ONE CONSERVATION LAW IN ONE SPACE VARIABLE

by

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Shock solutions are constructed for a conservation-law partial differential equation, given initial data satisfying mild conditions; stronger conditions guarantee finite-shock solutions. Shock structure is described, and the number of shocks is estimated in terms of the initial data. The solution and its shock structure are proven stable in various ways.
Dedicated To My Parents,

Oscar and Sarah Friedel
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1. Introduction

(1.1) Motivation

The mass of a fluid, viewed as a density-function depending on space and time, represents a type of physical entity which obeys a "conservation law"; this asserts that any change of the amount within an arbitrary region of space depends solely on the net inflow (including outflow) through the boundary of that region. In particular, if the inflow were zero, the quantity inside would remain constant, i.e., be "conserved".

The conservation law is expressed as an IDE\(^1\)

\[
(1.11) \quad \frac{d}{dt} \int_{\Omega} u(x,t) dx = \int_{\partial \Omega} \langle F, \bar{n} \rangle.
\]

Here \(\Omega\) is any open subset of \(\mathbb{R}^n\) (x-space), whose boundary \(\partial \Omega\) has inward unit normal \(\bar{n}\); \(u(x,t)\) is density and \(F(x,t)\) is flux. (Flux is a vector indicating the direction of flow and its rate: approximately the amount passing through a small area orthogonal to the direction of flow in a short time, divided by the product of the area and the time interval.) The left side of 1.11 is the rate-of-change of the amount of the conserved quantity in \(\Omega\), while the right side gives the

\(^1\) IDE = integro-differential equation, PDE = partial differential equation.
rate at which it enters $\Omega$.

If $u$ and $F$ are $C^1$ on $\Omega \times \{t\}$, then bring the derivative on the left inside the integral and apply the divergence theorem on the right to get

\begin{equation}
\int_{\Omega} u_t \, dx = -\int_{\Omega} \text{div}(F) \, dx.
\end{equation}

Thus, for $(x,t) \in \Omega \times \{t\}$ we have

\begin{equation}
u_t + \text{div}(F) = 0
\end{equation}

because $1.12$ holds when $\Omega$ is replaced by any small neighborhood of $x$.

In some physical processes\footnote{The basic example is isentropic fluid flow, in which mass and momentum are conserved.} several quantities (with densities $(u^1, u^2, \ldots, u^n) = \vec{u}$) are conserved and it may be possible to replace the unknown flux-functions $f^i(x,t)$ by known functions $f_i^i(x,t,u(x,t))$. (Here flux $f_i^i(x,t,\vec{u}(x,t))$ makes no sense if $\vec{u}$ is discontinuous at $(x,t)$.) The conservation law becomes a system of IDE's

\begin{equation}
\frac{d}{dt}\int_{t_0}^{t} \int_{\Omega} u_i \, dx = \int_{\partial \Omega} \langle f^i, \vec{n} \rangle (i = 1, \ldots, m)
\end{equation}

which $\vec{u}$ must satisfy whenever continuous on $\partial \Omega \times \{t_0\}$ minus a set of $\mathbb{R}^{n-1}$-measure zero (so the integral over $\partial \Omega$ makes sense). Where $C^1$, $\vec{u}$ solves
\[ u_t^i + \text{div}(f^i(x,t,u(x,t))) = 0 \quad (i = 1, \ldots, m). \]

An initial-value problem (IVP) for a nonlinear PDE such as 1.15 need not have a differentiable solution for all time \( t \geq 0 \). But because physical processes continue indefinitely, we would like some sort of global solution to an IVP for the conservation law 1.14. Various types of solution have been defined, the most desirable being the "shock solution".

1.2 Results

We study the simplest nonlinear case: a single conservation law (i.e., \( u(x,t) \) is scalar-valued) in one space-variable, with \( f(x,t,u) = f(u) \) not explicitly dependent on \( (x,t) \). Probably such an oversimplification cannot accurately represent any physical process (it is unlikely for flux to depend only on density) but it is a model for generalization.

We give an algorithm (3.42) for constructing shock solutions using the method of characteristics. To each PDE-IVP there corresponds an "integral surface" in \( (x,t,u) \)-space, formed from those characteristics (integral curves to a certain vector field) which pass through the graph of the initial data. Locally this surface is the graph of a solution to the PDE, but globally it may not represent a single-valued function \( u(x,t) \). The Rankine-Hugoniot jump
condition for shocks (discontinuity-paths) is an ordinary
differential equation for which a sequence of initial-
value problems tell how to carve the graph of a global shock-
solution out of the integral surface.

Assuming $f'$ increasing, we show existence of shock
solutions (3.43) for a class of initial data residual in
$C^3$ modulo mild conditions at infinity (5.1), with finite-
shock solutions for a large subclass.

Following Lax [2], Schaeffer [3] used the theory of
singularities to prove a similar result for rapidly-
decaying $C^\infty$ initial data assuming $f'' > \epsilon > 0$. We use only
basic analysis.

Our construction yields a description of the solution's
discontinuities (4) and an estimate of their number (4.1).
As initial data is perturbed in $C^3$ the shocks maintain
their mutual intersection-relationships; they vary continu-
ously, as does the solution locally in measure and in
$L^1$ (6.1). Golubitsky and Schaeffer [4] proved a different
form of shock stability.

2. **Definition of Shock Solution**

We establish notation, cf. Lax [2].

(2.1) **Various types of solution**

Consider a single conservation law in one space variable
with flux $f(u)$ depending ($C^1$) only on the unknown variable
\[ u(x, t) \text{. The IDE 1.14 here becomes} \]

\[ \frac{d}{dt} \int_{x_1}^{x_2} u(x, t) \, dx = \int_{\partial(x_1, x_2)} \langle f \cdot u, \eta \rangle \]

if \( u \) is \( C^0 \) at \( [x_1, x_2] \times \{t_0\} \). The inward normal \( \eta \) is +1 at \( x_1 \) and -1 at \( x_2 \) so

\[ \int_{\partial(x_1, x_2)} \langle f \cdot u, \eta \rangle = f \cdot u \left[ (x_1, t_0) \right] - f \cdot u \left[ (x_2, t_0) \right] = - \int u(x_1, t_0) f' \cdot \]

Set \( f' = a \). Then 2.12 changes 2.11 to

\[ \frac{d}{dt} \int_{x_1}^{x_2} u \, dx + \int u(x_2, t_0) - u(x_1, t_0) = 0 \]

so where \( u \) is \( C^1 \),

\[ u_t + a(u)u_x = 0 \]

Note: The IDE form of the conservation law, 2.13, may be recovered from the simpler form 2.14. Thus we regard the PDE 2.14 as the central object of study, for which we seek global generalized solutions. Any definition of "solution" ought not to depend on the particular antiderivative of (a) that \( f \) happens to be, because 2.14 (also 2.13) involves only \( f' = a \), not \( f \).

Notation: (i) \( H = \mathbb{R} \times \mathbb{R}_{\geq 0} = \text{space-time} \ (x, t) \). A function \( u(x, t) \) is "global" iff its domain is \( H \).

(ii) \( \mathcal{S} \) denotes the set of \( C^1 \) test-functions \( \varphi : H \rightarrow \mathbb{R} \) which vanish outside a bounded subset of \( \mathbb{R} \times \mathbb{R}_{\geq 0} \subset H \).
(iii) $u : H \to R$ is "piece-wise $C^1$" iff for each bounded subset $B \subset H$ we have $u(B)$ is bounded, and $u$ restricted to $B$ is $C^1$ on the complement of a finite union of graphs of $C^1$ paths.

(2.15) **Definition.** Let $\psi \in C^1(R)$. "IVP($\psi$)" denotes the problem of finding $u : H \to R$ such that $u(x,0) = \psi(x)$ and $u$ is a "generalized solution" of 2.14 in one of these four ways: $u$ is a

(i) "(global) $C^1$ solution" iff $u$ is $C^1$ and solves 2.14.

(ii) "Implicit solution" iff $u$ solves 2.13.

(iii) "weak solution"$^3$ iff $\int_H (\varphi_t u + \varphi_x f(u)) dx dt = 0$ for all $\varphi \in \mathcal{F}$.

(iv) "piece-wise $C^1$ solution" iff $u$ is piece-wise $C^1$ and, where $C^1$, solves 2.14.

(2.2) **Relative status of definitions**

Notation: Let $u(x,t)$ be piece-wise $C^1$, with a discontinuity-path $\gamma$. (i) Let $g$ be a function of $u : g = f$ or $g = a$ or $g = \text{identity}$. Define the "jump function" $[g]$ on graph($\gamma$) as $g^+u(\gamma(t)+0,t)$.

(ii) The average of a function $a(u)$ for $u$ between $u_1$ and $u_2$ is

$^3$Integrate by parts $\varphi(u_t + (f \varphi)_x) = 0$. "Weak solution" is independent of the particular antiderivative $f$ since $\int_R \varphi_x(x,t) dx = 0$. 

\[
\begin{align*}
\mathbf{u}_2 = \mathbf{u}_1 \mathbf{a}_1 = \\
\begin{cases} 
\int_{\mathbf{u}_1}^{\mathbf{u}_2} a(u) du & \text{if } \mathbf{u}_1 \neq \mathbf{u}_2 \\
\frac{\mathbf{u}_2 - \mathbf{u}_1}{\mathbf{u}_2} & \text{if } \mathbf{u}_1 = \mathbf{u}_2 \\
\end{cases}
\end{align*}
\]

Define \( \mathbf{a}_1 \) along \( \gamma \) as \( \mathbf{a}_1 \mathbf{u}(\gamma(t)+0,t) \).

(iii) The "Rankine-Hugoniot condition" is \( \gamma' = \mathbf{a}_1 \) at interior points of \( \gamma \), for all discontinuity paths \( \gamma \).

Note: \( \mathbf{a}_1 = \left[ \begin{array}{c} f \\ \|\mathbf{u}\| \end{array} \right] \).

(2.21) Proposition

(2.211) A \( C^1 \) solution is a solution in all 4 ways 2.15.

(2.212) A solution in any of the 4 ways 2.15 satisfies the FDE 2.14 on any open set on which it is \( C^1 \).

(2.213) Let \( \mathbf{u} \) be a piece-wise \( C^1 \) solution. Then the following 3 statements are equivalent:

(i) \( \mathbf{u} \) obeys the Rankine-Hugoniot condition.

(ii) \( \mathbf{u} \) is an IDE solution.

