THE WEAK CLOSURE OF THE UNITARY ORBIT OF OPERATORS

A Dissertation presented

by

Mahmoud Mohammed Kutkut

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STATE UNIVERSITY OF NEW YORK

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Abstract of the Dissertation

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In a beautiful paper [5] in 1973 Halmos studied the weak closure of the set of all shift operators defined on an infinite dimensional complex Hilbert space $\mathcal{H}$. We start this work trying to answer his question about the closure of the unitary orbit of a weighted shift. We give necessary and sufficient conditions for the weak closure of the unitary orbit of a given contractive weighted shift to be equal to the set of all contractions. Moreover we give sufficient conditions as well as necessary ones for two arbitrary weighted shifts to generate the same weak closure of their unitary orbits.

Then we characterize the weak closure of the unitary orbit of an arbitrary contraction, by proving the equivalence of the following, for any contraction T on $\mathcal{H}$.

a) Weak closure of the unitary orbit of T equals the set of all contractions;

b) The closure of the numerical range of T equals the closed unit disc;

c) The spectrum of T contains the boundary of the unit disc;

d) The essential spectrum of T contains the boundary of the unit disc.

We note that if a contraction T satisfies any one of the above statements then for any compact operator K such that norm of T + K is less than or equal to one, then T + K satisfies that statement. Moreover as an application we prove that the representation of the disc algebra as a subalgebra of the algebra of all bounded linear operators determined by a given contraction T which satisfies condition a), must be isometric.

More generally we characterize the weak closure of the unitary orbit of a given operator T having a convex spectral set X, by proving the equivalence of the following statements, provided that X is compact and its boundary is a C^2-class function and contains no straight line segments.

e) The closure of the numerical range of T equals X.

f) The spectrum of T contains the boundary of X.

g) The essential spectrum of T contains the boundary of X.

h) $\|(T-\lambda)^{-1}\|$ = 1/distance from λ to X, for all λ not in X.

In fact we prove that if the weak closure of the unitary orbit of T equals the set of all operators having X as a spectral set then T satisfies the above statements (e,f, g, and h). Conversely we prove that if any one of these statements hold for T then the set of all operators having X as a spectral set is included in the weak closure of the unitary orbit of T.

Moreover using the Riemann mapping Lemma and the functional calculus we show that (c) is equivalent to (f).

Finally, there are some corollaries, examples, and applications considered throughout the dissertation.

To those youths who understand ISLAM as a way of life.

To those poeple who struggle in the sake of ALLAH to re-establish Islamic state and society.

This work is dedicated.

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CHAPTER I

INTRODUCTION

This chapter is devoted to the terminology of this dissertation and Preliminaries; this is in Section 1. In Section 2 the description of the problem and the results of its solution are presented.

I-1. Terminology and Preliminaries.

Throughout this work we consider an infinite dimensional complex Hilbert space $\mathcal{H}$. We let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be a contraction if $\|A\| \le 1$.

For $S \in \mathcal{L}(\mathcal{H})$, S is said to be a shift operator if there is an orthonormal basis (e_i) in $\mathcal{H}$ such that

$$Se_i = e_{i+1} ,$$

for every $i \in \mathbb{N}$. If $i \in \mathbb{N}$ is the set of all positive integers, then S is said to be a unilateral shift and if $i \in \mathbb{Z}$, the set of all integers, then S is said to be a bilateral shift. The multiplicity of a shift is its co-rank.

An operator $T_\alpha \in \mathcal{L}(\mathcal{H})$ is said to be a weighted shift, if there is an orthonormal basis (e_i) in $\mathcal{H}$ and a sequence of complex numbers (α_i) called the sequence of weights, such that

$$T_\alpha e_i = \alpha_i e_{i+1},$$

for every i; we have a unilateral weighted shift or a bilateral weighted shift according to $i \in \mathbb{N}$ or $\mathbb{Z}$.

It is well known that $T_\alpha = DS$, where S is the shift and D is the diagonal operator whose diagonal entries are the (α_i). Equally it is known that

$$\|T_\alpha\| = \sup_i |\alpha_i| .$$

If T_β is another weighted shift with weight sequence (β_i), then T_α is unitarily equivalent to T_β if, and only if $|\alpha_i| = |\beta_i|$ for every i. Because of this fact we consider weight sequences of positive real numbers, without loss of generality. For these facts see for example Shields [16].

For an operator $A \in \mathcal{L}(\mathcal{H})$, $\sigma(A)$ will denote the spectrum of A, which is the set of all complex numbers λ for which $A - \lambda$ is not invertible. If $\mathcal{K}$ is the closed two-sided ideal of compact operators in the algebra $\mathcal{L}(\mathcal{H})$, then $\mathcal{L}(\mathcal{H})/\mathcal{K}$ denotes the Calkin algebra and π, the canonical map defined by

$$\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K} ,$$

$$\pi(A) = A + \mathcal{K}, \text{ for every } A \in \mathcal{L}(\mathcal{H}).$$

The spectrum $\sigma(\pi(A))$ of $\pi(A)$ in the Calkin algebra is called the essential spectrum of A and is denoted by $\sigma_e(A)$. It is known that $\sigma_e(A) \subset \sigma(A)$ for any operator $A \in \mathcal{L}(\mathcal{H})$, and both are compact subsets of the complex plane $\mathbb{C}$.

For the unilateral shift S, $\sigma(S)$ is the closed unit disc denoted by $\overline{\mathbb{D}}$ and for the bilateral shift S, $\sigma(S)$ is the entire unit circle denoted by $\partial\overline{\mathbb{D}}$, the boundary of the unit disc. For any contraction A, $\sigma(A) \subset \overline{\mathbb{D}}$. Let $\mathfrak{U}$ denote the group of all unitaries in $\mathcal{L}(\mathcal{H})$, i.e., $\mathfrak{U} = \{U \in \mathcal{L}(\mathcal{H}) :$ U is a unitary operator on $\mathcal{H}\}$. If $U \in \mathfrak{U}$, then $\sigma(U) \subset \partial\overline{\mathbb{D}}$. For these facts see [4].

For an operator $A \in \mathcal{L}(\mathcal{H})$, W(A) denotes the <u>numerical range</u> of A defined by ,

$$W(A) = \{\lambda \in \mathbb{C} : \exists e \in \mathcal{H}, \|e\| = 1, \lambda = (Ae,e)\},$$

where $(\cdot,\cdot)$ denotes the inner product defined on $\mathcal{H}$. The closure in $\mathbb{C}$ of the numerical range of A is denoted by $\overline{W(A)}$. It is known that W(A) is convex and not necessarily closed and that $\sigma(A) \subset \overline{W(A)}$. The <u>numerical radius</u> of A denoted by $|W(A)|$ is defined to be

$$|W(A)| = \text{Sup } \{|\lambda| : \lambda \in \overline{W(A)}\} ,$$

and the spectral radius of A = $|\sigma(A)| = \sup \{|\lambda| : \lambda \in \sigma(A)\}$. If A is a normal operator then $\overline{W(A)}$ is the convex hull of $\sigma(A)$ denoted by $ch(\sigma(A))$. For these facts see Halmos [7].

<u>Definition 1</u>. For an operator $A \in \mathcal{L}(\mathcal{H})$, X a subset of $\mathbb{C}$, we say that X is a <u>spectral set</u> of A if, and only if $\sigma(A) \subset X$, and for any φ a holomorphic function on X, we have,

$$\|\varphi(A)\| \leq \|\varphi\|_{\infty} = \sup_{x \in X} |\varphi(x)| \ .$$

Von Neuman [12] proved that $\overline{\mathbb{D}}$ is a spectral set for every contraction. If X, Y are subsets of $\mathbb{C}$ and if

$$\psi : X \to Y ,$$

is a homeomorphism which is holomorphic, then X is a spectral set of an operator A if, and only if Y is a spectral set of $\psi(A)$. See Riesz - Sz Nagy [14].

For X a subset of $\mathbb{C}$, $\mathcal{A}(X)$ denotes the algebra of all functions holomorphic in the interior of X and continuous on the closure of X, or, equivalently, the norm closure of all polynomials defined on ∂X, boundary of X (cf. Hoffman [8]). For $X = \overline{\mathbb{D}}$, $\mathcal{A}(\overline{\mathbb{D}}) \equiv \mathcal{A}$ is called the <u>disc algebra</u>.

Throughout the dissertation the work is classified into Theorems, propositions, lemmas, corollaries and so on and each category is numbered in a series.

I-2. The problem and results of its solution.

In a beautiful paper [5] in 1973, Halmos, studied the closure of the unitary orbit of the shift. More precisely, if S is a shift then the unitary orbit $\{U^*SU : U \in \mathfrak{U}\}$, is the set of all translates of S. The problem solved by Halmos in that paper, is what can we say about the closure of the unitary orbit of S in different topologies, namely, uniform operator topology (Norm), strong operator topology and weak operator topology. If a sequence (B_n) of operators converges in the Norm (resp. strong and weak) operator topology, we say that (B_n) converges uniformly (or in norm) (resp. strongly and weakly).

We present these results without proof.

Theorem A. The uniform closure of the unitary orbit of a given unilateral shift of multiplicity n, is the set of all isometries of corank n, where $1 \leq n \leq \infty$.

Theorem B. The uniform closure of the unitary orbit of a given bilateral shift of multiplicity n, is the set of all unitary operators whose spectrum is the entire unit circle.

Theorem C. The strong closure of the unitary orbit of a given shift (unilateral or bilateral) of multiplicity one is the set of all isometries.

Theorem D. The weak closure of the unitary orbit of a given shift (unilateral or bilateral) of multiplicity one is the set of all contractions.

Halmos asked what can one say about the closure of the unitary orbit if we replace the shift by a weighted shift, i.e., if T_α is a weighted shift what is the closure of the set $\{U^*T_\alpha U : U \in \mathcal{U}\}$, in different topologies. Our work considers mainly the weak closure part of the problem.

We start this work considering the problem for a contractive weighted shift. This case is discussed in Chapter II and we give sufficient and necessary condition for the weak closure of the unitary orbit of a contractive weighted shift to be the set of all contractions. We give also sufficient conditions as well as necessary ones for the strong and weak closure of the unitary orbits of two arbitrary weighted shifts to be equal.

The results of Chapter II encourage us to consider the same problem for an arbitrary contraction. This case is presented in Chapter III.

Not only do we obtain necessary and sufficient conditions for the weak closure of the unitary orbit of a given contraction to be the set all contractions, but we prove the equivalence of the following conditions for any contraction $A \in \mathcal{L}(\mathcal{H})$.

a) $WC\{U^*AU : U \in \mathcal{U}\}$ = the set of all contractions, where WC ≡ weak closure;

b) The closure of the numerical range $\overline{W(A)}$ of A is equal to $\overline{\mathbb{D}}$, the closed unit disc;

c) The spectrum $\sigma(A)$ of A contains $\partial\overline{\mathbb{D}}$, the entire unit circle;

d) The essential spectrum $\sigma_e(A)$ of A contains $\partial\overline{\mathbb{D}}$.

As an application we prove that condition c) implies that, the map ψ from the disc algebra $\mathfrak{a}$ into $\mathcal{L}(\mathcal{H})$ defined by $\psi(f) = f(A)$ for every $f \in \mathfrak{a}$, is an isometry. We could not prove the converse, namely, ψ being isometry implies condition c), but we show that if

$$|W(\psi(f))| = \|f\|_\infty ,$$

for every $f \in \mathfrak{a}$, then $\sigma(A) \supset \partial\overline{\mathbb{D}}$.

We observe that if a contraction $A \in \mathcal{L}(\mathcal{H})$ satisfies any one of the above conditions and if K is a compact operator in $\mathcal{L}(\mathcal{H})$ such that $\|A+K\| \leq 1$, then A + K satisfies these conditions.

In Chapter IV we consider the same problem in more generality. To be more precise, if X is a spectral set for a given operator $T \in \mathcal{L}(\mathcal{H})$, let $O(X)$ denote the set of all operators $A \in \mathcal{L}(\mathcal{H})$ for which X is a spectral set. We study the relation between $O(X)$ and the weak closure of the unitary orbit of T.

We prove the equivalence of the following statements for any $T \in O(X)$, provided that X satisfies certain conditions.

a) $\overline{W(T)} = X$,

b) $\sigma(T) \supset \partial X$,

c) $\sigma_e(T) \supset \partial X$,

d) $\|(T-\lambda)^{-1}\| = 1/d(\lambda,X)$, for every $\lambda \notin X$, where $d(\lambda,X) \equiv$ the distance from λ to X.

As in the contraction case we prove that statement b) implies that, the map

$$\psi : \mathfrak{a}(X) \to \mathcal{L}(\mathcal{H}),$$

defined by $\psi(f) = f(T)$ for every $f \in \mathfrak{a}(X)$, is an isometry; and if $|W(\psi(f))| = \|f\|_\infty$ for every $f \in \mathfrak{a}(X)$, then b) holds.

Moreover we prove that if $\overline{W(T)} = X$ then $O(X) \subset WC\{U^*TU : U \in \mathfrak{U}\}$, and if $O(X) = WC\{U^*TU : U \in \mathfrak{U}\}$ then $\overline{W(T)} = X$.

We investigate the relation between the contraction case considered in Chapter III and the more general case in Chapter IV, and they turn out to be equivalent.

CHAPTER II

"Weak closure of the Unitary Orbit of weighted shifts"

In this Chapter we study the weak closure of the unitary orbit of a contractive weighted shift. In section 1 we start by considering a specific weighted shift, namely, T_α with $\alpha = (\alpha_i)$, such that (α_i) converges to 1; then we show that the set $WC\{U^*T_\alpha U : U \in \mathcal{U}\}$, contains the set of all contractions, concluding that, equality holds if, and only if $\alpha_i \leq 1$, $\forall i$.

In section 2 we give necessary and sufficient conditions for the weak closure of the unitary orbit of a contractive weighted shift to be equal to the set of all contractions.

In section 3 we give necessary conditions as well as sufficient ones so that the strong (weak) closure of the unitary orbits of two arbitrary weighted shifts are equal. Moreover we give sufficient conditions for the limit of a sequence of weighted shifts to be a weighted shift, since it is not the case in general.

II-1. Weak closure of some weighted shifts.

We let $T_\alpha \in \mathcal{L}(\mathcal{H})$ be a weighted shift having the weight sequence (α_i), $i \in \mathbb{N}$, with respect to the orthonormal basis (e_i), $i \in \mathbb{N}$, of $\mathcal{H}$, such that $\alpha_i > 0$, $\forall i \in \mathbb{N}$, and we have

Lemma 1. If the sequence of weights (α_i) $i \in \mathbb{N}$ of the weighted shift T_α is such that $\alpha_0 = \alpha$ arbitrary positive number and $\alpha_i = 1$, for $i = 1,2,\cdots$, then there exists a sequence (U_N) of unitary operators such that

$$S = s - \lim U_N^* T_\alpha U_N ,$$

where S is the shift with respect to the same orthonormal basis of T_α and s-lim is the limit in the strong operator topology.

Proof: Define U_N, on (e_i), $i \in \mathbb{N}$, the orthonormal basis in $\mathcal{H}$ shifted by T_α, as follows

$$U_N e_i = e_{i+1}, \quad i \leq N - 1,$$

$$U_N e_N = e_0 \quad ,$$

and

$$U_N e_i = e_i \quad , \quad i > N,$$

it is clear from the construction that U_N is a unitary operator for every $N = 1,2,\cdots,$.

Now, we have,

$$U_N^* T_\alpha U_N e_i = e_{i+1}, \quad i \le N-1,$$

$$U_N^* T_\alpha U_N e_N = \alpha e_0 \quad ,$$

and

$$U_N^* T_\alpha U_N e_i = e_{i+1} \ , \quad i > N \ .$$

Thus on $\{e_0, e_1, \cdots, e_{N-1}\}$, we have

$$U_N^* T_\alpha U_N = S,$$

so that, as N increases, we have

$$\|(U_N^* T_\alpha U_N - S)e_i\| \to 0.$$

Q.E.D.

