CLASSIFICATION OF RELATIVISITIC n-PARTICLE DYNAMICS

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Charles Patton

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Abstract of the Dissertation

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A relativistic n-particle dynamical law is a collection, S of n-tuples of smooth, timelike, one-dimensional submanifolds of Minkowski space, such that the set S is invariant under the natural action of the Poincaré group. A classical n-particle dynamics of order k is a smooth one parameter family of diffeomorphisms $H : T_f^{k-2}(R^3 \times B^3)^n \times R \to T_f^{k-2}(R^3 \times B^3)^n$ ((T_f = fiber tangent)) satisfying the usual k-th order condition. (The unit open ball is taken for velocity since we do not allow particles to go faster than the speed of light = c = 1.)

The classification of relativistic n-particle dynamics which arise as the orbit-set of a classical n-particle dynamics of order k proceeds in two stages. First, the problem is reduced to classifying the solutions of certain systems of first order, non-linear partial differential equations. Although formally overdetermined, these are shown to have some remarkable covariance properties which allow local analytic

solutions to be completely determined. General solutions for various classes are explicitly determined, and for the second order, three particle case it is shown that there are no non-trivial solutions which conserve relativistic momenta The problem of conservation of generalized momenta is raised.

These results extend the results of Currie, Jordan, Sudarshan: Rev. Mod. Phys., 35, 350 (63), Cannon, Jordan: J. Math. Phys., 5 299 (64) and Leutwler: Nuo. Cim. 37, 556 (66) when $k = 2$ and H is taken to be Hamiltonian, although completely different techniques are involved, and Arens:Adv. Math. 10, 332 (73) when the generalized momenta are taken to be the standard relativist momenta.

To

Dr. John A. Wheeler

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Notation

To facilitate notation, we have omitted range specifications for subscripts and superscripts. We use the following ranges everywhere unless otherwise noted.

Subscripts

$i, j, \ell, m, i', j', \ell', m'$, etc. take all values in $\{1,2,3\}$.

Superscripts

n = number of particles considered.

s, r, s', r', etc. take all values in $\{1,\ldots,n\}$.

Order

k = order of the dynamics considered.

These allow us to deal with various number of particles and various orders in a uniform manner.

Example

the equation $p_i p_j = 0$ actually represents the system of equations $p_1 p_1 = 0$, $p_1 p_2 = 0$, $p_1 p_3 = 0, \ldots$.

CHAPTER I

§1. In non-relativistic physics, the description of a physical system is constituted in specifying a space, M (identified as the space of states of the system in question) and a one parameter family, $\{\varphi_t | t \in R\}$, of automorphisms of the space M. Automorphism here refers to that structure with which we have equipped M.

For example, in classical mechanics, for a single particle $M = T^*R^3$ equipped with the canonical symplectic structure. The φ_t are then symplectic diffeomorphisms. In quantum mechanics, $M = \mathcal{H}$, a separable Hilbert space and φ_t are unitary transformations.

The "t" in φ_t refers to time and the one parameter family of φ_t's represents the time development of the system.

There are usually fundamental symmetries of the system present, however. These symmetries are given as a homomorphism ρ, of a group G, into the group Γ, of all automorphisms of M. As an example, when $M = T^*R^3$, G is often R^3, O(3), or $R^3 \circledast O(3)$ and ρ gives the standard action of these on T^*R^3.

That these symmetries should remain present as time progresses is expressed in the requirement that $\forall g \in G$, $\forall t \in R$, $\varphi_t \circ \rho(g) = \rho(g) \circ \varphi_t$.

In relativistic physics, the symmetries and the dynamics become inextricably intertwined. The usual thing to do in this case is to incorporate the dynamics as "just one more symmetry". We thus arrive at a homomorphism $\rho : iL \to \Gamma$, where iL is the connected component of the group of isometries of Minkowski space, M^4 (Poincare group). However, reasonable is this abstraction, it is still a good deal removed from the physical meaning of these isometries. Let us then take a different point of view.

The space-time history of a point-particle in special relativistic physics is described by a time-like curve in Minkowski space (a one-dimensional, time-like submanifold). Correspondingly, the space-time history of an n-particle system would be described by an n-tuple of one-dimensional time-like submanifolds. For the purposes of classical mechanics, it would be completely adequate to describe a set $\mathcal{S}$ of such n-tuples, where each element of this set would correspond to an n-particle space-time history which can occur in nature. We would also need some criterion for distinguishing among elements of this set, especially one which is local in time, for the purposes of prediction. Local in time means, we could design experiments which take a uniformly bounded amount of time to perform and would distinguish trajectories.

Noting that every n-tuple of time-like 1-dimensional submanifold intersects the hyperplane $t = 0$ in n points (possibly coincident), it is reasonable to assume that we may parametrize our orbit set $\mathcal{S}$ by $(R^3)^n \times P$, where P represents the results of our experiments.

The requirement of relativistic invariance for this scheme is simply that the set $\mathcal{S}$ be invariant, as a set, under the natural action of the group of isometries of Minkowski space.

Note that if we have any bijective map $\sigma : S \to \mathcal{S}$ and $\mathcal{S}$ is relativistically invariant, then σ induces an action of iL on S. In particular, if $\mathcal{S}$ is relativistically invariant and has a parametrization of the type $(R^3)^n \times P$ given above, and P is a manifold, it is sensible to ask whether the induced action of iL on $(R^3)^n \times P$ is smooth (C^∞) or not. In case it is, we will call $\mathcal{S}$, together with its parametrization, an (n-particle) <u>relativistic dynamical law</u>, or R.D.L. for short.

Among the most interesting, from a physical standpoint, are those R.D.L whose parametrization space corresponds to the phase space of a k^{th} order system. That is to say, if $x_1,\ldots,x_{3n} : R^{3n} \to R$ are coordinate functions for R^{3n} and X_o is the vectorfield on $R^{3n} \times P$ corresponding to the infintesimal generator of time translations, then the functions $\tilde{x}_1,\ldots,\tilde{x}_{3n},X_o(\tilde{x}_1),\ldots,X_o(\tilde{x}_{3n}),\ldots,X_o^{k-1}(\tilde{x}_1),\ldots,X_o^{k-1}(\tilde{x}_{3n})$ are a system of coordinates on $R^{3n} \times P$. ($\tilde{x}_1,\ldots,\tilde{x}_{3n}$ are the natural extensions of $x_1,\ldots,x_{3n}$ to functions on $R^{3n} \times P$. We will drop the tildas henceforth.)

In this case, any vectorfield is determined by its action on these $k\cdot 3n$ functions. From the fact that each point in $R^{3n} \times P$ corresponds to an n-tuple of time-like, one-dimensional submanifolds of M^4 and the projection $\pi : R^{3n} \times P \to R^{3n}$ corresponds to intersecting these submanifolds with the hyperplane t = 0, we obtain certain relations which will now be explored.

§2. Let us take standard coordinates (t,x_1,x_2,x_3) for M^4. Then the following one parameter subgroups of iL generate

Time translation

$$\tau_\alpha(t,x_1,x_2,x_3) = (t+\alpha,x_1,x_2,x_3) \qquad \forall\alpha \in R$$

corresponding Lie algebra element called $\bar{X}_0$.

Space translations

$$\chi_\alpha^1(t,x_1,x_2,x_3) = (t,x_1+\alpha,x_2,x_3)$$
$$\chi_\alpha^2(t,x_1,x_2,x_3) = (t,x_1,x_2+\alpha,x_3) \qquad \forall\alpha \in R$$
$$\chi_\alpha^3(t,x_1,x_2,x_3) = (t,x_1,x_2,x_3+\alpha)$$

corresponding Lie algebra elements called $\bar{X}_1,\bar{X}_2,\bar{X}_3$ respectively.

Space Rotations

$$\psi_\alpha^{12}(t,x_1,x_2,x_3) = (t,x_1\cos(\alpha)-x_2\sin(\alpha),x_1\sin(\alpha)+x_2\cos(\alpha),x_3)$$
$$\psi_\alpha^{13}(t,x_1,x_2,x_3) = (t,x_1\cos(\alpha)-x_3\sin(\alpha),x_2,x_1\sin(\alpha)+x_3\cos(\alpha))$$
$$\psi_\alpha^{23}(t,x_1,x_2,x_3) = (t,x_1,x_2\cos(\alpha)-x_3\sin(\alpha),x_2\sin(\alpha)+x_3\cos(\alpha))$$

Lie algebra elements $\bar{Y}_{12},\bar{Y}_{13},\bar{Y}_{23}$ respectively.

Lorentz Boosts

$$\psi_\alpha^{01}(t,x_1,x_2,x_3) = (\cosh(\alpha)t+\sinh(\alpha)x_1,\sinh(\alpha)t+\cosh(\alpha)x_1,x_2,x_3)$$
$$\psi_\alpha^{02}(t,x_1,x_2,x_3) = (\cosh(\alpha)t+\sinh(\alpha)x_2,x_1,\sinh(\alpha)t+\cosh(\alpha)x_2,x_3)$$
$$\psi_\alpha^{03}(t,x_1,x_2,x_3) = (\cosh(\alpha)t+\sinh(\alpha)x_3,x_1,x_2,\sinh(\alpha)t+\cosh(\alpha)x_3)$$

Lie algebra elements respectively $\bar{Y}_{01}, \bar{Y}_{02}, \bar{Y}_{03}$.

With our choice of coordinates, any one-dimensional time-like submanifold, call it γ, can be written uniquely as

$$\gamma = \{(\tau, \varphi_1(\tau), \varphi_2(\tau), \varphi_3(\tau)) \mid \tau \in R\}$$

where $\varphi_1, \varphi_2, \varphi_3$ are smooth functions and

$$\left(\frac{d\varphi_1}{d\tau}\right)^2 + \left(\frac{d\varphi_2}{d\tau}\right)^2 + \left(\frac{d\varphi_3}{d\tau}\right)^2 < 1$$

for all $\tau \in R$. Hence if $\tilde{\gamma} \in \mathcal{S}$ then it can be written $\tilde{\gamma} = \{\gamma^i \mid i = 1, \ldots, n\}$ where the γ^i are of the above form. In the case (identifying $\mathcal{S}$ with its parametrization) we have

$$x_j^r(\tilde{\gamma}) = \varphi_j^r(0)$$

Clearly we have the following induced actions of the above mentioned one parameter subgroups of iL on $R^{3n} \times P$:

$$x_j^r(\tau_\alpha(\tilde{\gamma})) = \varphi_j^r(-\alpha) \qquad \alpha \in R$$

$$x_j^r(\chi_\alpha^i(\tilde{\gamma})) = \delta_{ij}\alpha + x_j^r(\tilde{\gamma}) \qquad \alpha \in R$$

$$x_j^r(\psi_\alpha^{im}(\tilde{\gamma})) = \delta_{jm}(x_i^r(\tilde{\gamma})\sin(\alpha) + x_m^r(\tilde{\gamma})\cos(\alpha))$$
$$+ \delta_{ij}(x_i^r(\tilde{\gamma})\cos(\alpha) - x_m^r(\tilde{\gamma})\sin(\alpha)) + (1-\delta_{ij}-\delta_{jm})x_j^r(\tilde{\gamma})$$

If we denote the vectorfields on $R^{3n} \times P$ induced by these one parameter groups of motions by the unbarred version of the corresponding Lie algebra elements, we thus have

$$X_0(x_j^r)(\tilde{\gamma}) = \frac{-d}{d\tau}\varphi_j^r\Big|_{\tau=0}$$

$$X_i(x_j^r)(\tilde{\gamma}) = \delta_{ij}$$

$$Y_{im}(x_j^r)(\tilde{\gamma}) = \delta_{jm}x_j^r(\tilde{\gamma}) - \delta_{ij}x_m^r(\tilde{\gamma}).$$

Finally, we have

$$x_j^r(\psi_\alpha^{oi}(\tilde{\gamma})) = \delta_{ij}(\sinh(\alpha)t+\cosh(\alpha)\varphi_j^r(t)) + (1-\delta_{ij})\varphi_j^r(t)$$

where t satisfies the equation

$$\cosh(\alpha)t + \sinh(\alpha)\varphi_i^r(t) = 0.$$

Suppose we have a one parameter family of t's, say $t(\alpha)$ satisfying

$$\cosh(\alpha)t(\alpha) + \sinh(\alpha)\varphi_i^r(t(\alpha)) = 0.$$

Then taking derivative with respect to α at $\alpha = 0$ we have

$$\frac{d}{d\alpha}t(\alpha)|_{\alpha=0} + \varphi_i^r(t(\alpha))|_{\alpha=0} = 0 \text{ but } t(0) = 0.$$

