, we shall have occasion to use that proof, when we study the spectrum of W.

CHAPTER III: INDEX OF A FREDHOLM TUPLE

We are now ready to introduce the index for a Fredholm tuple of almost commuting operators on an infinite dimensional Hilbert space H. As it is probably expected, we shall do that using II.Corollary 3.8. Naturally, index will be continuous, invariant under compact perturbations, onto Z. We devote the rest of Section 1 to obtain an alternative definition, similar to the Euler characteristic of a topological space. Section 2 is reserved for the examples, and we also calculate various spectra. In Section 3, a number of elementary propositions is given, which increase our list of known indices. In Section 4 we treat the case $A=(A_1,\dots,A_n)$ an essentially normal Fredholm tuple with all commutators in trace class and conclude that index$(A)=0$, whenever $n\geq 2$.

1. Let H be an infinite dimensional Hilbert space, $A=(A_1,\dots,A_n)$ be an almost commuting tuple of operators on H and $\hat{A}\in L(H\otimes C^{2^{n-1}})$ be the operator associated to A as in I.1(2). By II.Corollary 3.8, we know that $A\in F$ if and only if $\hat{A}$ is Fredholm.

Definition 1.1. Let $A=(A_1,\dots,A_n)$ be an almost commuting

Fredholm tuple of operators on H. Then index(A)=index($\hat{A}$).

Theorem 1.2. index: $F \longrightarrow Z$ is continuous, invariant under compact perturbations, onto Z. Consequently, index is constant on arcwise components of F.

Proof: Since $A \mapsto \hat{A}$ is continuous, it follows easily that index is continuous. The invariance under compact perturbations follows from the fact that $\widehat{A+K} - \hat{A} \in K(H \otimes C^{2^{n-1}})$ for $K \in K(H) \otimes C^n$. We shall see in Section 2 that

$$\operatorname{index}(W_1^{(k)}, W_2, \ldots, W_n) = -k$$

for all $k \in Z$, which proves ontoness.

Suppose (D) is a Fredholm Koszul system such that $D_k - \tilde{D}_k$ is compact (all k). According to II.Theorem 4.2, $(\tilde{D}) \in F$. Moreover, index(D)=index($\tilde{D}$) by Theorem 1.2.

Theorem 1.3. Let (D), ($\tilde{D}$) be as above. Then:

$$\begin{aligned}\operatorname{index}(D) &= \sum_k (-1)^{k+1} \dim(\ker \tilde{D}_k / \operatorname{ran} \tilde{D}_{k+1}) \\ &= \sum_k (-1)^{k+1} \{\dim(\ker D_k \cap (\operatorname{ran} D_{k+1})^{\perp}) \\ &\quad -\dim(\operatorname{ran} D_{k+1} \cap (\ker D_k)^{\perp})\}.\end{aligned}$$

Proof: Since $\operatorname{index}(\tilde{D}) = \operatorname{index}(\hat{\tilde{D}}) = \dim \ker \hat{\tilde{D}} - \dim \ker \hat{\tilde{D}}^*$, we shall compute both kernels.

Since $\tilde{D}_k \tilde{D}_{k+1} = 0$ (all k) we get:

$$\ker \hat{\tilde{D}} = \ker \hat{\tilde{D}}^* \hat{\tilde{D}} = \bigoplus_{\substack{k \\ \text{odd}}} \ker(\tilde{D}_k^* \tilde{D}_k + \tilde{D}_{k+1} \tilde{D}_{k+1}^*) \quad \text{and}$$

$$\ker \hat{\tilde{D}}^* = \ker \hat{\tilde{D}} \hat{\tilde{D}}^* = \bigoplus_{\substack{k \\ \text{even}}} \ker(\tilde{D}_k^* \tilde{D}_k + \tilde{D}_{k+1} \tilde{D}_{k+1}^*) .$$

But

(1) $\ker(\tilde{D}_k^*\tilde{D}_k+\tilde{D}_{k+1}\tilde{D}_{k+1}^*)=\ker \tilde{D}_k \cap (\operatorname{ran} \tilde{D}_{k+1})^{\perp}$ (all k).

Furthermore, $\ker \tilde{D}_k \supset \ker D_k \supset \operatorname{ran} \tilde{D}_{k+1}$, so that

(2) $\dim(\ker \tilde{D}_k / \operatorname{ran} \tilde{D}_{k+1})=\dim(\ker \tilde{D}_k / \ker D_k)$

$$+\dim(\ker D_k / \operatorname{ran} \tilde{D}_{k+1}) .$$

We now observe that:

(3) $\ker D_k \cap (\operatorname{ran} \tilde{D}_{k+1})^{\perp}=\ker D_k \cap (\operatorname{ran} D_{k+1})^{\perp}$, because $\tilde{D}_{k+1}=P_k D_{k+1}$, with P_k the projection onto $\ker D_k$ (O. PR6).

Finally, $\ker \tilde{D}_k=\ker(P_{k-1}D_k)=D_k^{-1}(\ker P_{k-1})$

$$=D_k^{-1}((\ker D_{k-1})^{\perp}\cap \operatorname{ran} D_k),$$

so that:

$$0\rightarrow \ker D_k \rightarrow \ker \tilde{D}_k \rightarrow \ker \tilde{D}_k / \ker D_k \rightarrow 0$$

and

$$0\rightarrow \ker D_k \rightarrow D_k^{-1}((\ker D_{k-1})^{\perp}\cap \operatorname{ran} D_k) \xrightarrow{D_k} (\ker D_{k-1})^{\perp}\cap \operatorname{ran} D_k \rightarrow 0$$

are both exact, from which is clear that

(4) $\dim(\ker \tilde{D}_k / \ker D_k) = \dim(\operatorname{ran} D_k \cap (\ker D_{k-1})^{\perp})$.

Combining all four equations, the theorem follows.

<u>Corollary 1.4</u>. If (D) is a Fredholm Koszul complex, then index(D)$=-\chi$(D), where χC denotes the Euler characteristic of the complex C.

<u>Proof</u>: Straightforward from the preceding theorem, since $(D)=(\tilde{D})$.

<u>Corollary 1.5</u>. Let $A=(A_1,\dots A_n)$ be a doubly commuting

Fredholm tuple of operators on H. Then $H_k = \ker D_k / \operatorname{ran} D_{k+1}$ is exactly $\bigoplus_{f\in I_k} (\bigcap_{i=1}^{n} \ker {}^fA_i)$, where the sum is orthogonal, $I_k = \{f: \{1,\dots,n\} \longrightarrow \{0,1\} / f(i)=0 \text{ exactly } k \text{ times}\}$, ${}^fA_i$, as in II.Corollary 3.4, is meant to be $A_i^*A_i$ or $A_iA_i^*$, according to $f(i)=0$ or 1. Therefore,

$$\operatorname{index}(A) = \sum_k (-1)^{k+1} \sum_{f\in I_k} \dim(\bigcap_{i=1}^{n} \ker {}^fA_i) .$$

<u>Proof</u>: We already know that $H_k = \ker(D_k^*D_k + D_{k+1}D_{k+1}^*)$. Since A is doubly commuting, by the proof of II.Corollary 3.4 we know that $L_k = D_k^*D_k + D_{k+1}D_{k+1}^*$ is a block diagonal matrix whose entries are precisely the $\binom{n}{k}$ different combinations $\sum_{i=1}^{n} {}^fA_i$ for $f \in I_k$. Since all ${}^fA_i$ are positive operators, we know that $\ker(\sum_{i=1}^{n} {}^fA_i) = \bigcap_{i=1}^{n} \ker {}^fA_i$, which completes the proof.

We shall now illustrate the case n=2. Here (D) is:

$$0 \longrightarrow H \xrightarrow{D_2} H\oplus H \xrightarrow{D_1} H \longrightarrow 0 ,$$

so that $\operatorname{index}(D) = -\dim \ker D_1^* + \dim(\ker D_1 \cap (\operatorname{ran} D_2)^{\perp}) - \dim(\operatorname{ran} D_2 \cap (\ker D_1)^{\perp}) - \dim \ker D_2$, or $-\dim \ker D_1^* + \dim(\ker D_1 \cap \ker D_2^*) - \dim(\operatorname{ran} D_1^* \cap \operatorname{ran} D_2) - \dim \ker D_2$.

The term $\dim(\operatorname{ran} D_2 \cap (\ker D_1)^{\perp})$ "measures" the lack of "complexity" at the middle stage, that is, since D_1D_2 need not be zero, but only a compact operator, in general there is an adjustment in what would be the natural way of computing the index, as minus the Euler characteristic of the complex. The minus sign is required to (a) fit the unidimensional theory (if T is Fredholm, then $\text{index}(T)=\dim\ker T-\dim\ker T^*$ $=\dim\ker D_1-\dim\ker D_1^*$) and (b) produce a uniform -1 as $\text{index}(W_1,\ldots,W_n)$ on $H^2(S^1\times\ldots\times S^1)$ as we shall see in the examples.

Observe that $\hat{D}=\binom{D_1}{D_2^*}$ is a two-by-two matrix with $\ker\hat{D}=\ker D_1\cap\ker D_2^*$ and $\ker D_1^*$, $\ker D_2\subset\ker\hat{D}^*$. The term $\text{ran}\,D_1^*\cap\text{ran}\,D_2$ does not directly appear in $\hat{D}$, but an isomorphic image is the piece which $\ker D_1^*$ and $\ker D_2$ need to fill $\ker\hat{D}^*$.

<u>Remark</u>: Although we have studied only the Fredholm case, II.Proposition 3.1 makes possible a reasonable definition of a semi-Fredholm n-tuple, i.e., an almost commuting tuple A is semi-Fredholm iff $\hat{A}$ is semi-Fredholm. Consequently, either all even dimensional homology modules are finite or so are the odd dimensional ones. Index is then well defined and Theorems 1.2 and 1.3 clearly extend to this case (Observe that $\dim(\text{ran}\,D_{k+1}\cap(\ker D_k)^{\perp})$ is always finite, since $\text{ran}\,D_k$ is closed and so $D_k(\text{ran}\,D_{k+1}\cap(\ker D_k)^{\perp})\subset$ $\text{ran}\,D_kD_{k+1}$ is finite dimensional, being a closed subspace of the range of a compact operator.).

2. Examples

(i) If $A=(A_1,\ldots,A_n)\in F$ and $(A_{i_1},\ldots,A_{i_k})\in F$ $(k<n)$, where $i:\{1,\ldots,k\}\longrightarrow\{1,\ldots,n\}$ is injective, then $\mathrm{index}(A)=0$.

We can assume that $1\notin i(\{1,\ldots,k\})$. Define $\gamma:[0,1]\longrightarrow F$ by $\gamma(t)=(t+(1-t)A_1,(1-t)A_2,\ldots,(1-t)A_n)$ (use I.Proposition 2.2 and II.Proposition 4.5 to see that $\gamma(t)\in F$ (all t)). Thus, γ is a path in F from A to $(I,0,\ldots,0)$. Therefore $\mathrm{index}(a)=\mathrm{index}(I,0,\ldots,0)$ $=\mathrm{index}(\widehat{I,0,\ldots,0})=\mathrm{index}(I_{H\otimes C^{2^{n-1}}})=0$.

(ii) On $H^2(S^1\times\ldots\times S^1)$, $\mathrm{index}(W_1,\ldots,W_n)=-1$ (all n).

We shall prove that $\ker D_k=\mathrm{ran}\, D_{k+1}$ for all $k\geq 1$ and that $\mathrm{ran} D_1$ has codimension 1. For n=1 this is obvious. If n=2, it is clear that D_2 is one-to-one. Assuming that $W_1x+W_2y=0$, we have $x=-W_1^*W_2y=-W_2(W_1^*y)$. Since $y=-W_1W_2^*x$, we see that $W_1(W_1^*y)=-W_1W_1^*W_1W_2^*x$ $=-W_1W_2^*x=y$, that is $x\oplus y$ is in the range of D_2. Finally, $\mathrm{ran}\,W_1+\mathrm{ran}\,W_2=\{f\in H^2(S^1\times S^1):\ \hat f(0,0)=0\}$ has codimension 1.

For n>2, we can give an inductive proof, exactly as in II.2(ii) to show exactness at the kth. stage $(k\geq 1)$, while, as before, $\mathrm{ran}\,D_1=\mathrm{ran}\,W_1+\ldots+\mathrm{ran}\,W_n=\{f\in H^2(S^1\times\ldots\times S^1):$ $\hat f(0,\ldots,0)=0\}$ has codimension 1.

(iii) On $H^2(S^{2n-1})$, $T_z=(T_{z_1},\ldots,T_{z_n})$ has index -1 (all n).

An easy way to see this is to appeal to a theorem of Venugopalkrisna [15, Theorem 1.5] on the index of a Toeplitz matrix. As we shall prove in IV.3, T_z can be connected (in F) to a copy of $(W_1,\dots,W_n)$, so that $\mathrm{index}(T_z)=\mathrm{index}(W)=-1$.

(iv) We consider $(W_1^{(m)},W_2,\dots,W_n)$ on $H^2(S^1\times\dots\times S^1)$. By $W_1^{(m)}$ we understand W_1^m for $m\geq 0$ and W_1^{*-m} for $m<0$. By the spectral mapping theorem (I. Proposition 2.6) and II. Corollary 3.10, $(W_1^{(m)},W_2,\dots,W_n)\in F$.