<u>Lemma 2</u>. If the weight α in Lemma 1 is less than or equal to one, then

$$WC\{U^* T_\alpha U : U \in \mathfrak{U}\} = \text{set of all contractions.}$$

<u>Proof</u>: By Lemma 1, we have, in particular, that

$$S = w - \lim U_N^* T_\alpha U_N,$$

where w - lim ≡ weak limit, S is the shift, and thus, we have

$$WC\{U^* T_\alpha U : U \in \mathfrak{U}\} \supset WC\{U^* S U : U \in \mathfrak{U}\} \ .$$

But the latter set is equal to the set of all contractions by Halmos [5] (Theorem D of Chapter I), so that,

$$WC\{U^* T_\alpha U : U \in \mathfrak{U}\} \supset \text{set of all contractions.}$$

Since $\alpha_0 = \leq 1$, we have $\|T_\alpha\| \leq 1$, and therefore

$$WC\{U^*T_\alpha U : U \in \mathcal{U}\} = \text{set of all contractions.}$$

Q.E.D.

Now, what about if N weights of the weight sequence are different from one and the rest of the weights are equal to one. The following Lemma gives a light.

<u>Lemma 3</u>. If T_α is a weighted shift with weight sequence $(\alpha_0, \alpha_1, \cdots, \alpha_{N-1}, 1, 1, \cdots)$. Where $\alpha_0, \alpha_1, \cdots, \alpha_{N-1}$ are arbitrary positive numbers, then there exists a sequence of unitary operators (U_N) and a shift S, such that

$$S = s - \lim U_N^* T_\alpha U_N .$$

<u>Proof</u>: Define U_N, on the orthonormal basis (e_i) $i \in \mathbb{N}$, shifted by T_α, as follows

$$U_N(e_i) = e_{N+i}, \quad 0 \leq i \leq N-1,$$

$$U_N(e_{N+i}) = e_i, \quad 0 \leq i \leq N-1,$$

$$U_N(e_i) = e_i, \quad i \geq 2N,$$

from the definition, it is clear that U_N is unitary for every $N \in \mathbb{N}$.

Now, on the span of $\{e_0, e_1, \cdots, e_{N-1}\}$, we have

$$U_N^* T_\alpha U_N e_i = U_N^* T_\alpha e_{N+i} = U_N^* e_{N+i+1}$$

$$= e_{i+1}, \quad 0 \leq i \leq N-1,$$

i.e.,

$$U_N^* T_\alpha U_N e_i = S e_i, \quad 0 \le i \le N-1,$$

where S is the shift determined by (e_i), $i \in \mathbb{N}$. Thus, for each fixed i, we have

$$\|(U_N^* T_\alpha U_N - S)e_i\| \to 0,$$

as N increases, for every i, i.e.,

$$S = s - \lim U_N^* T_\alpha U_N .$$

Q.E.D.

Lemma 4. If the sequence of weights in Lemma 3 is such that $\sup_{0 \le i \le N-1} \alpha_i \le 1$, then

$$WC\{U^* T_\alpha U : U \in \mathcal{U}\} = \text{set of all contractions.}$$

Proof: Similar to the proof of Lemma 2.

Q.E.D.

Now, we have the following

Proposition 1. If T_α is a weighted shift with weight sequence (α_i) $i \in \mathbb{N}$ such that $\lim_i \alpha_i = 1$, $(\alpha_i > 0, \forall_i \in \mathbb{N})$, then there exists a sequence (U_N) of unitary operators, and a shift S such that

$$S = s - \lim U_N^* T_\alpha U_N . \qquad (*)$$

Proof: Define T_β a weighted shift with weight sequence (β_i), $i \in \mathbb{N}$, such that

$$\beta_i = \alpha_i \ , \quad \text{for } i \le N,$$

$$\beta_i = 1 \ , \quad \text{for } i > N,$$

for some numbers $N \in \mathbb{N}$. By Lemma 3, there exists a sequence (U_N) of unitary operators and a shift S such that

$$S = s - \lim U_N^* T_\beta U_N .$$

Now, we want to show that $\|T_\alpha - T_\beta\|$ is arbitrarily small, indeed,

$$\|(T_\alpha - T_\beta)e_i\| = |\alpha_i - \beta_i| \ ,$$

since $T_\alpha - T_\beta$ is a weighted shift, we have

$$\|T_\alpha - T_\beta\| = \sup_i |\alpha_i - \beta_i|$$

$$= \sup_k |1 - \alpha_{N+k}| \ ;$$

since $\lim_i \alpha_i = 1$, for N large enough we have

$$\|T_\alpha - T_\beta\| = \sup_k |1 - \alpha_{N+k}| < \varepsilon \ . \quad (**)$$

By (*) and (**), we have

$$\|(U_N^* T_\alpha U_N - S)e_i\| \le \|(U_N^* T_\alpha U_N - U_N^* T_\beta U_N)e_i\| + \|(U_N^* T_\beta U_N - S)e_i\|$$

$$\le \|T_\alpha - T_\beta\| + \|(U_N^* T_\beta U_N - S)e_i\| \ ,$$

so that for N large enough, for a given e_i, we have

$$\|(U_N^* T_\alpha U_N - S)e_i\| < 2\varepsilon \ ,$$

that is,

$$S = s - \lim U_N^* T_\alpha U_N \ .$$

Q.E.D.

<u>Theorem 1</u>. If T_α is a weighted shift as in proposition 1, then $WC\{U^*T_\alpha U : U \in \mathfrak{U}\}$ contains the set of all contractions. Moreover if $\operatorname{Sup}_i \alpha_i \leq 1$, then equality holds.

<u>Proof</u>: Since by proposition 1 there is a sequence (U_N) of unitary operators and a shift S such that

$$S = w - \lim U_N^* T_\alpha U_N \ ,$$

we have,

$$WC\{U^*T_\alpha U : U \in \mathfrak{U}\} \supset WC\{U^*SU : U \in \mathfrak{U}\} \ ,$$

and thus,

$$WC\{U^*T_\alpha U : U \in \mathfrak{U}\} \supset \text{set of all contractions.}$$

If $\operatorname{Sup}_i \alpha_i \leq 1$, then T_α is a contraction, in which case we have the other inclusion.

Q.E.D.

<u>Remark 1</u>. The above results hold for a bilateral weighted shift as well as the unilateral weighted shift considered.

II-2. Necessary and sufficient condition.

In this section we give necessary and sufficient condition on the sequence of weights of a given weighted shift T_α in order that

$$WC\{U^*T_\alpha U : U \in \mathfrak{U}\} = \text{set of all cntractions.}$$

All the conditions considered so far were sufficient but not necessary. To see that consider the following example.

Example 1. Let T_α be the weighted shift with the weight sequence $(1,\varepsilon,1,1,\varepsilon,\cdots,)$ i.e. it consists of sequences of ones of arbitrary length and of ε such that $0 < \varepsilon \leq 1$, then

$$WC\{U^*T_\alpha U : U \in \mathfrak{U}\} = \text{set of all contractions.}$$

Indeed: Since we have chains of ones of arbitrary length, then, given a positive integer N, there is an integer M such that

$$\alpha_M = \alpha_{M+1} = \alpha_{M+2} = \cdots = \alpha_{M+N-1} = 1.$$

Consider U_N on $\mathcal{H}$ defined by

$$U_N e_i = e_{M+i}, \ i \leq N; \ U_N e_{N+i} = e_i, \ i \leq M$$

and

$$U_N e_{M+N+i} = e_{M+N+i} \text{ for every i, for the case } M \leq N.$$

For the case $M > N$, we have $U_N e_i = e_{M+i}$, $i \leq N$,

$$U_N e_{N+i} = e_{N+i},\ i \le M-N,\ U_N e_{M+i} = e_i\ ,\ i \le N,$$

and

$$U_N e_{M+N+i} = e_{M+N+i},\ \text{for every } i.$$

In both cases U_N is a unitary on $\mathcal{H}$.

Now on the span of $\{e_0, e_1, \cdots, e_{N-1}\}$

$$U_N^* T_\alpha U_N e_i = U_N^* T_\alpha e_{M+i} = U_N^* \alpha_{M+i} e_{M+i+1}$$

$$= e_{i+1}\ , \qquad i \le N-1,$$

so that

$$U_N^* T_\alpha U_N e_i = Se_i, \qquad i \le N-1,$$

and as N increases, we have

$$S = s - \lim U_N^* T_\alpha U_N,$$

and thus

$$WC\{U^* T_\alpha U : U \in \mathcal{U}\} = \text{set of all contractions.}$$

We conclude that $\lim \alpha_i = 1$ is more than enough. It is natural to see that we need only the weight sequence to have sequences of positive real numbers of arbitrary length and very near to one. The following Theorem formulate this condition.

<u>Theorem 2</u>. Let T_α be a contractive weighted shift, i.e., $\sup_i \alpha_i \le 1$, where (α_i) its weight sequence. Then,

$$WC\{U^* T_\alpha U : U \in \mathcal{U}\} = \text{set of all contractions}$$

if, and only if, for every positive integer N, $\varepsilon > 0$, there is an integer M such that

$$\alpha_{M+i} \geq 1 - \varepsilon, \quad i = 1,2,\cdots,N.$$

We divide the proof into several lemmas and propositions. The next proposition proves the sufficiency of the condition.

<u>Proposition 2</u>. If the weight sequence (α_i), $i \in \mathbb{N}$, of the weighted shift T_α is such that for every positive integer N, $\varepsilon > 0$ there is an integer M such that $\alpha_{M+i} > 1 - \varepsilon$, $i = 0,1,2,\cdots,N$, then

$$WC\{U^*T_\alpha U : U \in \mathcal{U}\} = \text{set of all contractions.}$$

<u>Proof</u>: By the assumption given N, $\varepsilon > 0, \exists M,$ such that $\alpha_{M+i} \geq 1 - \varepsilon$, $i = 0,1,2,\cdots,N$, or

$$1 - \alpha_{M+i} \leq \varepsilon, \quad i = 0,1,2,\cdots,N.$$

Define U_N on $\mathcal{H}$ such that

$$U_N : \{e_0,e_1,\cdots,e_N\} \to \{e_M,e_{M+1},\cdots,e_{M+N}\},$$

by

$$U_N e_i = e_{M+i}, i \leq N; \; U_N e_{N+i} = e_i, \; i \leq M,$$

and

$$U_N e_{N+M+i} = e_{N+M+i} \text{ for every } i, \text{ for the case } M \leq N.$$

For the case $M > N$, we have

$$U_N e_i = e_{M+i}, \; i \leq N; \; U_N e_{N+i} = e_{N+i}, \; i \leq M-N,$$

$$U_N e_{M+i} = e_i, \; i \leq N \text{ and } U_N e_{N+M+i} = e_{N+M+i}, \text{ for every } i.$$

It is clear that U_N is unitary on $\mathcal{H}$, $\forall N$, and

$$U_N^* T_\alpha U_N e_i = \alpha_{M+i} e_{i+1},$$

and if S is the shift determined by (e_i), then on the span of $\{e_0, e_1, \cdots, e_N\}$, we have

$$\|(S - U_N^* T_\alpha U_N)e_i\| = |1 - \alpha_{M+i}| < \varepsilon,$$

$i = 0, 1, \cdots, N$, for every $e_i \in \{e_0, e_1, \cdots, e_N\}$. Since N, ε are arbitrary, we take $\varepsilon = \frac{1}{N}$ to obtain

$$\lim_N \|(S - U_N^* T_\alpha U_N)e_i\| = 0,$$

for every $e_i \in \mathcal{H}$, i.e.,

$$S = s - \lim_N U_N^* T_\alpha U_N,$$

which implies that

$$WC\{U^* T_\alpha U : U \in \mathfrak{U}\} = \text{set of all contractions.}$$

Q.E.D.

The following Lemma asserts that a "big" contraction is almost a weighted shift, the meaning of the term "big" is clarified in the lemma.

<u>Lemma 5</u>. If T is a contraction such that $|1 - (Te_i, e_j)| < \varepsilon$, for some vectors e_i, e_j in an orthonormal basis of $\mathcal{H}$, then there is a positive real number λ and a vector h such that

$$Te_i = \lambda e_j - h,$$

where $\|h\| < \varepsilon$, $1 - \varepsilon < \lambda \leq 1$ and $|1 - \lambda| < \varepsilon$.

Proof: We have,

$$1 - \varepsilon < |(Te_i, e_j)| \leq \|Te_i\| \cdot \|e_j\| \leq 1.$$

We know that if equality holds in the Cauchy-Schwartz inequality, then $Te_i = \lambda e_j$ for some complex number λ. But in our case it is approximately equal so that there is a complex number λ and a vector h orthogonal to e_j such that

$$Te_i = \lambda e_j - h. \qquad (*)$$

This implies that

$$1 - \varepsilon \leq |(Te_i, e_j)| = |\lambda(e_j, e_j) - (h, e_j)| \leq 1$$

and by substituting (*) into the assumption we have $|1-\lambda| < \varepsilon$, and $1 - \varepsilon \leq |\lambda| \leq 1$, and thus

$$\|h\|^2 + \|\lambda e_j\|^2 = \|Te_i\|^2 \leq 1,$$

which implies,

$$\|h\|^2 \leq 1 - |\lambda|^2 \leq 1 - (1-\varepsilon)^2 \leq 2\varepsilon,$$

so that,

$$\|h\|^2 \leq 2\varepsilon.$$

We claim that λ is almost a positive real number indeed since, $|1-\lambda| < \varepsilon$, and $|\lambda| \leq 1$, we have

$$1 - \varepsilon < \text{Real } \lambda \leq 1,$$

and therefore

$$Te_i = \text{Real } \lambda\, e_j + (i\text{Im}\lambda e_j - h),$$

which implies

$$\|(i\text{Im}.\lambda e_j - h)\|^2 = \|Te_i - \text{Real } \lambda e_j\|^2$$

$$= \|Te_i\|^2 - 2\,\text{Real}(\text{Real } \lambda)\,(Te_i, e_j) + $$

$$+ |\text{Real } \lambda|^2 .$$

Since $\text{Real}(Te_i, e_j) = \text{Real}(\lambda e_j - h, e_j) = \text{Real } \lambda$, we have,

$$\|i\text{Im}.\lambda e_j - h\|^2 \leq 1 - 2(\text{Real } \lambda)^2 + (\text{Real } \lambda)^2$$

$$\leq 1 - (\text{Real } \lambda)^2 \leq 1 - (1-\varepsilon)^2$$

$$< 1 - (1 - 2\varepsilon + \varepsilon^2) = 2\varepsilon - \varepsilon^2$$

$$< 2\varepsilon ,$$

so that we can assume λ to be positive real number to obtain

$$Te_i = \lambda e_j - h,$$

where $1 - \varepsilon < \lambda \leq 1$, $\|h\|^2 < 2\varepsilon$ and $|1-\lambda| < \varepsilon$. Q.E.D.

<u>Theorem 3</u>. If S is a shift, and T is any contraction such that

$$|((S-T)e_i, e_{i+1})| < \varepsilon, \text{ for } i < L,$$

then for every positive integer N, we have

$$|((S^N - T^N)e_i, e_{i+N})| < \delta_N(\varepsilon), \quad i + N < L,$$

where (e_i) is the orthonormal basis shifted by S, $\delta_N(\varepsilon)$ depends on ε, N linearly.