(iii) \( \mathbf{u} \) is a weak solution.

Proof. See Appendix.

(2.3) The Shock Solution

Definition. A piece-wise \( C^1 \) solution obeys the "shock condition" iff \( \mathbf{a}_1 < 0 \) on the interior of each discontinuity-path ("shock"). A "shock solution" is a piece-wise \( C^1 \) solution which obeys both the Rankine-Hugoniot condition and
the shock condition.

(2.31) **Note.** If \( (a) \) is increasing, then the shock condition is equivalent to \( [u] < 0 \).

Recalling that for the PDE \( 2.14 \) the characteristic speed is \( \frac{dx}{dt} = a\cdot u(x,t) \), we see the shock condition says every shock is forced by the intersection of characteristics emanating from either side, carrying conflicting values for the solution.

A shock solution (if it exists) seems the best alternative to a global \( C^1 \) solution. It is a generalized solution in all senses except \( C^1 \) (by \( 2.213 \)). The discontinuities are simple (paths) and minimal ("forced"). Oleinik [1] proved that a shock solution is unique among weak solutions up to redefinition on a set of measure zero.

3. **Construction of Solution**

(3.1) **The integral surface**

Assume hence \( a(u) \) is \( C^1 \). The method of characteristics tries to build the graph of a \( C^1 \) solution to IVP(\( \psi \)) out of integral curves to the characteristic vector field

\( (x,t,u) \rightarrow (a(u),1,0) \) in \( H \times R \) which pass through graph \( \psi \).

Thus we seek functions \( (s,\tau) \rightarrow (\bar{x},\bar{t},\bar{u}) \) solving

\[
\begin{align*}
\bar{x}_\tau &= a(\bar{u}) \quad , \quad \bar{x}(s,0) = s \\
\bar{t}_\tau &= 1 \quad , \quad \bar{t}(s,0) = 0 \\
\bar{u}_\tau &= 0 \quad , \quad \bar{u}(s,0) = \psi(s) 
\end{align*}
\]
Here $s$ parametrizes the $x$-axis and $\tau$ parametrizes each characteristic curve, i.e., $\tau \rightarrow (\bar{x}, \bar{t}, \bar{u})|_{(s, \tau)}$ gives the characteristic through $(s, 0, \psi(s))$.

$\bar{t}_\tau = 1$ and $\bar{t}(s, 0) = 0$ imply $\bar{t}(s, \tau) = \tau$, so we identify $\tau$ with "time" $t$. Then 3.11 simplifies to

\[
\begin{align*}
\bar{x}_t &= a(\bar{u}) , & x(s, 0) &= s \\
\bar{u}_t &= 0 , & u(s, 0) &= \psi(s)
\end{align*}
\]

which has solution

\[
\bar{u}(s, t) = \psi(s), \quad \bar{x}(s, t) = t \cdot a\psi(s) + s.
\]

(3.12) Note: \[
\begin{align*}
\bar{u}_s &= \psi'(s) , & \bar{x}_s &= t \cdot (a\psi)'(s) + 1 = \\
t \cdot a'(\psi(s)) \cdot \psi'(s) + 1 , & \text{and} & \bar{x}_t &= a(\bar{u}) = a\psi(s) , \bar{u}_t &= 0.
\end{align*}
\]

Definition. $\mathcal{P} = \{(s, t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}\}$, parameter-space.

$\bar{\xi} : \mathcal{P} \rightarrow \mathbb{H} \times \mathbb{R}, \bar{\xi}(s, t) = (\bar{x}, t, \bar{u}); \mathcal{P} = \bar{\xi}(\mathcal{P})$ (parametrized integral surface).

(3.13) Proposition. $\bar{\xi}$ is a proper embedding.$^4$ Thus $\mathcal{P}$ is a surface in $\mathbb{H} \times \mathbb{R}$ for which the relative topology from $\mathbb{R}^3$ equals the topology induced via $\bar{\xi}$ from $\mathcal{P}$; $\bar{\xi}^{-1} : \mathcal{P} \rightarrow \mathcal{P}$ is continuous.

$^4$ i.e., $\bar{\xi}$ is $C^1$, injective, has injective differential, and the pre-image of compact sets is compact. Fact: A proper embedding maps its domain diffeomorphically onto its image, which is a submanifold of the range space.
Proof. $\phi$ is injective because integral curves of a $C^1$ vector field do not intersect. 3.12 gives $\phi$ is $C^1$, with

$$
d\phi = \begin{pmatrix}
t \cdot a'(\psi(s)) \cdot \psi'(s) + 1 & a \cdot \psi(s) \\
0 & 1 \\
\psi'(s) & 0
\end{pmatrix}.
$$

$d\phi$ has rank 2, because the first 2 or the last 2 rows are independent (corresponding to $\psi'(s) = 0$ or $\neq 0$).

We show $\phi^{-1}(K)$ is compact for compact $K \subset H \times R$. $K$ is closed and $\phi$ is $C^0$, so $\phi^{-1}(K)$ is closed. $K$ is bounded, so the last 2 slots $(t, \psi(s))$ are bounded, implying $t \cdot a \cdot \psi(s)$ is bounded for $(s, t) \in \phi^{-1}(K)$; but the first slot $(t \cdot a \cdot \psi(s) + s)$ also is bounded, so that $s$ is bounded in $\phi^{-1}(K)$. Thus $\phi^{-1}(K)$ is bounded and closed, hence compact.

QED 3.13.

**Definition.** $V = \{p \in \phi : (0, 0, 1) \text{ is tangent to } \phi \text{ at } p\}$, $V = \phi^{-1}(V)$, "vertical sets" in $\phi$ and in $P$.

(3.14) **Lemma.** $V = \{(s, t) \in \phi : \bar{x}_s(s, t) = 0\} = \{(s, t) \in \phi : t = \frac{-1}{(a \cdot \psi)'(s)}\} = \{(s, t) : (0, 0, 1) \text{ is tangent to curve } \phi(Rx(t)) \text{ at } \phi(s, t)\}$.

**Proof.** The tangent plane to $\phi$ at $\phi(s, t)$ is spanned by $\phi_s$ (tangent to $\phi(Rx(t))$) and $\phi_t$, so $(s, t) \in V$ iff
\[ \begin{array}{ccc} x_S & x_t & 0 \\ 0 & 1 & 0 \\ \bar{u}_S & \bar{u}_t & 1 \end{array} = 0 \]

iff \( 0 = x_S = t \cdot (a \cdot \psi)'(s) + 1. \) QED 3.14.

**Definition.** \( \pi : P \to H, \pi(x, t, u) = (x, t) \). A subset \( S \subset P \) is "1-valued" if \( \pi|_S \) is injective, i.e., \( S \) is the graph of a function \( u^S : \pi(S) \to R. \)

(3.15) **Proposition.**

(3.151) Let \( p \in P \). \( P \) contains a 1-valued neighborhood of \( p \) iff \( p \in P - \emptyset. \)

(3.152) Let \( S \subset P \) be 1-valued and open. Then \( u^S \) is a \( C^1 \) solution of 2.14 with open domain \( \pi(S) \).

(3.153) IVP(\( \psi \)) has a global \( C^1 \) solution iff \( \psi = \emptyset \), iff \( (a \cdot \psi)' \geq 0. \)

**Proof.** Suppose no neighborhood of \( p \) in \( P \) is 1-valued. Then write \( \psi^{-1}(p) = (s_o, t_o) \) and use 3.13 to ensure sequences \( \{(s_n, t_n)\}_{n=1}^\infty \), \( \{\hat{s}_n, \hat{t}_n\}_{n=1}^\infty \) in \( P \) converging to \((s_o, t_o),(s_o, t_o), \) with

\[ (s_n, t_n) = 0 \] for all \( n \). Then \( t_n = \hat{t}_n \), and \( x_j(s_n, t_n) = 0, \ (\hat{s}_n, \hat{t}_n) \) so the mean value theorem guarantees \( \psi \) \( \in (s_n, \hat{s}_n) \) such that \( x_j(s_n, t_n) = 0 \). Now \( \lim_{n \to \infty} s_n = s_o \) (because \( \lim s_n = s_o = \lim \hat{s}_n \)).
so \( \lim (s_n, t_n) = (s_0, t_0) \). Thus \( \bar{x}_g(s_0, t_0) = 0 \), and 3.14 gives \( p \in \nu \).

Now let \( p \in \mathcal{P} - \nu \). Jacobian \((\pi \circ \xi) = \bar{x}_g\), nonzero at \( \xi^{-1}(p) \) (by 3.14), so the inverse-function theorem promises a neighborhood \( N_p \) of \( \xi^{-1}(p) \) in \( \mathcal{P} \), which \( \pi \circ \xi \) maps diffeomorphically onto \( \pi \circ \xi(N_p) \).

In particular, \( \pi \) is injective on \( \xi(N_p) \), a neighborhood of \( p \) (by 3.13).

QED 3.151.

For \( p \in \mathcal{G} \), let \( N_p \) as above; we may assume \( N_p \subset \xi^{-1}(\mathcal{G}) \) since \( \xi^{-1}(\mathcal{G}) \) is open in \( \mathcal{P} - \nu \). Then \( \pi \circ \xi(N_p) \) is open, so

\[
\pi(\mathcal{G}) = \pi \circ \xi(\xi^{-1}(\mathcal{G})) = \pi \circ \xi(\bigcup_{p \in \mathcal{G}} N_p) = \bigcup_{p \in \mathcal{G}} \pi \circ \xi(N_p)
\]

is open.

Let \( \xi : \pi \circ \xi(N_p) \to N_p \) denote the inverse function such that \( \pi \circ \xi(\xi(x, t), t) = (x, t) \); then \( \xi \) is \( C^1 \), since so is \( \pi \circ \xi \). Note \( u^\mathcal{G}(x, t) = \bar{u}(\xi(x, t), t) \) for \((x, t) \in \pi \circ \xi(N_p)\), so \( u^\mathcal{G} \) is \( C^1 \) near \( \pi(p) \).

\( \mathcal{G} \) has normal vector field \((u^\mathcal{G}_x, u^\mathcal{G}_t, -1)\). But \( \mathcal{P} \) is tangent to the vector field \((a(u), 1, 0)\). \( 0 = \langle \text{tangent, normal} \rangle = \langle (a(u), 1, 0), (u^\mathcal{G}_x, u^\mathcal{G}_t, -1) \rangle = u^\mathcal{G}_t a(u) u^\mathcal{G}_x \). (Standard proof.)