Thus

$$\frac{d}{d\alpha}t(\alpha)|_{\alpha=0} = -\varphi_i^r(0) = -x_i^r(\tilde{\gamma}).$$

We can now write

$$x_j^r(\psi_\alpha^{oi}(\tilde{\gamma})) = \delta_{ij}(\sinh(\alpha)t(\alpha)+\cosh(\alpha)\varphi_j^r(t(\alpha))+(1-\delta_{ij})\varphi_j^r(t(\alpha)).$$

Differentiating at $\alpha = 0$, we have

$$Y_{oi}(x_j^r)(\tilde{\gamma}) = -x_i^r(\tilde{\gamma})\cdot\frac{d}{dt}\varphi_j^r|_{t=0} = x_i^r(\tilde{\gamma})\cdot X_o(x_j^r)(\tilde{\gamma}).$$

The vectorfields X_o, X_i, Y_{ij} and Y_{oi} inherit bracket relations from the Lie algebra of iL. Namely

(2.1) $\quad [X_o, X_i] = 0$

(2.2) $\quad [X_o, Y_{ij}] = 0$

$$(2.3)\qquad [X_o, Y_{oi}] = X_i$$

$$(2.4)\qquad [Y_{ij}, Y_{om}] = \delta_{jm}Y_{oi} - \delta_{im}Y_{oj}$$

$$(2.5)\qquad [Y_{oi}, Y_{oj}] = Y_{ij}$$

$$(2.6)\qquad [X_i, Y_{oj}] = \delta_{ij}X_o$$

From the previous we also must have

$$(2.7)\qquad X_i(x_j^r) = \delta_{ij}$$

$$(2.8)\qquad Y_{ij}(x_m^r) = \delta_{jm}x_i^r - \delta_{im}x_j^r$$

$$(2.9)\qquad Y_{oi}(x_j^r) = x_i^r X_o(x_j^r)$$

$$(2.10)\qquad (X_o(x_1^r))^2 + (X_o(x_2^r))^2 + (X_o(x_3^r))^2 < 1$$

Let us now combine equations 2.3, 2.7 and 2.9 into the following system:

$$(2.0)'\qquad Y_{oi}(x_j^r) = x_i^r X_o(x_j^r)$$

$$(2.1)'\qquad Y_{oi}X_o(x_j^r) = X_o(x_i^r)X_o(x_j^r) + x_i^r X_o X_o(x_j^r) - \delta_{ij}$$

$$(2.2)'\qquad Y_{oi}X_oX_o(x_j^r) = X_o^2(x_i^r)X_o(x_j^r) + 2X_o(x_i^r)X_o^2(x_j^r + x_i^r X_o^3(x_j^r)$$

$\vdots$

$$(2.s)'\qquad Y_{oi}X_o^s(x_j^r) = X_o^s(x_i^r)X_o(x_j^r) + sX_o^{s-1}(x_i^r)X_o^2(x_j^r) + \dots$$

$$\dots + \binom{s}{\ell}X_o^{s-1}(x_i^r)X_o^{\ell+1}(x_j^r) + \dots + x_i^r X_o^{s+1}(x_j^r)$$

$\vdots$

<u>Lemma</u>. Suppose we have an R.D.L. parametrized by $R^{3n} \times P$ and a $k \geq 3$ and an open subset $U \subset R^{3n} \times P$ together with a $j \in (1,2,3)$ and an $r \in (1,\dots,n)$ such that $X_o^k(x_j^r) \equiv 0$ on U.

Then $X_o^2(x_1^r) \equiv X_o^2(x_2^r) \equiv X_o(x_3^r) \equiv 0$ on U.

Proof: First note that if $X_o^k(x_j^r) \equiv 0$, then so is $X_o^k(x_1^r)$, $X_o^k(x_2^r)$ and $X_o^k(x_3^r)$ since they can be obtained by applying the appropriate infintesimal rotation. We will now proceed by induction on k.

Case k = 3. We have the equation

$$Y_{oi}X_o^3(x_j^r) = X_o^3(x_i^r)X_o(x_j^r) + 3X_o^2(x_i^r)X_o^2(x_j^r)$$
$$+ 3X_o(x_i^r)X_o^3(x_j^r) + x_i^r X_o^4(x_j^r)$$

which when restricted to U becomes

$$0 = 3X_o^2(x_i^r)X_o^2(x_j^r).$$

Taking i = j we find $X_o^2(x_j^r) \equiv 0$ on U.

Reducing Case k to Case k-1. Assume $X_o^k(x_j^r) \equiv 0$ on U. Restricing equation 2.k' to U we get

$$0 = kX_o^{k-1}(x_i^r)X_o^2(x_j^r) +\dots+ \binom{k}{k-2}X_o^2(x_i^r)X_o^{k-1}(x_j^r).$$

Apply X_o^{k-3} to both sides of this equation on U and obtain

$$0 = KX_o^{k-1}(x_i^r)X_o^{k-1}(x_j^r)$$

on U, where K is a positive integer. Thus we have $X_o^{k-1}(x_j^r) \equiv 0$ on U, completing the induction step.

The meaning of this proposition is that there can be no "constant acceleration" dynamics, other than straight line motion. This rules out the most intuitively plausible possibility for relativistically invariant dynamics.

§3. <u>k-th Order Dynamics.</u>

Let us now consider the case of a k-th order R.D.L. In this case, the functions

$$x_i^r, X_o(x_i^r), \ldots, X_o^{k-1}(x_i^r)$$

give a coordinatization of $R^{3n} \times P$. In particular any vectorfield is determined by its action on these functions. Thus X_o is completely determined by the functions $X^k(x_i^r)$, $r = 1, \ldots, n$; $i = 1,2,3$. Y_{oj} is determined by the functions

$$Y_{oj}(x_i^r), Y_{oj}X_o(x_i^r), \ldots, Y_{oj}X_o^{k-1}(x_i^r).$$

<u>Proposition.</u> Suppose X_o is a k-th order, $k \geq 2$, vectorfield on $R^{3n} \times P$. Define vectorfields Y_{oi}, $i = 1,2,3$, by equations (2.0)',...,k-1'. Then, if Y_{oi} satisfy equations (2.k') and (k+1)', they also satisfy equations (2.q)' for all q; moreover the vectorfields

$$Y_{oi} := Y_{oi}$$

$$X_o := X_o$$

$$X_i := [X_o, Y_{oi}]$$

$$Y_{ij} := [Y_{oi}, Y_{oj}]$$

satisfy equations (2.1),...,(2.9).

<u>Proof:</u> Let us first determine X_i. Since we are dealing with a k-th order system, it is sufficient to operate with X_i on the first k-1 X_o-derivatives of the x_j^r's.

3.1.0 $X_i(x_j^r) = [X_o, Y_{oi}](x_j^r) = X_o Y_{oi}(x_j^r) - Y_{oi} X_o(x_j^r) = \delta_{ij}$ by equations (2.0)' and (2.1)'.

3.1.1 $X_i X_o(x_j^r) = 0$ by equations (2.1)' and (2.2)'.

⋮

3.1.k-1 $X_i X_o^{k-1}(x_j^r) = 0$ by equations (2.k-1)' and k'.

Moreover

3.1.k $X_i X_o^k(x_j^r) = 0$ by equations (2.k)' and (k+1)'.

In particular, equation (2.7) is satisfied.

3.2.0 $[X_o, X_i]x_j^r = X_o X_i(x_j^r) - X_i X_o(x_j^r) = X_o(\delta_{ij}) - 0 = 0$

3.2.1 $[X_o, X_i]X_o(x_j^r) = X_o X_i X_o(x_j^r) - X_i X_o X_o(x_j^r) = 0$

⋮

3.2.k-1 $[X_o, X_i]X_o^{k-1}(x_j^r) = X_o X_i X_o^{k-1}(x_j^r) - X_i X_o^k(x_j^r) = 0.$

Thus, $[X_o, X_i] = 0$ and equation (2.1) is satisfied.

The right hand side of e.g., q' $(q > 1)$ can be obtained from the right hand side of equation (q-1)' by applying X_o to it. Moreover, applying X_o to the left hand side of equation (q-1)' gives $X_o Y_{oi} X_o^{q-1}(x_j^r) = Y_{oi} X_o X_o^{q-1}(x_j^r) + X_i X_o^{q-1}(x_j^r)$ $= Y_{oi} X^q(x_j^r)$, which is the left hand side of equation q'. Thus (q-1)' true implies equation q' true, hence they are all satisfied.

We can now compute $[X_i, Y_{oj}]$.

3.3.0 $[X_i, Y_{oj}]x_\ell^r = X_i Y_{oj}(x_\ell^r) - Y_{oj} X_i(x_\ell^r) = \delta_{ij} X_o(x_\ell^r)$ from equation (2.0)'.

3.3.1 $[X_i, Y_{oj}]X_o(x_\ell^r) = \delta_{ij} X_o X_o(x_\ell^r)$
from equation (2.1)'.

⋮

3.3.k-1 $[X_i, Y_{oj}]X_o^{k-1}(x_\ell^r) = \delta_{ij} X_o X_o^{k-1}(x_\ell^r)$
from equation (2.k-1)'.

Since the vectorfields $[X_i, Y_{oj}]$ and $\delta_{ij} X_o$ give the same values on the coordinate functions, we may conclude that they are the same. Thus equation (2.6) is satisfied.

Next note that since $Y_{ij} := [Y_{oi}, Y_{oj}]$,
$[X_o, Y_{ij}] = [X_o, [Y_{oi}, Y_{oj}]] = -[Y_{oi}, [Y_{oj}, X_o]] - [Y_{oj}, [X_o, Y_{oi}]]$
$= -\delta_{ij} X_o + \delta_{ij} X_o = 0$ and so equation (2.2) is satisfied.

Finally we have the computation of the components of Y_{ij}.

3.4.0 $Y_{ij}(x_\ell^r) = [Y_{oi}, Y_{oj}]x_\ell^r = Y_{oi} Y_{oj}(x_\ell^r) - Y_{oj} Y_{oi}(x_\ell^r)$
$= -\delta_{i\ell} x_j^r + \delta_{j\ell} x_i^r$
from equation (2.0)',(2.1)',(2.3),(2.6),(2.7)
((2.3) defines X_i).

3.4.1 $Y_{ij} X_o(x_\ell^r) = -\delta_{i\ell} X_o(x_j^r) + \delta_{j\ell}(x_i^r$
from equation (2.1)', (2.2)', (2.3),(2.6),(2.7).

⋮

3.4.k-1 $Y_{ij} X_o^{k-1}(x_\ell^r) = -\delta_{i\ell} X_o(x_j^r) + \delta_{j\ell} X_o(x_i^r)$
from equation (2.(k-1))', (2.k)', (2.3), (2.6), (2.7).

Note that 3.4.0 yields equation (2.8). We also have

3.5.0 $$[Y_{ij}, Y_{om}]x^r_\ell = Y_{ij}Y_{om}(x^r_\ell) - Y_{om}Y_{ij}(x^r_\ell)$$
$$= Y_{ij}(x^r_m X_o(x^r_\ell)) - Y_{om}(\delta_{j\ell}x^r_i - \delta_{i\ell}x^r_j)$$
$$= \delta_{jm}x^r_i X_o(x^r_\ell) - \delta_{im}x^r_j X_o(x^r_\ell)$$
$$= \delta_{jm}Y_{oi}(x^r_\ell) - \delta_{im}Y_{oj}(x^r_\ell)$$

from equations (2.0)', (2.2), (2.8).

3.5.1 $$[Y_{ij}, Y_{om}]X_o(x^r_\ell) = Y_{ij}(X_oY_{om} - X_m)(x^r_\ell) - (X_oY_{om} - X_m)Y_{ij}(x^r_\ell)$$
(from equation (2.3).)
$$= X_oY_{ij}Y_{om}(x^r_\ell) - Y_{ij}X_m(x^r_\ell) - X_oY_{om}Y_{ij}(x^r_\ell) + X_mY_{ij}(x^r_\ell)$$

(from equation (2.2).)

$$= X_oY_{ij}(x^r_m X_o(x^r_\ell)) - X_oY_{om}(\delta_{j\ell}x^r_i - \delta_{i\ell}x^r_j) + \delta_{im}\delta_{j\ell} - \delta_{jm}\delta_{i\ell}$$
$$= \delta_{jm}(X_oX_o(x^r_i)X_o(x^r_\ell) + 2X_o*x^r_j)X_oX_o(x^r_\ell) - \delta_{i\ell})$$
$$- \delta_{im}(X_oX_o(x^r_j)X_o(x^r_\ell) + 2X_o(x^r_j)X_oX_o(x^r_\ell) - \delta_{j\ell}$$
$$= \delta_{jm}Y_{oi}X_o(x^r_\ell) - \delta_{im}Y_{oj}X_o(x^r_\ell)$$

from equation (2.1)'.