<u>Proof</u>: From the assumption we have

$$\varepsilon > |(Se_i,e_{i+1}) - (Te_i,e_{i+1})| = |1 - (Te_i,e_{i+1})|, \quad \text{for } i < L; \qquad (*)$$

and since $(Te_i,e_{i+1}) = (e_i,\overset{*}{T}e_{i+1})$, we have

$$\varepsilon > |1-(e_i,T^*e_{i+1})|, \ i < L . \qquad (**)$$

Applying Lemma 5 to (*) and (**), we obtain

$$Te_i = \lambda_i e_{i+1} - h_i,$$
$$T^*e_{i+1} = \lambda_i' e_i - h_i', \quad i < L,$$

where λ_i, λ_i' are positive numbers with $|1-\lambda_i| < \varepsilon$, $|1-\lambda_i'| < \varepsilon$ and $1 - \varepsilon < \lambda_i, \lambda_i' \le 1$ and $\|h_i\|^2 < 2\varepsilon$, $\|h_i'\|^2 < 2\varepsilon$.
Now,

$$|((S^2-T^2)e_i,e_{i+2})| = |1 - (T^2e_i,e_{i+2})| <$$
$$< |1 - (Te_i,T^*e_{i+2})| = |1 - (Te_i,\lambda e_{i+1}-h)|$$
$$< |1 - \lambda(Te_i,e_{i+1})| + |(Te_i,h)|$$
$$< |1 - \lambda(\lambda' e_{i+1}-h',e_{i+1})| + \|h\|$$
$$< |1 - \lambda\lambda'| + \|h'\| + \|h\|$$
$$< |1 - (1-\varepsilon)^2| + 2\cdot\sqrt{3\varepsilon}$$
$$\le 1 - 1 + 2\varepsilon - \varepsilon^2 + 2\sqrt{3\varepsilon} = \delta_2(\varepsilon) .$$

This prove the proposition for N = 2, assume that the proposition is true for N, so that for N + 1, we have

$$|((S^{N+1}-T^{N+1})e_i, e_{i+N+1})| = |1 - (T^{N+1}e_i, e_{i+N+1})|$$

$$= |1 - (T^N e_i, T^* e_{i+N+1})|$$

$$= |1 - (T^N e_i, \lambda' e_{i+N} - h')|$$

$$\leq |1 - \lambda'(T^N e_i, e_{i+N})| + |(T^N e_i, h')| .$$

By the induction step we have

$$|1 - (T^N e_i, e_{i+N})| < \delta_N(\varepsilon) ,$$

so that applying Lemma 5, we obtain that

$$T^N e_i = \lambda e_{i+N} - h ,$$

where λ' is positive real number such that

$$1 - \delta_N(\varepsilon) < \lambda \leq 1,$$

and $\|h\|^2 < \delta_N(\varepsilon)$.

This implies that

$$|((S^{N+1}-T^{N+1})e_i, e_{i+N+1})| \leq |1 - \lambda(T^N e_i, e_{i+N})| + \|h\|$$

$$\leq |1 - \lambda(\lambda' e_{i+N} - h', e_{i+N})| + \|h\|$$

$$\leq |1 - \lambda\lambda'| + \lambda|(h', e_{i+N})| + \|h\|$$

$$\leq |1 - (1-\delta_N(\varepsilon))^2| + \|h'\| + \|h\|$$

$$< 1 - 1 + 2\delta_N(\varepsilon) - \delta_N^2(\varepsilon) + 2\delta_N(\varepsilon) \leq \delta_{N+1}(\varepsilon) .$$

Q.E.D.

Theorem 4. If S is a shift with respect to the orthonormal basis (φ_i) and T_α is a contractive weighted shift with respect to the orthonormal basis (e_i) and having the weight sequence (α_i), $i \in \mathbb{N}$; such that

$$|((S-T_\alpha)\varphi_i,\varphi_{i+1})| < \varepsilon, \qquad i < L,$$

then there are chains of weights of length less than or equal to L, each weight of which is $> 1 - \delta(\varepsilon)$.

Proof: First we want to show that there is at least one weight $\alpha_M > 1 - \varepsilon$. Consider φ_0,φ_1, and we have

$$|((S-T_\alpha)\varphi_0,\varphi_1)| < \varepsilon ,$$

so that

$$\varepsilon > |(S\varphi_0,\varphi_1) - (T_\alpha\varphi_0,\varphi_1)| > 1 - |(T_\alpha\varphi_0,\varphi_1)|,$$

or

$$1 - \varepsilon < |(T_\alpha\varphi_0,\varphi_1)|.$$

Let $\varphi_0 = \sum_0^\infty a_k e_k$, $\varphi_1 = \sum_0^\infty b_k e_k$; then (a_k), (b_k) are sequences of complex numbers such that

$$\sum_0^\infty |a_k|^2 = 1, \sum_0^\infty |b_k|^2 = 1.$$

This is from the fact that φ_0, φ_1 are unit vectors, then we have

$$1 - \varepsilon < |(T_\alpha \varphi_0, \varphi_1)| = |(\sum_0^\infty a_k T_\alpha e_k, \sum_0^\infty b_k e_k)| =$$
$$< |(\sum_0^\infty a_k \alpha_k e_{k+1}, \sum_0^\infty b_k e_k)| =$$
$$< |\sum_0^\infty \alpha_k a_k \overline{b}_{k+1}| \le \sum_0^\infty \alpha_k |a_k| |\overline{b}_{k+1}| .$$

This implies that

$$(1-\varepsilon)^2 < (\sum_0^\infty \alpha_k^2 |b_{k+1}|^2)(\sum_0^\infty |a_k|^2) =$$
$$< \sum_0^\infty \alpha_k^2 |b_{k+1}|^2 \le \sup_k \alpha_k^2 (\sum_0^\infty |b_{i+1}|^2) ,$$

i.e.,

$$(1-\varepsilon)^2 < \sup_k \alpha_k^2 ,$$

this implies that there is M such that

$$\alpha_M^2 > (1-\varepsilon)^2,$$

or

$$\alpha_M > 1 - \varepsilon .$$

The second step is to show there is a chain of weights of length two and each weight of this chain is $> 1 - \delta_2(\varepsilon)$. Consider φ_0, φ_1, and φ_2 and Theorem 3 implies that

$$|(T_\alpha^2 \varphi_0, \varphi_2)| > 1 - \delta_2(\varepsilon) .$$

If we let $\varphi_2 = \sum_0^\infty c_k e_k$, where $\sum_0^\infty |c_k|^2 = 1$, then we have,

$$1 - \delta_2(\varepsilon) < |(\sum_0^\infty a_k T_\alpha^2 e_k, \sum_0^\infty c_k e_k)| =$$

$$< |(\sum_0^\infty \alpha_k \alpha_{k+1} a_k e_{k+2}, \sum_0^\infty c_k e_k)| \leq$$

$$< \sum_0^\infty \alpha_k \alpha_{k+1} |a_k| |\overline{c}_{k+2}| .$$

This implies that

$$(1-\delta_2(\varepsilon))^2 < (\sum_0^\infty \alpha_k^2 \alpha_{k+1}^2 |\overline{c}_{k+2}|^2)(\sum_0^\infty |a_k|^2)$$

$$< \sup_k \alpha_k^2 \alpha_{k+1}^2 (\sum_0^\infty |\overline{c}_{k+2}|^2)$$

$$< \sup_k \alpha_k^2 \alpha_{k+1}^2 \quad ,$$

and thus, there is a positive integer M such that

$$\alpha_M^2 \alpha_{M+1}^2 > (1-\delta_2(\varepsilon))^2 ,$$

or

$$\alpha_M \alpha_{M+1} > (1-\delta_2(\varepsilon)),$$

which implies that

$$\alpha_M > 1 - \delta_2(\varepsilon) \text{ and } \alpha_{M+1} > 1 - \delta_2(\varepsilon) .$$

The last step is to show that there is a chain of weights of length $N \leq L$, such that each weight of this chain is $> 1 - \delta_N(\varepsilon)$. So consider $\varphi_0, \varphi_1, \cdots, \varphi_N$, $N \leq L$ and let $\varphi_N = \sum_0^\infty d_k e_k$ where $\sum_0^\infty |d_k|^2 = 1$, then by Theorem 4, we have

$$1 - \delta_N(\varepsilon) < |(T^N\varphi_0, \varphi_N)| = |(\sum_0^\infty a_k T^N e_k, \sum_0^\infty d_k e_k)|$$

$$< |(\sum_0^\infty \alpha_k\alpha_{k+1}\cdots\alpha_{k+N} a_k e_{k+N}, \sum_0^\infty d_k e_k)|$$

$$< \sum_0^\infty \alpha_k\alpha_{k+1}\cdots\alpha_{k+N}|a_k|\cdot|\overline{d}_{k+N}|$$

$$< (\sum_0^\infty \alpha_k^2\alpha_{k+1}^2\cdots\alpha_{k+N}^2|\overline{d}_{k+N}|^2)^{\frac{1}{2}}(\sum_0^\infty |a_k|^2)^{\frac{1}{2}}$$

$$< (\sup_k \alpha_k^2\alpha_{k+1}^2\cdots\alpha_{k+N}^2)^{\frac{1}{2}}(\sum_0^\infty |\overline{d}_{k+N}|^2)^{\frac{1}{2}}$$

$$< (\sup_k \alpha_k^2\alpha_{k+1}^2\cdots\alpha_{k+N}^2)^{\frac{1}{2}} .$$

This implies that there is a positive integer M such that

$$1 - \delta_N(\varepsilon) < (\alpha_M^2\alpha_{M+1}^2\cdots\alpha_{M+N}^2)^{\frac{1}{2}}$$

$$< \alpha_M\alpha_{M+1}\cdots\alpha_{M+N},$$

so that

$$\alpha_{M+i} > 1 - \delta_N(\varepsilon),\ i = 0, 1, \cdots, N.$$

Q.E.D.

<u>Proposition 3</u>. If T_α is a weighted shift whose sequence of weights is (α_i), $i \in \mathbb{N}$ such that

$$WC\{U^*T_\alpha U : U \in \mathfrak{U}\} = \text{set of all contractions}$$

then, $\forall\ \epsilon, N; \exists\ M$ such that $\alpha_{M+i} > 1 - \delta_N(\varepsilon), i \le N$.

<u>Proof</u>: Theorem D and the assumption implies that

$$WC\{U^*T_\alpha U : U \in \mathfrak{U}\} = WC\{U^*SU : U \in \mathfrak{U}\} ,$$

where S is a shift, this implies that for T_α, given any positive integer L there is a sequence (U_N) of unitary operators such that

$$S = w - \lim U_N^* T_\alpha U_N ,$$

which is equivalent to say that $\forall \varepsilon$, $\exists K$ such that $N > K$, we have

$$|((S-U_N^* T_\alpha U_N)\varphi_i,\varphi_{i+1})| < \varepsilon, \; 0 \le i < L.$$

Now, given $N \le L$, fix it; then $U_N^* T_\alpha U_N$ is a fixed weighted shift having the same sequence of weights and different orthonormal basis. Applying Theorem 4 to this weighted shift we obtain

$$\forall \varepsilon > 0, N, \; \exists M \text{ such that}$$

$$\alpha_{M+i} > 1 - \delta_N(\varepsilon), \quad i = 0,1,2,\cdots,N.$$

Q.E.D.

<u>Proof of Theorem 2</u>: Proposition 2 proves sufficiency and Theorem 4, proposition 3 prove necessity.

Q.E.D.

<u>Remark 2</u>. Theorem 2 holds for contractive bilateral weighted shift as well.

<u>Corollary 1</u>. If T_α is a weighted shift such that $WC\{U^*T_\alpha U : U \in \mathfrak{U}\}$ = set of all contractions and if T is

another weighted shift with positive weights such that $\|T_\alpha + T\| \le 1$, then

$$WC\{U^*(T_\alpha+T)U : U \in \mathcal{U}\} = \text{set of all contractions}$$

Proof: The assumption is equivalent to say

$$\forall \varepsilon > 0,\ N; \exists\ M \text{ such that}$$

$$\alpha_{M+i} > 1 - \varepsilon, \quad i = 0,1,\cdots,N.$$

If (γ_i) is the weight sequence of the weighted shift T then $\eta_i = \alpha_i + \gamma_i$ is the weight sequence of $T_\alpha + T$ and we have

$$\forall \varepsilon > 0,\ N; \exists\ M \text{ such that}$$

$$1 \ge \eta_{M+i} = \alpha_{M+i} + \gamma_{M+i} > 1 - \varepsilon,$$

for $i = 0,1,\cdots,N$, which is equivalent to

$$WC\{U^*(T_\alpha+T)U : U \in \mathcal{U}\} = \text{set of all contractions.}$$

Q.E.D.

Corollary 2. The set of all weighted shifts is weakly dense in $\mathcal{L}(\mathcal{H})$.

Proof: Given $A \in \mathcal{L}(\mathcal{H})$, let $\lambda = \|A\|$ then $T = A/\lambda$ is a contraction, then there is a weighted shift T_α such that

$$\forall L,\ \varepsilon > 0,\ \exists U \text{ unitary, such that}$$

$$|((T-U^*T_\alpha U)f_i,f_j)| < \varepsilon/\lambda \quad , i,\ j \le L,$$

or,

$$|((A-U^*\lambda T_\alpha U)f_i,f_j)| < \varepsilon, \quad i,j \le L$$

Since λT_α is weighted shift whose sequence of weights is $(\lambda\alpha_i)$, that approximate A weakly the corollary is proved.

Q.E.D.

We conclude this section by the following proposition not related directly to the theme of the section.

Proposition 4. If U is unitary operator and T_α is a weighted shift with respect to the orthonormal basis (e_i) having (α_i) as the weight sequence then there is a unitary operator U' and a weighted shift T_β such that

$$U \oplus T_\alpha = U'T_\beta .$$

Proof: Let $U' = \begin{pmatrix} 0 & U \\ I & 0 \end{pmatrix}$, $T_\beta = \begin{pmatrix} 0 & T_\alpha \\ I & 0 \end{pmatrix}$,

then

$$U'T_\beta = \begin{pmatrix} 0 & U \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & T_\alpha \\ I & 0 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & T_\alpha \end{pmatrix} = U \oplus T_\alpha .$$

It is clear that U' is unitary. It remains to show that T_β is a weighted shift. Indeed, T_β has $(1,\alpha_0,1,\alpha_1,1,\alpha_2,\cdots)$ as weight sequence with respect to the orthonormal basis $(e_0,0)$, $(0,e_0)$, $(e_1,0)$, $(0,e_1),\cdots$, since we have

$$\begin{pmatrix} 0 & T_\alpha \\ I & 0 \end{pmatrix}\begin{pmatrix} e_k \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e_k \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 0 & T_\alpha \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 \\ e_k \end{pmatrix} = \begin{pmatrix} T_\alpha e_k \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_k e_{k+1} \\ 0 \end{pmatrix} = \alpha_k \begin{pmatrix} e_{k+1} \\ 0 \end{pmatrix} .$$

Q.E.D.

II-3. Weak and strong closure of the unitary orbit of arbitrary weighted shifts.

In this section we consider T_α, T_β two weighted shifts having the sequences of weights (α_i), (β_i) respectively, which are arbitrary positive numbers. We give relations between (α_i) and (β_i) in order that T_α, T_β have the same strong closure of their unitary orbits. We start with the following simple case.

Proposition 5. If T_α, T_β are as above, if $\forall_i$, $\alpha_i = \alpha$, a constant, $\beta_i \le \alpha$, $\forall i$, and if $\forall \varepsilon > 0$, N, $\exists$ M such that $\beta_{M+i} \ge \alpha - \varepsilon$, $i \le N$, then

$$WC\{U^*T_\alpha U : U \in \mathcal{U}\} \subset WC\{U^*T_\beta U : U \in \mathcal{U}\} .$$

Proof: Given N, $\varepsilon = \frac{1}{N} > 0$, then M exists and define

$$U_N : \{e_1, e_2, \cdots, e_N\} \to \{e_{M+1}, \cdots, e_{M+N}\} ,$$

such that

$$U_N e_i = e_{M+i}, \ i \le N; \ U_N e_{N+i} = e_i, \ i \le M$$

and

$$U_N e_{N+M+i} = e_{N+M+i}, \ \forall_i, \text{ for the case } M \le N.$$

For the case M > N, we have

$$U_N e_i = e_{M+i}, \ i \le N; \quad U_N e_{N+i} = e_{N+i}, \ i \le M-N,$$

$$U_N e_{M+i} = e_i,\ i \le N; \quad U_N e_{M+N+i} = e_{M+N+i},\ \forall i,$$

so that in both cases U_N is unitary on $\mathcal{H}$, and

$$U_N^* T_\beta U_N(e_i) = \beta_{M+i} e_{i+1} .$$

This implies that for every $e_i \in \{e_1, \cdots, e_N\}$,

$$\|(U_N^* T_\beta U_N - T_\alpha)e_i\| = |\beta_{M+i} - \alpha| < \varepsilon = \frac{1}{N} ,$$

which implies that

$$T_\alpha = s\text{-}(w)\lim U_N^* T_\beta U_N ,$$

and hence

$$SC\{U^* T_\alpha U : U \in \mathfrak{U}\} \subset SC\{U^* T_\beta U : U \in \mathfrak{U}\} ,$$

and in particular,

$$WC\{U^* T_\alpha U : U \in \mathfrak{U}\} \subset WC\{U^* T_\beta U : U \in \mathfrak{U}\} . \qquad \text{Q.E.D.}$$

<u>Proposition 6</u>. If T_α, T_β are as above and if (α_i) converges to α, $\alpha_i \le \alpha$, $\forall i$, $\beta_i \le \alpha$, $\forall i$ and if $\forall \varepsilon > 0$, N, $\exists$ M such that $\beta_{M+i} > \alpha - \varepsilon$, $i \le N$, then $WC\{U^* T_\alpha U : U \in \mathfrak{U}\} \subset WC\{U^* T_\beta U : U \in \mathfrak{U}\}$.