QED 3.152.

3.14 shows \( \nu = \emptyset \) iff \( (a \circ \psi)' \geq 0 \).

If a global \( C^1 \) solution exists, then its graph is \( \mathcal{P} \).

Suppose \((s, t) \in V \neq \emptyset \). Then \( 0 = \bar{x}_g(s, t) = t \cdot a'(\psi(s)) \cdot \psi'(s) + 1 \)

\[
\text{(3.152)}
\]
(by 3.14), and $0 \neq \hat{u}_s(s,t) = \psi'(s)$ (else $\bar{x}_s(s,t) = 1$), so
\[
\frac{\partial u}{\partial x}(\pi \circ \#(s,t)) = \frac{\hat{u}_s}{\bar{x}_s}(s,t) = \pm \infty,
\]
contradicting $u$ is $C^1$.

Finally, suppose $v = \emptyset$. $\pi$ is l-valued (else the mean-value theorem yields a zero for $\bar{x}_s$, contradicting $v = \emptyset$) and a $C^1$ solution where defined (by 3.152). But $v = \emptyset$ implies $(a \circ \psi)' > 0$ so $t \cdot (a \circ \psi)'(s) + 1 = \bar{x}_s > 1$, so that
\[
\{\bar{x}(s,t) : s \in \mathbb{R}\} = \mathbb{R} \text{ for each } t \geq 0; \text{ thus } \pi \circ \#(\mathcal{P}) = H, \text{ and } u^\mathcal{P} \text{ is defined on all } H.
\]
QED 3.15.

(3.2) **Sheets**

We divide the integral surface into l-valued subsets.

**Definition.** $\mathcal{T} : \{s : (a \circ \psi)'(s) < 0\} \to \mathbb{R}$, $\mathcal{T}(s) = \frac{-1}{(a \circ \psi)'(s)}$.

**Note.** $v = \text{graph } (\mathcal{T})$ (by 3.14).

**Definition.** $\mathcal{P}^+ = \{(s,t) \in \mathcal{P} : t < \mathcal{T}(s)\}$. $\mathcal{P}^+ = \xi(\mathcal{P}^+)$. $\mathcal{P}^\dagger = \{(s,t) : \bar{x}_s(s,t) > 0\}$ (by 3.14).

**Note.** $\mathcal{P}^+ \subseteq C^2$. $\{s : (a \circ \psi)'(s) < 0, (a \circ \psi)''(s) = 0\}$ is the critical set of $\mathcal{T}$; assume here it is finite.

**Definition.** Suppose the local minima of $\mathcal{T}$ occur at $s^1 < s^2 < \ldots < s^n$, with $t^i = \mathcal{T}(s^i)$. Call each $t^i$ a "bifurcation-time", and $(s^i, t^i)$ a "bifurcation-point". Write
\[
\inf\{s : s > s^1, \mathcal{T}(s) = t^1\}, \text{ if this set is nonempty}
\]
\[
\{ -\infty \}, \text{ otherwise}
\]
\[ s^{-i} = \begin{cases} \max \{s : s \leq s^i, \tau(s) = t^i\}, & \text{if this set is nonempty} \\ -\infty, & \text{otherwise} \end{cases} \]

Define the "base" \( B^i_+ = (s^i_-, s^i_+) \times \{t^i\} \) and the base \( B^i_- = (s^i_-, s^i_+) \times \{t^i\} \). Also view the \( s \)-axis \( s \times t \) as a base, \( B^0 \).

A point \( p = (s, t) \in P^i \) is said to "lie over" a base \( (s_L^i, s_R^i) \times \{t_o\} \) iff \( s_L^i < s < s_R^i \) and \( t_o < t \). Among the bases over which \( P \) lies, there is one base \( B \) with largest \( t \)-coordinate and we say \( p \) "lies just over" \( B \).

Define the "sheet" over the base \( B^i_\pm \) as \( S^i_\pm = \{ p \in P^i : p \text{ lies just over } B^i_\pm \} \).

Order the sheets (except \( S^0 \)) by \( S^1_- < S^1_+ < \ldots < S^n_- < S^n_+ \).

Extend this terminology to \( P \) and to \( H \) via \( \psi \) and \( \pi \).
Notation. Fix sheets $S, S_1$. Write $S = \mathcal{S}(S), \sigma = \pi(S)$ and $S_1 = \mathcal{S}(S_1), \sigma_1 = \pi(S_1)$.

(3.21) Lemma. (Sheet structure in $P$).

(3.211) There exist $t_0, t_1$ (0 $\leq t_0 < t_1 \leq \infty$) such that $[t_0, t_1)$ is domain for $C^0$ paths $s_L(t) < s_R(t)$ (except we allow $s_L = -\infty$ or $s_R = +\infty$) with 

$S = \{(s, t) : t_0 \leq t < t_1, s_L(t) < s < s_R(t)\}.$

Call $s_L$ the "left side" of $s$, and $s_R$ the "right side".

(3.212) If $s_L$ (or $s_R$) is finite-valued, then it is $C^1$ on the complement of a finite set, with graph in $V$.

(3.213) $t_0$ is the bifurcation-time of $S$. Suppose $S_1 < S$ share a bifurcation-point $(s_0, t_0)$, and $\mathcal{S}_R : [t_0, t_2) \to R$ is the right side of $S_1$; then $s_{L}(t_0) = s_0 = s_R(t_0)$ and $x(s_L(t), t) < x(\mathcal{S}_R(t), t)$ for $t_0 < t < \min\{t_1, t_2\}$.

(3.214) $t_1$ is a bifurcation-time, or $t_1 = \infty$, or $s_{L}(t_1 - 0) = s_R(t_1 - 0)$.

Proof. If $S = S^0$, then $t_0 = 0$, $t_1 = \min\{t^i\}_1^n$, and $s_L = -\infty$, $s_R = +\infty$. Hence, assume $S = S^i_+$ for some $i > 0$ (similar proof for $S = S^i_-$). Recall $B^i_+ = (s^i_+, s^i_+ \times \{t^i\}$, and set $t_0 = t^i$ (bifurcation-time).

Set $\alpha = \sup\{s > s^i_+ : \tau \text{ increasing on } (s^i_+, s)\}$

$\beta = \inf\{s < s^i_+ : \tau \text{ decreasing on } (s^i_+, s^i_+)\}, \text{ if } s^i_+ < \infty$

$\beta = \infty, \text{ if } s^i_+ = \infty.$
\[ t_1 = \begin{cases} \infty, & \text{if } \text{domain}(\mathcal{T}) \cap [\alpha, \beta] \neq \emptyset \\ \inf \{ s : \alpha \leq s \leq \beta, \mathcal{T}(s) = t_1 \}, & \text{otherwise} \end{cases} \]

\[ \bar{\alpha} = \begin{cases} \alpha, & \text{if } t_1 < \infty \\ \infty, & \text{if } t_1 = \infty \end{cases} \]

\[ \bar{\beta} = \begin{cases} s^1, & \text{if } s^1 < \infty \\ \infty, & \text{if } s^1 = \infty \end{cases} \]

Note \( s^1 < \bar{\alpha} \leq \alpha \leq \beta \leq \bar{\beta} \leq s^1 < \infty \), and \( t_0 < t_1 \).
\( \tau \) is monotone on \([s^1, \bar{a}]\), hence has a \( C^0 \) inverse, \( s_L(t) \) for \( t_0 \leq t < t_1 \). If \( s^1_+ < \infty \), then also \( \bar{s} < s^1_+ \), and \( \tau \) is monotone on \((\bar{s}, s^1_1)\), with a \( C^0 \) inverse \( s_R(t) \) for \( t_0 \leq t < t_1 \); if \( s^1_1 = \infty \), set \( s_R(t) = \infty \) \((t_0 \leq t < t_1)\).

If \( s_L \) (or \( s_R \)) is finite-valued, then its graph is part of \( \text{graph}(\tau) = V \). It is \( C^1 \) except possibly at critical values of \( \tau \) (a finite set) by the inverse-function theorem.

QED 3.212.

Define 3 regions

\[ \Omega_1 = \{(s,t) : s^1 < s < \bar{a}, \ t_0 \leq t < \tau(s)\} \]
\[ \Omega_2 = \{(s,t) : \bar{a} \leq s \leq \bar{s}, \ t_0 \leq t < t_1\} \]
\[ \Omega_3 = \{(s,t) : \bar{s} < s < s^1_+, \ t_0 \leq t < \tau(s)\}. \]

It is easy to check \((k = 1,2,3)\) that \( p \in \Omega_k \) implies \( p \in P^+ \) and \( p \) lies just over \( B^+_1 \); thus, \( \cup \Omega_k \subset S \). Also note we may rewrite the sets \( \Omega_k \) as

\[ \Omega_1 = \{(s,t) : t_0 \leq t < t_1, \ s_L(t) < s < \bar{a}\} \]
\[ \Omega_2 = \{(s,t) : t_0 \leq t < t_1, \ \bar{a} \leq s \leq \bar{s}\} \]
\[ \Omega_3 = \{(s,t) : t_0 \leq t < t_1, \ \bar{s} < s < s_R(t)\}, \]

which shows that our advertised description of \( S \) is equivalent to \( "S = \cup \Omega_k" \), so we prove \( S \subset \cup \Omega_k \).

Let \( p = (\bar{s}, \bar{t}) \in P^+ \), \( p \) lies just over \( B^+_1 \) \((i.e., p \in S)\); we suppose \( p \notin \cup \Omega_k \) and derive a contradiction. "\( p \) lies
over \(B^1_+\) implies \(s^1 < \overset{\wedge}{s} < s^1_+\), \(\overset{\wedge}{s} \not\in (s^1, \overset{\wedge}{a})\), since otherwise \(t_0 < \overset{\wedge}{t} < \tau(\overset{\wedge}{s})\) (which follows from \(p \in P^+\) and \(p\) lying over \(B^1_+\)) contradicts \(p \not\in \Omega_1\). Similarly \(\overset{\wedge}{s} \not\in (\overset{\wedge}{s}, s^1_+)\), so we must have \(\overset{\wedge}{s} \in [\overset{\wedge}{a}, \overset{\wedge}{s}]\). Then \(p \not\in W_2\) implies \(t_1 < \overset{\wedge}{t}\), so \(t_1 < \infty\). Also note \(\overset{\wedge}{a} \neq \overset{\wedge}{s}\); otherwise \((\overset{\wedge}{a} = \overset{\wedge}{s} = \overset{\wedge}{s})\) \(\tau\) achieves a local maximum \((t_1)\) at \(\overset{\wedge}{s}\), and \(\tau(\overset{\wedge}{s}) = t_1 < \overset{\wedge}{t}\) contradicts \((\overset{\wedge}{s}, \overset{\wedge}{t}) \in P^+\).