From the components of X_m and Y_{ij}, and equations (2.1), (2.2) we calculate that $[Y_{ij}, X_m] = \delta_{jm}X_i - \delta_{im}X_j$. We can then find

$$[X_o, [Y_{ij}, Y_{om}]] = -[Y_{ij}, [Y_{om}, X_o]] - [Y_{om}, [X_o, Y_{ij}]]$$
$$= \delta_{jm}X_i - \delta_{im}X_j.$$

Hence by induction on $q > 1$,

3.5.q $$[Y_{ij},Y_{om}]X_o^{-q}(x_\ell^r) = X_o[Y_{ij},Y_{om}]X^{q-1}(x_\ell^r)$$
$$- (\delta_{jm}X_i - \delta_{im}X_j)X_o^{q-1}(x_\ell^r)$$

by induction
$$= X_o(\delta_{jm}Y_{oi}X_o^{q-1}(x_\ell^r) - \delta_{im}Y_{oj}X_o^{q-1}(x_\ell^r)$$
$$- (\delta_{jm}X_i - \delta_{im}X_j)X_o^{q-1}(x_\ell^r)$$

$(q > 1)$
$$= \delta_{jm}Y_{oi}X_i^q(x_\ell^r) - \delta_{im}Y_{oj}X_o^q(x_\ell^r).$$

Thus we find $[Y_{ij},Y_{om}] = \delta_{jm}Y_{oi} - \delta_{im}Y_{oj}$ completing the proof.

With the aid of this proposition, we may begin to study k-th order R.D.L.'s. Clearly, once we specify that we are considering a k-th order R.D.L. the only choice we have left is to specify the k-th order part of the vectorfield X_o, since then everything else is determined. The natural way to specify the k-th order part is to specify the functions $X_o^k(x_j^r)$. Then equations (2.p)' and (2.(p+1))' define a coupled system of non-linear p.d.e. on the functions $X_o^k(x_j^r)$ which must be satisfied for X_o to generate an R.D.L.

§4. Euclidean Invariance

We have been led to consider a system of coordinates x_j^r, $a_j^{r1}, a_j^{r2}, \ldots, a_j^{rk-1}$ for $R^{3n} \times P$. The problem of finding a generator for a k-th order ($k > 1$) R.D.L. has been reduced to finding 3n functions, f_j^r, of the coordinates so that the vectorfields

4.1 $$X_o = \sum_{r,j} a_j^{r1}\partial x_j^r + \sum_{r,j} a_j^{r2}\partial a_j^{r1} + \sum_{r,j} a_j^{r3}\partial a_j^{r2} + \ldots + \ldots + \sum_{r,j} f_j^r \partial a_j^{rk-1}$$

4.2 $$Y_{oi} = \sum_{r,j} x_i^r a_j^{r1}\partial x_j^r + \sum_{r,j} (a_i^{r1} - \delta_{ij} + x_i^r a_j^{r2})\partial a_j^{r1} + \sum_{r,j} (a_i^{r2} a_j^{r1} + 2a_j^{r2} a_i^{r1} + x_i^r a_j^{r3})\partial a_j^{r2} + \ldots + \sum_{r,j} (a_i^{rk-1} a_j^{r1} + \ldots + x_i^r f_j^r)\partial a_j^{rk-1}$$

satisfy

4.3.k $$(Y_{oi} - x_i^r X_o) f_j^r = f_i^r a_j^{r1} + k a_i^{rk-1} a_j^{r2} + \ldots + k a_i^{r1} f_j^r$$

4.3.k+1 $$(Y_{oi} - x_i^r X_o) X_o(f_j^r) = X_o(f_i^r) a_j^{r1} + (k+1) f_i^r a_j^{r2} + \ldots + (k+1) a_i^{r1} X_o(f_j^r).$$

(In these formulae, and throughout, we will use ∂x_j^r, etc., instead of $\frac{\partial}{\partial x_j^r}$, for the coordinate vectorfield tangent to the x_j^r.)

We have already shown that if equation 4.k and 4.k+1 are satisfied, then in particular, we have

4.4.1 $X_i := [X_o, Y_{oi}]$

satisfying

4.4.2 $X_i(x_j^r) = \delta_{ij},\ X_i(a_j^{r1}) = \ldots = X_i(a_j^{rk-1}) = 0$

as well as

4.4.3 $[X_i, X_o] = 0.$

We also have

4.5.1 $Y_{ij} := [Y_{oi}, Y_{oj}]$

satisfying

4.5.2
$$Y_{ij}(x_\ell^r) = \delta_{j\ell} x_i^r - \delta_{i\ell} x_j^r$$
$$Y_{ij}(a_\ell^{r1}) = \delta_{j\ell} a_i^{r1} - \delta_{i\ell} a_j^{r1}$$
$$\vdots$$
$$Y_{ij}(a_\ell^{rk-1}) = \delta_{j\ell} a_i^{rk-1} - \delta_{i\ell} a_j^{rk-1}$$

as well as

4.5.3 $[Y_{ij}, X_o] = 0.$

X_i and Y_{ij} are generators of the euclidean group, the subgroup of iL taking the hyperplane t = 0 to itself. The generators have the same form in terms of the adapted coordinates, independent of which k-th order R.D.L. we are considering. In this sense, the $[X_i, X_o] = [Y_{ij}, X_o] = 0$ are quite natural assumptions for the dynamics, even though consequences of 4.3.k and 4.3.k+1. Moreover, these are linear partial differential equations for the f_j^r's and consequently easier to handle.

We will next show that 4.3.k and 4.3.k+1 can be replaced by 4.3.k and $[X_i,X_o] = 0$.

Proposition. Suppose we have functions f_j^r of the coordinates $x_j^r, a_j^{r1},\ldots,a_j^{rk-1}$ so that the vectorfields X_o defined by 4.1 and Y_{oi} defined by 4.2 satisfy equation 4.3.k. If, in addition, we assume $[X_i,X_o] = 0$, where X_i is defined by 4.4.2, then 4.3.k+1 must be satisfied.

Proof. Comparing with equations 3.1.0,...,1.k-1 we find that as long as 4.3.k is satisfied, we have $[X_o,Y_{oi}] = X_i$ as we have defined it in 4.4.2. Assuming $[X_i,X_o] = 0$, then 4.3.k+1 can be obtained from 4.3.k by applying X_o to both sides of the equation. Hence, 4.3.k+1 is satisfied. This completes the proof.

This however, does not exhaust all the symmetries of our system of equations. Equations 4.3.k are euclidean covariant in a sense we will now explain.

Consider the operator D

$$\sum_{r,\ell} f_\ell^r \partial a_\ell^{rk-1} = A \overset{D}{\to} \sum_{\substack{i,j\\r}} [(Y_{oi}-x_i^r X_o)f_j^r - f_i^r a_j^{r1} \ldots k a_i^{r1} d_j^r] \; \partial a_i^{rk-1} \otimes \partial a_j^{rk-1}$$

from vectorfields "tangent to the a^{rk-1}-directions" to tensorfields "tangent to the a^{rk-1}-directions".

Notice that $[X_i, A] = 0 \Leftrightarrow [X_i, X_o] = 0$ and also $[Y_{ij}, A] = 0 \Leftrightarrow [Y_{ij}, X_o] = 0$ and finally $D(A) = 0 \Leftrightarrow$ the f_j^r's satisfy 4.3.k. We already know that if $D(A) = 0$ then $[X_i, X_o] = 0 = [Y_{ij}, X_o]$, however the converse is false. The euclidean covariance of the operator D is expressed in the following

<u>Proposition</u>. If A, D, X_i, Y_{ij} as above $(k > 1)$, then

$$\mathcal{L}_{X_i} A = 0 \qquad \mathcal{L}_{X_i} D(A) = 0$$

and

$$\mathcal{L}_{Y_{ij}} A = 0 \qquad \mathcal{L}_{Y_{ij}} D(A) = 0.$$

Here, $\mathcal{L}_{X_i}$ means Lie derivative with respect to X_i, etc. (for a vectorfield, A, $\mathcal{L}_{X_i} A = [X_i, A]$).

Since we are searching for solutions of $D(A) = 0$ and we know such A must be euclidean invariant, we may as well restrict ourselves to such A from the beginning. This proposition allows us to "factor out" the euclidean invariance of these equations. If $U \subset R^{3n} \times P$ is a subset which has the property that the union of all orbits of the euclidean group which pass through U is all of $R^{3n} \times P$ itself, then any euclidean invariant vector or tensorfield on $R^{3n} \times P$ is completely determined by its restriction to U. In particular, if it is zero on U, then it is zero everywhere. Thus, if $\mathcal{L}_{X_i} A = 0 = \mathcal{L}_{Y_{ij}} A$ then $D(A) = 0 \Leftrightarrow D(A)|_U = 0$. If U is an open set, there is not much advantage in this. However, if U is of lower dimension this allows us to reduce the number of variables in our equations. The procedure is to start with a vectorfield defined over U, extend it to a euclidean invariant vectorfield on $R^{3n} \times P$ (if possible), apply D to the vectorfield and restrict the result to D. In practice, what we will do is use euclidean invariance to solve for the derivatives "normal" to U in terms of derivatives along U and values of the vectorfields themselves.

Proof of Proposition. Comparing with equations 3.3.0,...,3.k-1 we find that $[X_i, Y_{oj}] = \delta_{ij} X_o$ independent of whether equations 4.3.k are satisfied or not, as long as $[X_i, X_o] = 0$ (hence $[X_i, A] = 0$). Thus we find

$$\mathcal{L}_{X_i} D(A) = \sum_{j,\ell,r} \{[X_i, Y_{oj} - x_j^r X_o] f_\ell^r + [Y_{oj} - x_j^r X_o] X_i(f_\ell^r)$$

$$- X_i(f_j^r p_\ell^r) - \ldots - (k-1) X_i(p_j^r f_\ell^r)\} \partial a_j^{rk-1} \otimes \partial a_\ell^{rk-1}$$

$$+ \sum_{j,r,\ell} \{[Y_{oj} - x_j^r X_o] f_\ell^r - f_j^r p_\ell^r - \ldots - (k-1) p_j^r f_\ell^r\}$$

$$\{[X_i, \partial a_j^{rk-1}] \otimes \partial a_\ell^{4k-1}$$

$$+ \partial a_j^{rk-1} \otimes [X_i, \partial a_\ell^{rk-1}].$$

But, since $[X_i, X_o] = 0$ by assumption, we have

$$[X_i, \partial a_j^{rq}] = 0 \text{ for } q = 2, \ldots, k-1$$

and also

$$X_i(a_j^{rq}) = X_i(p_j^r) = X_i(f_j^r) = 0.$$

We therefore find

$$\mathcal{L}_{X_i} D(A) = \sum_{j,r,\ell} [\delta_{ij} X_o - \delta_{ij} X_o] f^r_\ell \partial a^{rk-1}_j \otimes \partial a^{rk-1}_\ell = 0.$$

This gives the translation part of the proposition.

Similarly, direct computation shows (compare with 3.5) that if $[Y_{ij}, A] = 0$ (Y_{ij} defined by 4.5.2) then $[Y_{ij}, Y_{o\ell}] = \delta_{j\ell} Y_{oi} - \delta_{i\ell} Y_{oj}$ independent of whether 4.3.k is satisfied or not. It is also easily verified that if $[Y_{ij}, A] = 0$, then $Y_{ij}(f^r_\ell) = \delta_{j\ell} f^r_u - \delta_{i\ell} f^r_j$, and conversely. Thus we find that if $\mathcal{L}_{Y_{ij}}(A) = 0$ then

$$\begin{aligned}
\mathcal{L}_{Y_{ij}} D(A) &= \mathcal{L}_{Y_{ij}} \sum_{\ell,m,r} \{ [Y_{o\ell} - x^r_\ell X_o] f^r_m - f^r_\ell p^r_m - \ldots - (k-1) p^r_\ell f^r_m \} \partial a^{rk-1}_\ell \otimes \partial a^{rk-1}_m \\
&= \sum_{\ell,m,r} \{ \delta_{j\ell}([Y_{oi} - x^r_i X_o] f^r_m - f^r_i p^r_m - \ldots - (k-1) p^r_i f^r_m) \\
&\qquad - \delta_{i\ell}([Y_{oj} - x^r_j X_o] f^r_m - f^r_j p^r_m - \ldots - (k-1) p^r_j f^r_m) \\
&\qquad + \delta_{jm}([Y_{o\ell} - x^r_\ell X_o] f^r_i - f^r_\ell p^r_i - \ldots - (k-1) p^r_\ell f^r_i) \\
&\qquad - \delta_{im}([Y_{o\ell} - x^r_\ell X_o] f^r_j - f^r_\ell p^r_j - \ldots - (k-1) p^r_\ell f^r_j) \} \\
&\qquad \times \partial a^{rk-1}_\ell \otimes \partial a^{rk-1}_m \\
&\quad + \sum_{\ell,m,r} \{ [Y_{o\ell} - x^r_\ell X_o] f^r_m - f^r_i p^r_m - \ldots - (k-1) p^r_i f^r_m \} \\
&\qquad \times \{ (\delta_{j\ell} \partial a^{rk-1}_i - \delta_{i\ell} \partial a^{rk-1}_j) \otimes \partial a^{rk-1}_m \\
&\qquad + \partial a^{rk-1}_\ell \otimes (\delta_{jm} \partial a^{rk-1}_i - \delta_{im} \partial a^{rk-1}_j) \} \\
&= 0.
\end{aligned}$$

This completes the proof.