When $m>0$, one can see that the associated Koszul complex is exact at every kth. stage ($k\geq 1$) and that codimension of ran D_1 is m (in this case, ran D_1 is $\{f\in H^2(S^1\times\dots\times S^1)$: $\hat{f}(j,0,\dots,0)=0$ for all $j<m$). When $m<0$, it is easy to check that exactness holds everywhere but at k=1, where $\ker D_1/\mathrm{ran}\, D_2$ has dimension m. An easier way is to apply Proposition 3.1, yet to be proved. Thus, $\mathrm{index}(W_1^{(m)},W_2,\dots,W_n)=-m$.

<u>Theorem 2.1</u>. (a) $\mathrm{Sp}((W_1,\dots,W_n),H^2(S^1\times\dots\times S^1))=\prod_{i=1}^{n} D_i$, where D_i is the closed unit disc on the ith. coordinate.

(b) $\mathrm{Sp}((T_{z_1},\dots,T_{z_n}),H^2(S^{2n-1}))=B^{2n}$.

(c) $\mathrm{Sp}_e(W,H^2(S^1\times\dots\times S^1))=\mathrm{Fr}(\prod_{i=1}^{n} D_i)=$
$=(T\times D_2\times\dots\times D_n)\cup(D_1\times T\times\dots\times D_n)\cup\dots\cup(D_1\times D_2\times\dots\times T)$.

(d) $\mathrm{Sp}_e(T_z,H^2(S^{2n-1}))=S^{2n-1}$.

<u>Proof</u>: (d) Since $T_{z_1}^*T_{z_1}+\dots+T_{z_n}^*T_{z_n}=I$ and the T_{z_i}'s are essentially normal, we conclude that $\mathrm{Sp}_e(T_z,H^2(S^{2n-1})$ is

contained in S^{2n-1} (II.Corollary 3.6). Since index(T_z) is -1, we see that $Sp_e(T_z)=S^{2n-1}$ (otherwise contradicting the continuity of index).

(b) Being index constant on path-components, we conclude that $B^{2n}\subset Sp(T_z)$. By I.Proposition 2.5, $Sp(T_z)\subset Sp_B(T_z)$, where B is the Banach subalgebra of $C(S^{2n-1})$ generated by $z_1,\dots,z_n$. Since B can be identified with $P(B^{2n})$, the uniform closure on B^{2n} of the algebra of polynomials in $z_1,\dots,z_n$ and B^{2n} is polynomially convex, then the maximal ideal space of B is homeomorphic to B^{2n} and consequently, $Sp_B(T_z)=B^{2n}$ (see [6], Chapter III for the pertinent results).

Thus: $B^{2n}\subset Sp(T_z)\subset Sp_B(T_z)=B^{2n}$, which shows (b).

Remark: The index argument gave us only one containment, but it can actually be used to prove the other inclusion when n=2. For, it is clear that $\ker(T_{z_1}-\lambda_1)=0$ when $|\lambda|>1$. If we can show that $\operatorname{ran}(T_{z_1}-\lambda_1)+\operatorname{ran}(T_{z_2}-\lambda_2)=H^2(S^3)$ for $|\lambda|>1$, then, since index$(T_z-\lambda)=0$ outside B^4 (by continuity), we must have exactness at the middle stage as well. So, let us prove that $\ker(T_{z_1}^*-\bar{\lambda}_1)\cap\ker(T_{z_2}^*-\bar{\lambda}_2)=0$ for $|\lambda|>1$.

Assume $f\in H^2(S^3)$ and $T_{z_i}^*f=\bar{\lambda}_i f$ $(i=1,2)$.

Recall that $T_{z_1}e_k=\frac{c_k}{c_{k'}}e_{k'}$ $\quad (k'=(k_1+1,k_2))$,

$$T_{z_2}e_k=\frac{c_k}{c_{k^+}}e_{k^+} \qquad (k^+=(k_1,k_2+1))$$

and

$$c_k = \frac{1}{\sqrt{2}\,\pi} \sqrt{\frac{(|k|+1)!}{k!}} \; .$$

Then:

$$(f,e_{k'}) = \frac{c_{k'}}{c_k}(T_{z_1}{}^{*} f,e_k) = \bar{\lambda}_1 \frac{c_{k'}}{c_k}(f,e_k)$$

and

$$(f,e_{k''}) = \bar{\lambda}_2 \frac{c_{k''}}{c_k}(f,e_k).$$

Combining them,

$$(f,e_k) = \frac{c_k}{c_{00}} \bar{\lambda}_1{}^{k_1} \bar{\lambda}_2{}^{k_2} (f,e_{00}).$$

Therefore:

$$\|f\|_2^2 = \sum_k |(f,e_k)|^2 = \sum_k \frac{c_k^2}{c_{00}^2} |\lambda_1|^{2k_1} |\lambda_2|^{2k_2} |(f,e_{00})|^2$$

$$= \sum_{l=0}^{\infty} \sum_{|k|=l} \frac{(l+1)!}{k_1!k_2!} |\lambda_1|^{2k_1} |\lambda_2|^{2k_2} |(f,e_{00})|^2$$

$$= \sum_{l=0}^{\infty} (l+1) |\lambda|^{2l} |(f,e_{00})|^2 \; ,$$

so that $(f,e_{00})=0$ (being the series convergent and $|\lambda|>1$) or $f=0$.

(c) Assume that $\lambda \notin Fr(\prod_{i=1}^{n} D_i)$. If $|\lambda_1|>1$, $W_1-\lambda_1$ is invertible and so is $W-\lambda$, which implies $\lambda \notin Sp_e(W)$. If $|\lambda_1|=1$; then at least one of the λ_i's with $i \geq 2$ must have modulus greater than 1, showing again that $W-\lambda$ is invertible and $\lambda \notin Sp_e(W)$.

If $|\lambda_1|<1$, three possibilities occur: $|\lambda_2|>1$, $|\lambda_2|=1$ and $|\lambda_2|<1$. It is again clear that only the case $|\lambda_2|<1$ deserves comment. Continuing this reasoning for the remaining λ_i's, we conclude that only the situation $|\lambda_i|<1$ $(i=1,\ldots,n)$ presents some difficulty.

So suppose that $|\lambda_i|<1$ $(i=1,\ldots,n)$. Let $(D(\lambda))$ be the Koszul complex for $W-\lambda$. We shall prove that $W-\lambda$ is Fredholm by induction.

If $n=2$ and $W_i f=\lambda_i f$ $(i=1,2)$, it is clear that $f=0$. Assume that $(W_1-\lambda_1)f_1+(W_2-\lambda_2)f_2=0$ $(f_1,f_2\in H^2(S^1\times S^1))$. Then, evaluating at λ_2, we have $f_1(z_1,\lambda_2)=0$, which implies that $f_1(z_1,z_2)=(z_2-\lambda_2)g(z_1,z_2)$ for some g in $H^2(S^1\times S^1)$ (0. PR 7). Therefore $(W_1-\lambda_1)(W_2-\lambda_2)g+(W_2-\lambda_2)f_2$ is zero and consequently, $f_2=-(W_1-\lambda_1)g$. Thus:

$$f_1=-(W_2-\lambda_2)(-g)$$
$$f_2=(W_1-\lambda_1)(-g) \ .$$

We finally observe that ran $D(\lambda)_1=\{f\in H^2(S^1\times S^1):f(\lambda_1,\lambda_2)=0\}$, using the fact that $|\lambda_1|,|\lambda_2|<1$.

We wish to remark that the preceding fact for $n=2$ could have been easily obtained from the Corollary to Theorem 4 in [5]. However, since we can now make an inductive argument for $n>2$ as in Example (ii), it is clear that an algebraic proof is needed. By the way, it is interesting to point out that the kind of argument used in II.2(ii) and (ii) works for $H^2(S^1\times\ldots\times S^1)$ due to the special structure of this space, which allows one to prove exactness by algebraic methods.

We have thus proved that $Sp_e(W)\subset Fr(\prod_{i=1}^{n} D_i)$.

Since index(W)=-1 and index is continuous, we immediately get: $Sp_e(W)=Fr(\prod_{i=1}^{n} D_i)$.

(a) From (c) we obtain: $Sp(W)\supset\prod_{i=1}^{n} D_i$. Moreover, if $\lambda\notin\prod_{i=1}^{n} D_i$, then for at least one i, $|\lambda_i|>1$. Then $W_i-\lambda_i$ is invertible and so is $W-\lambda$. Thus $Sp(W)=\prod_{i=1}^{n} D_i$.

3. Indices of related tuples

The following propositions are rather elementary, though useful to find indices of several related tuples.

Proposition 3.1. Let $A=(A_1,\dots,A_n)\in F$, $\varphi:\{1,\dots,n\}\to\{1,*\}$ be a function and define $\varphi(A_i)=A_i^{\varphi(i)}$ as in II.Corollary 3.10. Assume $\varphi(A)=(\varphi(A_1),\dots,\varphi(A_n))$ is an almost commuting tuple. Then $\varphi(A)\in F$ and index $\varphi(A)=(-1)^{|\varphi|}$index($A$), where $|\varphi|=\#\{i:\ \varphi(i)=*\}$.

Corollary 3.2. If $A=(A_1,\dots,A_n)\in F$ and one of the A_i's is essentially self-adjoint, then index(A)=0.

Proof of the Proposition: Without loss of generality, we can restrict attention to the case $\varphi(1)=*$, $\varphi(i)=1$ $(i>1)$ (see proof of II.Corollary 3.10). We also know that $\widehat{\varphi(A)}$ is, up to some minus signs, $\hat{A}^*$. From this it follows that index $\varphi(A)$=$-$index(A), since those minus signs leave the homology modules unchanged.

Corollary 3.3. If A_1 and A_2 almost doubly commute, and $(A_1,A_2)\in F$, then (A_1^*,A_2), (A_1,A_2^*) and (A_1^*,A_2^*) are Fredholm and $\mathrm{index}(A_1,A_2)=\mathrm{index}(A_1^*,A_2^*)=-\mathrm{index}(A_1^*,A_2)$ $=-\mathrm{index}(A_1,A_2^*)$.

Proposition 3.4. Let $A=(A_1,\ldots,A_n)\in F$, V be a Fredholm operator such that there exists a path $\gamma:[0,1]\longrightarrow F$ with $\gamma(0)=V$, $\gamma(1)=I$ and $[\gamma(t),A_k]\in K(H)$ (all t, $k\geq 2$). Then $\mathrm{index}(A)=\mathrm{index}(VA)=\mathrm{index}(AV)$, where VA, AV are defined as in II.Corollary 4.7(i).

Proof: We already know, from II.Corollary 4.7(i), that VA, AV$\in F$. Since $[\gamma(t),A_k]\in K(H)$ $(k\geq 2)$ for all t and $\gamma(t)\in F$, we see that $(\gamma(t)A_1,A_2,\ldots,A_n)$, $(A_1\gamma(t),A_2,\ldots,A_n)\in F$. By continuity, $\mathrm{index}(A)=\mathrm{index}(VA)=\mathrm{index}(AV)$.

Corollary 3.5. If $A=(A_1,\ldots,A_n)\in F$ and $\lambda\in C-\{0\}$, then $(\lambda A_1,A_2,\ldots,A_n)\in F$ and $\mathrm{index}(A)=\mathrm{index}(\lambda A_1,A_2,\ldots,A_n)$.

Proof: Let $\gamma(t)=(1-t)\lambda+t$ if $\mathrm{Im}\,\lambda\neq 0$ and

$$\begin{cases} 2(i-\lambda)t+\lambda & 0\leq t\leq \frac{1}{2} \\ 2(1-i)t+2i-1 & \frac{1}{2}\leq t\leq 1 \end{cases}$$

when $\mathrm{Im}\,\lambda=0$.

Proposition 3.6. Let $A=(A_1,\ldots,A_n)\in F$ and $p\in S_n$ be a permutation. Then $p^*A=(A_{p(1)},\ldots,A_{p(n)})\in F$ and $\mathrm{index}(A)=\mathrm{index}(p^*A)$.

Proof: Since every permutation is a product of transpositions, we can restrict attention to the latter. Furthermore, it suffices to consider a transposition p with $p(1)=j$, $p(j)=1$, in virtue of the relation $(j\ k)=(1\ j)(1\ k)(1\ j)$. This can be done by using the same argument as in II.Proposition 4.5, thus showing that both Koszul systems are unitarily isomorphic (i.e., the isomorphism is given by unitaries). Therefore $\operatorname{index}(A)=\operatorname{index}(p^*A)$.

Remarks: Using the above definition of index, we can define the index of a nonsingular tuple of elements of the Calkin algebra $A(H)$, by lifting it to an almost commuting Fredholm tuple of operators on H. A classical result of E. Michael [10] on cross sections induces immediately a bijection of path components between F and $I(A(H))$=commuting invertible tuples on $A(H)$. It is also clear that $I(A(H))$ is open in the set of commuting tuples.

The above definition of index was given only for tuples of operators (that is, Fredholm Koszul systems), while we could have extended it to more general systems. One approach is to consider the same definition for systems with $\sum_{k\text{ even}} n_k = \sum_{k\text{ odd}} n_k$ in order to get a square matrix $\hat{D}$.

Other viewpoint would be to take the content of Theorem 1.3 as the starting point. We have not pursued this farther since our main interest is centered on Koszul systems.

4. Although it is not hard to see that a normal n-tuple $N=(N_1,\dots,N_n)$ (i.e., $N_iN_j=N_jN_i$ and $N_iN_i^*=N_i^*N_i$ for all i,j) which is Fredholm will have necessarily index zero (its associated N is normal), it is not completely obvious that the same is true for essentially normal tuples with all commutators in trace class ($n\geq 2$).

<u>Theorem 4.1</u>. Let $A=(A_1,\dots,A_n)$ ($n\geq 2$) be an essentially normal tuple (that is, $[A_i,A_j]$, $[A_i,A_i^*]\in K(H)$ for all i,j) with all commutators in trace class. Assume that A is Fredholm. Then index(A)=0.