Thus we are left with \(\overset{\wedge}{a} < \overset{\wedge}{s}\) and \(t_1 < \infty\). One may check that \(\tau \mid_{[\overset{\wedge}{a}, \overset{\wedge}{s}]\}\) assumes the minimum \(t_1\) at an interior point \(s_m \in (\overset{\wedge}{a}, \overset{\wedge}{s})\), and \((s_m, t_1)\) is a bifurcation-point generating 2 bases. \(p\) must lie over 1 of these bases, contradicting "\(p\) lies just over \(B^1_+\)."

QED 3.211.

If \(\overset{\wedge}{a} < \overset{\wedge}{s}\) and \(t_1 < \infty\), then we just saw \(t_1\) is a bifurcation-time. If \(\overset{\wedge}{a} = \overset{\wedge}{s}\), then \(s_L(t_1-0) = s_R(t_1-0)\) (if \(\tau\) continuous at \(\overset{\wedge}{a} = \overset{\wedge}{s}\)) or \(t_1 = \infty\) (if \(\tau\) discontinuous there).

QED 3.214.

Recall \(S = S^1_+\). Suppose now \(S^1_+ = S^1_-\). (We have written the bifurcation-point \((s^1, t^1)\) as \((s_0, t_0)\).) A mirror-image of the previous development for \(S\) applies to \(S^1_+\), so \(\overset{\wedge}{s}_R\) is the inverse of \(\tau\) on an interval \((\overset{\wedge}{a}, s^1)\) \(= (\overset{\wedge}{a}, s_0)\) where \(\tau\) is decreasing. Thus \(\overset{\wedge}{s}(t_0) = s_o = s_L(t_0)\). Also, if \(\overset{\wedge}{s}_R\), \(s_L\) both exist at
t_o < t_o, then τ is decreasing on \([s_R(t_3), s_o]\), increasing
on \([s_o, s_L(t_3)]\), so

\((s_R(t_3), s_L(t_3)) \times \{t_3\} \subseteq \{(s, t) \in P : t > \tau(s)\} = P - (P^t \cup V) =

\{(s, t) : \bar{x}_s(s, t) < 0\}, \text{ implying}

\bar{x}(s_R(t_3), t_3) > \bar{x}(s_L(t_3), t_3).

QED 3.213, 3.21.

(3.22) **Lemma.** \(s\) is 1-valued.

**Proof.** Otherwise \(\tau^s \circ \tau^q\) \(= 0\) for some \(p, q \in S\), \(p \neq q\).
\(\tau^s \circ \tau^q\) preserves the \(t\)-coordinate, so \(p = (s_1, t_1)\),
\(q = (s_2, t_1)\) with \(s_1 \neq s_2\) (say \(s_1 < s_2\)). 3.211 shows

\([s_1, s_2] \times \{t_1\} \subseteq S \subseteq P^t = \{(s, t) : \bar{x}_s > 0\}, \text{ so}

\bar{x}(s_1, t_1) < \bar{x}(s_2, t_1), \text{ contradicting } \tau^s \circ \tau^q\)^{(s_1, t_1)}(s_2, t_2) = 0.

QED 3.22.

(3.23) **Lemma.** (Sheet Structure in \(H\)).

(3.231) There exist \(t_o, t_1\) \((0 < t_0 < t_1 \leq \infty)\) such that
\([t_o, t_1]\) is domain for \(C^1\) paths \(x_L(t) < x_R(t)\) (except
we allow \(x_L = -\infty\) or \(x_R = +\infty\) with

\[\sigma = \{(x, t) : t_o < t < t_1, x_L(t) < x < x_R(t)\}\].

Call \(x_L\) the "left side" of \(\sigma\), and \(x_R\) the "right side".

(3.232) Extend \(u^3\) by continuity to \(\text{graph}(x_L) \cup \text{graph}(x_R)\).
Then $x_L'(t) = au(s)(x_L(t), t)$ and $x_R'(t) = au(s)(x_R(s), t)$.

(3.233) $t_o$ is the bifurcation-time of $s$. Suppose $s_1 < s$

share a bifurcation-point $(x_o, t_o)$, and $\hat{x}_R : [t_o, t_2) \to R$
is the right side of $s_1$; then $x_L(t_o) = x_o = \hat{x}_R(t_o)$
and $x_L(t) < \hat{x}_R(t)$ for $t_o < t < \min(t_1, t_2]$.

(3.234) $t_1$ is a bifurcation-time, or $t_1 = \infty$, or

$x_L(t_1-0) = x_R(t_1-0)$.

Proof. Use notation of 3.21. Define $x_L = -\infty$ if

$s_L = -\infty$, and $x_R = \infty$ if $s_R = \infty$; otherwise $x_L(t) = \bar{x}(s_L(t), t)$
and $x_R(t) = \bar{x}(s_R(t), t)$. Note for $t_3 \in (t_o, t_1)$ that

$\bar{x}((s, t_3) : s_L(t_3) < s < s_R(t_3)) = \{(x, t_3) : x_L(t_3) < x < x_R(t_3)\}$
since $\bar{x}_s > 0$ here. Thus

$\sigma = \pi \circ \bar{x} = \pi \circ \bar{x} \left( \bigcup_{t_3 \in [t_o, t_1]} \{(s, t_3) : s_L(t_3) < s < s_R(t_3)\} \right) =

\bigcup_{t_3} \pi \circ \bar{x}((s, t_3) : s_L(t_3) < s < s_R(t_3)) =

\bigcup_{t_3} (\bar{x}((s, t_3) : s_L(t_3) < s < s_R(t_3)), t_3) = \bigcup_{t_3} \{(x, t_3) : x_L(t_3) < x < x_R(t_3)\} =

\{(x, t) : t_o < t < t_1, x_L(t) < x < x_R(t)\}$, as promised.

Except for a finite set $\{t_1\} \in \mathbb{N}$ where $s_L(t)$ is not
differentiable, the chain rule gives

$x_L'(t) = \bar{x}_s(s_L(t), t) \cdot s_L'(t) + \bar{x}_t(s_L(t), t)$.

$\bar{x}_s(s_L(t), t) = 0$ (by 3.212 and 3.14), and 3.12 implies
\[ x_t(s_L(t), t) = a \circ \tilde{u}(s_L(t), t) = a \circ u^s(\tilde{x}(s_L(t), t), t) = a \circ u^s(x_L(t), t). \] Thus
\[ x_L'(t) = a \circ u^s(x_L(t), t) \quad \text{for} \quad t \in [t_0, t_1) - \{\tau_i\}_1^N. \]

But \( x_L(t) \) is \( C^0 \) (being composed of \( C^0 \) functions) and
\[ \lim_{t \to \tau_i} x_L'(t) = a \circ u^s(x_L(\tau_i), \tau_i) \quad \text{for each} \quad \tau_i, \quad \text{so that} \quad x_L' \]
exists and is continuous at \( \tau_i \).

QED 3.231, 3.232.


QED 3.23.

(3.24) Lemma. Assume (a) increasing. \( s_1 < s \) iff
\( u^s_1 > u^s \) wherever both defined, except a common bifurcation-point. If \( \sigma_1, \sigma \) share a bifurcation-point \( p \), then
\( u^s(p) = u^s_1(p) \).

Proof. Let \( x_L, x_R \) be the left and right sides of \( \sigma \), and
analogously \( \hat{x}_L, \hat{x}_R \) for \( \sigma_1 \). Recall
\[ \text{domain}(u^s) = \sigma \cup \text{graph}(x_L) \cup \text{graph}(x_R), \quad \text{and similarly for} \quad s_1. \]

Claim 1: Let \( p \in \sigma_1 \cap \sigma \). Then \( s_1 < s \) iff
\( u^s_1(p) > u^s(p) \).

Proof. Let \( q \in S, q_1 \in s_1 \) be the unique points such that
\( \tau \circ \delta(q) = p = \tau \circ \delta(q_1) \). These have the same \( t \)-coordinate:
\( q = (s_0, t_0), q_1 = (s_1, t_0) \). Note \( s_1 < s \) iff \( s_1 < s_0 \); also
\[ u^g(p) = \bar{u}(q) = \psi(s_0) \] and \[ u^g_1(p) = \bar{u}(q_1) = \psi(s_1). \] Thus, it suffices to show \( s_1 < s_0 \) iff \( \psi^g_1 > 0. \)

\[ \pi \circ \psi^g_1 q_1 - q = 0, \] so \( x^g_1 - q = 0, \) so \( t \cdot a^g(s_1) + s_1 = t_0 \cdot a^g(s_0) + s_0, \) so \( t_0 \cdot a^g_1 s_0 - s_1. \) Thus \( s_1 < s_0 \) iff \( t_0 \cdot a^g_1 s_0 > 0, \) iff \( \psi^g_1 > 0. \) (use \( t_0 > 0 \) and (a) increasing).

Claim 2: Suppose \( s_1 < s. \) Then \( u^g_1(p) = u^g(p) \) iff \( p \) is the bifurcation point of both \( \sigma_1, \sigma. \)

**Proof.** If \( p \in H \) is a common bifurcation point, then

(permitting \( p_0 \) be the corresponding bifurcation point in \( P \))

\[ u^g(p) = \bar{u}(p_0) = \bar{u}^g_1(p). \] Conversely, suppose \( p = (x_0, t_0) \)

with \( u^g_1(p) = u^g(p) = u_0. \) Then

\[ \xi^{-1}(x_0, t_0, u_0) \in (\text{graph}(s_L) \cup \text{graph}(s_R)) \cap (\text{graph}(\hat{S}_L) \cup \text{graph}(\hat{S}_R)), \]

which is possible only if \( \xi^{-1}(x_0, t_0, u_0) \) is a common bifurcation point (in \( P \)) for \( S_1, S. \)

QED 3.24.

(3.3) **Shocks**

The Rankine-Hugoniot condition will choose the discontinuity-path to mark off separate domains for 2 sheets which disagree about \( u(x, t). \)

**Definition.** Suppose \( (x_0, t_0) = p \in \sigma_1 \cap \sigma, \) or \( p \) is the common bifurcation point (in \( H \)) of \( \sigma_1, \sigma. \) Let \( \gamma(t) \) be the
unique maximal path solving
\[ \gamma'(t) = a^u \eta^{s}_t (\gamma(t), t) \] and \( \gamma(t_0) = x_0 \).