<u>Proof</u>: Define (U_N) as in the proof of the previous proposition, then we have

$$\|(U_N^* T_\beta U_N - T_\alpha)e_i\| = |\beta_{M+i} - \alpha_i|$$

$$\le |\beta_{M+i} - \alpha| + |\alpha - \alpha_i| \le 2\varepsilon = \frac{2}{N} ,$$

for $i \le N$, so that

$$T_\alpha = s - (w)\lim U_N^* T_\beta U_N \ ,$$

and we have the result. Q.E.D.

In order that T_α, T_β generate the same strong and weak closure of their unitary orbits, the preceeding propositions suggest that the sequences of weights must be very near to each other in a certain sense. This sense is explained in the following theorem.

Theorem 5. If T_α, T_β are weighted shifts having (α_i), (β_i) as their sequences of weights respectively and if $\forall \varepsilon > 0$, N, K; $\exists$ M such that $|\alpha_{K+i} - \beta_{M+i}| < \varepsilon$, $i \le N$, and if $\forall \varepsilon > 0$, N, M; $\exists$ K such that $|\beta_{M+i} - \alpha_{K+i}| < \varepsilon$, $i \le N$, then we have

$$SC\{U^* T_\alpha U : U \in \mathcal{U}\} = SC\{U^* T_\beta U : U \in \mathcal{U}\} \ ,$$

and in particular,

$$WC\{U^* T_\alpha U : U \in \mathcal{U}\} = WC\{U^* T_\beta U : U \in \mathcal{U}\} \ .$$

Proof: It is enough to show that the first condition implies that $SC\{U^* T_\beta U : U \in \mathcal{U}\}$ contains $SC\{U^* T_\alpha U : U \in \mathcal{U}\}$, since the other condition implies the second inclusion in a similar manner.

Given N, $\varepsilon > 0$, K so that $\exists$ M such that $|\alpha_{K+i} - \beta_{M+i}| < \varepsilon$, $i \le N$, then define the unitary operator U_N on $\mathcal{H}$ such that

$$U_N : \{e_{K+1}, \cdots, e_{K+N}\} \to \{e_{M+1}, \cdots, e_{M+N}\},$$

we have two cases $K \leq M$ and $K > M$, so for the case $K \leq M$, we define U_N to be,

$$U_N(e_{K+i}) = e_{M+i}, \qquad i \leq N,$$

$$U_N(e_i) = e_i, \qquad i \leq K,$$

$$U_N(e_{K+N+i}) = e_{K+i}, \qquad K + i \leq M,$$

and

$$U_N(e_{M+N+i}) = e_{M+N+i}, \qquad \forall i.$$

For the case $K > M$, we define U_N to be,

$$U_N(e_{K+i}) = e_{M+i}, \qquad i \leq N,$$

$$U_N(e_i) = e_i, \qquad i \leq M,$$

and

$$U_N e_{M+i} = e_{M+N+i}, \qquad M + i \leq K,$$

$$U_N e_{K+N+i} = e_{K+N+i}, \qquad \forall i .$$

In both cases we have

$$U_N^* T_\beta U_N e_{K+i} = \beta_{M+i} e_{K+i+1},$$

and thus,

$$\|(U_N^* T_\beta U_N - T_\alpha) e_{K+i}\| = |\beta_{M+i} - \alpha_{K+i}| < \varepsilon = \frac{1}{N},$$

for $i \leq N$, so that, for every element in $\{e_{K+1},\cdots,e_{K+N}\}$ we have,

$$U_N^* T_\beta U_N e_{K+i} \to T_\alpha e_{K+i}, \quad i \leq N,$$

as N increases. Since K is arbitrary, we obtain

$$T_\alpha = S(W) - \lim U_N^* T_\beta U_N \; ,$$

which implies that

$$SC\{U^*T_\beta U : U \in \mathfrak{U}\} \supset SC\{U^*T_\alpha U : U \in \mathfrak{U}\},$$

and in particular,

$$WC\{U^*T_\beta U : U \in \mathfrak{U}\} \supset WC\{U^*T_\alpha U : U \in \mathfrak{U}\} \; .$$

Q.E.D.

The following Theorem gives necessary conditions for the equality,

$$SC\{U^*T_\alpha U : U \in \mathfrak{U}\} = SC\{U^*T_\beta U : U \in \mathfrak{U}\} \; .$$

Theorem 6. If T_α, T_β are weighted shifts having (α_i), (β_i) as their respective weight sequences such that $SC\{U^*T_\alpha U : U \in \mathfrak{U}\} = SC\{U^*T_\beta U : U \in \mathfrak{U}\}$, then, we have

$$\forall \varepsilon > 0,\ N,\ M,\ \exists K \text{ such that } \prod_{i=1}^{N} \alpha_{K+i} > \prod_{i=1}^{N} \beta_{M+i} - \varepsilon,$$

and

$$\forall \varepsilon > 0,\ N,\ K,\ \exists M \text{ such that } \prod_{i=1}^{N} \beta_{M+i} > \prod_{i=1}^{N} \alpha_{K+i} - \varepsilon \; .$$

<u>Proof</u>: It is enough to show that if (U_n) is a sequence of unitary operators such that $U_n^* T_\alpha U_n$ converges strongly to T_β then, $\forall \varepsilon$, N,M,$\exists$K such that $\prod_{i=1}^{N} \alpha_{K+i} > \prod_{i=1}^{N} \beta_{M+i} - \varepsilon$, since if $U_n^* T_\beta U_n$ converges strongly to T_α implies the other condition in a similar manner.

Let (φ_i) be the orthonormal basis shifted by T_β and given ε,M, N = 1, and consider φ_{M+1}, we have

$$\|(U_n^* T_\alpha U_n - T_\beta)\varphi_{M+1}\| \to 0,$$

as n increases, i.e.,

$$\forall \varepsilon > 0,\ \exists L(\varphi_{M+1}) \text{ such that } n > L, \text{ we have}$$

$$\|(U_n^* T_\alpha U_n - T_\beta)\varphi_{M+1}\| < \varepsilon.$$

This implies that

$$\varepsilon > \|\beta_{M+1}\varphi_{M+2} - U_n^* T_\alpha U_n \varphi_{M+1}\| \geq$$

$$> \beta_{M+1} - \|U_n^* T_\alpha U_n \varphi_{M+1}\| ,$$

or

$$(\beta_{M+1} - \varepsilon)^2 < \|T_\alpha U_n \varphi_{M+1}\|^2 = (T_\alpha U_n \varphi_{M+1}, T_\alpha U_n \varphi_{M+1}).$$

If (e_i) is the orthonormal basis shifted by T_α and let n > L, but fixed, then

$$U_n : \varphi_i \to (\eta_i^n)$$

where (η_i^n) is an orthonormal basis, in particular let

$U_n\varphi_{M+1} = h^n_{M+1} = \sum_k a^n_k e_k$, where (a^n_k) if a sequence of complex numbers such that $\sum_k |a^n_k|^2 = 1$, and thus we have

$$(\beta_{M+1}-\varepsilon)^2 < (\sum_k a^n_k T_\alpha e_k, \sum_k a^n_k T_\alpha e_k)$$

$$< (\sum_k a^n_k \alpha_k e_{k+1}, \sum_k a^n_k \alpha_k e_{k+1})$$

$$< \sum_k \alpha^2_k |a^n_k|^2 \leq \sup_k \alpha^2_k \sum_k |a^n_k|^2$$

$$< \sup \alpha^2_k \ .$$

This implies that $\exists$ K such that

$$(\beta_{M+1}-\varepsilon)^2 < \alpha^2_{K+1} \ ,$$

i.e.,

$$\beta_{M+1} - \varepsilon < \alpha_{K+1} \ .$$

Now, since product is sequentially continuous in the strong operator topology, we have, for every positive integer N,

$$U^*_n T^N_\alpha U_n \text{ converges strongly to } T^N_\beta \ ,$$

so that, for $\varepsilon > 0$, M, N, consider φ_{M+1}, and we have for $n > L_{(\varphi M+1)}$,

$$\| T^N_\beta \varphi_{M+1} - U^*_n T^N_\alpha U_n \varphi_{M+1} \| < \varepsilon \ .$$

This implies that

$$\varepsilon > \|\beta_{M+1}\beta_{M+2}\cdots\beta_{M+N}\varphi_{M+N+1} - U_n^* T_\alpha^N U_n \varphi_{M+1}\|$$

$$> \prod_{i=1}^{N} \beta_{M+i} - \|U_n^* T_\alpha^N U_n \varphi_{M+1}\| ,$$

or, we have, for $U_n\varphi_{M+1} = \sum_k a_k^n e_k$, $n > L$, but fixed,

$$\prod_{i=1}^{N} \beta_{M+i} - \varepsilon < \|T_\alpha^N U_n \varphi_{M+1}\|$$

$$< \|\sum_k a_k^n T_\alpha^N e_k\|$$

$$< \|\sum_k a_k^n \alpha_k \alpha_{k+1} \cdots \alpha_{k+N-1} e_{k+N}\|$$

$$< \sup_k \alpha_k \alpha_{k+1} \cdots \alpha_{k+N-1} \|\sum_k a_k^n e_{k+N}\|$$

$$< \sup_k \alpha_k \alpha_{k+1} \cdots \alpha_{k+N-1} (\sum_k |a_k^n|^2)^{\frac{1}{2}}$$

$$< \sup \alpha_k \alpha_{k+1} \cdots \alpha_{k+N-1} .$$

This implies that $\exists$ K such that,

$$\prod_{i=1}^{N} \beta_{M+i} - \varepsilon < \alpha_{K+1}\alpha_{K+2}\cdots\alpha_{K+N} = \prod_{i=1}^{N} \alpha_{K+i} .$$

Q.E.D.

Remark 3. We cannot replace strong closure by weak closure in Theorem 6 simply, because product is not sequentially continuous in the weak operator topology.

The following Theorem gives sufficient conditions for the limit in the strong operator topology of a sequence of weighted shifts to be a weighted shift, since it is not the case in general.

<u>Theorem 7</u>. If (T_n) is a sequence of weighted shifts, with respect to the orthonormal basis $(e_i^n)_i$ and having $(\alpha_i^n)_i$ a sequence of positive numbers, as weight sequence; which converges strongly to T. If (e_i^n) converges strongly to e_i as n increases and (α_i^n) converges to α_i, $\forall i$, as n increases, then (e_i) is an orthonormal basis with respect to which T is a weighted shift having (α_i) as a weight sequence.

<u>Proof</u>: First we show that (e_i) is orthonormal basis, indeed,

$$\|e_i\| \leq \|e_i - e_i^n\| + \|e_i^n\| \leq \varepsilon + 1,$$

for n large enough, and,

$$1 = \|e_i^n\| \leq \|e_i^n - e_i\| + \|e_i\| \leq \varepsilon + \|e_i\|,$$

for n large enough, which implies that

$$1 - \varepsilon \leq \|e_i\| \leq 1 + \varepsilon,$$

for every i, since ε is arbitrary, $\|e_i\| = 1$, $\forall i$.

Now, for $i \neq j$, we have

$$|(e_i,e_j)| \leq |(e_i-e_i^n,e_j)| + |(e_i^n,e_j-e_j^m)| + \\ + |(e_i^n,e_j^m)|,$$

for n,m large enough take n = m and we have

$$|(e_i,e_j)| \leq \|e_i - e_i^n\| + \|e_j - e_j^n\| + |(e_i^n,e_j^n)|$$
$$\leq 2\varepsilon,$$

and thus $(e_i,e_j) = 0$, for $i \neq j$. This shows that (e_i) is an orthonormal sequence;still to show that (e_i) generates $\mathcal{H}$. For that let $f \in \mathcal{H}$, then $\exists (f_n)$ such that (f_n) converges strongly to f and $\forall n$, $\exists (a_k^n)$ such that $f_n = \sum_k a_k^n e_k^n$.

Since $e_k^n \to e_k$ strongly, $\forall k$, we have

$$a_k^n e_k^n \to a_k e_k, \text{ strongly, } \forall k,$$

and thus

$$\sum_k a_k^n e_k^n \to \sum_k a_k e_k, \text{ strongly}$$

i.e.,

$$f_n \to \sum_k a_k e_k, \text{ strongly,}$$

and thus $f = \sum_k a_k e_k$, and therefore (e_i) is orthonormal basis.

Now, since (T_n) converge strongly to T, i.e. $T_n e_i$ converges strongly to Te_i, $\forall i$, we need to show that $T_n e_i$ converges strongly to $\alpha_i e_{i+1}, \forall i$, indeed.

Since (T_n) is bounded, then we have $T_n e_i^m$ converges strongly to $T_n e_i$, $\forall i$, and,

$$\|T_n e_i - \alpha_i e_{i+1}\| \le \|T_n e_i - T_n e_i^m\| + \|T_n e_i^m - T_n e_i^n\| + $$
$$+ \|T_n e_i^n - \alpha_i e_{i+1}\|$$
$$\le \frac{\varepsilon}{4} + \|\alpha_i^n e_{i+1}^n - \alpha_i e_{i+1}\| + \|T_n\| \cdot \|e_i^m - e_i^n\|$$
$$\le \frac{\varepsilon}{2} + \|\alpha_i^n e_{i+1}^n - \alpha_i^n e_{i+1}\| + \|\alpha_i^n e_{i+1} - \alpha_i e_{i+1}\|$$
$$\le \frac{\varepsilon}{2} + \alpha_i^n \|e_{i+1}^n - e_{i+1}\| + |\alpha_i^n - \alpha_i|$$
$$\le \varepsilon .$$

This implies that $T_n e_i$ converges strongly to $\alpha_i e_{i+1}$, $\forall i$, and thus, we conclude that

$$Te_i = \alpha_i e_{i+1}, \ \forall i \ ;$$

i.e., T is a weighted shift with respect to (e_i) and having (α_i) as a weight sequence.

Q.E.D.

CHAPTER III

"Weak closure of the unitary orbit of an arbitrary contraction"

In this Chapter we study the weak closure of the unitary orbit of an arbitrary contraction; we give necessary and sufficient condition for that to be equal to the set of all contractions. This is presented in Section 1. In Section 2 we study the compact perturbation and the weak closure of the unitary orbit of a given contraction. In section 3 we introduce an application to the disc algebra.

III-1. Characterization of the weak closure of the unitary orbit of a contraction.

For T a contraction in $\mathcal{L}(\mathcal{H})$, recall that the numerical range of T is defined by $W(T) = \{\lambda \in \mathbb{C} : \lambda = (Te,e) \text{ for some } e \in \mathcal{H}, \|e\| = 1\}$. In this section we characterize the weak closure of $\{U^*TU : U \in \mathcal{U}\}$ by the following theorem.