We shall need the following lemma, which appears in [9].

<u>Lemma</u>: Let $T=(T_{ij})\in L(H^N)$ be a Fredholm operator and $[T_{ik},T_{lm}]\in C_1$ (all $i,k,l,m=1,\dots,N$), i.e., all commutators are in trace class. Then det(T) is well defined, det(T) is Fredholm and index(T)=index(det(T)).

<u>Proof of the Theorem</u>: We can apply the preceding lemma to $\hat{A}$ and thus conclude that $\text{index}(\hat{A})=\text{index}(\det(\hat{A}))$. An easy calculation shows that $\det(\hat{A})-(\sum_{i=1}^{n}A_i^*A_i)^{n-1}$ is

compact. Therefore, $\mathrm{index}(\det(A))=(n-1)\mathrm{index}(\sum_{i=1}^{n} A_i^* A_i)=0$, since the last operator is positive.

The preceding theorem has certain points of contact with a result of Helton and Howe. In [8, Part II, Theorem 2], they precisely state: Suppose that $X_1,\ldots,X_n$ is a family of self-adjoint operators on H such that $[X_i,X_j]\in C_1$(=trace class) for all i,j. Let B be the C^*-algebra they generate. Then the index of an operator U in $M_m(B)$ with unitary valued symbol (that is, its projection into $M_m(A(H))$ is a unitary m×m-matrix) having an extension to a matrix function $u=(u_{ij})$ with entries in $S(R^n)$ is

$$\mathrm{index}(U)=\sum_{j,k=1}^{n} h_l(du_{jk}\wedge d\bar{u}_{jk}) \quad ,$$

where h_l is the homology class induced by a certain linear functional l defined on closed 2-forms of R^n.

Thus, if we consider a tuple $X=(X_1,\ldots,X_n)$ of self-adjoint operators with $[X_i,X_j]\in C_1$ for all i,j, such that X is essentially unitary, we can apply that result (notice that $x_i\in S(R^n)$ for all i=1,...,n) to obtain:

$$\mathrm{index}(X)=\sum_{j,k=1}^{n} h_l(du_{jk}\wedge d\bar{u}_{jk})=0 \quad ,$$

since u_{jk} is real (all j,k). Of course, we can get this much easier applying Corollary 3.2.

We finally want to observe that for a doubly commuting

Fredholm tuple with a coordinate normal, index is also zero, a fact that follows immediately from I.Lemma 1.1 ($\hat{A}$ is normal iff A_1 is normal) and Proposition 3.6. When $n \leq 3$, the same holds without assuming doubly commutativity, because I.Lemma 1.1 works in that case.

CHAPTER IV: THE DEFORMATION PROBLEM

Let H be an infinite dimensional Hilbert space and $A=(A_1,\dots,A_n)$ be an almost commuting tuple of operators on H. If A is Fredholm, index(A) is a well-defined integer; by III.Theorem 1.2, index is an invariant for the path-components of F. In [4], Douglas raised the following question: Is it the only invariant? In other words, given two n-tuples $A=(A_1,\dots,A_n)$ and $B=(B_1,\dots,B_n)$ in F with same index, is it always possible to find a path $\gamma:[0,1]\longrightarrow F$, continuous, such that $\gamma(0)=A$ and $\gamma(1)=B$? This is the deformation problem. For n=1 the answer is known to be yes (cf. [3]) and for the case A, B essentially normal, Douglas himself gave a proof in [4], using the extension theory of Brown-Douglas-Fillmore. We shall give a detailed exposition of this fact in Section 2. Previously in Section 1 we solve the deformation problem in the easiest case: when A has a Fredholm coordinate; then A can be joined to $(I,0,\dots,0)$ in F. In Section 3 we consider again $(W_1,\dots,W_n)$ and $(T_{z_1},\dots,T_{z_n})$ and show that they lie in the same component. As a consequence, we obtain the non-obvious fact that $(W_1,\dots,W_n)$ can be connected to $(W_1^*,\dots,W_n^*)$ for n even. If A_1 is essentially normal with closed range, then index(A)=0 and in fact A can be joined to $(I,0,\dots,0)$. We show this in Section 4. We then describe how, in case the tuple is almost doubly commuting and at least one of the A_i's has closed range, the problem can be reduced to n-tuples with at least one partial isometry as a coordinate. In Section 6 we deal with the case in which that partial isometry is semi-Fredholm and solve the deformation problem there. We finish the chapter considering a rather technical case, when the essential spectrum of one of the coordinates has 0 as an accesible point from the essential resolvent set.

<u>Notation</u>: If $X \subset L(H)\otimes C^n$, $A=(A_1,\dots,A_n)$, $B=(B_1,\dots,B_n)\in X$ and there is $\gamma:[0,1]\longrightarrow X$ continuous, $\gamma(0)=A$ and $\gamma(1)=B$, we write $A\overset{X}{\sim}B$.

1. Let $A=(A_1,\dots,A_n)$ be an almost commuting tuple of operators on H. Assume $n\geq 2$ and A_i Fredholm for some i. Without loss of generality, it suffices to deal with the case i=n. Then $\gamma(t)=(t+(1-t)A_1,\dots,t+(1-t)A_{n-1},A_n)\in F$, $\gamma(0)=A$ and $\gamma(1)=(I,\dots,I,A_n)$, so that $A\overset{F}{\sim}\gamma(1)$. Now $\rho(t)=(I,1-t,\dots,1-t,(1-t)A_n)\in F$ $\rho(0)=\gamma(1)$ and $\rho(1)=(I,\dots,0)$, as expected; that is

<u>Proposition 1.1</u>. Let $A=(A_1,\dots,A_n)\in F$ $(n\geq 2)$ and assume A_i is Fredholm for some i. Then $A\overset{F}{\sim}(I,0,\dots,0)$.

Notation: Since $(I,0,\dots,0)$ gives $\widehat{(I,0,\dots,0)}=I_{H\otimes C^{2^{n-1}}}$ it is natural to think of that tuple as the "identity" tuple. We shall do that in what follows and simply write $I=(I,0,\dots,0)$. The above statement rephrases as $A \overset{F}{\sim} I$ $(n\geq 2)$.

2. The essentially normal case

We shall write EN for the set of essentially normal tuples of operators, i.e., those tuples producing commuting normal tuples in the Calkin algebra. We write ENF for $EN \cap F$. From the results of Chapter II we record:

Lemma 2.1. Let $A \in EN$. Then $A \in ENF$ iff $\sum_{i=1}^{n} A_i^* A_i$ is Fredholm.

If $A=(A_1,\dots,A_n) \in EN$, then the C^*-subalgebra B of $A(H)$ generated by $a_1,\dots,a_n$ (recall that we use capital letters for operators, small letters for their corresponding Calkin algebra projections)is abelian. By the Gelfand-Naimark theorem, B is $*$-isometrically isomorphic to $C(M)$, the continuous functions on M, the maximal ideal space of B. There is a natural embedding of M into C^n given by $\varphi \mapsto (\varphi(a_1),\dots,\varphi(a_n))$ that allows us to consider M as a subset of C^n.

Lemma 2.2. $A=(A_1,\dots,A_n) \in ENF$ iff $0 \notin M$.

We now compute M. The answer is expected.

Lemma 2.3. Let $A=(A_1,\ldots,A_n)\in EN$ and B be the C^*-subalgebra of A(H) generated by $a_1,\ldots,a_n$. Then the maximal ideal space of B, seen as a subset of C^n, is identical to $Sp_e(A)$.

Definition 2.4. An essentially normal tuple A is essentially unitary (in symbols $A\in EU$) in case $\hat{A}$ is essentially unitary.

The fact that for $A\in ENF$ the maximal ideal space M does not contain the origin was used by Douglas in [4] to deform it into S^{2n-1} and so get and element of $Ext(S^{2n-1})$[1]. The topological lemma needed is the invariance of Ext under homotopy. We shall now give an alternate approach, algebraic in nature, that avoids considerations of homotopy. It is based on the fact that the tuple $a=(a_1,\ldots,a_n)$ is normal and so is $\hat{a}$. A look at the polar decomposition of $\hat{a}$ shows that both factors lie in $M_n(A(H))$ since a is invertible. We can then reduce the problem to S^{2n-1}. We now give the details.

Lemma 2.5. Let $a=(a_1,\ldots,a_n)$ be a doubly commuting tuple on a C^*-algebra B. Then a_1 is normal iff $\hat{a}$ is normal.

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1 For a complete exposition on Ext, see [1].

<u>Remark</u>: The assertion (a_1, a_2) invertible iff (a_2, a_1) invertible was proved establishing an isomorphism between the complexes. Since we have a representation $\hat{a}=\begin{pmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{pmatrix}$, one would think that it is possible to find some unitary transformation U such that $U\hat{a}U^*=\begin{pmatrix} a_2 & a_1 \\ -a_1^* & a_2^* \end{pmatrix}$. Lemma 2.5 says that, in general, it is not possible (take for example a_1 normal, a_2 non-normal).

If $\phi: A(H) \longrightarrow L(R)$ is a faithful representation of A(H) into the algebra of bounded linear operators on a certain Hilbert space R, then $\phi_k: M_k(A(H)) \longrightarrow M_k(L(R))$ defined by $\phi_k(a)=(\phi(a_{ij}))$ is a faithful representation of $M_k(A(H))$ (actually, this is the way one defines a norm on $M_k(A(H))$ so as to make it a C^*-algebra). If a is a k×k-matrix over A(H), then $\phi_k(a)$ has a polar decomposition $\phi_k(a)=VP$, where V is a partial isometry and $0 \leq P=\phi_k((a^*a)^{\frac{1}{2}})$. Thus P belongs to the image of ϕ_k. But, in general, $V \notin \mathrm{im}\,\phi_k$ (V is only in the von Neumann algebra generated by $\phi_k(a)$). If a is invertible, however, $\phi_k(a)$ is invertible and then $V=\phi_k(a)P^{-1}=\phi_k(ap^{-1}) \in \mathrm{im}\,\phi_k$ $(p=(a^*a)^{\frac{1}{2}})$ (V is unitary in this case). So that $a=vp$, where v is unitary, $p>0$ and $v, p \in M_k(A(H))$.

<u>Lemma 2.6</u>. Let $a=(a_1,\ldots,a_n)$ be an invertible normal tuple on $A(H)$ and $\hat{a}$ be its associated $k\times k$-matrix ($k=2^{n-1}$) over $A(H)$. Then, if $a=vp$ is the polar decomposition of a with $v,p\in M_k(A(H))$, there exist $u_1,\ldots,u_n,q\in A(H)$ such that $q>0$, $u=(u_1,\ldots,u_n)$ is a commuting normal tuple, $\hat{u}=v$ and $\widehat{(q,0,\ldots,0)}=p$ (q is indeed $(\sum_{i=1}^{n}a_i^*a_i)^{\frac{1}{2}}$).

<u>Proof</u>: Define q to be $(\sum_{i=1}^{n}a_i^*a_i)^{\frac{1}{2}}$. It is almost obvious that (II.Corollaries 3.4 and 3.5)

$$\hat{a}^*\hat{a}=\widehat{(q,0,\ldots,0)}^2=p^2.$$

Since $\hat{a}$ is invertible, so is p and then

$$v=\hat{a}p^{-1}=\hat{a}\widehat{(q^{-1},0,\ldots,0)}\ .$$

Observe that $\widehat{(q^{-1},0,\ldots,0)}$ is diagonal, so that:

$$\hat{a}\widehat{(q^{-1},0,\ldots,0)} = \widehat{(a_1q^{-1},\ldots,a_nq^{-1})}.$$

Let $u_i=a_iq^{-1}$. Then $\widehat{(u_1,\ldots,u_n)}=v$ and $u=(u_1,\ldots,u_n)$ is a commuting normal tuple.

If $A=(A_1,\ldots,A_n)\in ENF$, then $a=(a_1,\ldots,a_n)$ is an invertible normal tuple and, by Lemma 2.6,

$$\widehat{(a_1q^{-1},\ldots,a_nq^{-1})} = \hat{a}p^{-1} = v = \hat{u}\ .$$ If $q_t=(1-t)q+t$, it is clear that $(a_1q_t^{-1},\ldots,a_nq_t^{-1})$ is a path of normal commuting tuples joining a to u. Since $\sum_{i=1}^{n}u_i^*u_i=1$, for any φ in the maximal ideal space M of the C^*-algebra generated by $u_1,\ldots,u_n$ we have $\sum_{i=1}^{n}|\varphi(u_i)|^2=1$, from which it follows that $M\subset S^{2n-1}$. We summarize this in the following

<u>Proposition 2.7</u>. Any $A \in ENF$ can be joined (in ENF) to an essentially unitary tuple. If $A \in EU$, then $Sp_e(A) \subset S^{2n-1}$.