CHAPTER II

§1. We will begin this chapter by considering the simplest case of these equations we have derived for generators of k-th order R.D.L.'s. This is the one particle case. Since there is only one particle involved, we may drop the superscript "r" in all the equations. With the notation of I.4, the relevant equations become, when $k > 1$

1.1 $$(Y_{oi}-x_iX_o)f_j = f_ip_j + ka_i^{k-1}a_j^2 +\dots+ kp_if_j$$

1.2 $$X_i(f_j) = 0$$

and if $k = 1$

1.3 $$(Y_{oi}-x_iX_o)f_j = f_if_j-\delta_{ij}$$

1.4 $$(Y_{oi}-x_iX_o)X_o(f_j) = X_o(f_i)f_j + 2f_iX_o(f_j)$$

The most important thing about the one particle equations is that the operator $(Y_{oi}-x_iX_o)$ is completely independent of the f_j's and these (except 1.4) equations are therefore linear first order partial differential equations.

Since we have $Y_{oi}(x_j) = x_iX_o(x_j)$, we find that in the first order case ($k = 1$) the operator $(Y_{oi}-x_iX_o)$, which is determined here by its action on the coordinate functions x_j, must be identically zero. We may then state

Proposition. There are no one particle, first order R.D.L.'s.

Proof. If there were, then 1.3 would be satisfied. However,

$(Y_{oi}-x_iX_o) \equiv 0$ here so that 1.3 reads

$$0 = f_if_j-\delta_{ij}.$$

Taking $i = j$ we have $(f_i)^2 = 1$ but taking $i \neq j$ we have $f_if_j = 0$. Squaring this equation, we have $(f_i)^2(f_j)^2 = 0 = 1\cdot 1 = 1$ a contradiction.

The next case to be considered is that of a one particle second order R.D.L.; the classical case. Computing the operator $Y_{oi}-x_iX_o$ from definition we find that it may be written

$$D_i \equiv Y_{oi}-x_iX_o = \sum_j(p_ip_j-\delta_{ij})\partial p_j.$$

In particular, $D_i(x_j) = 0$, $D_i(p_j) = p_ip_j-\delta_{ij}$ and $[D_i,D_j]x_\ell = 0$ and also

$$[D_i,D_j]p_\ell = D_i(p_jp_\ell-\delta_{j\ell}) - D_j(p_ip_\ell-\delta_{i\ell})$$

$$= p_\ell(p_ip_j-\delta_{ij}) + p_j(p_ip_\ell-\delta_{i\ell})$$

$$- p_\ell(p_jp_i-\delta_{ji}) - p_i(p_jp_\ell-\delta_{j\ell})$$

$$= p_jD_i(p_\ell) - p_iD_j(p_\ell).$$

Thus $[D_i,D_j] = p_jD_i-p_iD_j$.

<u>Proposition</u>. There is one and only one one particle, second order R.D.L.

<u>Proof</u>. Equation 1.1 may be written

1.5 $$D_i f_j = f_i p_j + 2p_i f_j.$$

Using 1.5 and the values of $D_i(p_j)$ computed above, we find

1.6 $$[D_i, D_j]f_\ell = -\delta_{i\ell} f_j + \delta_{j\ell} f_i.$$

On the other hand, we have found that the operator $[D_i, D_j]$ may be expressed as a linear combination of D_i and D_j, $[D_i, D_j] = p_j D_i - p_i D_j$. Consequently, we must also have

1.7 $$\begin{aligned}[D_i, D_j]f_\ell &= (p_j D_i - p_i D_j)f_\ell \\ &= p_j(p_\ell f_i + 2p_i f_\ell) - p_i(p_\ell f_j + 2p_j f_\ell) \\ &= p_j p_\ell f_i - p_i p_\ell f_j.\end{aligned}$$

Combining equations 1.6 and 1.7 we obtain

1.8 $$(p_j p_\ell - \delta_{j\ell})f_i - (p_i p_\ell - \delta_{i\ell})f_j = 0.$$

We can write the $i = \ell$ component of this system as the matrix equation

$$\begin{pmatrix} 1-(p_2)^2, & p_1 p_2 & ,0 \\ 0 & , 1-(p_3)^2, & p_2 p_3 \\ p_1 p_3 & , 0 & ,1-(p_1)^2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In order to have a non-zero solution to this system, we must have det( ) = 0 which is the same as

1.9 $$(1-(p_1)^2-(p_2)^2-(p_3)^2) + (p_1 p_2)^2 + (p_1 p_3)^2 + (p_2 p_3)^2 = 0.$$

Since we want speed less than 1, the left hand side is greater than zero. Thus $f_1 = f_2 = f_3 \equiv 0$. Moreover $f_1 = f_2 = f_3 \equiv 0$ is clearly a solution, corresponding to straight line motion of our particle. This completes the proof.

There are many one particle third, or higher, order solutions. We will now construct a quite general class of third order solutions. The higher order construction is analogous but involves more variables.

The third order equations may be written

1.10 $$\{\sum_{\ell}(p_i p_\ell - \delta_{i\ell})\partial p_\ell + (a_i p_\ell + 2p_i a_\ell)\partial a_\ell\} f_j$$
$$= f_i p_j + 3a_i a_j + 3p_i f_j \equiv D_i f_j.$$

The vectorfields D_i generate an action of the Lorentz group on the phase space. The domain of our functions f_j must be invariant under this action as well as that induced by the time translation generator $X_o = \sum_j p_j \partial x_j + a_j \partial p_j + f_j \partial a_j$ and the euclidean group.

<u>Proposition.</u> Let $\bar{f}_1, \bar{f}_2, \bar{f}_3$ be smooth functions defined on the closed ball of radius $K > 0$ in R^3 such that $\tilde{Y}_{ij}(\bar{f}_\ell) = \delta_{j\ell}\bar{f}_i - \delta_{i\ell}\bar{f}_j$, where $\tilde{Y}_{ij}$'s are the standard generators of the action of the rotation group on R^3. Finally, assume that the $\bar{f}$'s vanish on the boundary of the ball. Then there exists a domain U in the third order, one particle phase space which is invariant under the action of the euclidean group and the action of the Lorentz group

generated by the D_i's and functions f_1, f_2, f_3 on U so that the f's are translation invariant and satisfy 1.10. Moreover, U is invariant under X_o,

$$U \cap \{p_1 = p_2 = p_3 = x_1 = x_2 = x_3 = 0\} = U \cap V$$

$$= \{a_1^2 + a_2^2 + a_3^2 < K\} \cap V$$

and

$$f_i|_{U\cap V} = \bar{f}_i.$$

Proof. First note that the set $\bar{V} = \{p_2 = p_3 = x_1 = x_2 = x_3 = 0, -1 < p \leq 0\}$ is of the type mentioned in the proposition on euclidean invariance. Namely, the union of the orbits of the euclidean group passing through $\bar{V}$ is the whole phase space. Thus, if we can find f's which satisfy the "pullback" of 5.10 on $\bar{V}$ and which can be extended by euclidean invariance, then the extended functions will satisfy 1.10 everywhere.

In this case, however, D_1 is tangent to $\bar{V}$. Moreover, since $[Y_{ij}, D_\ell] = \delta_{j\ell}D_i - \delta_{i\ell}D_j$ and $[X_i, D_j] = 0$ the pullback system is the equation

1.11 $$D_i f_j = f_1 p_j + 3a_1 a_j + 3p_1 f_j$$

itself. In order for U to satisfy the invariance requirements, $U \cap \bar{V}$ must, at least, contain all the orbits of D_1 through $U \cap V$, and we will find that this will be enough to generate U.

Let us first find the orbits of D_1 in $\bar{V}$. If we let

$p_1(t) = -\tanh(t)$ $(t \geq 0)$ then, taking everything on $\bar{V}$ unless otherwise stated, we have

1.12 i) $p_1(t) = -\tanh(t) \quad p_2(t) = p_2(0) = 0 \quad p_3(t) = p_3(0) = 0$

ii) $a_1(t) = a_1(0)\cosh^{-3}(t)$

$a_2(t) = a_2(0)\cosh^{-2}(t)$

$a_3(t) = a_3(0)\cosh^{-2}(t)$

as a complete orbit set of D_1 with initial conditions $p_1(0) = 0 = p_2(0) = p_3(0)$, others as stated.

We now take $U \cap \bar{V}$ to be the union of all the orbits of D_1 through $U \cap V$, namely the subset of $\bar{V}$ given by

1.13 $$([1-p_1p_1]^{-3/2}a_1)^2 + ([1-p_1p_1]^{-1}a_2]^2 + ([1-p_1p_1]^{-1}a_3)^2 < K.$$

We can now solve equation 1.11 in parametrized form in a quite straightforward manner.

1.14 $$f_1 = \cosh^{-4}(t)[3(a_1(0))^2\tanh(t)+\bar{f}_1(a_1(0),a_2(0),a_3(0))]$$

$$f_2 = \cosh^{-3}(t)[3a_1(0)a_2(0)\tanh(t) + \bar{f}_2(a_1(0),a_2(0),a_3(o))]$$

$$f_3 = \cosh^{-3}(t)[3a_1(0)a_2(0)\tanh(t) + \bar{f}_3(a_1(0),a_2(0),a_3(0))].$$

In particular the $f_j = \bar{f}_j$ on $U \cap V$ and $f_j \to 0$ along every orbit of D_1.

If we set $G(x,p,a)$ to be the euclidean invariant extension of the function given by the left hand side of 1.13, the U defined by $G < K$ and, of course, $p_1^2+p_2^2+p_3^2 < 1$, clearly satisfies all the invariance requirements, and the euclidean invariant extension of $\Sigma f_j \partial a_j$ is then defined on U. (The possibility of such an extension is assured by the symmetry requirements of our initial functions $\bar{f}_j$).

All we have left to check is that the resulting vectorfield defines a global flow on our domain, i.e., U is invariant under X_o. There are three things to check under this heading. First, that the spacial coordinates of an integral curve of X_o don't run off to infinity in finite time, which is clear since $(X_o(x_1))^2 + (X_o(x_2))^2 + (X_o(x_3))^2 < 1$. Second, we must check that X_o is tangent to the boundary of U defined by $G = K$. Since $\bar{f}_j$ were taken to be defined and smooth on $a_1(0)^2 + a_2(0)^2 + a_3(0)^2 = K$, 1.14 actually defines smooth solutions up to and including $G = K$. We must only check that $X_o(G)|_{G=K} = 0$ to be assured of this tangency. Since both X_o and G are euclidean invariant, it suffices to check this on $U \cap \bar{V}$. There, taking into account the euclidean invariance of G, the pullback equation reads

1.15 $$[a_1\partial p_1 + \frac{a_2}{p_1}(a_2\partial a_1 - a_1\partial a_2) + \frac{a_3}{p_1}(a_3\partial a_1 - a_1\partial a_3) + f_1\partial a_1 + f_2\partial a_2 + f_3\partial a_3][[1-p_1p_1]^{-3}a_1^2 + [1-p_1p_1]^{-2}a_2^2 + [1-p_1p_1]^{-2}a_3^2] = 0$$

when

$$[1-p_1p_1]^{-3}a^2_1 + [1-p_1p_1]^{-2}(a_2^2+a_3^2) = K \text{ and } p_1 \neq 0.$$

When $p_1 = 0$, we have $-D_1 = \partial p_1$, etc., and the equation may be written

1.16 $$[-a_1D_1-a_2D_2-a_3D_3+\bar{f}_1\partial a_1+\bar{f}_2\partial a_2+\bar{f}_3\partial a_3]G = 0$$

when $G = K$ and $0 = p_1$ $(= p_2 = p_3)$.

Taking 1.16 first, G was constructed so that $D_jG = 0$ always, on the other hand $\bar{f}_1, \bar{f}_2, \bar{f}_3$ were required to vanish on $G = K$, thus 1.16 is satisfied.