We say \( \gamma \) "solves \( (s_1, s, p) \)."

(3.31) **Lemma.** Use above notation. Assume (a) increasing and \( s_1 < s \). Write \( x_L, x_R \) as left and right sides of \( \sigma \), and \( \hat{x}_L, \hat{x}_R \) for \( \sigma_1 \).

(3.311) \( \gamma(t) \) exists for small \( t - t_0 > 0 \).

(3.312) \( x_L(t) < \gamma(t) < \hat{x}_R(t) \) for \( t > t_0 \) when \( \gamma(t) \) exists.

(3.313) Suppose also \( s < s_2 \), \( \tilde{\gamma} \) solves \( (s, s_2, (x_1, t_0)) \), and \( x_0 < x_1 \). If \( \gamma(t_1) = x_L(t_1) \) or \( \tilde{\gamma}(t_1) = x_L(t_1) \) for \( t_1 > t_0 \), then \( \gamma(t_2) = \tilde{\gamma}(t_2) \) for some \( t_2 \in (t_0, t_1) \).

**Proof.** Let \( \text{domain}(x_R) = [a, \beta] \) and \( \text{domain}(\hat{x}_L) = [a_1, \beta_1] \).

Note \( t_0 \in [a, \beta] \cap [a_1, \beta_1] = [\max(a, a_1), \min(\beta, \beta_1)] = [\tilde{a}, \tilde{\beta}] \).

**Claim 1:** \( \sigma_1 \cap \sigma = \{(x, t) : \tilde{a} \leq t < \tilde{\beta}, \hat{x}_L(t) < x < \hat{x}_R(t), x_L(t) < x < x_R(t) \} \).

**Proof:** 3.231.

**Claim 2:** If \( p \in \sigma_1 \cap \sigma \), then 3.311 holds.

**Proof.** The vector \((dx/dt) = (a^1, 1)\) at \( p \) points into \( \sigma_1 \cap \sigma \) (by Claim 1) so the vector field \((a^1, 1)\) has an integral curve through \( p \) which runs for some time past \( t_0 \).

**Proof.** Define \( W = \{(x, t) : t_0 < t \leq t_2, x_L(t) < x < x_R(t) \} \).
Claim 2 shows $x_L \triangleleft \check X_R$ on $(t_0, t_2]$. Claims 2 and 3 give $W \subset \sigma_1 \cap \sigma_2$. For $n \in \mathbb{Z}_+$, $n > \frac{1}{t_2 - t_0}$, define

$$p_n = \frac{x_L(t_0 + \frac{1}{n}) + \check X_R(t_0 + \frac{1}{n})}{2}, \quad t_0 + \frac{1}{n} \in W. \quad \text{Claim 5 shows}$$

$$\lim_{n \to \infty} p_n = p.$$

$(g_1, g_2, p_n)$ has a solution $\gamma_n$ in $W$, which (by claim 2) runs for some $t > t_0 + \frac{1}{n}$, until it intersects the boundary, $\partial W \subset \text{graph}(x_L) \cup \text{graph}(\check X_R) \cup R \times \{t_2\}$. Claim 4 shows $\gamma_n$ intersects neither $x_L$ nor $\check X_R$, hence must exist until $t_2$.

Now define $\gamma : [t_0, t_2] \to R$ by

$$\gamma(t) = \begin{cases} 
 x_0, & \text{if } t = t_0 \\
 \lim_{n \to \infty} \gamma_n(t), & \text{if } t_0 < t \leq t_2
\end{cases}$$

It is easy to prove $\gamma$ is $C^1$, solves $(g_1, g_2, p)$, and $x_L(t) \leq \gamma(t) \leq \check X_R(t)$.

We show $x_L(t) < \gamma(t) < \check X_R(t)$ for $t_0 < t \leq t_2$. Suppose not, say $x_L(t_3) = \gamma(t_3)$, $t_0 < t_3 \leq t_2$. Then Claim 4 implies $x_L(t) = \gamma(t)$ for all $t \in [t_0, t_3]$, so

$$\gamma'(t_3) = x_L'(t_3) = a \circ u^g(x_L(t_3), t_3) = a = \gamma'(t_3),$$

contradiction.  

QED 3.311, 3.312.

Apply the intermediate-value theorem to $\gamma - \check \gamma$ on $[t_0, t_1]$. $(\gamma - \check \gamma)(t_0) < 0$ since $x_0 < x_1$. In case

$\gamma(t_1) = \check X_R(t_1)$, then $(\gamma - \check \gamma)(t_1) = (x_R - \check \gamma)(t_1) > 0$ by 3.312 applied to $(g, g(t_1), (x_1, t_0))$.  

QED 3.31.
Claim 3: \( a \cdot u_1 > a \cdot u_2 > a \cdot u_3 \), except equality holds at a common bifurcation-point.

Proof: 3.24 and (a) increasing.

Claim 4: If \( p \in \sigma_1 \cap \sigma \), then 3.312 holds.

Proof. We show \( x_L < \gamma (\gamma < \hat{x}_R \) is similar). Suppose not; let \( t_1 = \min \{ t \geq t_0 : x_L(t) = \gamma(t) \} > t_0 \). Note \( x_L(t) < \gamma(t) \) for \( t_0 \leq t < t_1 \), by Claim 1. Thus

\[
\gamma'(t_1) = \lim_{t \to t_1 - 0} \frac{\gamma(t_1) - \gamma(t)}{t_1 - t} \leq \lim_{t \to t_1 - 0} \frac{x_L(t_1) - x_L(t)}{t_1 - t} = \frac{u_3}{u_1}(x_L(t_1), t_1) = \gamma'(t_1),
\]

contradiction! (We used 3.232 and Claim 3.)

Claim 5: Suppose \( p \) is a bifurcation-point for both \( \sigma_1, \sigma \). Then \( x_L(t_0) = x_o = \hat{x}_R(t_0) \). Also there exists \( t_2 > t_0 \) such that on \( [t_0, t_2] \) all \( \{ x_L, x_R, \hat{x}_L, \hat{x}_R \} \) are defined and on \( (t_0, t_2) \) they satisfy \( \hat{x}_L < x_L < \hat{x}_R < x_R \).

Proof. 3.233 gives \( x_L(t_0) = x_o = \hat{x}_R(t_0) \), and 3.231 gives \( x_L < x_R, \hat{x}_L < \hat{x}_R \); thus \( \hat{x}_L(t_0) < x_L(t_0), \hat{x}_R(t_0) < x_R(t_0) \).

By continuity these inequalities persist for small time past \( t_0 \). Finally, 3.233 gives \( x_L < \hat{x}_R \).

Claim 6: Suppose \( p \) is a bifurcation-point for both \( \sigma_1, \sigma \). Let \( t_2 \) as in Claim 5. Then \( (s_1, s_2, p) \) has a solution \( \gamma \).
with \([t_0, t_2] \subseteq \text{domain}(\gamma)\) and \(x_L(t) < \gamma(t) < x_R(t)\) for 
\(t_0 < t \leq t_2\).

(3.4) Algorithm for Constructing Solution

(3.41) **Lemma.** \(\lim\inf_{|s| \to \infty} \frac{a \circ \phi(s)}{s} \geq 0\) implies \(\pi \circ \phi : \mathcal{P} \to \mathcal{H}\) is onto and proper.

**Proof.** \(\bar{x}(s, t) = s(t \cdot \frac{a \circ \phi(s)}{s} + 1)\), so for \(t\) fixed

\[
\lim_{s \to \infty} \bar{x}(s, t) = \infty, \quad \lim_{s \to -\infty} \bar{x}(s, t) = -\infty.\]

Thus \(\pi \circ \phi(\mathcal{P}) = \mathcal{H}\).

Showing \(\pi \circ \phi\) proper reduces to proving that for \(M > 0\) there exists \(\bar{M}\) such that if \(t < M\) and \(|s| > \bar{M}\), then

\(|\bar{x}(s, t)| > M\). Indeed, choose \(\bar{M} > 2M\) such that \(|s| > \bar{M}\) implies \(\frac{a \circ \phi(s)}{s} > -\frac{1}{2\bar{M}}\). If \(t < M\) and \(|s| > M\), then

\[-\frac{1}{2\bar{M}} > -\frac{1}{2t}, \quad \frac{a \circ \phi(s)}{s} > -\frac{1}{2t}, \quad \text{and} \quad t \frac{a \circ \phi(s)}{s} + 1 > \frac{1}{2};\]

thus

\(|\bar{x}(s, t)| = |s| |t \frac{a \circ \phi(s)}{s} + 1| > \bar{M} \cdot \frac{1}{2} > M.\)

QED 3.41.

Fix \(a \in \mathcal{C}^2\), increasing.

**Definition.** \(\mathcal{J}_0\) (respectively, \(\mathcal{J}\)) is the set of \(\psi \in \mathcal{C}^2\) such that \(\lim\inf_{|s| \to \infty} \frac{a \circ \psi(s)}{s} \geq 0\) and

\[\text{5Given } \epsilon > 0, \text{ there exists } M \text{ such that } |s| > M \text{ implies } \frac{a \circ \psi(s)}{s} > -\epsilon.\]
\[ \{ s : (a \psi)'(s) < 0, (a \psi)''(s) = 0 \} \text{ is finite (resp., discrete)}. \]

(3.42) **Theorem.** IVP(\(\psi\)) has a finite-shock solution for \(\psi \in \mathfrak{F}_0\), generated in finitely many steps by the following algorithm.

Define inductively a sequence \((n = 0, 1, \ldots)\) of pairs

\[
\begin{cases}
T_n \in [0, \infty] \\
a \text{finite-shock solution } u(x, t) \text{ for } t < T_n \text{ (for } t \leq T_n \text{ if } T_n < \infty) \text{ with graph in } \mathbb{R}^+.
\end{cases}
\]

\(T_0 = 0, u(x, 0) = \psi(x)\). Given the \(n^{th}\) pair with \(T_n < \infty\), construct the \((n+1)^{st}\) as follows:

Let \(x_1 < x_2 < \ldots < x_k\) be the points at which \(u(\cdot, T_n)\) discontinuous or \(u_x(\cdot, T_n) = \pm \infty\); \((x_i, T_n)\) belongs to one or more previous shocks, or is a bifurcation-point in \(\mathcal{H}\).