Calling $i: Sp_e(A) \hookrightarrow S^{2n-1}$ the inclusion map, then $Ext(Sp_e(A)) \xrightarrow{i_*} Ext(S^{2n-1}) \cong Z$. Consequently, any $A \in EU$ produces an integer $i(A)$ and, if $A, B \in EU$ are such that $i(A)=i(B)$, then they induce equivalent extensions, so that there exists a unitary $U \in L(H)$ satisfying $UA_iU^* = B_i + K_i$ $(i=1,\dots,n)$, $K_i \in K(H)$. Since U is unitary we can find a path U_t of unitaries such that $U_0=U$, $U_1=I$. Therefore, $A \overset{EU}{\sim} (UA_1U^*,\dots,UA_nU^*)$ (use U_t and II.Proposition 4.6), which in turn can be joined with B (take the line segment). To complete the argument, we have to show that $index(A)=i(A)$ or $-i(A)$. Since H is infinite dimensional, $H \cong H \oplus H^2(S^{2n-1})$, so that a unitary copy of A can be joined to $(I \oplus T_{z_1}^{(-i(A))},\dots,I \oplus T_{z_n})$. Consequently, it suffices to check the formula for the tuple $(T_{z_1}^{(k)},\dots,T_{z_n})$ on $H^2(S^{2n-1})$, which can be easily done. (Of course, $(T_{z_1}^{(k)},\dots,T_{z_n})$ does not define immediately an element of $Ext(S^{2n-1})$, in general, but in that case we consider its unitary part, after Lemma 2.6.)

3. We now consider (W_1,W_2) on $H^2(S^1 \times S^1)$ and (T_{z_1},T_{z_2}) on $H^2(S^3)$. We already know that both pairs are Fredholm of index -1. The purpose of this section is to show that they can be joined by a path of Fredholm pairs.

More precisely, if $S_i (i=1,2)$ is defined on $H^2(S^3)$ so as to be unitarily equivalent to W_i $(i=1,2)$, that is: $S_1 e(k_1,k_2)=e(k_1+1,k_2)$, $S_2 e(k_1,k_2)=e(k_1,k_2+1)$, where e_k is the natural basis for $H^2(S^3)$ as defined in 0.Notation, then $(T_{z_1},T_{z_2}) \overset{DF}{\sim} (S_1,S_2)$, where DF stands for the almost doubly commuting Fredholm tuples. To show this we first notice that S_i is the partial isometry in the polar decomposition for $T_{z_i}=S_i P_i$, where

$$P_1 e_k = \frac{c_k}{c_{k'}} e_k \quad (k'=(k_1+1,k_2)$$

$$P_2 e_k = \frac{c_k}{c_{k^\dagger}} e_k \quad (k^\dagger=(k_1,k_2+1)) \ .$$

We now define $T_t^{(i)}=S_i((1-t)P_i+t)$ $(i=1,2)$ and show that $(T_t^{(1)},T_t^{(2)})\in DF$ (all $t\in[0,1]$).

First at all, we have to verify almost doubly commutativity. This amounts to showing that $[S_i,T_{z_j}]\in K$, $[S_i,T_{z_j}^*]\in K$ $(i\neq j)$.

Now,

$$[S_1,T_{z_2}]=S_1 S_2 P_2 - S_2 P_2 S_1 = S_2 S_1 P_2 - S_2 P_2 S_1 .$$

Recall that $e_k=c_k Z^k$, $c_k=\dfrac{1}{\sqrt{2}\,\pi}\sqrt{\dfrac{(|k|+1)!}{k!}}$,

so that

$$S_1 P_2^2 e_k = \frac{c_k^2}{c_{k^\dagger}^2} e_{k'} \quad , \quad P_2^2 S_1 e_k = \frac{c_{k'}^2}{c_{k'^\dagger}^2} e_{k'} \quad , \text{ or}$$

$$(S_1 P_2^2 - P_2^2 S_1) e_k = \Big(\frac{c_k^2}{c_{k^\dagger}^2} - \frac{c_{k'}^2}{c_{k'^\dagger}^2} \Big) e_{k'} = \Big(\frac{k_2+1}{|k|+2} - \frac{k_2+1}{|k|+3} \Big) e_{k'}$$

$$= \frac{k_2+1}{(|k|+2)(|k|+3)} e_{k'} ,$$

so that $[S_1,P_2^2]\in K$ and then $[S_1,P_2]\in K$ and $[S_1,T_{z_2}]\in K$.
Similarly

$$[S_1,T_{z_2}^*] = S_1P_2S_2^*-P_2S_2^*S_1=S_1P_2S_2^*-P_2S_1S_2^*$$
$$=(S_1P_2-P_2S_1)S_2^*\in K .$$

Analogously, $[S_2,T_{z_1}]$ and $[S_2,T_{z_1}^*]$ are compact.
Since $[(1-t)P_1+t,T_{z_2}]\in K$, an application of II.Corollary 4.7 shows that $(T_t^{(1)},T_{z_2})\in DF$. Since $[(1-t)P_2+t,T_t^{(1)}]$ is compact, a second application of the same corollary implies that $(T_t^{(1)},T_t^{(2)})\in DF$.
Thus, $(T_{z_1},T_{z_2}) \overset{DF}{\sim} (S_1,S_2)$.

We now show that $(W_1,W_2) \overset{DF}{\sim} (W_1^*,W_2^*)$, or equivalently, that $(S_1,S_2) \overset{DF}{\sim} (S_1^*,S_2^*)$. Since $(S_1,S_2) \overset{DF}{\sim} (T_{z_1},T_{z_2})$ and (T_{z_1},T_{z_2}), $(T_{z_1}^*,T_{z_2}^*)$ give rise to the same element of $\mathrm{Ext}(S^3)$, we know that there exists a unitary $U\in L(H^2(S^3))$ such that $T_{z_i}=U^*T_{z_i}^*U+K_i$ (K_i K, $i=1,2$). It is now clear that $(T_{z_1},T_{z_2}) \overset{DF}{\sim} (T_{z_1}^*,T_{z_2}^*)$.
Using $(T_t^{(1)*},T_t^{(2)*})$ instead of $(T_t^{(1)},T_t^{(2)})$, we can connect (S_1^*,S_2^*) to $(T_{z_1}^*,T_{z_2}^*)$ in DF. Therefore, $(S_1,S_2) \overset{DF}{\sim} (S_1^*,S_2^*)$, as desired.

<u>Remarks</u>: An obvious extension of the preceding proof shows that $(S_1,\ldots,S_n) \overset{DF}{\sim} (T_{z_1},\ldots,T_{z_n})$ (all n). However, the statement $(S_1,S_2) \overset{DF}{\sim} (S_1^*,S_2^*)$ will extend only for even n,

because $(T_{z_1},\ldots,T_{z_n})$ and $(T_{z_1}^*,\ldots,T_{z_n}^*)$ must produce equivalent extensions. Easier: index $(S_1^*,S_2^*,\ldots,S_n^*)=$ $=(-1)^n\text{index}(S_1,S_2,\ldots,S_n)=(-1)^{n+1}$ by III.Proposition 3.1, so that $\text{index}(S_1^*,\ldots,S_n^*)=-1$ iff n is even.

By taking "powers" of T_t, i.e., $T_t^{(m)}=(T_t^{(1)^{(m_1)}},T_t^{(2)^{(m_2)}})$ where $T^{(m)}$ is T^m or T^{*-m} according to $m\geq 0$, $m<0$, we can easily prove that $(S_1^{(m_1)},S_2^{(m_2)})\overset{DF}{\sim}(T_{z_1}^{(m_1)},T_{z_2}^{(m_2)})$ (as we remarked in III.2(iv), we need the spectral mapping theorem to assure that $(T_t^{(1)^{(m_1)}},T_t^{(2)^{(m_2)}})$ is Fredholm), so that $(S_1^{(m_1)},S_2^{(m_2)})\overset{DF}{\sim}(S_1^{(k_1)},S_2^{(k_2)})$ iff $m_1m_2=k_1k_2$.

We now deal with the case $(T_\varphi\otimes I,I\otimes T_\psi)$ on $H^2(S^1\times S^1)$, where $\varphi,\psi\in C(T)$ and T_φ,T_ψ are their associated Toeplitz operators.

Theorem 3.1. Let $\varphi,\psi\in C(T)$ and assume that neither T_φ nor T_ψ is invertible. Then $(T_\varphi\otimes I,I\otimes T_\psi)\in F$ iff T_φ and T_ψ are Fredholm. If $\text{index}(T_{\varphi_1}\otimes I,I\otimes T_{\psi_1})=\text{index}(T_{\varphi_2}\otimes I,I\otimes T_{\psi_2})$, there is a path $(T_{\varphi_t}\otimes I,I\otimes T_{\psi_t})$ of Fredholm pairs joining them.

Proof: Let $\Phi(z_1,z_2)=\varphi(z_1)$ and $\Psi(z_1,z_2)=\psi(z_2)$. Then $(T_\varphi\otimes I,I\otimes T_\psi)$ is (T_Φ,T_Ψ). By the Corollary to Theorem 4 in [5], we know that $(T_\Phi,T_\Psi)\in F$ iff $(T_{\Phi(z_1,\cdot)},T_{\Psi(z_1,\cdot)})$ and $(T_{\Phi(\cdot,z_2)},T_{\Psi(\cdot,z_2)})$ are invertible for all $z_1,z_2\in T$, i.e., iff $(\varphi(z_1),T_\psi)$ and $(T_\varphi,\psi(z_2))$ are invertible pairs for all $z_1,z_2\in T$. A moment's thought shows that this is the same as having $\varphi\neq 0$, $\psi\neq 0$, which in turn is equivalent to T_φ, T_ψ both Fredholm.

Now, if T_φ is Fredholm and $\mathrm{index}(T_\varphi)=-n$, then T_φ can be connected to $T_z^{(n)}$ by a path of Fredholm Toeplitz operators (see, for instance, [3]). From this and what we proved before, the second part of the theorem follows.

<u>Remark</u>: Notice that we can add the fact:

$$\mathrm{index}(T_\varphi\otimes I, I\otimes T_\psi)=-(\mathrm{index}(T_\varphi))(\mathrm{index}(T_\psi)) \quad ,$$

a result that comes right out of the deformation statement.

The remark suggests the following

<u>Problem</u>: Let $A=(A_1,A_2)\in F$. We know that $A^{(m)}=(A_1^{(m_1)},A_2^{(m_2)})$ is also Fredholm (spectral mapping theorem). Is it true that $\mathrm{index}(A^{(m)})=m_1m_2\,\mathrm{index}(A)$?

It is clear that an affirmative answer to the deformation problem would show this to be true, since it is true for the class $\left\{(W_1^{(m_1)},W_2^{(m_2)})\right\}_{m_1,m_2\in\mathbb{Z}}$.

4. <u>Theorem 4.1</u>. Let $A=(A_1,\ldots,A_n)\in DF\,(n\geq 2)$, where A_1 is an essentially normal operator with closed range. Then $\mathrm{index}(A)=0$ and indeed $A\overset{DF}{\rightsquigarrow}I$, keeping the first coordinate essentially normal with closed range.

<u>Proof</u>: Consider $H=\ker A_1\oplus \mathrm{ran}\,A_1^*$. Then

$$A_1=\begin{pmatrix}0 & B_1\\ 0 & C_1\end{pmatrix},\ A_2=\begin{pmatrix}D_2 & B_2\\ E_2 & C_2\end{pmatrix},\dots,\ A_n=\begin{pmatrix}D_n & B_n\\ E_n & C_n\end{pmatrix}.$$

Since $A\in DF$, a direct calculation using the Open Mapping Theorem for A_1 shows that $B_2,\dots,B_n,E_2,\dots,E_n\in K(H)$. Observe that $A_1^*A_1$ has only one nonzero entry, the lower right hand corner. By II.Corollary 3.4, one can easily show that $(0,D_2,\dots,D_n)$ is a Fredholm tuple, i.e., $(D_2,\dots,D_n)$ is a Fredholm (n-1)-tuple. (We should notice at this stage that, in case ker A_1 or ran A_1^* is finite dimensional, the theorem follows at once, because A_1 is then Fredholm or finite rank (forcing $(A_2,\dots,A_n)\in F$).)

We now claim that $B_1\in K(H)$, $[C_1^*,C_1]\in K$ and C_1 is Fredholm. From $A_1^*A_1-A_1A_1^*\in K(H)$ we get $B_1B_1^*\in K(\text{ran } A_1^*, \ker A_1)$ and

$$B_1^*B_1+C_1^*C_1-C_1C_1^*\in K(\text{ran } A_1^*).$$

Therefore, B_1 is compact and $[C_1^*,C_1]\in K(\text{ran } A_1^*)$.

Finally, since $\ker A_1=\ker A_1^*A_1$ and ran A_1 is closed (so that ran $A_1^*A_1$ is also closed), we see that $B_1^*B_1+C_1^*C_1$ is invertible. Then $C_1^*C_1$ is Fredholm and, being C_1 essentially normal, C_1 is Fredholm.

We now connect A to $\left(\begin{pmatrix}0 & 0\\ 0 & C_1\end{pmatrix},\begin{pmatrix}D_2 & 0\\ 0 & C_2\end{pmatrix},\dots,\begin{pmatrix}D_n & 0\\ 0 & C_n\end{pmatrix}\right)$

(by the line segment) and then to

$$\left(\begin{pmatrix}0 & 0\\ 0 & C_1\end{pmatrix},\begin{pmatrix}D_2 & 0\\ 0 & 0\end{pmatrix},\dots,\begin{pmatrix}D_n & 0\\ 0 & 0\end{pmatrix}\right)$$

(by the line segment again, since C_1 is Fredholm). We now use the proof of Proposition 1.1.

Remark: We shall see in the next section that, if $A=(A_1,\dots,A_n)\in F$, $[A_1,A_k^*]\in K(H)$ $(k\geq 2)$ and ran A_1 is closed, then $A \overset{F}{\sim} (V,A_2,\dots,A_n)$ where V is the partial isometry in the polar decomposition $A_1=VP$. One might expect that a slight perturbation of a tuple $A\in F$ would provide one with first coordinate (or any other coordinate, of course) having closed range. It is clear that a compact perturbation will not do it. Theorem 4.1 tells us that, unless index(A)=0 or we can afford to lose important algebraic properties (like A_1 being essentially normal), we shall not succeed.