Theorem 8. For any contraction T in $\mathcal{L}(\mathcal{H})$, the following statements are equivalent

a) $WC\{U^*TU : U \in \mathcal{U}\}$ = set of all contractions ;

b) $\overline{W(T)} = \overline{\mathbb{D}}$, the closed unit disc ;

c) $\sigma(T) \supset \partial\overline{\mathbb{D}}$, the boundary of $\overline{\mathbb{D}}$;

d) $\sigma_e(T) \supset \partial\overline{\mathbb{D}}$.

We split the proof into propositions and theorems.

Theorem 9. For any contraction $T \in \mathcal{L}(\mathcal{H})$, $WC\{U^*TU : U \in \mathcal{U}\}$ = set of all contractions, if and only if, $\overline{W(T)} = \overline{\mathbb{D}}$.

Proof: Since T is contraction, we have $\overline{W(T)} \subset \overline{\mathbb{D}}$. To show that $\overline{\mathbb{D}} \subset \overline{W(T)}$, we need to show that the assumption is equivalent to saying that any contraction on a finite dimensional subspace of $\mathcal{H}$ is approximable by the compression of T to a subspace of the same dimension. Indeed, the assumption means that given any contraction A then there is a sequence of unitary operators(U_N)such that

$$A = W - \lim U_n^* T U_n,$$

that is, $\forall \varepsilon$, L, $\exists$ N such that n > N, we have

$$|((A - U_n^* T U_n) f_i, g_j)| < \varepsilon, \quad i,j \leq L. \quad (*)$$

Since L is finite arbitrary positive integer, we conclude that if rank A = L, then A is approximable by compression of T to a subspace of dimension L. Conversely, if the last statement holds for any L, then (*) holds and thus

$$A = W - \lim U_n^* T U_n ,$$

and the assumption is true.

Now, consider all contractions on one-dimensional subspaces of $\mathcal{H}$, those contractions can be identified by all complex numbers of $\overline{\mathbb{D}}$, thus for $\lambda \in \overline{\mathbb{D}}$, we have,

$$\|P_h T P_h - \lambda\| < \varepsilon,$$

for some one-dimensional subspace h = span $\{e\}$ with $\|e\| = 1$, where P_h is the projection of $\mathcal{H}$ onto h.

This implies that

$$|(Te,e) - \lambda| \leq \|P_{\mathfrak{h}}TP_{\mathfrak{h}} - \lambda\| < \varepsilon,$$

which implies that $\lambda \in \overline{W(T)}$, and hence $\overline{\mathbb{D}} = \overline{W(T)}$.

Conversely, we assume that $\overline{W(T)} = \overline{\mathbb{D}}$, and we want to show that $WC\{U^*TU : U \in \mathfrak{U}\}$ = the set of all contractions. Since T is a contraction, U^*TU is a contraction for every $U \in \mathfrak{U}$; thus $\{U^*TU : U \in \mathfrak{U}\} \subset$ the set of all contractions; hence $WC\{U^*TU : U \in \mathfrak{U}\} \subset$ set of all contractions.

To prove the other inclusion given any contraction A on a finite dimensional subspace, we have to show that A is approximable by the compression of T to a subspace of the same dimension. To do so we need two steps of approximation. For the first step there is a unitary operator U' that approximates A weakly. To be more precise let A be a contraction on a subspace $\mathfrak{h}$ of $\mathfrak{H}$ of finite dimension, since $\dim \mathfrak{H} \ominus \mathfrak{h} = \infty$, we can find a subspace $\mathfrak{m}$ of $\mathfrak{H} \ominus \mathfrak{h}$, of the same dimensions of $\mathfrak{h}$, and a unitary u that maps $\mathfrak{h}$ onto $\mathfrak{m}$. By Halmos Theorem [6] there is a unitary dilation of A to $\mathfrak{h} \oplus \mathfrak{h}$ given by

$$u' = \begin{pmatrix} A & \sqrt{1-AA^*} \\ \sqrt{1-A^*A} & -A^* \end{pmatrix},$$

Since $\mathfrak{h}$ is isomorphic to $\mathfrak{m}$, we can define U on $\mathfrak{H}$ by

$$U = \begin{pmatrix} A & \sqrt{1-AA^*}\,u & 0 \\ u^*\sqrt{1-A^*A} & -uA^*u^* & 0 \\ 0 & 0 & I \end{pmatrix} = U' \oplus I$$

and we have $(Uf,g) = (Af,g), \cdots (1)$ for every $f, g \in \mathfrak{h}$.

In fact U' approximate A weakly and since U' is unitary of finite dimension, it is diagonalizable, so let U' = D, where D is diagonal operator on $\mathfrak{n} \oplus \mathfrak{m}$.

The second step is to approximate any such diagonal operator D on a finite dimensional subspace by the compression of T to a subspace of the same dimension. So let $D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, on $\mathfrak{n} \oplus \mathfrak{m}$ such that $|\lambda_u| = 1$, (because $\sigma(D) = \sigma(U')$), $i = 1,2,\cdots,n$; in fact, we have $\lambda_i = (D\varphi_i, \varphi_i)$, $i = 1,2,\cdots,n$, for some orthonormal basis (φ_i), $i=1,2,\cdots,n$ of $\mathfrak{n} \oplus \mathfrak{m}$. Since $\lambda_i \in \overline{\mathbb{D}} = \overline{W(T)}$, $i = 1,2,\cdots,n$, we have

$$\forall \varepsilon, \exists\ e_i \in \mathcal{H} \text{ with } \|e_i\| = 1 \text{ such that}$$

$$|(Te_i, e_i) - \lambda_i| < \varepsilon,\ i = 1,2,\cdots,n,$$

i.e.,

$|(\overline{\lambda}_i Te_i, e_i) - 1| < \varepsilon$, $i \le n$, and by Lemma 5, we have

$$\overline{\lambda}_i Te_i = \eta_i e_i + h_i',\ i \le n,$$

where η_i is positive number, $|1-\eta_i| < \varepsilon$, $\|h_i'\|^2 < 2\varepsilon$, and thus

$$Te_i = \lambda_i \eta_i e_i + \lambda_i h_i',\ i \le n.$$

Let $\gamma_i = \lambda_i \eta_i$, $h_i = \lambda_i h_i'$, we have

$$Te_i = \gamma_i e_i + h_i,\ i \le n,$$

where $\gamma_i \in \mathbb{C}$, $|\gamma_i - \lambda_i| < \varepsilon$ and $\|h_i\|^2 < 2\varepsilon$, $i \le n$, i.e.,

$$\|Te_i - \gamma_i e_i\| < \varepsilon,\ i \le n, \quad (*)$$

This means that γ_i is an approximate eigenvalue.

If γ_i, $i \leq n$, are eigenvalues then the corresponding eigenvectors e_i, $i \leq n$, are orthogonal; indeed, it is enough to show for $n = 2$. Since T is a contraction then T has a unitary dilation B, on a Hilbert space $\mathcal{K} \supset \mathcal{H}$; B is given by

$$B = \begin{pmatrix} T & \sqrt{1-TT^*} \\ \sqrt{1-T^*T} & -T^* \end{pmatrix} ,$$

(for this see Halmos [7]).

Since $|\gamma_i| = |\lambda_i| = 1$, we have

$$(1-T^*T)e_i = e_i - \gamma_i\overline{\gamma}_i e_i = 0, \; i \leq 2,$$

and thus $(\sqrt{1-T^*T})e_i = 0$ (since $\ker P = \ker P^2$, for any positive operator P), (See Halmos [7]). This implies that $Be_i = \gamma_i e_i$, $i \leq 2$, i.e. (e_i), $i \leq 2$, are eigenvectors for the unitary operator B, so that we have

$$(e_1,e_2) = (Be_1,Be_2) = \gamma_1\overline{\gamma}_2(f_1,f_2),$$

i.e.,

$$(1-\gamma_1\overline{\gamma}_2)(e_1,e_2) = 0.$$

Since $\gamma_1 \neq \gamma_2$, $|\gamma_i| = 1$, $i \leq 2$, we have $\gamma_1\overline{\gamma}_2 \neq 1$ and $\gamma_2\overline{\gamma}_1 \neq 1$ and thus

$$(e_1,e_2) = 0,$$

i.e., (e_i) are orthogonal.

If γ_i, $i \leq n$, are not eigenvalues but approximate eigenvalues for T then there is a sequence of vectors (f_k)

such that

$$\|(T-\gamma_i)f_k\| \to 0, \text{ as } k \to \infty .$$

This implies that for k > N, we have

$$\|(T-\gamma_i)f_k\| < \varepsilon\|f_k\| \qquad (**) \quad .$$

Claim, there is a subspace η_i of vectors f, for which (**) is true, of infinite dimension. Indeed, let $T - \gamma_i = V_i P_i$ be the polar decomposition. Since γ_i is not an eigenvalue but an approximate eigenvalue, we have $\ker(T-\gamma_i) = \{0\}$ and thus $\ker P_i = \{0\}$. The spectral representation of P_i is given by

$$P_i = \int_0^{\|P_i\|} t dE_i(\Delta),$$

where for $\Delta = (0,\varepsilon)$, $\varepsilon > 0$, we have $E_i(\Delta) = E_i((0,\varepsilon))$ is an infinite dimensional subspace of $\mathcal{H}$. Since for every $f \in E_i((0,\varepsilon))$, we have

$$\|(T-\gamma_i)f\| = \|V_i P_i f\| \le \|P_i f\| \le \varepsilon\|f\| \ ,$$

we take $\eta_i = E_i((0,\varepsilon))$.

It is easy now, to choose the approximate eigenvectors $e_1,\cdots,e_n$ corresponding to the approximate eigenvalues $\gamma_1,\cdots,\gamma_n$ to be orthonormal. Indeed, it is enough to show that for n = 2, let e_1,e_2 be the approximate eigenvectors corresponding to γ_1,γ_2. If $(e_1,e_2) \neq 0$, choose e_2' from η_2, if $(e_1,e_2') \neq 0$, then consider e_2'' defined by

$$e_2'' = \frac{e_2}{(e_1,e_2)} - \frac{e_2'}{(e_1,e_2')} \quad .$$

Now, we have $(e_1, e_2'') = 0$ and since η_2 is a subspace we have $e_2'' \in \eta_2$, i.e.,

$$\|Te_2'' - \gamma_2 e_2''\| < \varepsilon,$$

i.e. e_2'' satisfies (*).

Thus, we can assume $e_1, e_2, \cdots, e_n$ to be orthonormal. Let $\mathfrak{L}$ be the subspace of $\mathfrak{H}$ generated by $e_1, \cdots, e_n$ and define the unitary operator

$$u : \mathfrak{L} \rightarrow \mathfrak{n} \oplus \mathfrak{m}$$

by $u(e_i) = \varphi_i$, $i \leq n$. Define U on $\mathfrak{H}$ to be $U = u$ on $\mathfrak{L}$ and U maps $\mathfrak{H} \ominus \mathfrak{L}$ onto $\mathfrak{H} \ominus (\mathfrak{n} \oplus \mathfrak{m})$ isometrically, so that U is unitary on $\mathfrak{H}$ and we have

$$|(Te_i, e_i) - (UDU^* e_i, e_i)| < \varepsilon, \ i \leq n. \cdots \qquad (2)$$

Now, for $i \neq j$, we have

$$|((T-UDU^*)e_i, e_j)| = |(Te_i, e_j) - (D\varphi_i, \varphi_j)|$$

$$= |(Te_i, e_j)| = |(\gamma_i e_i, e_j) - (h_i, e_j)|$$

$$= |(h_i, e_j)| \leq \|h_i\| < \delta(\varepsilon), \cdots \qquad (3)$$

Combining (2) and (3), we obtain, $\forall \varepsilon$, $\exists$ U such that

$$|((T-UDU^*)e_i, e_j)| < \delta(\varepsilon), \ i, j \leq n,$$

if $P_{\mathfrak{L}}$ is the projection of $\mathfrak{H}$ onto $\mathfrak{L}$, then $\forall \varepsilon > 0$, $\exists$ U such that

$$|((P_{\mathfrak{L}} T P_{\mathfrak{L}} - UDU^*)e_i, e_j)| < \delta(\varepsilon), \ i, j \leq n, \cdots (4)$$

i.e. D is approximable by the compression of T to $\mathfrak{L}$.

To end the proof of the Theorem combine the three approximations (1), (2) and (4), to obtain that for any contraction A on $\mathbb{C}^n$, to every $\varepsilon > 0$ there is U unitary operator such that

$$|((P_n T P_n - UAU^*)e_i, e_j)| < \varepsilon,$$

for $i,j \leq n$, i.e. $A \varepsilon\, WC\{U^*TU: U \in \mathcal{U}\}$, i.e., set of all contractions $\subset WC\{U^*TU : U \in \mathcal{U}\}$, which implies that

$$WC\{U^*TU : U \in \mathcal{U}\} = \text{set of all contractions.}$$

Q.E.D.

<u>Proposition 7</u>. If T is a contraction in $\mathcal{L}(\mathcal{H})$, then, $\overline{W(T)} = \overline{\mathbb{D}}$ if, and only if, $\sigma(T) \supset \partial\overline{\mathbb{D}}$.

<u>Proof:</u> Since T is a contraction we have

$$\overline{W(T)} \subset \overline{\mathbb{D}}. \quad \text{If we assume that } \sigma(T) \supset \partial\overline{\mathbb{D}},$$

then, we have

$$\partial\overline{\mathbb{D}} \subset \sigma(T) \subset \overline{W(T)} \subset \overline{\mathbb{D}},$$

since $\overline{W(T)}$ is convex and it contains the boundary of $\overline{\mathbb{D}}$, it must contain the interior of $\overline{\mathbb{D}}$, i.e., $\mathbb{D} \subset \overline{W(T)}$ and therefore $\overline{W(T)} = \overline{\mathbb{D}}$.

Conversely, assuming that $\overline{W(T)} = \overline{\mathbb{D}}$, let $\lambda \in \partial\overline{\mathbb{D}}$, and suppose that λ is not in $\sigma(T)$, so, we have $T - \lambda$ is bounded below, i.e., there is a real number $\delta > 0$ such that

$$\|(T-\lambda)f\| > \delta\|f\|,$$

for any f in $\mathcal{H}$.

Since $\lambda \in \overline{W(T)}$, we can say that, for every $\varepsilon > 0$ there is a unit vector e such that

$$|\lambda - (Te,e)| < \varepsilon,$$

which implies that $|1 - (\overline{\lambda}Te,e)| < \varepsilon$, so that by Lemma 5 there is a positive number γ', and a vector h' with $\|h'\|^2 < 2\varepsilon$ such that

$$\overline{\lambda}Te = \gamma' e - h' ,$$

or, $Te = \lambda\gamma' e - \lambda h'$, if we let $\gamma = \gamma'\lambda$, $h = \lambda h'$ then, we have

$$Te = \gamma e - h,$$

$$\|h\|^2 \leq 2\varepsilon, \; 1 - \varepsilon < |\gamma| \leq 1 \text{ and}$$

$$|\gamma - \lambda| < \varepsilon \qquad (*)$$

Now, for the unit vector e, we have

$$\delta^2 < \|(T-\lambda)e\|^2 = ((T-\lambda)e,(T-\lambda)e)$$

$$< (Te,Te) - \lambda(e,Te) - \overline{\lambda}(Te,e) + |\lambda|^2(e,e)$$

$$< (\gamma e-h,\gamma e-h) - \lambda(e,\gamma e-h) - \overline{\lambda}(\gamma e-h,e) + |\lambda|^2$$

$$< \gamma^2 - \lambda\overline{\gamma} - \overline{\lambda}\gamma + |\lambda|^2 + \|h\|^2$$

$$< |\gamma - \lambda|^2 + 2\varepsilon ,$$

i.e.,

$$\delta < |\gamma - \lambda| + 2\varepsilon . \qquad (**)$$

For ε small enough, (**) contradicts (*) so that

$\lambda \in \sigma(T)$, i.e.,

$$\partial \mathbb{D} \subset \sigma(T).$$

Q.E.D.