For each \(i \in \{0, 1, \ldots, k\}\), the restriction \(u(\cdot, T_n) \big|_{(x_i, x_{i+1})}\) has graph belonging to a unique sheet \(s_i\) (we used \(x_0 = -\infty, x_{k+1} = \infty\)). Let the path \(\gamma_i\) solve \((s_{i-1}, s_i, (x_i, T_n))\), for \(1 \leq i \leq k\). Set

\[
T_{n+1} = \begin{cases}
\min( \bigcup_{i=1}^{k} \{ t > T_n : \gamma_i(t) = \gamma_{i-1}(t) \} ) \cup \{ \text{bifurcation-times } > T_n \} \\
\infty, \text{ if above set is } \emptyset.
\end{cases}
\]

Take \(u(x, t)\) as defined for \(t \leq T_n\). For \(T_n < t < T_{n+1}\), set
\[
\begin{align*}
    u(x, t) = \begin{cases} 
    u^0(x, t), & \text{if } x < \gamma_1(t) \\
    u^1(x, t), & \text{if } \gamma_1(t) < x < \gamma_2(t) \\
    \vdots \\
    u^k(x, t), & \text{if } \gamma_k(t) < x
    \end{cases}
\end{align*}
\]

If \( T_{n+1} \leq \infty \), define \( u(x, T_{n+1}) = u(x, T_{n+1}^-) \) for all but finitely many \( x \) where this limit or \( u(x, T_{n+1}^-) \) doesn't exist.

**Proof.** Suppose given the \( n \)th solution-pair with \( T_n < \infty \), "\( x_1 < x_2 < \ldots < x_k \)" is really defined since \( u : \mathbb{R} \times [0, T_n] \to \mathbb{R} \) is a finite-shock solution so only finitely many shocks hit \( \mathbb{R} \times \{T_n\} \). Also, because \( \text{graph}(u) \subset P^+ \), by 3.24 see \( (x_1, T_n) \) is a bifurcation-point if not part of a shock. Claims 1 \( \rightarrow \) 7 show the algorithm does produce the \((n+1)\)st solution-pair.

**Claim 1:** \( \{s_i\}_{i=1}^{k} \) are well-defined.

**Proof.** Let \( G = \text{graph}(u)|_{(x_1, x_{i+1})} \). \( G \) is connected, and so is \( \Phi^{-1}(G) \), by 3.13. Note \( \Phi^{-1}(G) \subset \mathbb{R} \times \{T_n\} \). A connected subset of \( \mathbb{R} \times \{T_n\} \) is an interval:

\( \Phi^{-1}(G) = I \times \{T_n\} \). \( G \subset P^+ \) (use 3.14) so \( I \times \{T_n\} \subset P^+ \) and belongs to a single sheet (use 3.21).

**Claim 2:** \( s_{i-1} < s_1 \).
Proof. The shock condition 2.31 holds for \( u : R \times [0,T_n] \to R \), implying \( u(x-0,T_n) \geq u(x+0,T_n) \) for all \( x \), in particular \( s_i^{-1}(x_i,T_n) \geq s^-i(x_i,T_n) \). If \( s_i^{-1}(x_i,T_n) > s^-i(x_i,T_n) \), then 3.24 gives \( s_i^{-1} < s_i \).

There remains \( u(x_i,T_n) = \hat{u} \). Then

Then \( u_x(x_i,T_n) = \pm \infty \), so \( (x_i,T_n,\hat{u}) \in \mathcal{V} \). Write

\[(\hat{s},T_n) = s_i^{-1}(x_i,T_n,\hat{u}) \in \mathcal{V}.\]Because \( s^{-i},s_i \subset \mathbb{P}^+ \)

\[\{(s,t) : x_s(s,t) > 0\},\]we see for small \( \varepsilon > 0 \) that

\[(\hat{s} - \varepsilon, T_n) \in s_i^{-1}, (\hat{s} + \varepsilon, T_n) \in s_i.\]Thus \( s_i^{-1}, s_i \) are

distinct sheets (use 3.21l and \( (\hat{s},T_n) \in \mathcal{V}) \). \((\hat{s},T_n) \) is

the bifurcation-point of both \( s_i^{-1}, s_i \) (by 3.24);

\( s_i^{-1}, s_i \) are respectively the minus-sheet and plus-sheet

attached to this bifurcation-point, so \( s_i^{-1} < s_i \).

QED Claim 2.

Let \( \tau = \sup(\text{domain}(\gamma_i)). \) Use 3.23 to write

\[\sigma_i = \pi(s_i) = \{(x,t) : t_0 \leq t < t_1, x_L(t) < x < x_R(t)\}\]

\[\sigma_i^{-1} = \pi(s_i^{-1}) = \{(x,t) : \hat{t}_0 \leq t < \hat{t}_1, \hat{x}_L(t) < x < \hat{x}_R(t)\}\]

\[\sigma_i^{-1} \cap \sigma_i = \{(x,t) : \alpha < t < \beta, x_L(t) < x < x_R(t), \hat{x}_L(t) < x < \hat{x}_R(t)\}\]

where \( \alpha = \max(t_0,\hat{t}_0), \beta = \min(t_1,\hat{t}_1) \). Note \( \sigma_i^{-1} \cap \sigma_i \neq \emptyset \),

since \( (x_i,T_n) \in \sigma_i^{-1} \cap \sigma_i \), or is a bifurcation-point for

both \( \sigma_i^{-1}, \sigma_i \).
Claim 3: If $\tau < \infty$, then $\sigma_{i-1} \cap \sigma_i \cap R \times [T_n, \tau)$ is bounded.

Proof. Otherwise both $[x_L, \hat{x}_L]$ or both $[x_R, \hat{x}_R]$ are unbounded on $[T_n, \tau)$. Suppose $x_L, \hat{x}_L$ unbounded on $[T_n, \tau)$; at least one is finite-valued, say $x_L$. Recall $x_L(t) = \bar{x}(s_L(t), t)$, where $s_L$ is inverse to a monotone portion of $\bar{y}_0$. $s_L([T_n, \tau))$ is an unbounded interval $I$. $\bar{y}$ is monotone on $I$ and $\bar{y}(I) = [T_n, \tau)$; but then

$I \times \{\tau\} \subset \{(s, t): t > \bar{y}(s)\} = \{(s, t): \bar{x}_s(s, t) < 0\}$, violating 3.42.

Claim 4: $T_{n+1} \leq \tau$.

Proof. True if $\tau = \infty$, hence assume $\tau < \infty$. Claim 4 shows $(y_1(\tau-0), \tau) \in \text{boundary}(\sigma_{i-1} \cap \sigma_i)$, so three cases arise.

Case 1: $\tau = t_1$ or $\hat{t}_1$.

Use 3.234. If $\tau$ is a bifurcation-time, then $T_{n+1} \leq \tau$.

Else $x_L(\tau-0) = y_1(\tau-0) = x_R(\tau-0)$, or $\hat{x}_L(\tau-0) = y_1(\tau-0) = \hat{x}_R(\tau-0)$; but $x_L(\tau-0) < y_1(\tau-0) < \hat{x}_R(\tau-0)$ by the same proof as for 3.312 (use Claim 3).

Case 2: $y_1(\tau) = x_L(\tau)$ or $\hat{x}_R(\tau)$.

Impossible by 3.312.

Case 3: $y_1(\tau) = x_R(\tau)$ or $\hat{x}_L(\tau)$.

3.313 shows $y_1$ intersects $y_{i-1}$ or $y_{i+1}$ at some time before $\tau$.

Claim 5: $u$ is defined on $R \times (T_n, T_{n+1})$. 
Proof. 3.22 and Claim 4 show $u$ defined on
\[ \{(x,t) : T_n < t < T_{n+1}, \gamma_1(t) < x < \gamma_k(t)\} \] 3.41 shows
$S_1, S_k$ cover $(x,t)$ for $|x|$ large.

Claim 6: Suppose $T_{n+1} < \infty$. $u(x,T_{n+1})$ is defined and
$(x,T_{n+1}, u(x,T_{n+1})) \in \mathcal{P}^+$, for all but finitely many $x$.

Proof. $u(x,T_{n+1} - 0)$ exists for $x \in \mathbb{R} - \{\gamma_1(T_{n+1} - 0)\}^k_{i=1}$.
$\mathcal{P}^+$ is open in $\mathcal{P}$ with boundary $\mathcal{V}$, and
\[ \text{graph}(u : \mathbb{R} \times [0,T_n] \to \mathbb{R}) \subset \mathcal{P}^+ \], so that (if defined)
$(x,T_{n+1}, u(x,T_{n+1} - 0)) \in \mathcal{P}^+ \cup \mathcal{V}$. 3.24 shows that, this
limit is in $\mathcal{V}$ (i.e., $u_x = \pm \infty$) iff it is a bifurcation-
point in $\mathcal{P}$; there are only finitely many such points.

Claim 7: $u$ is a finite-shock solution for $t < T_{n+1}$
(for $t \leq T_{n+1}$ if $T_{n+1} < \infty$) with graph in $\mathcal{P}^+$.

Proof. Each sheet $S_1 \subset \mathcal{P}^+$, and Claim 6, give
\[ \text{graph}(u) \subset \mathcal{P}^+ \]. graph(u) was constructed as an open
subset of $\mathcal{P}^+$, so use 3.152 off the discontinuity set
(a finite union of graphs of paths) to see $u$ is a piece-
wise $C^1$ solution. By construction, each discontinuity
path obeys the Rankine-Hugoniot condition ($\gamma' = a_1$). The
shock condition 2.31 holds, by Claim 2 and 3.24.

Claim 8: Algorithm ends ($T_N = \infty$ for some $N \in \mathbb{Z}_+$) yielding
a global, finite-shock solution.
Proof. The number of bifurcation points (and bifurcation-times) is finite, by 3.411. A shock may start only at a bifurcation-point, or at the intersection of 2 shocks. No shocks exist before the first (smallest) bifurcation-time and only finitely many appear at the first such time. Between one bifurcation-time and the next, only finitely many shocks arise by intersection because only 1 shock emerges from the intersection of several. Similarly, only finitely many arise after the last bifurcation-time.

QED 3.42.

(3.43) Corollary. IVP(ψ) has a shock solution for ψ ∈ Φ.