5. Theorem 5.1. Let $A=(A_1,\dots,A_n)\in F$, $[A_1,A_k^*]\in K(H)$ for $k\geq 2$ and assume that ran A_1 is closed. Let $A_1=VP$ be the polar decomposition for A_1. Then $[V,A_k^*]\in K(H)$, $(V,A_2,\dots,A_n)\in F$ and $A \overset{F}{\sim} (V,A_2,\dots,A_n)$, while the first coordinate continues to almost doubly commute with $A_k (k\geq 2)$.

We shall need the following

Lemma 5.2. Let $S,T\in L(H)$, $[S,T]\in K$, $[S,T^*]\in K$ and $T=VP$ be the polar decomposition for T. Assume that ran T is closed. Then $[V,S]$, $[V,S^*]\in K$.

Proof: We know that ker T=ker V=ker P.

Consider $H=\ker T\oplus \operatorname{ran} T^*$. Then:

$$T=\begin{pmatrix} 0 & T_1 \\ 0 & T_2 \end{pmatrix}, \quad V=\begin{pmatrix} 0 & V_1 \\ 0 & V_2 \end{pmatrix}, \quad P=\begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}$$

and

$$S=\begin{pmatrix} S_1 & K_1 \\ K_2 & S_2 \end{pmatrix}.$$

Since ran T is closed, an application of the Open Mapping Theorem shows that $K_1, K_2 \in K$. Moreover, P_2 is invertible, $T_1=V_1P_2$, $T_2=V_2P_2$, $T_1S_2-S_1T_2 \in K$ and $T_2S_2-S_2T_2 \in K$, or $V_1P_2S_2-S_1V_2P_2$, $V_2P_2S_2-S_2V_2P_2 \in K$. But P belongs to the C^*-algebra generated by T, and T almost doubly commutes with S, so that $[P,S] \in K$, or $[P_2,S_2] \in K$. Thus

$$(V_1S_2-S_1V_2)P_2 \in K \text{ and } (V_2S_2-S_2V_2)P_2 \in K.$$

Since P_2 is invertible, we conclude that $V_1S_2-S_1V_2 \in K$ and $[V_2,S_2] \in K$, which implies that $[V,S] \in K$.
Similarly, $[V,S^*] \in K$ (this time using the fact that $[P,S^*] \in K$).

<u>Proof of the Theorem</u>: Once we know that $(V,A_2,\ldots,A_n)$ is an almost commuting tuple with $[V,A_k^*] \in K(H)$ $(k\geq 2)$, we proceed to show that it is Fredholm.

Since F is an open subset of the set of almost commuting tuples, there exists $\varepsilon>0$ such that $(A_1+\lambda V,A_2,\ldots,A_n) \in F$ whenever $|\lambda|<\varepsilon$.

Now $A_1+\frac{\varepsilon}{2}V=VP+\frac{\varepsilon}{2}V=V(P+\frac{\varepsilon}{2})$. By II.Corollary 4.7, $(V,A_2,\ldots,A_n)$ is Fredholm.

It is now clear that $\gamma(t)=(V((1-t)P+t),A_2,\ldots,A_n)$ joins

A to $(V,A_2,\dots,A_n)$ satisfying all the requirements.

<u>Note</u>: The preceding theorem is not obvious, since in general the partial isometry lies in the von Neumann algebra generated by T. Some condition on T is needed, and one can find examples where ran T is not closed and $[V,S]\notin K$.

6. We now turn to study those $A=(V,A_2)\in F$ such that $[V,A_2^*]\in K(H)$, where V is a partial isometry with finite dimensional kernel or cokernel. By Section 1, it suffices to consider the case when exactly one of dim ker V, dim ker V^* is finite.

<u>Lemma 6.1</u>. With A as above and dim ker V finite, $A \overset{DF}{\sim} (S,T)$, where S is a unilateral shift of infinite multiplicity.

<u>Proof</u>: Taking a finite rank perturbation, if necessary, we can assume that V is an isometry. By the Wold decomposition $V=U\oplus S$, where U is unitary and S is a shift of multiplicity equal to dim ker V^*. Now S can be written as a direct sum of shifts of multiplicity one. By Corollary 2.3 in [1], the first summand "absorbs" U up to unitary equivalence modulo the compacts, so that $U\oplus S$ is unitarily equivalent to a compact perturbation of S. II.Corollary 4.7 and the connectedness of the unitary

group complete the proof.

We shall need the following algebraic lemma in dealing with the (S,T) situation.

Lemma 6.2. Let B be a C^*-algebra, $s\in B$ be an isometry, $a_2\in B$: $sa_2=a_2s$ and $sa_2^*=a_2^*s$. Then (s,a_2) is invertible if and only if $\ker s^*\cap\ker a_2=0$ and $\operatorname{ran} s+\operatorname{ran} a_2=B$ (ker and ran understood to be kernel and range of the left multiplications induced by s and a_2).

Proof: The "only if" part is trivial.
For the "if" part we need to prove exactness of the Koszul complex for (s,a_2) at stages 2 and 1. But since s is an isometry, $\ker s=0$ and stage 1 is done.
Assume: $sa+a_2b=0$. Then $a=-s^*a_2b=-a_2s^*b$. Let $c=s^*b$.
Then $s^*(b-sc)=s^*b-s^*sc=c-c=0$ and $a_2(b-sc)=a_2b-a_2sc=$
$-(sa+sa_2c)=-(sa+s(a_2s^*b))=-(sa-sa)=0$.
Thus $b-sc\in\ker s^*\cap\ker a_2=0$, or $b=sc$, as desired.

Let M be a Hilbert space and $N=M\oplus M\oplus\ldots$ For $T\in L(M)$ we define $\hat{T}=T\oplus T\oplus\ldots$.

Lemma 6.3. Assume that $(S,T)\in DF$, where S is a unilateral shift of infinite multiplicity acting on $N=M\oplus M\ \ldots$
Let T_{00} be the $(0,0)$ entry of T. Then $(S,T)\overset{DF}{\sim}(S,\widehat{T_{00}})$.

Proof: Since SS^*+T^*T and SS^*+TT^* are both Fredholm, and

$M\oplus 0\oplus 0\oplus\ldots=\ker S^*$, we conclude that T_{00} is Fredholm. Also, $(T_{01}\ T_{02}\ T_{03}\ \ldots)$ is compact. Analogously, $(T_{10}^*\ T_{20}^*\ T_{30}^*\ \ldots)$ is compact. Consequently

(1) $\quad \operatorname{ran} s + \operatorname{ran} t_\lambda = A(N)$,

where, as usual, small letters are used for the projections in the Calkin algebra and $t_\lambda=(1-\lambda)t+\lambda t_{00}$. Suppose that $s^*a=0$ and $t_\lambda a=0$. Then

$$\begin{pmatrix} A_{10} & A_{11} & A_{12} & \\ A_{20} & A_{21} & A_{22} & \\ A_{30} & A_{31} & A_{32} & \\ & & & \ddots \end{pmatrix}$$

is compact, so that A can be chosen as

$$\begin{pmatrix} A_{00} & A_{01} & A_{02} & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & \ddots \end{pmatrix} .$$

Since $t_\lambda a=0$,

$$\begin{pmatrix} T_{00}A_{00} & T_{00}A_{01} & T_{00}A_{02} & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & \ddots \end{pmatrix}$$

is compact. But then

$$\begin{pmatrix} T_{00} & T_{01} & T_{02} & \\ T_{10} & T_{11} & T_{12} & \\ T_{20} & T_{21} & T_{22} & \\ & & & \ddots \end{pmatrix}\begin{pmatrix} A_{00} & A_{01} & A_{02} & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & \ddots \end{pmatrix}$$

is compact, or $ta=0$. Since (s,t) is invertible and $s^*a=ta=0$, we have $a=0$. We have thus proved:

(2) $\qquad \ker s^* \cap \ker t=0$

Combining (1), (2) and Lemma 6.2, we obtain that (s,t_λ) is invertible for every λ. Taking $\lambda\in[0,1]$, we have a path from (s,t) to $(s,\widehat{t_{00}})$.

<u>Lemma 6.4</u>. Let S be a unilateral shift on $N=M\oplus M\oplus\ldots$ and $C\in L(M)$. Then $(S,\hat{C})\in DF$ and $\operatorname{ran}\hat{C}$ is closed iff C is Fredholm. In that case, $\operatorname{index}(S,\hat{C})=\operatorname{index}(C)$.

<u>Proof</u>: "if". Clearly $[S,\hat{C}]=[S^*,\hat{C}]=0$. If $s^*a=\hat{c}a=0$, the argument in the preceding lemma again shows that $a=0$. Similarly, $\operatorname{ran} s + \operatorname{ran}\hat{c} = A(N)$. By Lemma 6.2, $(s,\hat{c})$ is invertible. Finally, $\operatorname{ran} C$ closed implies $\operatorname{ran}\hat{C}$ closed.

"Only if": $\operatorname{ran}\hat{C}$ closed $\Rightarrow \operatorname{ran} C$ closed. Furthermore $\ker\hat{C}=\ker C\oplus\ker C\oplus\ldots$ and $\ker\hat{C}^*\cap\ker S^*$, $\ker\hat{C}\cap\ker S^*$ are both finite dimensional. Thus: $\ker C\oplus 0\oplus 0\oplus\ldots$ and $\ker C^*\oplus 0\oplus 0\oplus\ldots$ are finite dimensional, which completes the proof that C is Fredholm.

Now, by III.Corollary 1.5, we know that

$$\begin{aligned}\operatorname{index}(S,\hat{C})&=\dim(\ker S^*\cap\ker C)-\dim(\ker S^*\cap\ker C^*)\\&=\dim\ker C - \dim\ker C^* = \operatorname{index}(C).\end{aligned}$$

We are now in a position to prove

<u>Theorem 6.5</u>. Let $A=(V,A_2)$, $B=(W,B_2)\in DF$, V,W be semi-Fredholm partial isometries. Assume that H is separable and index(A)=index(B). Then $A \overset{DF}{\sim} B$.

<u>Proof</u>: If dim ker V is finite then, by Lemma 6.1, $A \overset{DF}{\sim} (S,T)$. If dim ker V^* is finite then, taking the adjoint of the first coordinate of the path from (V^*,A_2) to (S,T), we get $A \overset{DF}{\sim} (S^*,T)$. Similarly, $B \overset{DF}{\sim} (S_1,T_1)$ or $B \overset{DF}{\sim} (S_1^*,T_1)$.
Since H is separable, any two unilateral shifts of infinite multiplicity are unitarily equivalent. By II.Corollary 4.7 and the connectedness of the unitary group, we can assume that $S=S_1$.
Without loss of generality our situation is:
$H=H^2(S^1\times S^1)$, $S=W_1$, $A=(W_1,\hat{T})$ or $(W_1^*,\hat{T})$ and $B=(W_1,\hat{R})$ or $(W_1^*,\hat{R})$.
Four possibilities arise:

(i) $A=(W_1,\hat{T})$, $B=(W_1,\hat{R})$
(ii) $A=(W_1^*,\hat{T})$, $B=(W_1,\hat{R})$
(iii) $A=(W_1,\hat{T})$, $B=(W_1^*,\hat{R})$
(iv) $A=(W_1^*,\hat{T})$, $B=(W_1^*,\hat{R})$.

Case (i): index(T)=index(A)=index(B)=index(R) by Lemma 6.4. Consequently, there is a path of Fredholm operators joining T and R. Using the "if" part of Lemma 6.4, $A \overset{DF}{\sim} B$.

Case (ii): Let $m=\text{index}(A)=-\text{index}(T)$. Then $T \overset{F}{\sim} U_+^{(m)}$ (U_+ is the unilateral shift of multiplicity one).

Thus, $A \overset{DF}{\sim} (W_1^*, U_+^{(m)})$, since $(W_1, \hat{T}^*) \overset{DF}{\sim} (W_1, \hat{U}_+^{(-m)})$.

Similarly, $\text{index}(R)=m$ implies $R \overset{F}{\sim} U_+^{(-m)}$, so that $B \overset{DF}{\sim} (W_1, \hat{U}_+^{(-m)})$.

It is easy to see that $\hat{U}_+=W_2$, so that we actually have

$$A \overset{DF}{\sim} (W_1^*, W_2^{(m)}) \text{ and } B \overset{DF}{\sim} (W_1, W_2^{(-m)}) .$$

By the remarks preceding Theorem 3.1, $(W_1^*, W_2^{(m)}) \overset{DF}{\sim} (W_1, W_2^{(-m)})$, completing the proof.

Case (iii) is completely analogous to (ii).

Case (iv): Consider $(W_1, \hat{T})$, $(W_1, \hat{R})$, use (i) to find a path in DF and then take adjoints in the first coordinate.

7. <u>Theorem 7.1</u>. Let $A=(A_1,\ldots,A_n)\in F$. Assume $n\geq 2$ and that 0 is an accesible point from the essential resolvent set for A_1 (i.e., there exists a path $\gamma:[0,1]\longrightarrow C$ with $\gamma(0)=0$, $\gamma(t)\notin\sigma_e(A_1)$ for $0<t\leq 1$). Then $\text{index}(A)=0$ and $A \overset{F}{\sim} I$.

<u>Proof</u>: Let $\Gamma:[0,1]\longrightarrow F$ given by $\Gamma(t)=(A_1+\gamma(t),A_2,\ldots,A_n)$. observe that for $t>0$, $A_1+\gamma(t)\in F$, so that $\Gamma([0,1])\subset F$. Thus $A \overset{F}{\sim} (A_1+\gamma(1),A_2,\ldots,A_n)$. By Proposition 1.1, $(A_1+\gamma(1),A_2,\ldots,A_n) \overset{F}{\sim} I$. Therefore, $A \overset{F}{\sim} I$ and $\text{index}(A)=0$.