Converting 1.14 to unparametrized form we have

1.17 $$f_1 = -3p_1a_1^2(1-p_1p_1)^{-1} + (1-p_1p_1)^2\bar{f}_1([1-p_1p_1]^{-3/2}a_1, ,[1-p_1p_1]^{-1}a_2,[1-p_1p_1]^{-1}a_3)$$

$$f_2 = -3p_1a_1a_2(1-p_1p_1)^{-1} + (1-p_1p_1)^{3/2}\bar{f}_2([1-p_1p_1]^{-3/2}a_1, ,[1-p_1p_1]^{-1}a_2,[1-p_1p_1]^{-1}a_3)$$

$$f_3 = -3p_1a_1a_3(1-p_1p_1)^{-1} + (1-p_1p_1)^{3/2}\bar{f}_3([1-p_1p_1]^{-3/2}a_1, ,[1-p_1p_1]^{-1}a_2,[1-p_1p_1]^{-1}a_3).$$

Plugging these into 1.15, we find that everything but the contributions of the $\bar{f}_j$'s cancel identically and then the $\bar{f}_j$'s themselves vanish on $G = K$ by assumption.

Finally, we must check that the integral curves of X_o do not reach $p_1^2+p_2^2+p_3^2 = 1$ in finite time. On the other hand $(X_o(p_1))^2 + (X_o(p_2))^2 + (X_o(p_3))^2 < (1-p_1p_1-p_2p_2-p_3p_3)^2K.$

Thus the speed goes up like tanh(t), in particular, never reaches 1 in finite time. This completes the proof.

We might note at this point that the Dirac solutions of the one body electrodynamics problem (1) are of this type. This is the best evidence that the "true" n-body electrodynamics is at least third order. This procedure may be mimicked for any higher order dynamics we wish, giving us a large, essentially exhaustive class of solutions for the single particle dynamics.

§2. Let us now consider two and three particle, first order dynamics. For first order dynamics, we may begin by considering equation 4.3.1 which reads

2.1 $$(Y_{oi}-x_i^r X_o)f_j^r = f_i^r f_j^r - \delta_{ij}$$

In this case $X_o = \sum_{s,j} f_j^r \partial x_j^r$ and $Y_{oi} = \sum_{s,j} x_i^s f_j^s \partial x_j^s$ reflecting the fact that we are dealing with a first order dynamics. We thus have

2.2 $$(Y_{oi}-x_i^r X_o) = \sum_{s,j} (x_i^s - x_i^r) f_j^r \partial x_j^r := D_i^r.$$

Proposition. There are no two particle, first order dynamics.

Proof. First note that on the submanifold given by $x_2^1 = x_3^1 = x_2^2 = x_3^2 = 0$, the operators D_2^r and D_3^r vanish and are thus tangent to this submanifold. If we had f_j^r's which defined a first order R.D.L. they would have to satisfy

2.3 $$0 = f^r_2 f^r_j - \delta_{2j} \text{ and } 0 = f^r_3 f^r_j - \delta_{3j}.$$

On this submanifold. Taking $j = 2$ and $j = 3$ in these equation we find $f^r_2 f^r_2 = 1 = f^r_3 f^r_3$ but $f^r_2 f^r_3 = 0$, an impossible situation. Moreover, this submanifold generates the entire space under the action of the euclidean group so that not even local solutions can exist.

Proposition. There are no (subluminal) three particle, first order R.D.L.

Proof. We again consider equations 2.2. This time we restrict to the submanifold $x^1_3 = x^2_3 = x^3_3 = 0$. Again, D^r_3 vanishes there and the f^r_j must satisfy

2.4 $$0 = f^r_3 f^r_j - \delta_{3j}.$$

Hence $(f^r_3)^2 = 1$ and $f^r_2 = f^r_1 = 0$. However, since we want speeds less than 1, this rules this out as a possible solution, completing the proof.

It is interesting to note that if we are willing to include speed = 1 solutions, then this actually gives us one. Namely, if we exclude from $R^3 \times R^3 \times R^3$ the various diagonals, then we can give the prescription that three particles travel at the speed of light in the direction (euclidean) normal to the plane they span with the sense given by the right hand rule applied to their ordering. This defines a three particle, first order R.D.L.

§3. The second order equations.

The second order case is the most important, physically, for in most physical systems, second order is assumed, and Hamiltonian (hence quantum) mechanics is intrinsically second order.

We will first construct a class of solutions which, in a sense, skirts the non-linearity of the equations. With our previous notation, the relevant equations are

3.1 $$\Big[\sum_{s,\ell}(x_i^s - x_i^r)(p_\ell^s \partial x_\ell^s + f_\ell^s \partial p_\ell^s) + (p_i^s p_\ell^s - \delta_{i\ell})\partial p_\ell^s\Big] f_j^r = f_i^r p_j^r + 2p_i^r f_j^r.$$

3.2 $$X_i(f_j^r) = 0 = \Big[\sum_s \partial x_i^s\Big] f_j^r$$

For the purpose of our construction, we will define the linearized operators F_i^r.

3.3 $$F_i^r = \sum_{s,\ell}(x_i^s - x_i^r)p_\ell^s \partial x_\ell^s + (p_i^s p_\ell^s - \delta_{i\ell})\partial p_\ell^s.$$

Proposition. Suppose we have 3n smooth functions $\tilde{f}_j^r$ satisfying the linear equations $F_i^r(\tilde{f}_j^r) = \tilde{f}_i^r p_j^r + 2p_i^r \tilde{f}_j^r$ and $X_i(\tilde{f}_j^r) = 0$. Suppose, moreover, that we have n smooth functions g^r satisfying $g^r g^s \equiv 0$ for $r \neq s$ and $F_i^r(g^r) = 0$ and $X_i(g^r) = 0$. Then the functions $f_j^r := g^r \tilde{f}_j^r$ will satisfy equations 3.1 and 3.2.

Proof. With our assumptions 3.2 reads

$$3.4 \qquad X_i f^r_j = X_i(g^r \tilde{f}^r_j) = X_i(g^s)\tilde{f}^r_j + g^s X_i(\tilde{f}^r_j) = 0.$$

Thus 3.2 is satisfied.

On the other hand 3.1 reads

$$3.5 \qquad [\sum_{s,\ell}(x^s_i - x^r_i)(p^s_\ell \partial x^s_\ell + f^s_\ell \partial p^s_\ell) + (p^s_i p^s_\ell - \delta_{i\ell})\partial p^s_\ell] f^r_j$$

$$= F^r_i(f^r_j) + [\sum_{s,\ell}(x^s_i - x^r_i) f^s_\ell \partial p^s_\ell] f^r_j = F^r_i(g^r \tilde{f}^r_j)$$

$$+ [\sum_{s,\ell}(x^s_i - x^r_i) g^s \tilde{f}^s_\ell \partial p^s_\ell](g^r \tilde{f}^r_j)$$

$$= F^r_i(g^r)\tilde{f}^r_j + g^s F^r_i(\tilde{f}^r_j) + [\sum_{s,\ell}(x^s_i - x^r_i) g^s \tilde{f}^s_\ell \partial p^s_\ell](g^r \tilde{f}^r_j)$$

$$= g^r(\tilde{f}^r_i p^r_j + 2p^r_i \tilde{f}^r_j) + [\sum_{s,\ell}(x^s_i - x^r_i) g^s \tilde{f}^s_\ell \partial p^s_\ell](g^r \tilde{f}^r_j)$$

$$= f^r_i p^r_j + 2p^r_i f^r_j + [\sum_{s,\ell}(x^s_i - x^r_i) g^s \tilde{f}^s_\ell \partial p^s_\ell](g^r \tilde{f}^r_j).$$

Thus, as long as the last term is zero, 3.1 will be satisfied. However, since we have assumed $g^s g^r \equiv 0$ for $s \neq r$

$$3.6 \quad [\sum_{s,\ell}(x^s_i - x^r_i) g^s \tilde{f}^s_\ell \partial p^s_\ell](g^r \tilde{f}^r_j)$$

$$= [\sum_{s,\ell}(x^s_i - x^r_i) g^r g^s \tilde{f}^s_\ell \partial p^s_\ell] \tilde{f}^r_j$$

$$+ \tilde{f}^r_j [\sum_{s,\ell}(x^s_i - x^r_i) g^s \tilde{f}^s_\ell \partial p^s_\ell] g^r.$$

Now at any point we have one of the following situations

i) $g^s = 0$ all s, in which case the last term of 3.6 also vanishes.

ii) $g^r \neq 0$, in which case $g^s = 0$ for $s \neq r$ and again the last term is zero.

iii) $g^s \neq 0$ some $s \neq r$, in which case $g^s \neq 0$ in a neighborhood of the point in question. This would then imply that $g^r = 0$ in that same neighborhood, thus all derivatives of g^r would vanish at the point in question and again the last term vanishes.

This concludes the proof.

Let us consider a procedure for constructing solutions of this type. We will take $n = 3$ for specificity, although this type of construction works for any number of particles but it will differ in details.

To find the $\tilde{f}_j^r$'s, we first note that, like the non-linear equations, the linearized equations

$$3.7 \qquad F_i^r \tilde{f}_j^r = \tilde{f}_j^r p_j^r + 2 p_i^r \tilde{f}_j^r$$

have the Euclidean covariance property outlined in Proposition of Section I.4. Moreover, if $\tilde{f}_j^r$ satisfy 3.7 and $X_i(\tilde{f}_j^r) = 0$ then they must also be Euclidean covariant, i.e.

$$3.8 \qquad Y_{ij} \tilde{f}_\ell^r = \delta_{j\ell} \tilde{f}_i^r - \delta_{i\ell} \tilde{f}_j^r .$$

These can be derived from the corresponding statements for the nonlinear equation by setting $g^r \equiv 1$ and $g^s \equiv 0$ for $s \neq r$ and applying the above proposition.

Consider the submanifold given by

$$3.9 \qquad x_1^1 = x_2^1 = x_3^1 = x_2^2 = x_3^2 = x_3^3 = 0.$$

Certainly, the union of the orbits of the Euclidean group through this submanifold is all of the three particle, second order space. We will then pull back 3.7 to this submanifold, solve there and extend by Euclidean covariance.

The only derivatives in the f^r_i's which are not tangential to this submanifold are, of course, $\partial x^1_1, \partial x^1_2, \partial x^1_3, \partial x^2_2, \partial x^2_3, \partial x^3_3$. On this submanifold, we can solve for these in terms of the Y_{ij}, X_i and tangential derivatives.

On the "transverse strata" given by $x^3_2 > 0$, $x^2_1 > 0$ (we will consider the lower strata separately since there are subgroups of the Euclidean group leaving them invariant). We can write

3.10.1 $$\partial x^3_3 \tilde{f}^r_j = \frac{1}{x^3_2}(Y_{23} - \sum_s p^s_2 \partial p^s_3 - p^s_3 \partial p^s_2)\tilde{f}^r_j$$

$$= \frac{1}{x^3_2}(\delta_{3j}\tilde{f}^r_2 - \delta_{2j}\tilde{f}^r_3 - \sum_s (p^s_2 \partial p^s_3 - p^s_3 \partial p^s_2)\tilde{f}^r_j)$$

3.10.2 $$\partial x^2_3 \tilde{f}^r_j = \frac{1}{x^2_1}(Y_{13} - x^3_1 \partial x^3_3 - \sum_s p^s_1 \partial p^s_3 - p^s_3 \partial p^s_1)\tilde{f}^r_j$$

$$= \frac{1}{x^2_1}(\delta_{3j}\tilde{f}^r_1 - \delta_{1j}\tilde{f}^r_3 - \frac{x^3_1}{x^3_2}(\delta_{3j}\tilde{f}^r_2 - \delta_{2j}\tilde{f}^r_3 - \sum_s (p^s_2 \partial p^s_3 - p^s_3 \partial p^s_2)\tilde{f}^r_j)$$

$$-\sum_s (p^3_1 \partial p^s_3 - p^s_3 \partial p^s_1)\tilde{f}^r_j)$$

3.10.3 $$\partial x^1_3 \tilde{f}^r_j = (X_3 - \partial x^2_3 - \partial x^3_3)\tilde{f}^r_j = -(\partial x^2_3 + \partial x^3_3)\tilde{f}^r_j$$

3.10.4 $$\partial x_2^2 \tilde{f}_j^r = \frac{1}{x_1^2}(Y_{12} - x_1^3 \partial x_2^3 - \sum_s p_1^s \partial p_2^s - p_2^s \partial p_1^s)\tilde{f}_j^r$$

$$= \frac{1}{x_1^2}(\delta_{2j}\tilde{f}_1^r - \delta_{1j}\tilde{f}_2^r - x_1^3 \partial x_2^3 \tilde{f}_j^r + x_2^3 \partial x_1^3 \tilde{f}_j^r$$

$$- \sum_s (p_1^s \partial p_2^s - p_2^s \partial p_1^s)\tilde{f}_j^r)$$

3.10.5 $$\partial x_2^1 \tilde{f}_j^r = (X_2 - \partial x_2^2 - \partial x_2^3)\tilde{f}_j^r = -(\partial x_2^2 + \partial x_2^3)\tilde{f}_j^r$$