Proof. We define the solution on each compact subset $K_n = \{(x,t) : |x| = n, t ≤ n\} \subset H$, n > 0. By 3.11 $(π \circ \psi)^{-1}(K_n)$ is compact; the algorithm 3.42 may be modified to apply only to this set, producing a solution over $K_n$ with only finitely many shocks there.

4. Shock Structure

Let $u$ solve IVP(ψ), ψ ∈ Φ.

By construction, $u$ is as smooth off shocks as aψ; shocks are also this smooth. All directional derivatives exist at each $p \in H$, except possibly in the direction $\frac{dx}{dt} = γ'$ if $p$ belongs to a shock γ. The solution and its
derivatives are bounded on bounded sets, except \( u_x = \infty \) at isolated bifurcation points.

A shock begins at a bifurcation point or at the intersection of 2 previous shocks. When 2 or more shocks meet, just 1 new shock emerges. Thus a shock may be identified by its first endpoint.

**Definition.** "Shock structure" is the partially ordered collection of shocks: \( \gamma < \gamma' \) if there is a sequence of shocks \( \gamma = \gamma_1, \gamma_2, \ldots, \gamma_n = \gamma' \) such that the last endpoint of \( \gamma_i \) is the first endpoint of \( \gamma_{i+1} \) \((1 \leq i < n)\). Shock structure is "normal" if no 3 shocks share a last endpoint.

(4.1) **Theorem.** If \( \psi \in \mathcal{F}_o \), define

\[
\begin{align*}
d(t) &= \# \{ x : u(\cdot,t) \text{ or } u_x(\cdot,t) \text{ discontinuous at } x \} \\
b(t) &= \# \{ s : (a \circ \psi)'(s) \leq \frac{-1}{t}, \quad (a \circ \psi)''(s) = 0 \} \quad (b(0) = 0), \quad \text{and} \\
\bar{b} &= \# \{ s : (a \circ \psi)'(s) < 0, \quad (a \circ \psi)''(s) = 0 \}. 
\end{align*}
\]

Then \( d(t) \leq b(t) \leq \bar{b} \), with \( d(t) \) eventually constant. The total number of shocks is \( < 2^\bar{b} \).

**Proof.** \( b(t) \) counts the bifurcation-points occurring before \( t \), eventually constant at \( \bar{b} \), the total number of such points.

By induction on \( n \) we show \( d(t) \leq b(t) \) for \( 0 \leq t \leq T_n \) (notation of 3.42). True for \( n = 0 \). Assume it for \( n \), we prove it for \( n+1 \). \( d(t), b(t) \) both constant for \( T_n \leq t < T_{n+1} \), by our algorithm. \( T_{n+1} \) is a bifurcation-
time or the intersection-time of 2 shocks. For each bifurcation-point appearing then, \( b(t) \) grows by 1 while \( d(t) \) grows by 1 or 0. Each shock-intersection doesn’t change \( b(t) \) but decreases \( d(t) \) by 1.

\( d(t) \) becomes constant after all bifurcation-times and shock intersections.

The minimal elements of the shock structure are those shocks emerging from bifurcation-points. Each shock determines a set of minimal elements \( \leq \) that shock; distinct shocks determine distinct sets of minimal elements. There are \( \leq 5 \) minimal elements, hence \( \leq 2^5 \) possible sets, which is a gross upper bound on the number of shocks.

QED 4.1.

5. **Genericity**

**Notation.** (i) A subset of a topological space is "generic" if it contains a dense open set; it is "residual" if it is a countable intersection of generic sets.

(ii) \( C^3 \) is topologized by uniform convergence on compacta of the function and 3 derivatives. \( C^3_c \) (compact support) and \( C^3_{[\alpha, \beta]} \) (support in \([\alpha, \beta]\)) are subspaces.

**Note.** Residual subsets of such function-spaces are dense, by Baire’s theorem.

(5.1) **Theorem.** Let \( a \in C^3 \), increasing. IVP(\( \psi \)) has a
(5.11) finite-shock solution for generic \( \psi \in C_{\alpha, \beta}^3 \),
with normal shock-structure for a residual subset.

(5.12) normal, finite-shock solution for residual \( \tilde{\psi} \in C_{\beta}^3 \).
(5.13) normal, shock solution for residual
\( \tilde{\psi} \in \{ \tilde{\psi} \in C^3 : \lim_{|s| \to \infty} \inf \frac{a \tilde{\psi}(s)}{s} \geq 0 \} \).

Proof. \( n = \{ \psi \in \mathcal{S} : \text{the solution to IVP}(\psi) \text{ is normal} \} \)
is residual in \( \mathcal{S} \). Proof: \( n = \cap_{j=1}^{\infty} n_j \), where \( n_j = \{ \psi \in \mathcal{S} : \psi \mid [-j,j] \)
generates normal shock structure),

\( n \) generic in \( \mathcal{S} \).

The map \( A, A(\psi) = a \psi \), is a homeomorphism onto its image, if domain(\( A \)) is any subspace of \( C^2 \).

Proof: \( A \) is continuous. So is \( A^{-1} \), because if \( \{ \psi_n \}^\infty_1 \)
is such that \( a \psi_n \) approaches \( a \psi \) in \( C^2 \) (or \( C^3 \)), then \( \psi_n \)
approaches \( \psi \) in \( C^2 \) (or \( C^3 \)); use \( a' \) positive almost every-
where, and \( (a \psi_n)'(s) = a' \psi_n(s) \cdot \psi_n'(s) \) and similarly for \((a \psi_n)''\), \((a \psi_n)'''\).

\( S = \{ g : (g^{-1}(0) \cap [\alpha, \beta] \text{ is finite} \} \) is generic in \( C^3 \), hence \( A^{-1}(S) \) is generic in \( C^3 \), and in \( C_{\alpha, \beta}^3 \). But
\( A^{-1}(S) \subset \mathcal{S}_o \).

\( \text{QED } 5.11. \)

\( S_n = \{ g : (g^{-1}(0) \cap [-n,n] \text{ is finite} \} \) is generic in \( C^3 \), hence \( A^{-1}(S_n) \) is generic in \( C_{o}^3 \), and \( \cap_{n=1}^{\infty} A^{-1}(S_n) \)
residual in $C^3_c$. But $A^{-1}(S_n) \subset \mathfrak{F}_o$, all $n$.

QED 5.12.

Similarly, $A^{-1}(S_n)$ is generic in $C^3$, and
\[ \bigcap_{1}^{\infty} A^{-1}(S_n) \text{ residual; but} \]
\[ \{ \psi \in A^{-1}(S_n) : \liminf_{|s| \to \infty} \frac{a\psi(s)}{s} = 0 \} \subset \mathfrak{F}. \]

QED 5.13.

6. Stability

6.1 Theorem. Let $a \in C^3$, increasing. Let $(\psi_n)_{n=1}^{\infty}$,
$\psi \in \mathfrak{F} \cap C^3$ with solutions $u_n$, $u$ to IVP($\psi_n$), IVP($\psi$).
Suppose $\psi_n$ approaches $\psi$ uniformly, with derivatives converging uniformly on compacta. Fix compact $K = [-k,k] \times [0,k] \subset \mathbb{R}$.

6.11 Let $\epsilon > 0$. For large $n$, shock structures of $u,u_n$ restricted to $K$ are isomorphic, i.e., shocks of $u,u_n$ intersecting $K$ correspond one-to-one $(\gamma_1^n) \leftrightarrow (\gamma_1^n)$, and $\gamma_1 < \gamma_j$ iff $\gamma_1^n < \gamma_j^n$. Also corresponding shock-endpoints are less than $\epsilon$ distance apart in $K$, and
$|\gamma_1(t) - \gamma_1^n(t)| < \epsilon$ where defined.

6.12 $\lim_{n \to \infty} (\text{measure}\{ (x,t) \in K : |(u-u_n)(x,t)| > \epsilon \}) = 0$.

6.13 $\lim_{n \to \infty} \int_{K} |u-u_n| \, dx \, dt = 0$.

Proof. Because $\psi_n \to \psi$ uniformly, there exists $M$ such that
for large $n$, we have $u_n |_{K}, u |_{K}$ depending only on $\psi_n |_{(-M,M)}$, $\psi |_{(-M,M)}$ (recall characteristic speed is $a \circ \psi_n$, $a \circ \psi$). List

$$\{s_i \} \downarrow_1 = \{ s : |s| < M, (a \circ \psi)'(s) < 0, (a \circ \psi)''(s) = 0 \}$$

with $s_i < s_i^{i+1}$; similarly list $\{ s_n \} \downarrow_{i=1}$ for $\psi_n$, $j = j_n$ for $n$ large, and the relevant bifurcation points $\{(s_i, \frac{-1}{(a \circ \psi)'(s_i)}) \} \downarrow_1$, for $\psi$ correspond to those for $\psi_n$, as do their sheets (which cover $K$). Corresponding sheets cover almost the same subset of $K$, and give almost equal $u$-values where both defined. The algorithm 3.42 for IVP($\psi_n$), IVP($\psi$) gives initial-value problems for ordinary differential equations which (over $K$) correspond, with initial data and equation coefficients almost equal. 6.11, 6.12 follow from stability of ODE's. 6.13 follows from 6.12 and $\{ u_n \}$, $u$ being uniformly bounded on $K$ (since $\{ \psi_n \}$, $\psi$ uniformly bounded on $(-M,M)$).

QED 6.1.

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Appendix: Proof of 2.21

2.211 and 2.212 are easy. We prove 2.213. Hence, let $u$ denote a piece-wise $C^1$ solution.

The discontinuity-set of $u$ is a countable union of graphs of discontinuity paths and their endpoints:

$$ D = \bigcup_i \text{graph}(\gamma_i) \cup (\bigcup \text{endpoints of } \gamma_i). $$

For each $\gamma_i$, let

$$ W_i = \{(\gamma_i(t), t) : \text{sign } (\gamma') \text{ is not constant in any neighborhood of } t\}. $$

Define

$$ E = \left( \bigcup_i W_i \right) \cup \{\text{endpoints}\}. $$

Note $E$ is countable.

Temporarily we say a set of the form

$$ B = (x_1, x_2) \times (t_1, t_2) \subset H $$

is a "l-box" if at most one discontinuity path $(\gamma)$ intersects $B$, with $\gamma$ defined on all $(t_1, t_2)$, sign$(\gamma')$ constant on $(t_1, t_2)$, and $\gamma((t_1, t_2)) = (x_1, x_2)$ unless $\gamma' = 0$ on $(t_1, t_2)$. Note every point in $H - E$ has a l-box containing it.