Proposition 8. For any contraction $T \in \mathcal{L}(\mathcal{H})$, $\sigma(T) \supset \partial \mathbb{D}$ if, and only if, $\sigma_e(T) \supset \partial \mathbb{D}$.

Proof: One direction is clear since $\sigma_e(T) \subset \sigma(T)$ for any $T \in \mathcal{L}(\mathcal{H})$. For the other direction, let $\lambda \in \partial \mathbb{D} \subset \sigma(T)$, then $T - \lambda$ is not invertible. If we assume that λ is not in $\sigma_e(T)$, then $\pi(T) - \lambda$ is invertible in the Calkin algebra, i.e., $T - \lambda$ is Fredholm. Now applying the theorem of Gohberg (see Douglas [4]), to conclude that

$$\dim \ker(T-\lambda) = \alpha, \text{ a constant,}$$

on a punctured disc of radius ε and center λ; except for isolated points on which it might be bigger. Since $\lambda \in \partial \mathbb{D}$, λ is not an isolated point, but since $T - \lambda$ is not invertible $\dim.\ker.(T-\lambda) \neq 0$, and for any $|\gamma| > 1$ such that $|\lambda - \gamma| < \varepsilon$, we have

$$\dim.\ker.(T-\lambda-\gamma) = 0,$$

while for any $|\gamma| = 1$ with $\lambda \neq \gamma$, we have

$$\dim.\ker.(T-\lambda-\gamma) \neq 0.$$

So that it is not true that $\dim.\ker.(T-\lambda)$ is locally constant and thus assumption is false.

Therefore $\lambda \in \sigma_e(T)$, and hence $\partial \mathbb{D} \subset \sigma_e(T)$.

Q.E.D.

Proof of Theorem 8: Theorem 9, Proposition 7 and Proposition 8 prove the theorem.

Q.E.D.

III-2. Compact perturbation and the unitary orbit of a contraction.

In this section we prove the following corollaries.

Corollary 3. If T is a contraction that satisfies condition a) of Theorem 8, and if K is a compact operator such that $\|T+K\| \le 1$, then T + K satisfies that condition also.

Proof: From the assumption we have

$$\partial\overline{\mathbb{D}} \subset \sigma_e(T),$$

since, $\sigma_e(T) = \sigma_e(T+K)$, for any compact operator K, we have,

$$\sigma_e(T+K) \supset \partial\overline{\mathbb{D}},$$

which is equivalent to condition a), since T + K is a contraction.

Q.E.D.

Corollary 4. If T_α is a weighted shift whose sequence of weights is (α_i), such that $0 < \alpha_i \le 1$, $\forall i \in \mathbb{Z}$, then

$$\sigma(T) \supset \partial\overline{\mathbb{D}} \text{ if, and only if, } \forall N,\ \varepsilon > 0,$$
$$\exists\ M \text{ such that } \alpha_{M+i} > 1 - \varepsilon, \quad i = 0,1,2,\cdots,N.$$

Proof: Theorem 2 and Theorem 8 imply the corollary.

Q.E.D.

<u>Corollary 5</u>. If T is a contraction, S is a shift, then,

$$WC\{U^*(T\oplus S)U : U \in \mathcal{U}\} = \text{set of all contractions.}$$

<u>Proof</u>: Since T is a contraction, $T \oplus S$ is a contraction. Since $\sigma(A\oplus B) = \sigma(A)\cup\sigma(B)$ we have

$$\sigma(T\oplus S) \supset \sigma(S) = \overline{\mathbb{D}},$$

and hence,

$$\sigma(T\oplus S) \supset \partial\overline{\mathbb{D}} .$$

So that by Theorem 8, this is equivalent to saying that,

$$WC\{U^*(T\oplus S)U : U \in \mathcal{U}\} = \text{set of all contractions.}$$

Q.E.D.

<u>Corollary 6</u>. If T is a contraction satisfies a) of Theorem 8, then $T \oplus T$ satisfies a) also.

<u>Proof</u>: If $S = W - \lim U_N^* T U_N$, where S is a shift, then

$$S \oplus T = w - \lim W_N^*(T\oplus T)W_n, \text{ where}$$

$W_N = U_N \oplus I$ and thus by corollary 5, $T \oplus T$ satisfies a).

Q.E.D.

<u>Remark 4</u>. Corollary 6 extends to any finite direct sum as well.

We end this section with the following proposition.

<u>Proposition 9</u>. If a sequence (T_n) of operators converges weakly to an operator T, and if Δ is a compact subset of $\mathbb{C}$

such that $\overline{W(T_n)} \subset \Delta$, for every n, then $\overline{W(T)} \subset \Delta$.

Proof: If $\lambda \in \overline{W(T)}$ then for every $\varepsilon > 0$ there is a unit vector e such that

$$|((T-\lambda)e,e)| < \varepsilon ,$$

which implies that

$$|((T_n-\lambda)e,e)| \leq |((T_n-T)e,e)| + |((T-\lambda)e,e)|$$
$$\leq 2\varepsilon .$$

Thus, we have

$$d(\lambda,\overline{W(T_n)}) \leq 2\varepsilon$$

for every $n > N$, where $d(\cdot,\cdot)$ is the distance function, so that

$$d(\lambda,\Delta) \leq 2\varepsilon,$$

since Δ is compact, $\lambda \in \Delta$ or $\overline{W(T)} \subset \Delta$.

Q.E.D.

III-3. An application to the disc algebra.

Recall that the disc algebra $\mathcal{A}$ is the norm closure of the set of all polynomials defined on the boundary $\partial\overline{\mathbb{D}}$ of the unit disc. For a given contraction $T \in \mathcal{L}(\mathcal{H})$, define the map $\psi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, by $\psi(f) = f(T)$ for every $f \in \mathcal{A}$. Von-Neumann [12] proved that ψ is a contractive homomorphism. In this section we prove the following results.

Proposition 10. If T is a contraction such that $WC\{U^*TU : U \in \mathcal{U}\} =$ = set of all contractions, then ψ is an isometry.

Proof: By von-Neumann theorem we need only to show that $\|f(T)\| \geq \|f\|_\infty \equiv \sup_{z\in\overline{\mathbb{D}}} |f(z)|$ for any $f \in \mathcal{A}$. Let $f \in \mathcal{A}$, since f is a norm limit of polynomials we apply the spectral mapping theorem to obtain that

$$\sigma(f(T)) = f(\sigma(T)),$$

by the assumption, we have $\sigma(T) \supset \partial\overline{\mathbb{D}}$, thus

$$\sigma(f(T)) \supset f(\partial\overline{\mathbb{D}}) \ ..$$

This implies that

$$\begin{aligned}\|f(T)\| &\geq |\sigma(f(T))|, \text{ the spectral radius of } f(T)\\ &\geq \sup\{|\lambda| : \lambda \in \sigma(f(T))\}\\ &\geq \sup\{|\lambda| : \lambda \in f(\partial\overline{\mathbb{D}})\}\\ &\geq \sup_{z\in\partial\overline{\mathbb{D}}} |f(z)| \equiv \|f\|_\infty .\end{aligned}$$

Q.E.D.

It is reasonable that one ask whether or not the converse of proposition 10 holds. We could not prove the converse but we prove the following proposition.

<u>Proposition 11</u>. If T, ψ are as above, and if $|\overline{W(f(T))}| = \|f\|_\infty$, for every $f \in \mathcal{G}$, then $WC\{U^*TU : U \in \mathfrak{U}\}$ = set of all contractions.

<u>Proof</u>: Let $\lambda \in \partial\mathbb{D}$, $\lambda \notin \sigma(T)$. There is no loss of generality if we take $\lambda = 1$. Let $f \in \mathcal{G}$, defined by $f(z) = \frac{1}{2}z + \frac{1}{2}$, then $f(1) = 1$ and f maps $\overline{\mathbb{D}}$ into a proper subset of itself. Since $\sigma(f(T)) = f(\sigma(T)) \subset f(\overline{\mathbb{D}})$, $1 = f(1) \notin \sigma(f(T))$, so that we can find a disc of radius $r < 1$ such that $\sigma(f(T)) \subset D_r$, the disc of radius r and thus

$$|\sigma(f(T))| \le r < 1,$$

since $\|f\|_\infty = 1$, we have

$$|\sigma(f(T))| < \|f\|_\infty \qquad (*)$$

Now, we have that, since $|\sigma(f(T))| \le |\overline{W(f(T))}| \le \|f(T)\|$, and by Williams Theorem [18] which asserts that $|\sigma(T-\lambda)| = \|T-\lambda\|$ if, and only if, $|\overline{W(T-\lambda)}| = \|T-\lambda\|$, for every λ, we have

$$|\sigma(f(T))| = \|f\|_\infty . \qquad (**)$$

But $(**)$ contradicts $(*)$ and thus $\lambda \in \sigma(T)$, or $\partial\overline{\mathbb{D}} \subset \sigma(T)$ which is equivalent to saying

$$WC\{U^*TU : U \in \mathfrak{U}\} = \text{set of all contractions.}$$

Q.E.D.

CHAPTER IV

"Operators having a convex spectral set"

In this chapter we study weak closure of the unitary orbit of an arbitrary operator not necessarily a contraction. In section 1 we introduce some properties of convex spectral sets; many of these results are well known. In section 2 we study the relations between the weak closure of the unitary orbit of an operator having a convex spectral set and the set of all operators having that convex set as a spectral set. We obtain results similar to the case of a contraction investigated in the previous chapter.

IV-1. Properties of convex spectral set.

In this section we present some results about convex spectral sets; most of these results are well known but we provide many proofs different from the original ones. We start with an important version of von Neumann theorem about spectral sets.

Proposition 12. A closed half plane P is a spectral set of an operator T if, and only if, P contains the closure of the numerical range $\overline{W(T)}$ of T.

For the proof see Williams [18].

Corollary 7. If X is a compact convex spectral set of an operator $T \in \mathcal{L}(\mathcal{H})$, then $\overline{W(T)} \subset X$.

Proof: Let $P = \{z \in \mathbb{C} : \text{Real } z \geq 0\}$, be the right half plane and define the map

$$\varphi : P \to X,$$

to be holomorphic and homeomorphism, then by von Neumann theorem [12] P is a spectral set of an operator A if, and only if, X is a spectral set of $\varphi(A)$. Let $A = \varphi^{-1}(T)$, then by proposition 12, P contains $\overline{W(A)}$. Now, using Kato Theorem [9] (which asserts that, if f is holomorphic on P, $W(A) \subset P$, then $W(f(A))$ is a subset of the closed convex hull of $f(P)$), we have

$$\overline{W(\varphi(A))} \subset ch(\varphi(P)),$$

i.e.,

$$\overline{W(T)} \subset X,$$

since X is compact convex.

Q.E.D.

Remark 5. A direct proof of Corollary 7 goes as follows. Since $X = \bigcap_\alpha H_\alpha$, where H_α is a closed half plane which contains $\overline{W(T)}$ by proposition 12, for every α, and thus $\overline{W(T)} \subset \bigcap_\alpha H_\alpha = X$.

For the next corollary, we need the following definition.

Definition 2. The generalized Minkowski distance functionals $w_r(\cdot)$, $0 \le r < \infty$ on $\mathcal{L}(\mathcal{H})$ is defined to be

$w_r(T) = \inf\{\lambda : \lambda > 0$ and $\lambda^{-1}T$ has a unitary r-dilation$\}$. The dilation here is in the sense of Sz-Nagy and Foias [11].

Definition 3. The generalized numerical range $W_r(\cdot)$ is defined to be

$$W_r(T) = \bigcap_\lambda \{z \in \mathbb{C} : |z-\lambda| \le w_t(T-\lambda)\} .$$

It is shown in Patel [13] that $W_r(T)$ is compact convex subset of the plane and for $1 \le r \le 2$, $W_r(T) = \overline{W(T)}$ and for $r \ge 1$ $w_r(T-\lambda) \le \|T-\lambda\|$ for every $\lambda \in \mathbb{C}$. For more about the properties of $w_r(\cdot)$ and $W_r(\cdot)$, see Patel [13].

Corollary 8. For any operator T, we have

$$\overline{W(T)} = \text{intersection of all convex spectral sets of T.}$$
$$= \bigcap_\lambda \{z \in \mathbb{C} : |z-\lambda| \le \|T-\lambda\|\}.$$

Proof: By corollary 7, $\overline{W(T)}$ is included in the intersection of all convex spectral sets of T. Since $\overline{W(T)}$ is convex, we have $\overline{W(T)} = \bigcap_\alpha H_\alpha$, (H_α) are closed half planes, so that H_α contains $\overline{W(T)}$ for every α and thus by proposotion 12, H_α is a spectral set of T, for every α. Now, the intersection of all convex spectral sets of T is included in $\bigcap_\alpha H_\alpha = \overline{W(T)}$. This proves the first part of the corollary.

For the second part, we have, from Definitions 1, 2 and the comments which follow it, that, for $1 \le r \le 2$,

$$\overline{W(T)} = W_r(T)$$

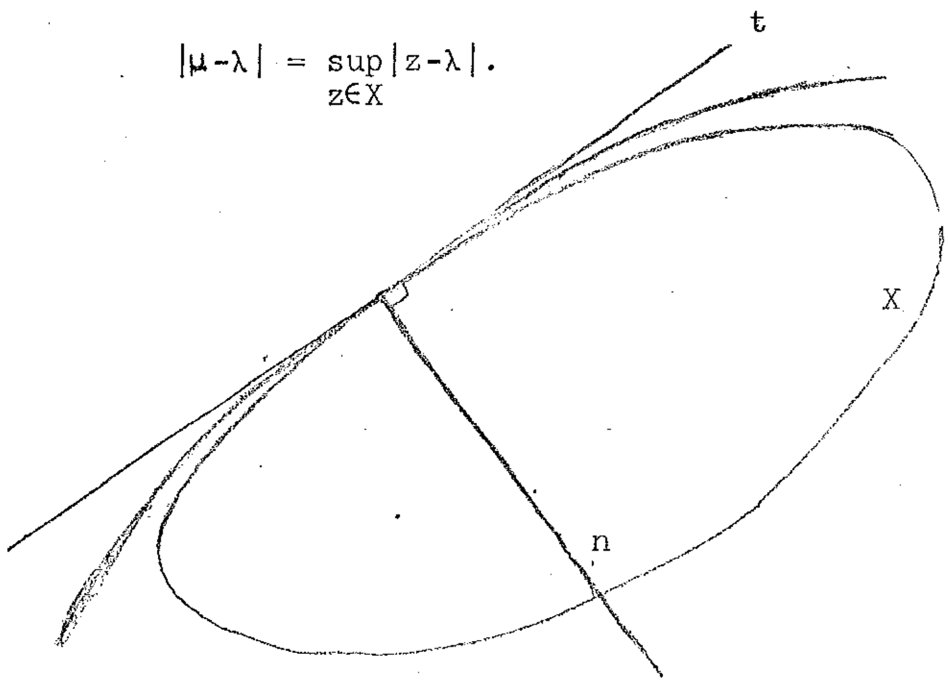
$$= \bigcap_{\lambda} \{z \in \mathbb{C} : |z-\lambda| \le w_r(T-\lambda)\}$$

$$= \bigcap_{\lambda} \{z \in \mathbb{C} : |z-\lambda| \le \|T-\lambda\|\} .$$

Q.E.D.

<u>Proposition 13</u>. Let X be a compact convex subset of $\mathbb{C}$ such that the boundary ∂X, of X belongs to the space C^2, of all functions having continuous second derivative, and contains no straight line segments. Let $\mu \in \partial X$, be an extreme point such that the curvature of ∂X at μ is not zero;then there is λ in $\mathbb{C}$ such that

$$|\mu-\lambda| = \sup_{z \in X} |z-\lambda| .$$



Proof: Given an extreme point $\mu \in \partial X$, since X is convex, the tangent t of ∂X at the point μ, will be in the complement of X. Draw the tangent t and draw the normal line n to ∂X at μ. It is known that every point on the line n is a center of a circle tangent to ∂X at μ, and the difference between these circles is the value of the curvature. Let r be the radius of curvature of ∂X at μ, which is given by

$$r = \frac{[1 + (f'(s)^2]^{3/2}}{f''(s)} ,$$

where f(s) represents ∂X locally as a C^2-function.