<u>Remark</u>: Continuity of index allows us to conclude that, if $(A_1)_m$ is a sequence of Fredholm operators, $(A_1)_m \longrightarrow A_1$

as $m \to \infty$, $\left[(A_1)_m, A_k\right] \in K(H)$ $(k \geq 2)$ and $(A_1, \ldots, A_n) \in F$ $(n \geq 2)$, then $\mathrm{index}(A_1, \ldots, A_n) = 0$.

CHAPTER V: CONNECTEDNESS OF INVERTIBLE TUPLES

Throughtout this chapter, we shall only deal with commuting tuples of operators on a Hilbert space H. The central problem is connectedness of invertible tuples. In Section 1 we present the algebraic machinery needed for the subsequent sections. It involves several manipulations with invertibility and the Koszul complex, and many of the results are important by their own. The connectedness of invertible tuples in case dim(H) is finite is proved in Section 2. It is a direct consequence of the upper triangular form for commuting matrices. We also show that for pairs in finite dimension, invertibility is equivalent to exactness in any stage of its Koszul complex.

We then proceed to the normal case, done in Section 3. We finally attack and solve the doubly commuting case in Section 4. This amounts to using a transfinite induction argument that reduces the problem to tractable pieces, for which we had already solved the problem. Both the normal case and the doubly commuting situation are treated with no assumption on dim(H). Although the proof we give in Section 2 cannot be extended to the case of infinite dimension, we strongly believe that the result does hold.

Notation: $I=I(H)$, $D=D(H)$ and $N=N(H)$ denote invertible, doubly commuting invertible and invertible normal tuples, respectively.

(Although we use the same symbol for the identity operator $Ix=x$ (all $x \in H$), the "identity" tuple $(I,0,\ldots,0)$ and the set of commuting invertible tuples, there should not be any confusion; they are, after all, three distinct entities.)

1. We begin by summarizing a series of facts from Chapter II.

Lemma 1.1. $A \in I(H) \Longrightarrow \sum_{i=1}^{n} A_i^* A_i$ and $\sum_{i=1}^{n} A_i A_i^*$ are invertible.

Lemma 1.2. Let A be a doubly commuting tuple. Then $A \in D$ iff $\sum_{i=1}^{n} {}^f A_i$ is invertible for each $f: \{1,\ldots,n\} \to \{0,1\}$, where

$$ {}^f A_i = \begin{cases} A_i^* A_i & \text{if } f(i)=0 \\ A_i A_i^* & \text{if } f(i)=1 \end{cases} . $$

Lemma 1.3. Let A be a commuting tuple. Then $A \in N$ iff A_i is normal and $\sum_{i=1}^{n} A_i^* A_i$ is invertible.

Lemma 1.4. If $A=(A_1,\ldots,A_n) \in N$ and $B=C^*(A_1,\ldots,A_n)$, then the maximal ideal space M of B, seen as a subset of C^n, is $Sp(A,H)$.

Lemma 1.5. Let A be a commuting tuple. Then $A \in I$ iff $\hat{A}$ is invertible.

Lemma 1.6. (i) $I(H)$ is an open subset of the set of commuting tuples on H.

(ii) $D(H)$ is an open subset of the set of doubly commuting tuples on H.

(iii) $N(H)$ is an open subset of the set of normal tuples.

Lemma 1.7. (i) If $A=(A_1,\dots,A_n)$ is a commuting tuple and $A_{i_1},\dots,A_{i_k}$ $(1\leq i_j\leq n,\ j=1,\dots,k)$ form an invertible k-tuple, then $A\in I$.

(ii) If $A=(A_1,\dots,A_n)\in I$ and $A_{i_1}=\dots=A_{i_k}=0$ $(1\leq i_j\leq n,\ j=1,\dots,k,\ k<n)$, then the tuple formed with the remaining coordinates is invertible.

(We notice that $i:\{1,\dots,k\}\longrightarrow\{1,\dots,n\}$ need not be injective; it is easy to see that if an invertible tuple has a coordinate repeated, then the tuple obtained by deleting all but one of those coordinates is still invertible.)

Lemma 1.8. Let $A\in I$, V be invertible, $VA_k=A_kV$ for $k\geq 2$. Then $VA=(VA_1,A_2,\dots,A_n)\in I$ and $AV=(A_1V,A_2,\dots,A_n)\in I$.

Lemma 1.9. Let $A\in I$, V be invertible and $A_V=(VA_1V^{-1},\dots,VA_nV^{-1})$. Then $A_V\in I$.

Lemma 1.10. Let $A=(A_1,\dots,A_n)$ be a doubly commuting tuple. Then $\hat{A}$ is normal iff A_1 is normal.

Lemma 1.11. Let $A=(A_1,\dots,A_n)\in I$, $\varphi:\{1,\dots,n\}\longrightarrow\{1,*\}$ and $\varphi(A_i)=A_i^{\varphi(i)}$. Assume that $\left[\varphi(A_i),\varphi(A_j)\right]=0$ $(i\neq j)$. Then $\varphi(A)=(\varphi(A_1),\dots,\varphi(A_n))\in I$.

We now prove some new auxiliary facts.

Lemma 1.12. (i) If $A=(A_1,\dots,A_n)$ and

$$\left(\begin{pmatrix} A_1 & 0 \\ B_1 & C_1 \end{pmatrix},\dots,\begin{pmatrix} A_n & 0 \\ B_n & C_n \end{pmatrix}\right)$$

are invertible, so is $C=(C_1,\dots,C_n)$.

(ii) If $A=(A_1,\dots,A_n)$ and

$$\left(\begin{pmatrix} A_1 & B_1 \\ 0 & C_1 \end{pmatrix},\dots,\begin{pmatrix} A_n & B_n \\ 0 & C_n \end{pmatrix}\right)$$

are invertible, so is $C=(C_1,\dots,C_n)$.

Proof: (i) Let H_1 be the space the A_i's act on, H_2 be the one the C_i's act on and $H=H_1\oplus H_2$. By identifying $H^{\binom{n}{k}}$ with $H_1^{\binom{n}{k}}\oplus H_2^{\binom{n}{k}}$, we obtain:

$$D\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \begin{pmatrix} D(A) & 0 \\ D(B) & D(C) \end{pmatrix},$$

where $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ stands for the n-tuple of 2×2-matrices and, as usual, D is the Koszul complex.

Assume that $D_k(C)x=0$. Then

$$\begin{pmatrix} D_k(A) & 0 \\ D_k(B) & D_k(C) \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix}=0.$$

Since $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ is invertible, there exist u and v such that

$$D_{k+1}(A)u=0 \ , \ D_{k+1}(B)u+D_{k+1}(C)v=x.$$

Since A is invertible, $u=D_{k+2}(A)y$ for some y.

Therefore:

$$D_{k+1}(B)D_{k+2}(A)y+D_{k+1}(C)v=x \ .$$

It is elementary to show that $D_{k+1}(B)D_{k+2}(A)$ is equal to $-D_{k+1}(C)D_{k+2}(B)$. Consequently:

$$D_{k+1}(C)(-D_{k+2}(B)y+v)=x \ ,$$

as desired.

(ii) Being A invertible, we can apply II.Corollary 3.10 to conclude that $A^*=(A_1^*,\dots,A_n^*)$ is invertible. Similarly, $\left(\begin{smallmatrix} A^* & 0 \\ B^* & C^* \end{smallmatrix}\right)$ is invertible. Now, (i) says that C^* is invertible, which in turn implies that C is invertible.

<u>Lemma 1.13</u>. (i) If $A=(A_1,\dots,A_n)$ and $C=(C_1,\dots,C_n)$ are invertible and $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ is a commuting tuple, then $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ is invertible.

(ii) If A and C are invertible and $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$ is a commuting tuple, then $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$ is invertible.

<u>Proof</u>: (i) Assume that

$$D_k\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)=\begin{pmatrix} D_k(A) & D_k(B) \\ 0 & D_k(C) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0 \ .$$

Then

$$D_k(A)x+D_k(B)y=0 \text{ and } D_k(C)y=0 \ .$$

Since C is invertible, there exists z: $y=D_{k+1}(C)z$.

Thus

$$D_k(A)x+D_k(B)D_{k+1}(C)z=0 .$$

But $D_k(B)D_{k+1}(C)=-D_k(A)D_{k+1}(B)$. Therefore

$$D_k(A)(x-D_{k+1}(B)z)=0 .$$

Being A invertible, we can find u:

$$x-D_{k+1}(B)z=D_{k+1}(A)u .$$

Thus

$$\begin{pmatrix} D_{k+1}(A) & D_{k+1}(B) \\ 0 & D_{k+1}(C) \end{pmatrix}\begin{pmatrix} u \\ z \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix},$$ showing that

$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ is invertible.

(ii) follows, as in Lemma 1.12, by taking adjoints.

<u>Notation</u>: As in Chapter IV, we shall write $A \overset{X}{\sim} B$ to indicate that the tuples A and B can be arcwise connected in $X \subset L(H \otimes \mathbb{C}^n)$.

<u>Lemma 1.14</u>. Let $A=(A_1,\dots,A_n)\in I$ and assume that A_1 commutes with A_k^* for $k\geq 2$. Let $A_1=VP$ be the polar decomposition for A_1. Then V commutes with A_k^* $(k\geq 2)$ and $A \overset{I}{\sim} (V,A_2,\dots,A_n)$.

<u>Proof</u>: That $VA_k^*=A_k^*V$ $(k\geq 2)$ follows from the fact that V belongs to the von Neumann algebra generated by A_1. Now, since I is open (as a subset of the set of commuting tuples), there is an $\varepsilon>0$ such that $(A_1+\lambda V,A_2,\dots,A_n)$ is in I $(|\lambda|\leq\varepsilon)$. But $A_1+\varepsilon V=V(P+\varepsilon)$ and $(P+\varepsilon)A_k=A_k(P+\varepsilon)$ $(k\geq 2)$. Observe that $P+\varepsilon$ is invertible. By II.Proposition 4.5,

$(V,A_2,\ldots,A_n)\in I$. We now define $\gamma:[0,1]\longrightarrow I$ by $\gamma(t)=(V(t+(1-t)P),A_2,\ldots,A_n)$.

Corollary: Let $A=(A_1,\ldots,A_n)\in D(H)$ and $A_i=V_iP_i$ be the polar decomposition for A_i $(i=1,\ldots,n)$. Then $A\overset{D}{\sim}(V_1,\ldots,V_n)$.

Lemma 1.15. Let $A=(A_1,\ldots,A_n)\in I(H)$ with A_1 invertible. Then $A\overset{I}{\sim}(I,0,\ldots,0)$.

Proof: Since A_1 is invertible, $A\overset{I}{\sim}(A_1,0,\ldots,0)$. Now, the set of invertible elements of $L(H)$ is arcwise connected (cf. [3], 5.30), so that $(A_1,0,\ldots,0)\overset{I}{\sim}I$.

2. Throughout this section, $\dim(H)$ will be finite. The following standard fact is crucial to obtain the upper triangular form for commuting n-tuples.

Lemma 2.1. Let $A=(A_1,\ldots,A_n)$ be a commuting tuple of operators on H $(\neq(0))$. Then there exists $\lambda=(\lambda_1,\ldots,\lambda_n)\in C^n$ such that $\bigcap_{i=1}^{n}\ker(A_i-\lambda_i)\neq(0)$.

Proof: By induction: $n=1$ is obvious. Assume that there exists $'\lambda=(\lambda_1,\ldots,\lambda_{n-1})$ such that $\bigcap_{i=1}^{n-1}\ker(A_i-\lambda_i)\neq(0)$.
Consider A_n on $L=\bigcap_{i=1}^{n-1}\ker(A_i-\lambda_i)$.
Since $A_nA_k=A_kA_n$ for $k=1,\ldots,n-1$, it follows that $A_nL\subset L$.
Let B_n be the restriction of A_n to L. Then (by the case $n=1$) there exists $\lambda_n\in C$ such that $\ker(B_n-\lambda_n)\neq(0)$.
Set $\lambda=('\lambda,\lambda_n)\in C^n$. Then
$\bigcap_{i=1}^{n}\ker(A_i-\lambda_i)=L\cap\ker(A_n-\lambda_n)=\ker(B_n-\lambda_n)\neq(0)$, as desired.

The preceding lemma says that a commuting tuple of operators on a finite dimensional Hilbert space always possess a common eigenvector. We can now generalize the triangular form for matrices.

Proposition 2.2. (Upper triangular form) Let $A=(A_1,\dots,A_n)$ be a commuting tuple of operators on H and let $N=\dim(H)$. Then there exist $(N+1)$ subspaces $M_0, M_1,\dots, M_N$ satisfying:

(i) each M_j $(j=0,\dots,N)$ is invariant under A, i.e., $A_i M_j \subset M_j$ for all $i=1,\dots,n$

(ii) $\dim(M_j)=j$

(iii) $0=M_0 \subset M_1 \subset \dots \subset M_{N-1} \subset M_N=H$.

Proof: For $N=0,1$, the result is trivial. Assume it is true for $\dim(H)=N-1$. Consider $A^*=(A_1^*,\dots,A_n^*)$. By the preceding lemma, there exists $x\in H$, $x\neq 0$, and $\lambda_i \in C$ $(i=1,\dots,n)$ such that $A_i^* x=\bar{\lambda}_i x$ $(i=1,\dots,n)$.