Claim 1: Let $B$ as above and $\phi \in \mathcal{I}$.

$$ \iint_B (\phi_t u + \phi_x f \cdot u) \, dx \, dt = $$

$$ \int_{t_1}^{t_2} \left( \phi (u) \, dx - [f] \, dt \right) + \int_{x_1}^{x_2} \left( \phi u \right) (x, t_1) \, dx + \int_{x_1}^{x_2} \left( \phi u \right) (x, t_2) \, dx $$

$$ + \int_{t_1}^{t_2} \left( \phi f \cdot u \right) (x_2, t) \, dt + \int_{t_1}^{t_2} \left( \phi f \cdot u \right) (x_1, t) \, dt. $$
Proof. Assume $\gamma' > 0$, the other cases are similar. Note $\gamma$ has a local inverse $\gamma^{-1} : (x_1, x_2) \rightarrow (t_1, t_2)$. If 

$(x, t) = (\gamma(t), t) = (x, \gamma^{-1}(x))$, then $u(\gamma(t) \pm 0, t) = u(x, \gamma^{-1}(x) \pm 0)$.

We calculate

$$
\iint_B \varphi u_t \, dxdt = \int_{x_1}^{x_2} \left( \int_{t_1}^{t_2} \varphi u_t \, dt \right) dx =
$$

$$
\int_{x_1}^{x_2} \left( \int_{t_1}^{t_2} \varphi u_t \, dt + \int_{\gamma^{-1}(x)}^{t_2} \varphi u \, dt \right) dx =
$$

$$
\int_{x_1}^{x_2} (\varphi u)_{\gamma^{-1}(x) - 0} \, dx
$$

$$
\int_{x_1}^{x_2} \varphi u \, dx = \int \varphi u \, dx
$$

$$
\int \varphi u \, dx - \iint_B \varphi u_t \, dxdt + \int_{x_1}^{x_2} (x, t_2) (x, t_1)
$$

$$
\iint_B (f \circ u) \, dxdt = \int \varphi \, df \, dt + \iint_B \varphi_x f \circ u \, dx dt + \int_{t_1}^{t_2} \varphi f \circ u \, dt.
$$

Add these two equations and recall $u_t + (f \circ u)_x = 0$, to get Claim 1. An immediate consequence is

Claim 2: If $u$ obeys the Rankine-Hugoniot condition, then

$$
\iint_B (\varphi_t u + \varphi_x f \circ u) \, dxdt = \int_{x_1}^{x_2} (x, t_2) \varphi u \, dx + \int_{t_1}^{t_2} \varphi f \circ u \, dt.
$$
Claim 3: If \( u \) obeys the Rankine-Hugoniot condition, then \( u \) is a weak solution.

Proof. Fix \( \varphi \in \mathcal{J} \). There exists \( \alpha > 0 \) such that \( \text{support}(\varphi) \subseteq [-\alpha, \alpha] \times [0, \alpha] = K \). We show

\[
\iint_K (\varphi_t u + \varphi_x f' u) = 0.
\]

Choose \( \varepsilon > 0 \). Write \( E \cap K = \{(s_i, t_i) : i \in \mathbb{Z}_+\} \).

For \( i \in \mathbb{Z}_+ \) let

\[
B_i = (s_i - \frac{\varepsilon}{2^i}, s_i + \frac{\varepsilon}{2^i}) \times (t_i - \frac{\varepsilon}{2^i}, t_i + \frac{\varepsilon}{2^i}).
\]

For \( p \in K - E \), let \( B^p \) be a \( 1 \)-box with \( p \in B^p \).

The open sets \( \{B_i : i \in \mathbb{Z}_+\} \cup \{B^p : p \in K - E\} \) cover the compact set \( K \), so there exist finite sets \( F_1 \subseteq \mathbb{Z}_+ \) and \( F_2 \subseteq K - E \) such that \( \{B_i : i \in F_1\} \cup \{B^p : p \in F_2\} \) cover \( K \). It is easy to see that there is a finite sequence of disjoint \( 1 \)-boxes \( \{R_j : 1 \leq j \leq N\} \) and a set \( Z \) of measure zero such that \( \bigcup_{j} R_j \cup Z = K - \bigcup_{F_1} B_i \). Thus

\[
\iint_K (\varphi_t u + \varphi_x f' u) = \iint_{\bigcup_{F_1} B_i} + \iint_{\bigcup_{j} R_j}.
\]

Set \( M = \sup_K (|\varphi_t u + \varphi_x f' u| + |\varphi u| + |\varphi f' u|) < \infty \). Thus

\[
\iint_{\bigcup_{F_1} B_i} (\varphi_t u + \varphi_x f' u) < M \sum_{i=1}^{\infty} \text{area}(B_i) = O(\varepsilon). \quad \text{Also}
\]

\[
\iint_{\bigcup_{F_1} B_i} (\varphi_t u + \varphi_x f' u) < M \sum_{i=1}^{\infty} \text{area}(B_i) = O(\varepsilon). \quad \text{Thus}
\]

\[
\iint_K (\varphi_t u + \varphi_x f' u) = 0.
\]

Thus
\[ \iint (\varphi_t u + \varphi_x f^0 u) = \sum \int_{R_j}^{j} \varphi_x f^0 u \] may be evaluated by Claim 2; cancellation occurs except along the boundary of \( \bigcup B_i \), a rectilinear curve with length \( O(\varepsilon) \), so that the integral along this curve of a function, whose absolute value is less than \( M \), is \( O(\varepsilon) \).

Thus \( \iint_K (\varphi_t u + \varphi_x f^0 u) = O(\varepsilon) \). Since we can choose \( \varepsilon \) arbitrarily small, Claim 3 holds.

**Claim 4:** If \( u \) is a weak solution, then \( u \) obeys the Rankine-Hugoniot condition.

**Proof.** Let \( p = (x_0, t_0) \) be an interior point of a discontinuity path \( \gamma \).

Assume first that \( p \in D-E \). Let \( B = (x_1, x_2) \times (t_1, t_2) \) be a 1-box containing \( p \). For any \( \varphi \in \mathcal{F} \) with support in \( B \), apply Claim 1 to get

\[ 0 = \int_{\gamma | (t_1, t_2)} \varphi([u] dx - [f] dt) = \int_{t_1}^{t_2} \varphi([u] \gamma' - [f]) dt. \]

The arbitrariness of \( \varphi \) gives \( ([u] \gamma' - [f]) |_{t=t_0} = 0 \).

Because \( \gamma' \) is \( C^0 \) and \( E \) is only countable, we see

\[ \gamma' = \frac{[f]}{[u]} \] at all interior points of \( \gamma \).

**Claim 5:** If \( u \) obeys the Rankine-Hugoniot condition, then
u is an IDE solution.

Proof. Let \((x_1, x_2) \times \{t_0\} = I\) be given. Finitely many discontinuity-paths intersect \(I\); let \(\{\gamma_i : 1 \leq i \leq N\}\) (respectively, \(\{\delta_i : 1 \leq i \leq N\}\)) parametrize those defined for some \(t > t_0\) (resp., \(t < t_0\)) and assume that for small \(\varepsilon > 0\) (resp., \(\varepsilon < 0\)) the paths are ordered at \((t_0 + \varepsilon)\) by \(\gamma_i < \gamma_{i+1}\) (resp., \(\delta_i < \delta_{i+1}\)). Let \((\frac{d}{dt})^+\) (resp., \((\frac{d}{dt})^-\)) denote the right (resp., left) derivative. Then

\[
\left. \left( \frac{d}{dt} \right)^+ \int_{x_1}^{x_2} u(x, t)dx \right|_{t_0}^{t_1} = \left( \frac{d}{dt} \right)^+ \left( \int_{x_1}^{x_2} \gamma_1(t) \right) + \left( \int_{x_1}^{x_2} \gamma_2(t) \right) + \ldots + \left( \int_{x_1}^{x_2} \gamma_N(t) \right) = \\
\int_{x_1}^{x_2} \left[ u_{t}(x, t)dx + \gamma'_1(t_0)u(\gamma_1(t_0)-0, t_0) + \right.
\int_{x_1}^{x_2} u_{t}(x, t)dx + \gamma'_1(t_0)u(\gamma_1(t_0)-0, t_0) - \gamma'_1(t_0)u(\gamma_1(t_0)+0, t_0) + \\
\int_{x_1}^{x_2} u_{t}(x, t)dx + \gamma'_2(t_0)u(\gamma_2(t_0)-0, t_0) - \gamma'_2(t_0)u(\gamma_2(t_0)+0, t_0) + \\
\int_{x_1}^{x_2} u_{t}(x, t)dx - \gamma'_N(t_0)u(\gamma_N(t_0)+0, t_0) = \\
\int_{x_1}^{x_2} \left[ a(u)u_{x}dx + \right. \int_{x_1}^{x_2} a(u)u_{x}dx + \ldots \int_{x_1}^{x_2} a(u)u_{x}dx = \\
- \sum_{i=1}^{N} \gamma'_i(t_0)u(\gamma_i(t_0)+0, t_0) - \gamma'_i(t_0)u(\gamma_i(t_0)-0, t_0) =
\]
A similar calculation with the paths \( \{ \delta_1 \}_1^N \) gives

\[
\left( \frac{d}{dt} \right)|_{t_0}^{x_2} u(x, t) dx = - \int_{x_1}^{x_2} u(x, t_0) \, dx.
\]

Thus (2.13) holds, so \( u \) is an IDE solution.

**Claim 6:** If \( u \) is an IDE solution, then \( u \) satisfies the Rankine-Hugoniot condition.

**Proof.** Let \( (x_0, t_0) \) be an interior point of the discontinuity-path \( \gamma \). For small \( \varepsilon > 0 \) we have

\[
\int_{u(x_0 - \varepsilon, t_0)}^{u(x_0 + \varepsilon, t_0)} u(x, t_0) \, dx = - \frac{d}{dt} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} u(x, t) \, dx = - \frac{d}{dt} \left( \int_{t_0}^{t} \gamma(t) \, dt + \int_{t_0}^{x_0 + \varepsilon} \gamma(t) \, dt \right) = \\
\gamma'(t_0) \cdot [u] - \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} u_t(x, t_0) \, dx.
\]

Let \( \varepsilon \) approach 0 to get \([f] = \gamma'[u]\).

QED 2.21.
BIBLIOGRAPHY