3.10.6 $$\partial x_1^1 \tilde{f}_j^r = (X_1 - \partial x_1^2 - \partial x_1^3)\tilde{f}_j^r = -(\partial x_1^2 + \partial x_1^3)\tilde{f}_j^r$$

Restricted to the manifold, equation 3.7 becomes

3.11.1 $$[(x_1^2 \sum_\ell p_\ell^2 \partial x_\ell^2 + x_1^3 \sum_\ell p_\ell^3 \partial x_\ell^3 + \sum_{s,\ell}(p_1^s p_\ell^s - \delta_{1\ell})\partial p_\ell^s]\tilde{f}_j^1$$

$$= \tilde{f}_1^1 p_j^1 + 2p_2^1 f_j^1.$$

3.11.2 $$[x_2^3 \sum_\ell p_\ell^3 \partial x_\ell^3 + \sum_{s,\ell}(p_2^s p_\ell^s - \delta_{2\ell})\partial p_\ell^s]\tilde{f}_j^1$$

$$= \tilde{f}_2^1 p_j^1 + 2p_2^1 \tilde{f}_j^1.$$

3.11.3 $$[\sum_{s,\ell}(p_3^s p_\ell^s - \delta_{3\ell})\partial p_\ell^s]\tilde{f}_j^1 = \tilde{f}_3^1 p_j^1 + 2p_3^1 \tilde{f}_j^1.$$

3.12.1 $$[-x_1^2 \sum_\ell p_\ell^1 \partial x_\ell^1 + (x_1^3 - x_1^2) \sum_\ell p_\ell^3 \partial x_\ell^3 + \sum_{s,\ell}(p_1^s p_\ell^s - \delta_{1\ell})\partial p_\ell^s]\tilde{f}_j^2$$

$$= \tilde{f}_1^2 p_j^2 + 2p_1^2 \tilde{f}_j^2.$$

3.12.2 $$[x_2^3 \sum_{\ell} p_{\ell}^3 \partial x_{\ell}^3 + \sum_{s,\ell} (p_2^s p_{\ell}^s - \delta_{2\ell}) \partial p_{\ell}^s] \tilde{f}_j^2 = \tilde{f}_2^2 p_j^2 + 2p_2^2 \tilde{f}_j^2.$$

3.12.3 $$[\sum_{s,\ell} (p_3^s p_{\ell}^s - \delta_{3\ell}) \partial p_{\ell}^s] \tilde{f}_j^2 = \tilde{f}_3^2 p_j^2 + 2p_3^2 \tilde{f}_j^2.$$

3.13.1 $$[-x_1^3 \sum_{\ell} p_{\ell}^1 \partial x_{\ell}^1 + (x_1^2 - x_1^3) \sum_{\ell} p_{\ell}^2 \partial x_{\ell}^2 + \sum_{s,\ell} (p_1^s p_{\ell}^s - \delta_{1\ell}) \partial p_{\ell}^s] \tilde{f}_j^3 = p_j^3 \tilde{f}_1^3 + 2p_1^3 \tilde{f}_j^3.$$

3.13.2 $$[-x_2^3 \sum_{\ell} p_{\ell}^1 \partial x_{\ell}^1 - x_2^3 \sum_{\ell} p_{\ell}^2 \partial x_{\ell}^2 + \sum_{s,\ell} (p_2^s p_{\ell}^s - \delta_{2\ell}) \partial p_{\ell}^s] \tilde{f}_j^3 = \tilde{f}_2^3 p_j^3 + 2p_2^3 \tilde{f}_j^3.$$

3.13.3 $$[\sum_{s,\ell} (p_3^s p_{\ell}^s - \delta_{3\ell}) \partial p_{\ell}^s] \tilde{f}_j^3 = \tilde{f}_3^3 p_j^3 + 2p_3^3 \tilde{f}_j^3.$$

Note that equation 3.11.3, 3.12.3, 3.13.3 all involve the same operator $L_3 := \sum_{s,\ell} (p_3^s p_{\ell}^s - \delta_{3\ell}) \partial p_{\ell}^s$. Also note that this operator is tangent to the submanifold we are considering. We will show that L_3 is an additional "symmetry" of the system. In particular, if $\tilde{f}_j^r$ satisfy $L_3 \tilde{f}_j^r = \tilde{f}_3^r p_j^r + 2p_3^r \tilde{f}_j^r$ (equation 3.11.3, 3.12.3, 3.13.3) then the functions given by the tangential versions of the other equations transform under L_3 in such a way that if they vanish at a point, then they vanish along the entire orbit of L_3 through the point.

This will allow us to factor out one more dimension.

To make the calculation easier, let us adopt the notation

3.14.1 $\tilde{Y}_{23} := \sum_s p_2^s \partial p_3^s - p_3^s \partial p_2^s$

3.14.2 $\tilde{Y}_{13} := \sum_s p_1^s \partial p_3^s - p_3^s \partial p_1^s$

3.14.3 $\tilde{Y}_{12} := x_1^3 \partial x_2^3 - x_2^3 \partial x_1^3 + \sum_s p_1^s \partial p_2^s - p_2^s \partial p_1^s$.

Each of these is tangent to the submanifold. With this notation, equation 3.11, 3.12, 3.13 may be written on the transverse strata as

3.15.1
$$\{x_1^2[p_1^2 \partial p_2^2 \frac{1}{x_1^2}(Y_{12}-\tilde{Y}_{12}) + p_3^2 \frac{1}{x_1^2}(Y_{13} - \frac{x_1^3}{x_2^3}(Y_{23}-\tilde{Y}_{23}) - \tilde{Y}_{13})]$$
$$= x_1^3[p_1^3 \partial x_1^3 + p_2^3 \partial x_2^3 + p_3^3 \frac{1}{x_2^3}(Y_{23}-\tilde{Y}_{23})]$$
$$+ \sum_{s,\ell}(p_1^s p_\ell^s - \delta_{1\ell})\partial p_\ell^s\}\tilde{f}_j^1 - \tilde{f}_1^1 p_j^1 - 2p_1^1 \tilde{f}_j^1 = 0$$

3.15.2
$$\{x_2^3[p_1^3 \partial x_1^3 + p_2^3 \partial x_2^3 + p_3^3 \frac{1}{x_2^3}(Y_{23}-\tilde{Y}_{23})]$$
$$+ \sum_{s,\ell}(p_2^s p_\ell^s - \delta_{2\ell})\partial p_\ell^s\}\tilde{f}_j^1 - \tilde{f}_2^1 p_j^1 - 2p_2^1 \tilde{f}_j^1 = 0.$$

3.16.1
$$\{-x_1^2[-p_1^1(\partial x_1^2 + \partial x_1^3) - p_2^1(\frac{1}{x_1^2}(Y_{12}-\tilde{Y}_{12}) + \partial x_2^3)$$
$$-p_3^1(\frac{1}{x_1^2}(Y_{13} - \frac{x_1^3}{x_2^3}(Y_{23}-\tilde{Y}_{23}) - \tilde{Y}_{13}) + \frac{1}{x_2^3}(Y_{23}-\tilde{Y}_{23}))]$$

$$+ (x_1^3-x_1^2)[p_1^3\partial x_1^3 + p_2^3\partial x_2^3 + p_3^3 \frac{1}{x_2^3}(Y_{23}-\tilde{Y}_{23})]$$

$$+ \sum_{s,\ell}(p_1^s p_\ell^s-\delta_{1\ell})\partial p_\ell^s\}\tilde{f}_j^2 - \tilde{f}_1^2 p_j^2 - 2p_1^2\tilde{f}_j^2 = 0.$$

3.16.2
$$\{x_2^3[p_1^3\partial x_1^3 + p_2^3\partial x_2^3 + p_3^3 \frac{1}{x_2^3}(Y_{23}-\tilde{Y}_{23})]$$

$$+ \sum_{s,\ell}(p_2^s p_\ell^s-\delta_{2\ell})\partial p_\ell^s\}\tilde{f}_j^2 - \tilde{f}_2^2 p_j^2 - 2p_2^2\tilde{f}_j^2 = 0.$$

3.17.1
$$\{-x_1^3[-p_1^1(\partial x_1^2+\partial x_1^3) - p_2^2(\frac{1}{x_1^2}(Y_{12}-\tilde{Y}_{12}) + \partial x_2^3)$$

$$- p_3^1(\frac{1}{x_1^2}(Y_{13} - \frac{x_1^3}{x_2^3}(Y_{23}-\tilde{Y}_{23})-\tilde{Y}_{13}) + \frac{1}{x_2^3}(Y_{23}-\tilde{Y}_{23}))]$$

$$+ (x_1^2-x_1^3)[p_1^2\partial x_1^2+p_2^2 \frac{1}{x_1^2}(Y_{12}-\tilde{Y}_{12})+p_3^2 \frac{1}{x_1^2}(Y_{13} - \frac{x_1^3}{x_2^3}$$

$$(Y_{23}-\tilde{Y}_{23})-\tilde{Y}_{13})]$$

$$+ \sum_{s,\ell}(p_1^s p_\ell^s-\delta_{1\ell})\partial p_\ell^s\}\tilde{f}_j^3 - \tilde{f}_1^3 p_j^3 - 2p_1^3\tilde{f}_j^3 = 0.$$

3.17.2
$$\{-x_2^3[-p_1^1(\partial x_1^2+\partial x_1^3) - p_2^1(\frac{1}{x_1^2}(Y_{12}-\tilde{Y}_{12}) + \partial x_2^3)$$

$$- p_3^1(\frac{1}{x_1^2}(Y_{13} - \frac{x_1^3}{x_2^3}(Y_{23}-\tilde{Y}_{23})-Y_{13}) + \frac{1}{x_2^3}(Y_{23}-\tilde{Y}_{23}))$$

$$+ p_1^2\partial x_1^2 + p_2^2 \frac{1}{x_1^2}(Y_{12}-\tilde{Y}_{12}) + p_3^2 \frac{1}{x_1^2}(Y_{13} - \frac{x_1^3}{x_2^3}(Y_{23}-\tilde{Y}_{23})-\tilde{Y}_{13}$$

$$+ \sum_{s,\ell}(p_2^s p_\ell^s-\delta_{2\ell})\partial p_\ell^s\}\tilde{f}_j^3 - \tilde{f}_2^3 p_j^3 - 2p_2^3\tilde{f}_j^3 = 0.$$

It is straightforward to check that if we define functions G_1^1, G_2^1, G_1^2, G_2^2, G_1^3, G_2^3 by the left hand sides of 3.15.1, 3.15.2, 3.16.1, 3.16.2, 3.17.1, 3.17.2, where we have used $Y_{ij}\tilde{f}_\ell^r = \delta_{j\ell}\tilde{f}_i^r - \delta_{i\ell}\tilde{f}_j^r$, and if the $\tilde{f}_j^r$ satisfy $L_3\tilde{f}_j^r = \tilde{f}_3^r p_j^r + 2p_3^r\tilde{f}_j^r$, then

3.18.1 $\quad L_3(G_2^1) = p_3^3 G_2^1$

3.18.2 $\quad L_3(G_2^2) = p_3^3 G_2^2$.

In particular, since $L_3(p_3^3) \neq 0$ everywhere, we see that if $G_2^1 = 0$ at some point (i.e., 3.15.2 is satisfied, then $G_2^1 = 0$ along the whole orbit of L_3 through that point, and similarly for G_2^2.

We also find that if we let $\tilde{G}_1^1 = G_1^1 - \frac{x_1^3}{x_2^3} G_2^1$ and $\tilde{G}_1^2 = G_1^2 - \frac{x_1^3 - x_1^2}{x_2^3} G_2^2$ then we have, with the above assumptions

3.19.1 $\quad L_3\tilde{G}_1^1 = p_3^2\tilde{G}_1^1$

3.19.2 $\quad L_3\tilde{G}_1^2 = p_3^1\tilde{G}_1^2$.

In particular, if $\tilde{G}_1^1 = 0$ at some point, then it is zero along the whole orbit of L_3 through that point ($L_3p_3^2 \neq 0$, $L_3p_3^1 \neq 0$ everywhere). Similarly for $\tilde{G}_1^2$.