Choose λ on the normal line n such that $|\mu-\lambda|$ is sufficiently greater than r, the radius of curvature of ∂X at μ, since ∂X is a C^2-function, the second derivative exists everywhere, and different from zero almost everywhere since ∂X does not contain straight line segments. Thus r is defined almost everywhere and thus λ exists such that the circle determined by λ contains X and hence

$$|\mu-\lambda| \geq |z-\lambda|,$$

for every z in the circle whose center is λ.

In particular, we have

$$|\mu-\lambda| > |z-\lambda|,$$

for every $z \in X$, and therefore

$$|\mu-\lambda| \geq \sup_{z \in X}|z-\lambda| .$$

But since $\mu \in X$, we have

$$|\mu-\lambda| \leq \sup_{z \in X}|z-\lambda|,$$

and thus,

$$|\mu-\lambda| = \sup_{z\in X}|z-\lambda|.$$

Still one point to investigate, namely if there is an open set E in ∂X such that, if $y \in E$, then the second derivative at $s = f^{-1}(y)$ is zero, then E must be a straight line.

Indeed: Consider $f : I \to E$, such that $f(I) = E$, the open subset with that property, where I is some interval. Consider $s_1, s_2 \in I$, then we have, $f''(s) \equiv 0 \ \ \forall s \in [s_1,s_2]$, so that by the Mean Value Theorem, we conclude (since $f'(s)$ is continuous on $[s_1,s_2]$) that

$$f'(s_1) = f'(s_2),$$

since s_1,s_2 are arbitrarily chosen in I, we have $f'(s) \equiv$ a constant on I, and thus by integrating $f'(s)$ over I we conclude that $E=f(I)$ is a straight line segment. This cannot occur by the assumption.

Q.E.D.

Remark 6. If $f''(s) = 0$ for some $z = f(s) \in \partial X$, then we can approximate by some other point $z' = f(s') : f''(s') \neq 0$.

IV-2. Weak closure of the unitary orbit of an operator having a convex spectral set.

For X a compact convex subset of the plane, recall that

$$O(X) = \{A \in \mathcal{L}(\mathcal{H}) : X \text{ is a spectral set of } A\}.$$

For a given $T \in O(X)$, the unitary orbit of T is a subset of O(X), since for any unitary operator U, U*TU has X as a spectral set, thus

$$\{U^*TU : U \in \mathcal{U}\} \subset O(X).$$

In this section we investigate the relations between O(X) and weak closure of the unitary orbit of a given operator $T \in O(X)$. In case X is the closed unit disc $\overline{\mathbb{D}}$ we know that the set of all contractions $O(\overline{\mathbb{D}})$ is weakly closed, but in case X is not $\overline{\mathbb{D}}$ we could not prove that O(X) is weakly closed, but we conjecture that O(X) is weakly closed. We prove the following theorem.

Theorem 10. If X is a compact convex subset of $\mathbb{C}$ such that its boundary ∂X belongs to the space C^2, and ∂X contains no straight line segments, then for any $T \in O(X)$ the following are equivalent

a) $\overline{W(T)} = X$,

b) $\sigma(T) \supset \partial X$,

c) $\sigma_e(T) \supset \partial X$,

d) $\|R_\lambda(T)\| = 1/d(\lambda,X)$, $\forall\lambda \notin X$, where $R_\lambda(T) = (T-\lambda)^{-1}$ and $d(\lambda,X)$ is the distance from λ to the set X.

We devide the proof into some lemmas and propositions, we start with the following

Lemma 6. If $\mu \in \overline{W(T)}$ such that $|\mu| = \|T\|$, then $\mu \in \sigma(T)$.

Proof: If $\mu \in \overline{W(T)}$ then there is a sequence (e_i) of unit vectors such that

$$(Te_i,e_i) \xrightarrow[i\to\infty]{} \mu,$$

i.e.,

$$|((T-\mu)e_i,e_i)| < \varepsilon, \quad \forall i > N,$$

for some positive integer N.

This implies that,

$$\begin{aligned}\|(T-\mu)e_i\|^2 &= \|Te_i\|^2 + |\mu|^2 - \bar{\mu}(Te_i,e_i) - \mu(e_i,Te_i)\\ &\le \|T\|^2 + |\mu|^2 - 2\bar{\mu}\ \mathrm{Real}(Te_i,e_i)\\ &\le 2|\mu|^2 - 2\bar{\mu}\ \mathrm{Real}(Te_i,e_i)\\ &\le 2|\bar{\mu}|\,|(\mu-(Te_i,e_i))|\\ &\le 2|\bar{\mu}|\cdot\varepsilon\ ,\end{aligned}$$

which implies that μ is an approximate eigenvalue and thus $\mu \in \sigma(T)$.

Q.E.D.

Proposition 14. Under the assumption of Theorem 10, we have

$$\overline{W(T)} = X \text{ if, and only if, } \sigma(T) \supset \partial X.$$

Proof: By proposition 13, we have, for any extreme point μ of ∂X, such that the curvature of ∂X at μ is not zero, there is $\lambda \in \mathbb{C}$ such that

$$|\mu-\lambda| = \sup_{z\in X}|z-\lambda| ,$$

since $\overline{W(T)} = X$, we have, for any extreme point μ of $\partial\overline{W(T)}$, there is $\lambda \in \mathbb{C}$ such that

$$|\mu-\lambda| = \sup_{z\in\overline{W(T)}} |z-\lambda| = |\overline{W(T-\lambda)}| ,$$

the numerical radius of $T - \lambda$.

Williams [18] proved that if $\overline{W(T)}$ is a spectral set of T, then

$$|\overline{W(T-\lambda)}| = \|T-\lambda\| ,$$

for every $\lambda \in \mathbb{C}$, this implies that, since $\overline{W(T)}$ is a spectral set of T,

$$|\mu-\lambda| = \|T-\lambda\|,$$

and thus by Lemma 6, $\mu - \lambda \in \sigma(T-\lambda)$, which is equivalent to saying that $\mu \in \sigma(T)$. Since ∂X is smooth and contains no straight line segments, the set of all extreme points μ of ∂X, with curvature of ∂X at μ is different from zero, is dense in ∂X and since $\sigma(T)$ is compact we have $\sigma(T) \supset \partial X$. This proves one direction.

For the other direction, we have

$$\partial X \subset \sigma(T) \subset \overline{W(T)},$$

and by convexity of $\overline{W(T)}$, we have $X \subset \overline{W(T)}$. On the other hand since X is convex spectral set of T, we have, by Corollary 7, $\overline{W(T)} \subset X$ i.e., $\overline{W(T)} = X$.

Q.E.D.

<u>Proposition 15</u>. If X is a simply connected subset of $\mathbb{C}$, and is a spectral set of T, then

$$\sigma(T) \supset \partial X \text{ if, and only if, } \sigma_e(T) \supset \partial X .$$

<u>Proof</u>: Same proof as Proposition 8, with $\overline{\mathbb{D}}$, replaced by X.

Q.E.D.

<u>Proposition 16</u>. Under the assumption of Theorem 10, we have,

$$\overline{W(T)} = X \text{ if, and only if, } \|R_\lambda(T)\| = 1/d(\lambda,X),$$

for every $\lambda \notin X$.

<u>Proof</u>: By Proposition 14, we have

$$\overline{W(T)} = X \text{ if, and only if, } \sigma(T) \supset \partial X ,$$

i.e. $\sigma(T) \supset \partial\overline{W(T)}$, and Patel [13] proved that this is equivalent to saying

$$\|R_\lambda(T)\| = 1/d(\lambda,\overline{W(T)}),$$

for every $\lambda \notin \overline{W(T)}$. Since $\overline{W(T)} = X$, we have

$d(\lambda,\overline{W(T)}) = d(\lambda,X)$, $\forall\lambda \notin X$, and thus, we have

$$\|R_\lambda(T)\| = 1/d(\lambda,X), \quad \forall\lambda \notin X.$$

Conversely, let $\|R_\lambda(T)\| = 1/d(\lambda,X)$, $\forall\lambda \notin X$. Since X is a convex spectral set for T, we have, by Crollary 7, $\overline{W(T)} \subset X$, so that for $\lambda \notin \overline{W(T)}$ we have,

$$d(\lambda,X) \leq d(\lambda,\overline{W(T)}).$$

This implies that,

$$1/d(\lambda,\overline{W(T)}) \leq 1/d(\lambda,X) = \|R_\lambda(T)\| .$$

But, for any T, we have

$$\|R_\lambda(T)\| \leq 1/d(\lambda,\overline{W(T)}), \text{ for all } \lambda \notin \overline{W(T)},$$

so that,

$$\|R_\lambda(T)\| = 1/d(\lambda,\overline{W(T)}), \quad \forall\lambda \notin \overline{W(T)} ,$$

which is equivalent to (by Patel's Theorem),

$$\sigma(T) \supset \overline{W(T)} ,$$

and since we have proved now that

$$1/d(\lambda,\overline{W(T)}) = \|R_\lambda(T)\| = 1/d(\lambda,X) ,$$

for every $\lambda \notin \overline{W(T)}$, $\lambda \notin X$, we conclude that $\partial X = \partial\overline{W(T)} \subset \overline{W(T)}$, and thus by convexity of $\overline{W(T)}$, $X \subset \overline{W(T)}$, i.e. $X = \overline{W(T)}$.

Q.E.D.

Proof of Theorem 10: Proposition 14, 15, and 16 prove the theorem.

Q.E.D.

Now we are going to investigate the relation between conditions of Theorem 8, and those of Theorem 10. It turns out to be equivalent, using the Riemann mapping lemma.

Lemma 7. If X is a simply connected subset of $\mathbb{C}$, then there exists an analytic homeomorphism $\gamma : \overline{\mathbb{D}} \to X$, from the closed unit disc onto X, such that $\gamma(\partial\overline{\mathbb{D}}) = \partial X$.

For the proof see (for example) Alfors [1].

Lemma 8. If $\varphi : X \to Y$ is an analytic homeomorphism, if X is a spectral set for T and $\gamma = \varphi^{-1}$, then $\sigma(\varphi(T)) = \varphi(\sigma(T))$, moreover, if Y is a spectral set for $B = \varphi(T)$, then

$$\sigma(\gamma(B)) = \gamma(\sigma(B)).$$

Proof: Since X is a spectral set for T, we have

$$\sigma(\varphi(T)) \subset \varphi(\sigma(T)),$$

and since γ is 1 - 1 and onto, we have

$$\gamma(\sigma(\varphi(T))) \subset \gamma(\varphi(\sigma(T))) = \sigma(T),$$

i.e.,

$$\gamma(\sigma(B)) \subset \sigma(\gamma(B)),$$

since Y is a spectral set for B, we have

$$\sigma(\gamma(B)) \subset \gamma(\sigma(B)),$$

and thus,

$$\sigma(\gamma(B)) = \gamma(\sigma(B)).$$

The other equality is proved similarly.

Q.E.D.

The following two propositions show the equivalence of the conditions of Theorem 8 and those of Theorem 10.

Proposition 17. If $\varphi : X \to \overline{\mathbb{D}}$ is an analytic homeomorphism and $T \in O(X)$, then

$$\sigma(T) \supset \partial X \text{ if, and only if, } \sigma(\varphi(T)) \supset \partial\overline{\mathbb{D}} .$$

Proof: Since $\partial X \subset \sigma(T)$, we have

$$\varphi(\partial X) \subset \varphi(\sigma(T)) = \sigma(\varphi(T)),$$

by Lemma 8, and by Lemma 7, we have

$$\partial\overline{\mathbb{D}} \subset \sigma(\varphi(T)) .$$

Conversely, if $\partial\overline{\mathbb{D}} \subset \sigma(\varphi(T))$, then

$$\gamma(\partial\overline{\mathbb{D}}) \subset \gamma(\sigma(\varphi(T))) = \sigma(T),$$

where $\gamma = \varphi^{-1}$, and thus $\partial X \subset \sigma(T)$.

Q.E.D.

Proposition 18. If X, ∂X satisfy the assumption of Theorem 10, $T \in O(X)$, and $\varphi : X \to \overline{\mathbb{D}}$ is an analytic homeomorphism, then

$$\overline{W(T)} = X \text{ if, and only if } \overline{W(\varphi(T))} = \overline{\mathbb{D}} .$$

Proof: It is clear by Proposition 17.

Remark 7. In Proposition 18, it is not enough to assume that X is a convex spectral set only we need also that ∂X is a C^2-function and contains no straight line segments.

Indeed: Consider the following example.

Example 2. Let $T = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. Since T is normal, $\overline{W(T)} = ch(\sigma(T))$ = the triangle whose vertices are $\lambda_1, \lambda_2, \lambda_3$. Call this triangle X, and let $\varphi : X \to \overline{\mathbb{D}}$, be analytic homeomorphism. Then $\varphi(T) = \text{diag}(\mu_1, \mu_2, \mu_3)$ where $\mu_i = \varphi(\lambda_i)$, $i = 1,2,3$ and thus, we have

$$\overline{W(\varphi(T))} = ch(\sigma(\varphi(T))) \neq \overline{\mathbb{D}} .$$

This example shows that if the X is a convex set but contains straight line segments then Proposition 18 is false.

We present now some corollaries of Theorem 10.

Corollary 9. If X, ∂X and T are the same as in Theorem 10, then

$$\overline{W(T)} = X \Rightarrow |W_\alpha(R_\lambda(T))| = 1/d(\lambda, W_r(T))$$

for all $\lambda \notin W_r(T)$, $\alpha \geq 1$ and $r \geq 1$.

Proof: By Theorem 10, we have

$$\overline{W(T)} = X \Leftrightarrow \|R_\lambda(T)\| = 1/d(\lambda,X),$$

for all $\lambda \notin X$, which implies that

$$\|R_\lambda(T)\| = 1/d(\lambda,\overline{W(T)}),$$

for all $\lambda \notin \overline{W(T)}$.

Patel [13] proved that the last equality is equivalent to saying that

$$|W_\alpha(R_\lambda(T))| = 1/d(\lambda,W_r(T)),$$

for all $\lambda \notin W_r(T)$, $\alpha \geq 1$, $r \geq 1$.

Q.E.D.

Corollary 10. If X, ∂X, and T are the same as in Theorem 10 then

$$\sigma(T) \supset \partial X \Rightarrow |\overline{W(T-\lambda)}| = \|T-\lambda\|$$

for every $\lambda \in \mathbb{C}$.

Proof: Theorem 10 says that

$$\sigma(T) \supset \partial X \Leftrightarrow \overline{W(T)} = X,$$

i.e. $\overline{W(T)}$ is a spectral set of T, and thus (by Williams [18]) we have

$$|\overline{W(T-\lambda)}| = \|T-\lambda\| ,$$

for every $\lambda \in \mathbb{C}$.

Q.E.D.

So far we studied some equivalent conditions to the condition $\overline{W(T)} = X$, and some of the corollaries. One can ask when such condition can be valid? The following proposition answers this question.

Proposition 19. If $WC\{U^*TU : U \in \mathcal{U}\} = O(X)$, then we have $\overline{W(T)} = X$.

Proof: Since X is convex spectral set of T, we have $W(T) \subset X$. For the other inclusion, we can identify points of X as (scalar) operators in $O(X)$, on one-dimensional subspaces of $\mathcal{H}$. The assumption asserts that any operator in $O(X)$ on a finite-dimensional subspace of $\mathcal{H}$ is approximable by the compression of T to that subspace, thus given $\lambda \in X$, then $\forall \varepsilon > 0$ there is $e \in \mathcal{H}$ with $\|e\| = 1$ (generator of one-dimensional subspace) such that

$$|(Te,e)-\lambda| < \varepsilon ,$$

which implies that $\lambda \in \overline{W(T)}$, i.e., $X \subset \overline{W(T)}$ and we have $\overline{W(T)} = X$.

Q.E.D.