Let L be the orthogonal complement of Cx. Then $\dim(L)$ is $N-1$. We claim that $A_i L \subset L$. For, if $y\in L$, then y belongs to $\mathrm{ran}(A_1-\lambda_1)+\dots+\mathrm{ran}(A_n-\lambda_n)$, so that $y=y_1+\dots+y_n$, where $y_i \in \mathrm{ran}(A_i-\lambda_i)$ $(i=1,\dots,n)$.

Then $A_j y=\sum_{i=1}^{n} A_j y_i$. Since $A_j A_i=A_i A_j$ (all i,j), A_j leaves $\mathrm{ran}(A_i-\lambda_i)$ invariant. Therefore, $A_j y$ is in L (all j).

By induction hypothesis, there exist $M_0,\dots,M_{N-1}$ such that (i), (ii) and (iii) hold with N replaced by N-1.

Write $M_N=H$ and the result follows.

We now prove

Theorem 2.3. $I(H)$ is arcwise connected.

Proof: Let $A=(A_1,\ldots,A_n)\in I$. By Proposition 2.2, we can write the A_i's in simultaneous upper triangular form:

$$A_1=\begin{pmatrix} \lambda_1^{(1)} & a_{12}^{(1)} & \cdots & a_{1N}^{(1)} \\ 0 & \lambda_2^{(1)} & \cdots & a_{2N}^{(1)} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_N^{(1)} \end{pmatrix},\ldots,\ A_n=\begin{pmatrix} \lambda_1^{(n)} & a_{12}^{(n)} & \cdots & a_{1N}^{(n)} \\ 0 & \lambda_2^{(n)} & \cdots & a_{2N}^{(n)} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_N^{(n)} \end{pmatrix},$$

where $N=\dim(H)$ and $\lambda_j^{(i)},\ a_{kl}^{(i)}\in C$ (all i,j,k,l).

Since $A\in I$, at least one of $\lambda_1^{(i)}$ $(i=1,\ldots,n)$ is nonzero (otherwise $\bigcap_{i=1}^{n}\ker A_i=(0)$). Consequently,

$$\lambda_1=(\lambda_1^{(1)},\ldots,\lambda_1^{(n)})\in C^n \text{ is an invertible tuple.}$$

By Lemma 1.12(ii), so is

$$\left(\begin{pmatrix} \lambda_2^{(1)} & a_{23}^{(1)} & \cdots & a_{2N}^{(1)} \\ 0 & \lambda_3^{(1)} & \cdots & a_{3N}^{(1)} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_N^{(1)} \end{pmatrix},\ldots,\begin{pmatrix} \lambda_2^{(n)} & a_{23}^{(n)} & \cdots & a_{2N}^{(n)} \\ 0 & \lambda_3^{(n)} & \cdots & a_{3N}^{(n)} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_N^{(n)} \end{pmatrix}\right).$$

Therefore, at least one of the $\lambda_2^{(i)}$ $(i=1,\ldots,n)$ is nonzero, which implies that $\lambda_2=(\lambda_2^{(1)},\ldots,\lambda_2^{(n)})\in C^n$ is invertible.

Continuing this process we conclude the following:

for each $j=1,\ldots,N$ there exists k_j such that $\lambda_j^{(k_j)}\neq 0$.

Then, each of $\lambda_1=(\lambda_1^{(1)},\ldots,\lambda_1^{(n)}),\ldots,\lambda_N=(\lambda_N^{(1)},\ldots,\lambda_N^{(n)})$

is an invertible tuple and, by Lemma 1.13, so is

$$\lambda=\lambda_1\oplus\ldots\oplus\lambda_N=$$

$$=\left(\begin{pmatrix}\lambda_1^{(1)} & & & \\ & \lambda_2^{(1)} & & \\ & & \ddots & \\ & & & \lambda_N^{(1)}\end{pmatrix},\ldots,\begin{pmatrix}\lambda_1^{(n)} & & & \\ & \lambda_2^{(n)} & & \\ & & \ddots & \\ & & & \lambda_N^{(n)}\end{pmatrix}\right)=$$

$$=(\lambda^{(1)},\ldots,\lambda^{(n)}).$$

Let

$$B_i=\begin{pmatrix}0 & a_{12}^{(i)} & a_{13}^{(i)} & \ldots & a_{1N}^{(i)}\\ 0 & 0 & 0 & \ldots & 0\\ \cdot & \cdot & \cdot & \ldots & \cdot\\ 0 & 0 & 0 & \ldots & 0\end{pmatrix},$$

$C_i=A_i-\lambda^{(i)}-B_i$ and $B=(B_1,\ldots,B_n)$, $C=(C_1,\ldots,C_n)$.

Define $\gamma:[0,1]\longrightarrow L(H)\otimes C^n$ by $\gamma(t)=\lambda+(1-t)B+C=$

$$=(\lambda^{(1)}+(1-t)B_1+C_1,\ldots,\lambda^{(n)}+(1-t)B_n+C_n).$$

γ is certainly continuous and $\gamma(0)=A$. Since $\gamma(t)$ is upper triangular, Lemma 1.13 will say that $\gamma(t)\in I(H)$ in case we can show that $\gamma(t)$ is a commuting tuple. But we know that A is commuting, that is:

$$\left[\lambda^{(i)}+B_i+C_i,\lambda^{(j)}+B_j+C_j\right]=0\ (i,j=1,\ldots,n),\text{ or}$$

$$\left[\lambda^{(i)},\lambda^{(j)}\right]+\left[\lambda^{(i)},B_j\right]+\left[\lambda^{(i)},C_j\right]+\left[B_i,\lambda^{(j)}\right]+\left[B_i,B_j\right]$$
$$+\left[B_i,C_j\right]+\left[C_i,\lambda^{(j)}\right]+\left[C_i,B_j\right]+\left[C_i,C_j\right]=0.$$

Observe that $\left[\lambda^{(i)},\lambda^{(j)}\right]=0$ and $B_iB_j=B_jB_i=0$.

In the preceding matrix identity, we multiply the first row by (1-t). Then:

$$\left[\lambda^{(i)},(1-t)B_j\right]+\left[\lambda^{(i)},C_j\right]+\left[(1-t)B_i,\lambda^{(j)}\right]+\left[(1-t)B_i,C_j\right]$$
$$+\left[C_i,\lambda^{(j)}\right]+\left[C_i,(1-t)B_j\right]+\left[C_i,C_j\right]=0 .$$

But the left hand side is precisely

$$\left[\lambda^{(i)}+(1-t)B_i+C_i,\lambda^{(j)}+(1-t)B_j+C_j\right] ,$$

so that $\gamma(t)$ is a commuting tuple.

Therefore: $A \overset{I}{\sim} \gamma(1)$. Let us look at $\gamma(1)$.

$$\gamma(1)_i=\begin{pmatrix} \lambda_1^{(i)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{(i)} & a_{23}^{(i)} & \cdots & a_{2N}^{(i)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_N^{(i)} \end{pmatrix} .$$

It is by now clear that a similar argument will lead us to

$$\begin{pmatrix} \lambda_1^{(i)} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^{(i)} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^{(i)} & \cdots & a_{3N}^{(i)} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_N^{(i)} \end{pmatrix}$$

and, eventually, to

$$\begin{pmatrix} \lambda_1^{(i)} & & & & \\ & \lambda_2^{(i)} & & & \\ & & \lambda_3^{(i)} & & \\ & & & \ddots & \\ & & & & \lambda_N^{(i)} \end{pmatrix} .$$

That is, we can join A to λ, in I(H).
Since each of the λ_i's is invertible with an invertible coordinate, $\lambda_i \overset{I}{\sim} (I,0,\ldots,0)$, by Lemma 1.15. Therefore, $=\lambda_1\oplus\ldots\oplus\lambda_N \overset{I}{\sim} I$, by Lemma 1.13.

Remark: The preceding proof relies heavily on the upper triangular form obtained in Proposition 2.2, which requires finite dimensionality of H. Although a different proof, based on the decomposition $H=\ker A_1 \oplus (\ker A_1)^{\perp}$ can be given, finite dimensionality is again required to prove that $(0,B_2,\ldots,B_n)\in L(\ker A_1)\otimes C^n$ is invertible, where $B_i=A_i /_{\ker A_1}$ $(i=2,\ldots,n)$. One then uses induction along with Lemmas 1.12 and 1.13.
We conjecture, however, the following:

I(H) is always arcwise connected.

We conclude this section with an interesting fact about commuting pairs of operators on H.

Theorem 2.4. Let $A=(A_1,A_2)$ be a commuting pair. Then the following conditions are equivalent:

(i) $A\in I$

(ii) $\ker A_1 \cap \ker A_2=0$

(iii) $\ker D_1=\operatorname{ran} D_2$, where D is the Koszul complex for A

(iv) $\ker A_1^* \cap \ker A_2^*=0$.

We shall need the following lemma, whose proof can be found in Halmos [7, Problem 56].

<u>Lemma</u>: (J. Schur) Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a matrix on a finite dimensional vector space, with $CD=DC$. Then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\det(AD-BC) .$$

In particular, $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is invertible iff $AD-BC$ is invertible.

<u>Proof of the Theorem</u>: (iv)$\Longrightarrow$(i) We assume that D_1 is onto, so that $A_1A_1^*+A_2A_2^*$ is invertible. By lemma, so is

$$\hat{A}=\begin{pmatrix} A_1 & A_2 \\ -A_2^* & A_1^* \end{pmatrix},$$

that is, A is invertible.

(ii)$\Longrightarrow$(i) $\ker A_1 \cap \ker A_2=0$ implies that $A_1^*A_1+A_2^*A_2$ is invertible. Therefore, so is

$$\hat{A}^*=\begin{pmatrix} A_1^* & A_2^* \\ -A_2 & A_1 \end{pmatrix},$$

that is $A^*=(A_1^*,A_2^*)$ is invertible. By Lemma 1.11 ($\varphi(1)=\varphi(2)=*$), A is invertible.

(iii)$\Longrightarrow$(i) Since $\ker \hat{A}=\ker D_1 \cap (\operatorname{ran} D_2)^{\perp}=0$, we see that $\hat{A}$ is one-to-one. Since $\dim(H)$ is finite, we conclude that $\hat{A}$ is invertible, or $A\in I(H)$.

(i)$\Longrightarrow$(ii), (i)$\Longrightarrow$(iii) and (i)$\Longrightarrow$(iv) follow trivially.

3. Connectedness of normal tuples

In this and next sections, no assumption on $\dim(H)$ is being made.

Theorem 3.1. $N(H)$ is arcwise connected.

Proof: Let $N=(N_1,\dots,N_n)$ be an invertible tuple of normal operators. Consider the decomposition $H=\ker N_1 \oplus (\ker N_1)^{\perp}$. We have:

$$N_1=\begin{pmatrix} 0 & 0 \\ 0 & B_1 \end{pmatrix},\ N_2=\begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix},\dots,\ N_n=\begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix}.$$

(recall that N_1 commutes with N_k^* (all k) by Fuglede's theorem, so that $\ker N_1$ reduces each of the N_k's.)

We now observe that $(0,A_2,\dots,A_n)$ and $(B_1,\dots,B_n)$ belong to $N(\ker N_1)$, $N((\ker N_1)^{\perp})$, respectively.

By Lemma 1.7, $(A_2,\dots,A_n)\in N$, so that $\gamma:[0,1]\longrightarrow N$ defined by $\gamma(t)=(t,(1-t)A_2,\dots,(1-t)A_n)$ connects $(0,A_2,\dots,A_n)$ to $(I,0,\dots,0)$ in $N(\ker A_1)$.

On the other hand, let $B_1=VP$ be the polar decomposition for B_1. Since B_1 is normal and $\ker B_1=0$, we can derive that V is unitary. By Lemma 1.14, and its proof, $(B_1,B_2,\dots,B_n)\overset{N}{\sim}(V,B_2,\dots,B_n)$, and $(V,B_2,\dots,B_n)\overset{N}{\sim} I$, by Lemma 1.15.

Finally, Lemma 1.13 glues the pieces together.

Corollary 3.2. Let $A=(A_1,\dots,A_n)\in I(H)$ with A_1 normal.

Then $A \overset{I}{\rightsquigarrow} I$, maintaining the first coordinate normal.

Proof: Straightforward from the preceding proof.

4. The doubly commuting case

We shall begin with a series of lemmas, which will enable us to solve some special cases. We shall then show that those cases are indeed characteristic.

Lemma 4.1. Let S be the unilateral shift, acting on $R=M\oplus M\oplus\ldots$ and $C=(C_1,\ldots,C_k)\in L(M)\otimes\mathbb{C}^k$. Then $(S,\hat{C})\in D(R)$ iff $C\in D(M)$. (Here $\hat{C}=(\hat{C}_1,\ldots,\hat{C}_k)$ with $\hat{C}_j=C_j\oplus C_j\oplus\ldots(1\le j\le k)$.)

Proof: If C is invertible, so is $\hat{C}$. By Lemma 1.7(i), $(S,\hat{C})\in I(R)$. But $S^*\hat{C}=\hat{C}S^*$, so that $(S,\hat{C})\in D(R)$. Conversely, assume that $f:\{1,\ldots,k\}\to\{0,1\}$ and $y\in M$ are given. Since $(S,\hat{C})\in D(R)$, $SS^*+\sum_{i=1}^{k}{}^{f}\hat{C}_i$ is invertible, by Lemma 1.2. Thus, there exists $x=(x_0,x_1,\ldots)$ such that

$$SS^*x+\sum_{i=1}^{k}{}^{f}\hat{C}_i x=(y,0,\ldots)\ .$$

In particular,

$$(SS^*x)_0+\sum_{i=1}^{k}({}^{f}\hat{C}_i x)_0=y\ ,$$

or

$$\sum_{i=1}^{k}{}^{f}C_i x_0=y\ .$$

That is, $\sum_{i=1}^{k}{}^{f}C_i$ is invertible. By Lemma 1.2 again, $C\in D(M)$.