Finally, if we define $\tilde{G}_1^3 = G_1^3 - \frac{x_1^3}{x_2^3} G_2^3$ and $\tilde{G}_2^3 = G_2^3 + \frac{x_1^2 - x_1^3}{x_2^3} G_2^3$ we find

3.20.1 $\quad L_3(\tilde{G}_1^3) = p_3^2\tilde{G}_2^3$

3.20.2 $\quad L_3(\tilde{G}_2^3) = p_3^1\tilde{G}_2^3$.

We, therefore, have

Proposition. If $L_3\tilde{f}_j^r = \tilde{f}_3^r p_j^r + 2p_3^r\tilde{f}_j^r$ and equation 3.15.1, 3.15.2, 3.16.1, 3.16.2, 3.17.1, 3.17.2 are satisfied at a point, then they are satisfied on the whole orbit of L_3 through that point.

Since they have nicer covariance properties under L_3, we will use, instead of equation 3.15, 3.16, 3.17 the equivalent system

3.21.1 $\quad \tilde{G}_1^1 = 0$

3.21.2 $\quad G_2^1 = 0$

3.22.1 $\quad \tilde{G}_1^2 = 0$

3.22.2 $\quad G_2^2 = 0$

3.23.1 $\quad \tilde{G}_1^3 = 0$

3.23.2 $\quad \tilde{G}_2^3 = 0$

(assuming, $L_3\tilde{f}_j^r = \tilde{f}_3^r p_j^r + 2p_3^r\tilde{f}_j^r$).

Because of the above proposition, we may pull back these equations to a subset of one lower dimension. We will consider in detail only the system 3.21.1 and 3.21.2, the others are similar. In fact, since the systems are completely decoupled we could have chosen different sections for the Euclidean action on each one so that the resulting equations would look exactly like 21.1 and 21.2 for the other systems as well. This nice feature will not appear in the full nonlinear equations.

Writing out 3.21.1 and 3.21.2 in full, we have

3.24.1
$$
\begin{aligned}
&\{x_1^2p_1^2\partial x_1^2+p_2^2x_2^3\partial x_1^3-p_2^2x_1^3\partial x_2^3+[p_2^2p_2^1+p_3^2p_3^1+p_1^1p_1^1-1-\tfrac{x_1^3}{x_2^3}p_2^1]\partial p_1^1\\
&+[-p_2^2p_1^1-p_3^2\tfrac{x_1^3}{x_2^3}p_3^1+p_1^1p_2^1-\tfrac{x_1^3}{x_2^3}(p_2^1p_2^1-1)]\partial p_2^1\\
&+[p_3^2\tfrac{x_1^3}{x_2^3}p_2^1-p_3^2p_1^1+p_1^1p_3^1-\tfrac{x_1^3}{x_2^3}p_2^1p_3^1]\partial p_3^1\\
&+[p_2^2p_2^2+p_3^2p_3^2+p_1^2p_1^2-1-\tfrac{x_1^3}{x_2^3}p_1^2p_2^2]\partial p_1^2\\
&+[-p_2^2p_1^2-p_3^2\tfrac{x_1^3}{x_2^3}p_3^2+p_1^2p_2^2-\tfrac{x_1^3}{x_2^3}(p_2^2p_2^2-1)]\partial p_2^2\\
&+[p_2^2p_2^3+p_3^2p_3^3+p_1^3p_1^3-1-\tfrac{x_1^3}{x_2^3}p_2^3p_1^3]\partial p_1^3\\
&+[-p_2^2p_1^3-p_3^2\tfrac{x_1^3}{x_2^3}p_3^3+p_1^3p_2^3-\tfrac{x_1^3}{x_2^3}(p_2^3p_2^3-1)]\partial p_2^3\\
&+[p_3^2\tfrac{x_1^3}{x_2^3}p_2^3-p_3^2p_1^3+p_1^3p_3^3-\tfrac{x_1^3}{x_2^3}p_2^3p_3^3]\partial p_3^3\}\tilde{f}^1_j\\
&-\tilde{f}^1_1p^1_j-2p_1^1\tilde{f}^1_j+p_2^2(\delta_{2j}\tilde{f}^1_1-\delta_{1j}\tilde{f}^1_2)+p_3^2(\delta_{3j}\tilde{f}^1_1-\delta_{1j}\tilde{f}^1_3)\\
&-p_3^2\tfrac{x_1^3}{x_2^3}(\delta_{3j}\tilde{f}^1_2-\delta_{2j}\tilde{f}^1_3)+\tfrac{x_1^3}{x_2^3}(\tilde{f}^1_2p^1_j+2p_2^1\tilde{f}^1_j-p_3^3(\delta_{3j}\tilde{f}^1_2-\delta_{2j}\tilde{f}^1_j))\\
&=0.
\end{aligned}
$$

3.24.2 $$\{x_2^3p_1^3\partial x_1^3+x_2^3p_2^3\partial x_2^3+p_2^1p_1^1\partial p_1^1+[p_3^3p_3^1+p_2^1p_2^1-1]\partial p_2^1$$

$$+ [-p_3^3p_2^1+p_2^1p_3^1]\partial p_3^1+p_2^2p_1^2\partial p_1^2+[p_3^3p_3^2+p_2^2p_2^2-1]\partial p_2^2$$

$$+ [-p_3^3p_2^2+p_2^2p_3^2]\partial p_3^2+p_2^3p_1^3\partial p_1^3+[p_3^3p_3^3+p_2^3p_2^3-1]\partial p_2^3\}\tilde{f}_j^1$$

$$-\tilde{f}_2^1p_j^1-2p_2^1\tilde{f}_j^1+p_3^3(\delta_{3j}\tilde{f}_2^1-\delta_{2j}\tilde{f}_3^1) = 0.$$

Notice that 3.24.1 is tangent to the submanifolds given by p_3^2 = constant and 3.24.2 is tangent to the submanifolds p_3^3 = constant. Also note that the orbits of L_3 through either of these fills up the whole space. We can thus choose $p_3^3 = 0$ as defining the reduced space.

On $p_3^3 = 0$ we have

3.25 $$L_3 = -\partial p_3^3 + \sum_{\substack{s\neq 3\\ j}} (p_3^sp_j^s-\delta_{3j})\partial p_j^s.$$

On $p_3^3 = 0$, 3.24.2 simplifies to

3.26 $$\{x_2^3p_1^3\partial x_1^3+x_2^3p_2^3\partial x_2^3+p_2^1p_1^1\partial p_1^1+(p_2^1p_2^1-1)\partial p_2^1$$

$$+ p_2^1p_3^1\partial p_3^1+p_2^2p_1^2\partial p_1^2+(p_2^2p_1^2-1)\partial p_2^2+p_2^2p_3^2\partial p_3^2$$

$$+ p_2^3p_1^3\partial p_1^3+(p_2^3p_2^3-1)\partial p_2^3\}\tilde{f}_j^1-\tilde{f}_2^1p_j^1-2p_2^1\tilde{f}_j^1 = 0.$$

On the other hand, writing $\tilde{L}_3 = \sum_{\substack{j\neq 3\\ \ell}} (p_3^sp_\ell^s-\delta_{3\ell})\partial p_\ell^s$ on $p_3^3 = 0$, 3.24.1 becomes

3.27 $$\{x_1^2 p_1^2 \partial x_1^2 + p_2^2 x_2^3 \partial x_1^3 - p_2^2 x_1^3 \partial x_2^3 + [p_2^2 p_2^1 + p_3^2 p_3^1 + p_1^1 p_1^1 - 1 - \frac{x_1^3}{x_2^3} p_2^1] \partial p_1^1$$

$$+ [-p_2^2 p_1^1 - p_3^2 \frac{x_1^3}{x_2^3} p_3^1 + p_1^1 p_2^1 - \frac{x_1^3}{x_2^3}(p_2^1 p_2^1 - 1)] \partial p_2^1$$

$$+ [p_3^2 \frac{x_1^3}{x_2^3} p_2^1 - p_3^2 p_1^1 + p_1^1 p_3^1 - \frac{x_1^3}{x_2^3} p_2^1 p_3^1] \partial p_3^1$$

$$+ [p_2^2 p_2^2 + p_3^2 p_3^2 + p_1^2 p_1^2 - 1 - \frac{x_1^3}{x_2^3} p_1^2 p_2^2] \partial p_1^2$$

$$+ [-p_2^2 p_1^2 - p_3^2 \frac{x_1^3}{x_2^3} p_3^2 + p_1^2 p_2^2 - \frac{x_1^3}{x_2^3}(p_2^2 p_2^2 - 1)] \partial p_2^2$$

$$+ [p_2^2 p_3^2 + p_1^3 p_1^3 - 1 - \frac{x_1^3}{x_2^3} p_2^3 p_1^3] \partial p_1^3$$

$$+ [-p_2^2 p_1^3 + p_1^3 p_2^3 - \frac{x_1^3}{x_2^3}(p_2^3 p_2^3 - 1)] \partial p_2^3$$

$$+ [p_3^2 \frac{x_1^3}{x_2^3} p_2^3 - p_3^2 p_1^3](\tilde{L}_3)\} \tilde{f}^1_j - \tilde{f}^1_1 p^1_j - 2p^1_1 \tilde{f}^1_j + p^2_2(\delta_{2j} \tilde{f}^1_1 - \delta_{1j} \tilde{f}^1_2)$$

$$+ p^2_3(\delta_{3j} \tilde{f}^1_1 - \delta_{1j} \tilde{f}^1_3) - p^2_3 \frac{x_1^3}{x_2^3}(\delta_{3j} \tilde{f}^1_2 - \delta_{2j} \tilde{f}^1_3) + \frac{x_1^3}{x_2^3}(\tilde{f}^1_2 p^1_j + 2p^1_2 \tilde{f}^1_j)$$

$$- [p_3^2 \frac{x_1^3}{x_2^3} p_2^3 - p_3^2 p_1^3](\tilde{f}^1_3 p^1_j + 2p^1_3 \tilde{f}^1_j) = 0.$$

If we define F to be the differential operator in 3.26 we may write 3.26 as $F\tilde{f}^1_j - \tilde{f}^1_2 p^1_j - 2p^1_2 \tilde{f}^1_j = 0$. If we let $\tilde{\tilde{G}}^1_1$ be the function defined by the left hand side of 3.27 then if $F\tilde{f}^1_j - \tilde{f}^1_2 p^1_j - 2p^1_2 \tilde{f}^1_j = 0$ we have

3.28 $$F\tilde{\tilde{G}}^1_1 = p^2_2 \tilde{\tilde{G}}_1.$$

These equations 3.18.1, 3.18.2 and 3.28 are the complete analogues of $[D_i, D_j] = p_j D_i - p_i D_j$ shown in the first section of this chapter. (See Section 4 for general calculation of 3.28.)

Equation 3.28 allows us to factor out one further dimension. For example, the orbits of F through $p^2_2 = 0$ fill out the whole space at hand. On $p^2_2 = 0$, F may be written as

3.29 $$F = -\partial p^2_2 + \sum_{\ell}(p^1_2 p^1_\ell - \delta_{2\ell})\partial p^1_\ell + p^3_2 p^3_1 \partial p^3_1 + (p^3_2 p^3_2 - 1)\partial p^3_2 + x^3_2 p^3_1 \partial x^3_1 + x^3_2 p^3_2 \partial x^3_2 .$$

Thus, using 3.29 and $F\tilde{f}^1_j - \tilde{f}^1_2 p^1_j - 2p^1_2 \tilde{f}^1_j = 0$ we may pull back 3.27 to $p^2_2 = 0$ and get the final equation.