It is reasonable and natural to ask if the converse of Proposition 19 holds, since it holds for the contraction case. We prove a version of the converse, namely, if $\overline{W(T)} = X$, and X, ∂X and T are the same as in Theorem 10 then $O(X)$ is included in $WC\{U^*TU : U \in \mathcal{U}\}$. In order to obtain the full converse of Proposition 19, we need to show that $O(X)$ is weakly closed, the thing we could not prove.

Theorem 11. If T, X, and ∂X are the same as in Theorem 10, then we have

$$\overline{W(T)} = X \Rightarrow O(X) \subset WC\{U^*TU : U \in \mathcal{U}\}.$$

For the proof, we need two propositions and a lemma.

Lemma 9. If X is simply connected and $\varphi : \overline{\mathbb{D}} \to X$ is analytic homeomorphism from the closed unit disc to X. If $\|A\| \le 1$ and $\gamma \in \overline{\mathbb{D}}$ such that $\|Ae_i - \gamma e_i\| \to 0$, for some sequence of unit vectors then

$$\|\varphi(A)e_i - \varphi(\gamma)e_i\| \to 0,$$

as i increases.

Proof: Since φ is holomorphic on the interior of $\overline{\mathbb{D}}$ and continuous on $\overline{\mathbb{D}}$, then φ belongs to the disc algebra, in which the set of all polynomials is norm dense. So that there is a sequence (P_n) of polynomials such that

$$\varphi = n - \lim P_n,$$

and thus it is enough to prove the lemma for polynomials, in which case it is enough to prove it for integral powers of

... A, namely for A^k, for any positive integer k.

Now, for k = 1, it is the assumption, so, assuming the induction step for k, then for k + 1, we have

$$\|A^{k+1}e_i - \gamma^{k+1}e_i\| \le \|A^{k+1}e_i - \gamma A^k e_i\| + \|\gamma A^k e_i - \gamma^{k+1}e_i\|$$

$$\le \|A^k\| \cdot \|Ae_i - \gamma e_i\| + |\gamma| \cdot \|A^k e_i - \gamma^k e_i\| .$$

Since the right-hand side converges to 0 as i increases, we have

$$\|\varphi(A)e_i - \varphi(\gamma)e_i\| \to 0,$$

as i increases.

Q.E.D.

Proposition 20. Under the hypothesis of Theorem 11, if $\lambda \in \partial X$ such that $\forall \varepsilon$, $\exists$ $e \in \mathcal{H}$ with $\|e\| = 1$ such that $|(Te,e) - \lambda| < \varepsilon$, then there is a number μ such that

$$Te = \mu e + h,$$

where $|\mu-\lambda| < \varepsilon$ and $\|h\| < \varepsilon$.

Proof: Let $\varphi : X \to \overline{\mathbb{D}}$ be analytic homeomorphism; then $\varphi(\partial X) = \partial\overline{\mathbb{D}}$, so if $\lambda \in \partial X$ then there is $\delta \in \partial\overline{\mathbb{D}}$ such that $\delta = \varphi(\lambda)$. The assumption implies that $\lambda \in \overline{W(T)}$ and since $\overline{W(T)} = X \Leftrightarrow \overline{W(\varphi(T))} = \overline{\mathbb{D}}$, (by Proposition 18), we conclude that $\delta \in \overline{W(\varphi(T))}$. Denote $A = \varphi(T)$, which is contraction (since X is a spectral set for T), so that, $\forall \varepsilon, \exists$ e a unit vector such that

$$|(Ae,e) - \delta| < \varepsilon,$$

since ε is arbitrary we match them to get the same unit vector e of the assumption. By Lemma 5, as in the proof of Proposition 7, there is a complex number $\hbar$ and a vector $h' \in \mathcal{H}$ such that $Ae = \hbar e - h'$, where $\|h'\| < \varepsilon$, and $|\hbar-\delta| < \varepsilon$.

This implies that $\|Ae - \hbar e\| < \varepsilon$, so by Lemma 9 we have $\|\psi(A)e - \psi(\hbar)e\| < \varepsilon$, where $\psi = \varphi^{-1}$ and thus

$$\psi(A)e = \psi(\hbar)e + h,$$

where $\|h\| < \varepsilon$. Define $\mu = \psi(\hbar)$ and we have

$$Te = \mu e + h,$$

where, $|\lambda-\mu| = |\psi(\delta)-\psi(\hbar)| < \varepsilon'$, by continuity of ψ and $|\delta-\hbar| < \varepsilon$.

Q.E.D.

Proposition 21. If X is simply connected spectral set for operator $A \in \mathcal{L}(\mathcal{H})$, then there exists a normal operator $N \in \mathcal{L}(\mathcal{H})$ such that N approximates A weakly and $\sigma(N) \subset \partial X$.

Proof: Under the hypothesis Sarason, [15] proved that there exists a normal dilation M of A on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\sigma(M) \subset \partial X$ and if $P_{\mathcal{H}} : \mathcal{K} \to \mathcal{H}$, is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, then

$$A^n P_{\mathcal{H}} = P_{\mathcal{H}} M^n P_{\mathcal{H}} ,$$

for every integer n.

So that, given $f_1, \cdots, f_k$, $g_1, \cdots, g_k \in \mathcal{H}$, and let $\mathfrak{h} = \operatorname{span}\{f_1, \cdots, f_k, g_1, \cdots, g_k\}$, then $\dim \mathfrak{h} < \infty$. Let $U : \mathcal{H} \to \mathcal{K}$ be the unitary operator such that $U|_{\mathfrak{h}} = I$ and $U|_{\mathcal{H}\ominus\mathfrak{h}}$ from $\mathcal{H}\ominus\mathfrak{h}$ to $\mathcal{K}\ominus\mathfrak{h}$ be isometric and onto and define $N = U^*MU$, then by definition of U, $N \in \mathcal{L}(\mathcal{H})$ and N is normal, and since $\sigma(M) \subset \partial X$, we have $\sigma(N) \subset \partial X$.

Still to show that N approximate A weakly, indeed:

$$|((A-N)f_i, g_j)| = |(Af_i, g_j) - (Mf_i, g_j)|$$
$$= |((AP_{\mathcal{H}} - P_{\mathcal{H}}MP_{\mathcal{H}})f_i, g_j)| < \varepsilon,$$

for $i, j < k$.

Moreover, since N is normal, $\sigma(N) \subset \partial X$, we have X is a spectral set for N as well.

Q.E.D.

<u>Proof of Theorem 11</u>. We want to show that $O(X) \subset WC\{U^*TU : U \in \mathfrak{U}\}$, so given $A \in O(X)$, then X is convex spectral set for A, thus, by Proposition 21, there is a normal operator N such that N approximates A weakly and $\sigma(N) \supset \partial X$. Moreover X is spectral set for N, i.e., $N \in O(X)$. This is the first step, we approximate A weakly by a normal $N \in O(X)$, the second step is to approximate such operator N by a diagonal operator.

By Berg's theorem [2], for any normal operator N, $\varepsilon > 0$, there is a diagonal operator D and a compact operator K such that $\|K\| < \varepsilon$ and,

$$U^*NU = D + K,$$

for some unitary operators U, and $\sigma(D) \subset \sigma(N)$. So that if $N \in O(X)$, normal then

$$\|U^*NU - D\| < \varepsilon ,$$

and $\sigma(D) \subset \sigma(N) \subset \partial X$, which implies that X is spectral set of D, i.e., $D \in O(X)$.

The third step of approximation is to approximate such operator D, weakly by some compressions of T. So given such D in O(X) on a finite-dimensional subspace of $\mathcal{H}$ such that $\sigma(D) \subset \partial X$, then

$$D = \mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n) ,$$

where $\lambda_i \in \partial X$, $i \le n$.

Since $X = \overline{W(T)}$, we have $\lambda_i \in \overline{W(T)}$ $i \le n$, so that $\forall \lambda_i$, $\varepsilon > 0$, $\exists$ e_i a unit vector such that

$$|(Te_i,e_i)-\lambda_i| < \varepsilon , \quad i \le n. \qquad (*)$$

So that it is possible to define a unitary operator U (as we did in the proof of Theorem 9) such that

$$|((T-U^*DU)e_i,e_i)| < \varepsilon, \quad i \le n.$$

Applying Proposition 20 to (*), we have

$$Te_i = \mu_i e_i + h_i, \quad i \le n$$

where $\|h_i\|^2 \le 2\varepsilon$, $|\mu_i-\lambda_i| < \varepsilon$, $i \le n$, so that for $i \ne j$, we have

$$|((T-U^*DU)e_i,e_j)| = |(Te_i,e_j)|$$
$$\leq |\mu_i(e_i,e_j)| + |(h_i,e_j)|$$
$$\leq \|h_i\| < \delta(\varepsilon), \quad i,j \leq n$$

and this implies that

$$|((T-U^*DU)e_i,e_j)| < \delta(\varepsilon), \quad i,j \leq n.$$

Combining the three steps of approximation together, we conclude that, $\forall n$, $\varepsilon > 0$ $\exists$ a unitary operator U such that

$$|((T-U^*AU)e_i,e_j)| < \delta(\varepsilon), \quad i,j \leq n,$$

which implies that $A \in WC\{U^*TU : U \in \mathfrak{U}\}$ i.e., $O(X) \subset WC\{U^*TU : U \in \mathfrak{U}\}$.

Q.E.D.

Now, we conjecture the following

Conjecture: If X is a convex spectral set of T, then O(X) is weakly closed.

If the conjecture is correct then we would obtain the full converse of Proposition 19.

We have the following proposition.

Proposition 22. The set $R = \{A \in \mathcal{L}(\mathcal{H}) : \overline{W(A)}$ is a spectral set of $A\}$ is norm closed and arcwise connected as a subset of $\mathcal{L}(\mathcal{H})$.

Proof: To see that the set is norm closed, let $(A_n) \subset R$

... such that (A_n) converges uniformly to A, then $\overline{W(A_n)}$ is a spectral set to A_n, for every n. Since the numerical range is uniformly continuous (see Halmos [7]), we have $\overline{W(A_n)} \to \overline{W(A)}$. Since $\sigma(A) \subset \overline{W(A)}$, to show that $\overline{W(A)}$ is a spectral set for A, it is enough to show that

$$\|P(A)\| \leq \|P\|_\infty ,$$

for any polynomial P on $\overline{W(A)}$. So that for any polynomial P, we have,

$$\|P(A)\| \leq \|P(A)-P(A_n)\| + \|P(A_n)\| , \quad n > N$$

$$\leq \varepsilon + \|P(A_n)\|,$$

since product is uniformly continuous, and since

$\|P(A_n)\| \leq \|P\|_{\overline{W(A_n)}} = \sup_{z \in \overline{W(A_n)}} |P(z)|$, we have

$$\|P(A)\| \leq \varepsilon + \sup_{z \in \overline{W(A_n)}} |P(z)| , \quad n > N.$$

Let $z_n = \sup_{z \in \overline{W(A_n)}} |P(z)|$, the $z_n \to z_0 \in \overline{W(A)}$ and since

$|z_0| \leq \sup_{z \in \overline{W(A)}} |P(z)|$, we have,

$$\|P(A)\| \leq \varepsilon + \|P\|_{\overline{W(A)}} \leq \|P\|_\infty$$

This proves the first part of the proposition for the second part, it is enough to show that everything is connected to zero, i.e., if $A \in R$, then $\lambda A \in R$ for any complex number λ.

Indeed: to say that if $A \in R$ then $\lambda A \in R$, for every $\lambda \in \mathbb{C}$ is equivalent to saying that if $\overline{W(A)}$ is a spectral set of A; then $\overline{W(\lambda A)}$ is a spectral set of λA for any $\lambda \in \mathbb{C}$. First, since $\lambda(Ae,e) = (\lambda Ae,e)$, for any vector e, and any $\lambda \in \mathbb{C}$, then $\lambda W(A) = W(\lambda A)$ and hence $\lambda\overline{W(A)} = \overline{W(\lambda A)}$. Let P be a polynomial, let $q(z) = P(\lambda z)$, then

$$\|P(\lambda A)\| = \|q(A)\| \le \sup_{z\in\overline{W(A)}} |q(z)|$$

$$\le \sup_{\frac{z}{\lambda}\in\overline{W(A)}} |P(\lambda z)| \le \sup_{z\in\lambda\overline{W(A)}} |P(\lambda z)|$$

$$\le \sup_{z\in\overline{W(\lambda A)}} |P(\lambda z)|,$$

which implies that $W(\lambda A)$ is a spectral set for λA and hence $\lambda A \in R$, i.e., R is arcwise connected.

Q.E.D.

Now we have some results as an application to Theorem 10.

<u>Corollary 11</u>. If X, ∂X and T are the asame as in Theorem 10 such that $\overline{W(T)} = X$, and if K is a compact operator with $T + K \in O(X)$ then $\overline{W(T+K)} = X$.

<u>Proof</u>: It is easy to modify the proof of Corollary 3 to suit this corollary.

Q.E.D.

Recall that $G(X)$ is the algebra of all function continuous on X and holomorphic on the interior of X, where X is a compact subset of $\mathbb{C}$. Recall also that $\mathcal{A}(X)$ is the uniform closure of all polynomials defined on ∂X. As the case of the disc algebra, we have the following results.

Proposition 23. If X, ∂X, and T are the same as in Theorem 10, such that $\overline{W(T)} = X$, then the map $\psi : \mathcal{A}(X) \to \mathcal{L}(\mathcal{H})$, defined by $\psi(f) = f(T)$, for any $f \in \mathcal{A}(X)$ is an isometry.

Proof: Since X is a spectral set of T, then for any $f \in \mathcal{A}(X)$, $f(T)$ is well defined and

$$\|f(T)\| \leq \|f\|_\infty \equiv \sup_{z \in X} |f(z)| ,$$

and

$$\|f(T)\| \geq |\sigma(f(T))|, \text{ the spectral radius of } f(T)$$
$$\geq \sup\{|\lambda| : \lambda \in \sigma(f(T))\} .$$

Since $\overline{W(T)} = X$ equivalent to $\sigma(T) \supset \partial X$, and (by functional Calculus) $f(\sigma(T)) = \sigma(f(T))$, we have $\sigma(f(T)) \supset f(\partial X)$ and thus

$$\|f(T)\| \geq \sup\{|\lambda| : \lambda \in f(\partial X)\}$$
$$\geq \sup_{\gamma \in \partial X} |f(\gamma)| = \|f\|_\infty$$

and therefore $\|f(T)\| = \|f\|_\infty$ i.e.,

$$\|\psi(f)\| = \|f\|_\infty, \quad \text{i.e.,}$$

ψ is an isometry.

Q.E.D.

Proposition 24. If X, ∂X and T are the same as in Theorem 10, and if the map $\psi : \mathfrak{a}(X) \to \mathfrak{L}(\mathfrak{H})$ is such that

$$|\overline{W(f(T))}| = \|f\|_\infty ,$$

for every $f \in \mathfrak{a}(X)$, then $\overline{W(T)} = X$.

Proof: It is enough to show that $\partial X \subset \sigma(T)$, so let $\lambda \in \partial X$, $\lambda \notin \sigma(T)$ then let $f \in \mathfrak{a}(X)$ such that $f(\lambda) = \lambda$, $\|f\|_\infty = |\lambda|$ and maps X into a subset of itself, since λ is not in $\sigma(T)$, $\lambda \notin \sigma(f(T))$, for any $f \in \mathfrak{a}(X)$ so that $\sigma(f(T))$ is included properly in a set does not contain ∂X, and thus

$$|\sigma(f(T))| \neq |\lambda| = \|f\|_\infty \qquad (*) .$$

Now since $|\overline{W(f(T))}| = \|f\|_\infty$, we have

$$|\overline{W(f(T))}| = \|f(T)\|,$$

and thus

$$|\sigma(f(T))| = \|f\|_\infty ,$$

for every $f \in \mathfrak{a}(X)$, but this contradicts (*) and hence $\lambda \in \sigma(T)$, i.e. $\partial X \subset \sigma(T)$.

Q.E.D.

Remark 8. The function f in the proof above can be defined in a similar manner as its analogue is defined in the proof of Proposition 11.

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