Corollary 4.2. Let S be as above, $B\in L(R)\otimes\mathbb{C}^k$. Then

$(S,B)\in D(R)$ implies $B\in D(R)$.

<u>Proof</u>: It is easy to check that any operator in $L(R)$ doubly commuting with S is of the form $\hat{C}$ for some $C\in L(M)$.

<u>Lemma 4.3</u>. If $A=(A_1,\ldots,A_n)\in D$ and A_1 is invertible, then $A \overset{D}{\sim} I$.

<u>Proof</u>: This result is already contained in the proof of Lemma 1.15.

<u>Lemma 4.4</u>. If $A=(V,A_2,\ldots,A_n)\in D$ and V is an isometry, then $A \overset{D}{\sim} I$.

<u>Proof</u>: By the Wold decomposition, V is the direct sum of a unitary operator U acting on $\bigcap_{n\geq 0} \operatorname{ran} V^n$ and a unilateral shift S. Let $Q=\bigcap_{n\geq 0} \operatorname{ran} V^n$ and write $H=Q\oplus R$. Since A_k doubly commutes with V $(k=2,\ldots,n)$, Q reduces A_k (all k). We then have:

$$V=\begin{pmatrix} U & 0 \\ 0 & S \end{pmatrix},\quad A_2=\begin{pmatrix} B_2 & 0 \\ 0 & C_2 \end{pmatrix}\quad ,\ldots,\quad A_n=\begin{pmatrix} B_n & 0 \\ 0 & C_n \end{pmatrix}.$$

Consequently, $(U,B_2,\ldots,B_n)$ and $(S,C_2,\ldots,C_n)$ both are doubly commuting invertible tuples. By Lemma 4.3, $(U,B_2,\ldots,B_n) \overset{D(Q)}{\sim} I$. Now, by Corollary 4.2, $C=(C_2,\ldots,C_n)$ is in $D(R)$. Therefore, $(S,C) \overset{D(R)}{\sim} I$. Then $A \overset{D(H)}{\sim} I$.

<u>Theorem 4.5</u>. $D(H)$ is arcwise connected.

Proof: Let $A=(A_1,\dots,A_n)\in D(H)$. We want to show that $A \underset{\smile}{D(H)} (I,0,\dots,0)$. By Lemma 1.14, we can connect A to $(V,A_2,\dots,A_n)$, where V is the partial isometry in the polar decomposition $A_1=VP$.

We now define, by transfinite induction, a chain of subspaces H_α as follows:

$$H_1=\ker V$$

$$H_\alpha=0 \text{ if } \alpha \text{ is a limit ordinal}$$

$H_{\alpha+1}=\ker B_\alpha$, where B_α is the compression of V to $(H^\alpha)^\perp$ and $H^\alpha=\bigvee_{\beta\leq\alpha} H_\beta$.

By construction, the family $\{H_\alpha\}$ is orthogonal. We shall agree to write

$$V=\begin{pmatrix} V_\alpha & C_\alpha \\ 0 & B_\alpha \end{pmatrix},\ A_2=\begin{pmatrix} A_2^{(\alpha)} & 0 \\ 0 & D_2^{(\alpha)} \end{pmatrix},\dots,\ A_n=\begin{pmatrix} A_n^{(\alpha)} & 0 \\ 0 & D_n^{(\alpha)} \end{pmatrix},$$

with respect to the decomposition $H=H^\alpha\oplus(H^\alpha)^\perp$. The fact that H^α is invariant under V, $A_2,\dots,A_n,A_2^*,\dots,A_n^*$ is the content of the following

Claim 1: $VH^\alpha\subset H^\alpha$

$$A_kH^\alpha\subset H^\alpha \quad (k=2,\dots,n)$$

$$A_k^*H^\alpha\subset H^\alpha \quad (k=2,\dots,n)$$

Proof: $\alpha=1$) $H^\alpha=H_1=\ker V$. Since V doubly commutes with all A_k $(k=2,\dots,n)$, the claim follows.

α a limit ordinal) $H^\alpha=\bigvee_{\beta<\alpha} H_\beta=\bigvee_{\beta<\alpha} H^\beta$. By assumption, $H^\beta(\beta<\alpha)$ is invariant under V,A_k,A_k^* $(k=2,\dots,n)$. Therefore, so is H^α, by linearity and continuity.

$\alpha=\beta+1$) Here $H^{\alpha}=H^{\beta}\oplus H_{\alpha}=H^{\beta}\oplus\ker B_{\beta}$. Since H^{β} is invariant under V and reduces A_k $(k=2,\dots,n)$, we have:

$$V=\begin{pmatrix} V_{\beta} & C_{\beta} \\ 0 & B_{\beta} \end{pmatrix}, \quad A_k=\begin{pmatrix} A_k^{(\beta)} & 0 \\ 0 & D_k^{(\beta)} \end{pmatrix},$$

with respect to the decomposition $H=H^{\beta}\oplus(H^{\beta})^{\perp}$.

Let $x\in\ker B_{\beta}$. Then $Vx=\begin{pmatrix} C_{\beta}x \\ 0 \end{pmatrix}\in H^{\beta}$, and $A_kx=\begin{pmatrix} 0 \\ D_k^{(\beta)}x \end{pmatrix}$. Since $B_{\beta}D_k^{(\beta)}=D_k^{(\beta)}B_{\beta}$, $D_k^{(\beta)}x\in\ker B_{\beta}$.

Similarly, $A_k^*x\in\ker B_{\beta}$.

Therefore, $VH^{\alpha}\subset H^{\alpha}$, $A_kH^{\alpha}\subset H^{\alpha}$ and $A_k^*H^{\alpha}\subset H^{\alpha}$.

<u>Claim 2</u>: For all α, $'A^{(\alpha)}=(A_2^{(\alpha)},\dots,A_n^{(\alpha)})\in D(H^{\alpha})$.

<u>Proof</u>: Consider the decomposition $H=\ker V\oplus(H^1)^{\perp}$.

Then

$$V=\begin{pmatrix} 0 & C_1 \\ 0 & B_1 \end{pmatrix}, \ A_2=\begin{pmatrix} A_2^{(1)} & 0 \\ 0 & D_2^{(1)} \end{pmatrix}, \dots, A_n=\begin{pmatrix} A_n^{(1)} & 0 \\ 0 & D_n^{(1)} \end{pmatrix}.$$

By Lemma 1.2,

$$V^*V+\sum_{k=2}^{n} {}^{f}A_k \text{ is invertible, for all possible}$$

$f:\{2,\dots,n\}\longrightarrow\{0,1\}$.

That is,

$$\begin{pmatrix} \sum_{k=2}^{n} {}^{f}A_k^{(1)} & 0 \\ 0 & I+\sum_{k=2}^{n} {}^{f}D_k^{(1)} \end{pmatrix}$$

is invertible (all f).

Therefore, Lemma 1.2 says that $'A^{(1)}\in D(H^1)$, and claim is true for $\alpha=1$.

Next we consider the case α a limit ordinal. Being the family $\{H_\alpha\}$ orthogonal, we can write $H^\alpha = \bigoplus_{\beta<\alpha} H_\beta$. The A_k's admit a matrix representation with respect to such a decomposition. We now assert that all these matrices are block diagonal. For, if $\gamma>\delta$, the (γ,δ)-entry of $A_k^{(\alpha)}$ is precisely the (γ,δ)-entry of the matrix associated with $A_k^{(\gamma)}$, and $A_k H^\delta \subset H^\delta$, or $A_k^{(\gamma)} H^\delta \subset H^\delta$, so that that entry is zero. Similarly for (γ,δ)-entries with $\gamma<\delta$ (this time using A_k^*, $A_k^{(\delta)*}$).

Once we know that all those matrices are block diagonal, it is almost obvious that $'A^{(\alpha)}$ is in $D(H^\alpha)$.

We proceed to the case $\alpha=\beta+1$. We decompose H into $H^\beta \oplus \ker B_\beta \oplus (H^\alpha)^\perp$ and then get

$$V=\begin{pmatrix} V_\beta & C_{\beta,1} & C_{\beta,2} \\ 0 & 0 & C_{\beta,3} \\ 0 & 0 & B_\alpha \end{pmatrix}, \quad A_k=\begin{pmatrix} A_k^{(\beta)} & 0 & 0 \\ 0 & E_k^{(\alpha)} & 0 \\ 0 & 0 & D_k^{(\alpha)} \end{pmatrix}.$$

We are assuming that $(A_2^{(\beta)},\dots,A_n^{(\beta)}) \in D(H^\beta)$. By Lemma 1.7, so is $(V_\beta, A_2^{(\beta)},\dots,A_n^{(\beta)})$. We now appeal to Lemma 1.12(ii) to conclude that

$$\left(\begin{pmatrix} 0 & C_{\beta,3} \\ 0 & B_\alpha \end{pmatrix}, \begin{pmatrix} E_2^{(\alpha)} & 0 \\ 0 & D_2^{(\alpha)} \end{pmatrix}, \dots, \begin{pmatrix} E_n^{(\alpha)} & 0 \\ 0 & D_n^{(\alpha)} \end{pmatrix}\right)$$

is a doubly commuting invertible tuple on $(H^\beta)^\perp$.

By Lemma 1.2, both $\sum_{k=2}^{n} {}^f E_k^{(\alpha)}$ and $C_{\beta,3}^* C_{\beta,3} + B_\alpha^* B_\alpha + \sum_{k=2}^{n} {}^f D_k^{(\alpha)}$ are invertible (all $f:\{2,\dots,n\} \longrightarrow \{0,1\}$).

Consequently, $E^{(\alpha)}=(E_2^{(\alpha)},\dots,E_n^{(\alpha)})\in D(\ker B_\beta)$. By Lemma 1.13, $(V_\alpha,A_2^{(\alpha)},\dots,A_n^{(\alpha)})\in D(H^\alpha)$. This completes the proof of Claim 2.

The following lemma is well-known in the theory of ordinal numbers.

<u>Lemma</u>: Let α be an ordinal number, $[1,\alpha]$ its initial segment and $X=\{\beta\leq\alpha:\ \beta$ is not a limit ordinal$\}$. Then $\mathrm{card}(\alpha)=\mathrm{card}(X)$.

<u>Proof</u>: $G:[1,\alpha]\longrightarrow X$ given by $\begin{cases} G(\beta)=\beta+1 & (\beta<\alpha) \\ G(\alpha)=1 \end{cases}$ is a one-to-one map onto X.

We can now finish the proof of the theorem. Assume first that for some α, $H^\alpha=H$. Then $V=V_\alpha$, $A_k=A_k^{(\alpha)}$ $(k=2,\dots,n)$. Claim 2 then says that $'A=(A_2,\dots,A_n)\in D(H)$. Thus, A can be connected to $(I,0,\dots,0)$ in D(H). (Notice that we are not disregarding the n=1 situation, since $\ker V=(0)$ in that case)

The other possibility is $H^\alpha\neq H$ for all α. We claim that then there exists β not a limit ordinal such that $H_\beta=(0)$. For, if $H_\beta\neq(0)$ for all β not a limit ordinal and if $\aleph_{\beta_0}$ is the cardinality of H, then $\bigoplus_{\beta\leq\aleph_{\beta_0+1}} H_\beta$ would be a subspace of H with cardinality $\geq\aleph_{\beta_0+1}$ (by the Lemma), a contradiction. Let β_0 be the first non-limit ordinal such that $H_{\beta_0}=(0)$. By definition, this means that B_{β_0} is one-to-one. Since $'A^{(\beta_0)}\in D(H^{\beta_0})$ by Claim 2, the tuple $(V_{\beta_0},A_2^{(\beta_0)},\dots,A_n^{(\beta_0)})$

is in $D(H^{\beta_0})$ and can be joined to $(I,0,\ldots,0)$.

Now, an application of Lemma 1.12(ii) shows that $(B_{\beta_0},D_2^{(\beta_0)},\ldots,D_n^{(\beta_0)})$ is a doubly commuting invertible tuple on $(H^{\beta_0})^{\perp}$. Let V' be the partial isometry appearing in the polar decomposition $B_{\beta_0}=V'P'$. Then V' is an isometry. By Lemmas 1.14 and 4.4,

$$(B_{\beta_0},D_2^{(\beta_0)},\ldots,D_n^{(\beta_0)}) \overset{D(H^{\beta_0})^{\perp}}{\frown} (V',D_2^{(\beta_0)},\ldots,D_n^{(\beta_0)})$$

and

$$(V',D_2^{(\beta_0)},\ldots,D_n^{(\beta_0)}) \overset{D(H^{\beta_0})^{\perp}}{\frown} (I,0,\ldots,0) .$$

Finally, we first connect $(V,A_2,\ldots,A_n)$ to

$$\left(\begin{pmatrix} V_{\beta_0} & 0 \\ 0 & B_{\beta_0}\end{pmatrix},\begin{pmatrix} A_2^{(\beta_0)} & 0 \\ 0 & D_2^{(\beta_0)}\end{pmatrix},\ldots,\begin{pmatrix} A_n^{(\beta_0)} & 0 \\ 0 & D_n^{(\beta_0)}\end{pmatrix}\right)$$

by the line segment, and then use the preceding facts along with Lemma 1.13 to obtain that $A \overset{D(H)}{\frown} I$.

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