3.30 $$\begin{aligned} \{ & x^2_1 p^2_1 \partial x^2_1 + [p^2_3 p^1_3 + p^1_1 p^1_1 - 1 - \frac{x^3_1}{x^3_2} p^1_2]\partial p^1_1 \\ & + [-p^2_3 \frac{x^3_1}{x^3_2} p^1_3 + p^1_1 p^1_2 - \frac{x^3_1}{x^3_2}(p^1_2 p^1_2 - 1)]\partial p^1_2 \\ & + [p^2_3 \frac{x^3_1}{x^3_2} p^1_2 - p^2_3 p^1_1 + p^1_1 p^1_3 - \frac{x^3_1}{x^3_2} p^1_2 p^1_3]\partial p^1_3 \\ & + [p^2_3 p^2_3 + p^2_1 p^2_1 - 1]\partial p^2_1 \\ & + [-p^2_3 p^2_3 \frac{x^3_1}{x^3_2} + \frac{x^3_1}{x^3_2}][x^3_2 p^3_1 \partial x^3_1 + x^3_2 p^3_2 \partial x^3_2 + \sum_{\ell}(p^1_2 p^1_\ell - \delta_{2\ell})\partial p^1_\ell \\ & + p^3_2 p^3_1 \partial p^3_1 + (p^3_2 p^3_2 - 1)\partial p^3_2] \end{aligned}$$

$$+ [p_1^3 p_1^3 - 1 - \frac{x_1^3}{x_2^3} p_2^3 p_1^3] \partial p_1^3$$

$$+ [p_1^3 p_2^3 - \frac{x_1^3}{x_2^3}(p_2^3 p_2^3 - 1)] \partial p_2^3$$

$$+ [p_3^2 \frac{x_1^3}{x_2^3} p_2^3 - p_3^2 p_1^3] \tilde{L}_3 \} \; \tilde{f}_j^1 - \tilde{f}_1^1 p_j^1 - 2p_1^1 \tilde{f}_j^1 + p_3^2(\delta_{3j} \tilde{f}_1^1 - \delta_{1j} \tilde{f}_3^1)$$

$$- p_3^2 \frac{x_1^3}{x_2^3}(\delta_{3j} \tilde{f}_2^1 - \delta_{2j} \tilde{f}_3^1) + \frac{x_1^3}{x_2^3}(\tilde{f}_2^1 p_j^1 + 2p_2^1 \tilde{f}_j^1)$$

$$- [p_3^2 \frac{x_1^3}{x_2^3} p_2^3 - p_3^2 o_1^3](\tilde{f}_3^1 p_j^1 + 2p_3^1 \tilde{f}_j^1) - [-p_3^2 p_3^2 \frac{x_1^3}{x_2^3} + \frac{x_1^3}{x_2^3}]$$

$$[\tilde{f}_2^1 p_j^1 + \tilde{f}_j^1 p_2^1] = 0.$$

We can solve 3.29 by parametrization of the orbits of the vectorfield involved, e.g. noting that $p_3^2 p_3^2 \frac{x_1^3}{x_2^3}$: = k must be constant along the orbits, $p_1^2(\tau) = \sqrt{1+k}\tanh(\sqrt{1+k}\,\epsilon + \eta_f^2)$ etc. We can propagate functions defined on $p_1^2 = 0$ to a solution of 3.29.

We may complete this section by noting that the functions

$$h_{12} = (M_1M_2)^{-1}(1 - p_1^1p_1^2 - p_2^1p_2^2 - p_3^1p_3^2)$$

$$h_{13} = (M_1M_3)^{-1}(1 - p_1^1p_1^3 - p_2^1p_2^3 - p_3^1p_3^3)$$

$$h_{23} = (M_2M_3)^{-1}(1 - p_1^2p_1^3 - p_2^2p_2^3 - p_2^2p_3^3)$$

where $M_i = (\sum_{\ell} 1 - p_\ell^i p_\ell^i)^{\frac{1}{2}}$

(the Lorentz inner products in our coordinate system) are invariant under the X_i and F_2^r of the proposition. Thus, if v^s are functions on R^3 with mutually disjoint supports, then $g^s := v^s(h_{12}, h_{13}, h_{23})$ give us functions of the type required for the proposition.

§4. Local Existence of Solutions.

We will now return to the full non-linear equations.

4.1 $$\{\sum_{s,\ell}(x_i^s-x_i^r)(p_\ell^s\partial x_\ell^s+f_\ell^s\partial p_\ell^s)+(p_i^s p_\ell^s-\delta_{i\ell})\partial p_\ell^s\} f_j^r = f_i^r p_j^r+2p_i^r f_j^r.$$

4.2 $$X_i f_j^r = 0.$$

As with the linearized versions, we will factor out more and more of the symmetries. The difference in this case being that, after the initial Euclidean invariance, the symmetries are generated by quasi-linear rather than linear operators and thus we can get only local extensions under these "symmetries". Because, in the general case, we are not interested in the exact details of the equation, but rather only that at each stage they are symmetric hyperbolic and analytic, we will deal with them rather more abstractly than in the linearized case.

If we denote by D_i^r, the differential operator given by the left hand side of 4.1, we may write 4.1 as
$D_i^r f_j^r - f_i^r p_j^r - 2p_j^r f_j^r = 0.$

We will find functions $Q_{i\ell n}^r$ and $S_{i\ell}^r$ so that the operators

4.3 $$D_i^r - \sum_{\ell,m} Q_{i\ell m}^r Y_{\ell m} - \sum_\ell S_{i\ell}^r X_\ell$$

are tangent to a submanifold transverse to the orbits of the Euclidean group.

This is the "pullback" of D_i^r to this submanifold. We then write 4.1 as

$$4.4 \qquad (D_i^r - \sum_{\ell,m} Q_{i\ell m}^r Y_{\ell m} - \sum_\ell S_{i\ell}^r X_\ell) f_j^r + (\sum_{\ell,m} Q_{i\ell m}^r Y_{\ell m} + \sum_\ell S_{i\ell}^r X_\ell) f_j^r - f_i^r p_j^r - 2p_i^r f_j^r = 0.$$

Then use Euclidean covariance to write this as

$$4.5 \qquad (D_i^r - \sum_{\ell,m} Q_{i\ell m}^r Y_{\ell m} - \sum_\ell S_{i\ell}^r X_\ell) f_j^r + \sum_{\ell,m} Q_{i\ell m}^r (\delta_{mj} f_\ell^r - \delta_{\ell j} f_m^r) - f_i^r p_j^r - 2p_i^r f_j^r = 0.$$

4.5 is then tangent to the submanifold.

We then pick out a particular subset of the equations of 4.5, e.g. $i = 3$, and think of this set as generating a symmetry of the others. This can be done by showing that (writing $F_i^r = (D_i^r - \sum_{\ell,m} Q_{i\ell m}^r Y_{\ell m} - \sum_\ell S_{i\ell}^r X_\ell)$) Euclidean covariance of the f_j^r's together with

$$4.6 \qquad F_3^r f_j^r + \sum_{\ell,m} Q_{3\ell m}^r (\delta_{mj} f_\ell^r - \delta_{\ell j} f_m^r) - f_3^r p_j^r - 2p_3^r f_j^r = 0$$

assumed on the transverse submanifold implies that

$$4.7 \qquad F_3^r (F_i^r f_j^r + \sum_{\ell,m} Q_{i\ell m}^r (\delta_{mj} f_\ell^r - \delta_{\ell j} f_m^r) - f_3^r p_j^r - 2p_3^r f_j^r) = \sum_{\ell'} T_{i\ell'}^r (F_{\ell'}^r f_j^r + \sum_{\ell,m} Q_{\ell' \ell m}^r (\delta_{mj} f_\ell^r - \delta_{\ell j} f_m^r) - f_3^r p_j^r - 2p_3^r f_j^r)$$

on the submanifold. Here $T_{i\ell'}^r$ are functions. This will show that if $F_3^r f_j^r + \sum_{\ell,m} Q_{3\ell m}^r(\delta_{mj}f_\ell^r - \delta_{\ell j}f_m^r) - f_3^r p_j^r - 2p_3^r f_j^r = 0$ on the submanifold and $F_i^r f_j^r + \sum_{\ell,m} Q_{i\ell m}^r(\delta_{mj}f_\ell^r - \delta_{\ell j}f_m^r) - f_i^r p_j^r - 2p_i^r f_j^r = 0$ at a point of the submanifold, then these are zero along the whole orbit of F_3^r (approrpiate r) through this point. Thus if we can then find a sub-submanifold transverse to the orbits of all the F_3^r and can solve 4.7 starting with data on this submanifold (as we can with the Euclidean action) then we can reduce the problem to one on this sub-submanifold and repeat the procedure with, say, the pull-back of F_2^r.

Let us first collect some facts about the operators D_j^r. Assuming only the Euclidean covariance of the f_j^r's ($X_i f_j^r = 0$, $Y_{ij} f_\ell^r = \delta_{j\ell} f_i^r - \delta_{i\ell} f_j^r$) we find

4.8 $$[X_i, D_j^r] = 0, \ [Y_{ij}, D_\ell^r] = \delta_{j\ell} D_i^r - \delta_{i\ell} D_j^r.$$

Assuming nothing about the functions f_j^r, direct computation yields

4.9 $$[D_i^r, D_j^r] x_\ell^s = \delta_{j\ell}(x_i^s - x_i^r) - \delta_{i\ell}(x_j^s - x_j^r)$$

4.10 $$[D_i^r, D_j^r] p_\ell^s = (\delta_{j\ell} p_i^s - \delta_{i\ell} p_j^s) + (x_j^s - x_j^r)[D_i^r(f_\ell^s) - f_i^s p_\ell^s - 2f_\ell^s p_i^s] - (x_i^s - x_i^r)[D_j^r(f_\ell^s) - f_j^s p_\ell^s - 2p_j^s f_\ell^s].$$

Noting also that $D_i^r = D_i^s + (x_i^s - x_i^r) \sum_{s',\ell} p_\ell^{s'} \partial x_\ell^{s'} + f_\ell^{s'} \partial p_\ell^{s'}$ we may write

4.11 $$[D_i^r, D_j^r] = Y_{ij} - x_i^r X_j + x_j^r X_i + \sum_{s,\ell} \{(x_j^s - x_j^r)[D_i^s(f_\ell^s) - f_i^s p_\ell^s - 2p_i^s f_\ell^s] - (x_i^s - x_i^r)[D_j^s(f_\ell^s) - f_j^s p_\ell^s - 2p_j^s f_\ell^s]\} \partial p_\ell^s.$$

We may now begin to carry out this program. Given a point z in the second order, n-particle phase space, by Euclidean covariance we may assume that it lies on the submanifold given by $x_1^1 = x_2^1 = x_3^1 = x_2^2 = x_3^2 = x_3^3 = 0$. (For notational convenience we will consider the case $n = 2$ separately. Since $n = 1$ has already been considered, we will take $n \geq 3$.) As we go along, we will exclude certain submanifolds from consideration. The first of these is embodied in the assumption $x_1^2(z) > 0$, $x_2^3(z) > 0$. If these are non-zero, we can make them positive by an action of the Euclidean group.

We find that the following operators are tangential.

4.12.1 $$D_1^1 - p_2^2 Y_{12} - p_3^2\left(Y_{13} - \frac{x_1^3}{x_2^3} Y_{23}\right) - \frac{x_1^3}{x_2^3} p_3^3 Y_{23}$$

4.12.2 $$D_1^2 + p_1^1 x_1^2 X_1 + p_2^1 x_1^2 X_2 - p_2^1 Y_{12} + x_1^2 p_3^1 X_3 - p_3^1\left(Y_{13} - \frac{x_1^3}{x_2^3} Y_{23}\right) - p_3^1 \frac{x_1^2}{x_2^3} Y_{23} - (x_1^3 - x_1^2) p_3^3 \frac{1}{x_2^3} Y_{23}$$

4.12.3 $$D_1^3 + p_1^1 x_1^3 X_1 + x_1^3 p_2^1 X_2 - \frac{x_1^3}{x_1^2} Y_{12} + x_1^3 p_3^1 X_3 - \frac{x_1^3}{x_2^3}\left(Y_{13} - \frac{x_1^3}{x_2^3} Y_{23}\right) - p_3^1 \frac{x_1^3}{x_2^3} Y_{23} - (x_1^2 - x_1^3) p_2^2 \frac{1}{x_1^2} Y_{12} - (x_1^2 - x_1^3) p_3^2 \frac{1}{x_1^2}\left(Y_{23} - \frac{x_1^3}{x_2^3} Y_{23}\right)$$

4.12.4 $$D_1^4+x_1^4p_1^1X_1+x_1^4p_2^1X_2-p_2^1\frac{x_1^4}{x_1^2}Y_{12}+x_1^4p_3^1X_3-x_1^4p_3^1\frac{1}{x_1^2}(Y_{13}-\frac{x_1^3}{x_2^3}Y_{23})$$
$$-x_1^4\frac{1}{x_2^3}p_3^1Y_{23}-(x_1^2-x_1^4)p_2^2\frac{1}{x_1^2}Y_{12}$$
$$-(x_1^2-x_1^4)p_3^2\frac{1}{x_1^2}(Y_{13}-\frac{x_1^3}{x_2^3})-(x_1^3-x_1^4)p_3^3\frac{1}{x_2^3}Y_{23}$$

⋮

4.12.n $$D_1^n+x_1^n(p_1^1X_1+p_2^1X_2-p_2^1\frac{1}{x_1^2}Y_{12}+p_3^1X_3-p_3^1\frac{1}{x_1^2}(Y_{13}-\frac{x_1^3}{x_2^3}Y_{23})-p_3^1\frac{1}{x_2^3}Y_{23})$$
$$-(x_1^2-x_1^n)(p_2^2\frac{1}{x_1^2}Y_{12}+p_3^2\frac{1}{x_1^2}(Y_{13}-\frac{x_1^3}{x_2^3}Y_{23}))$$
$$-(x_1^3-x_1^n)(p_3^3\frac{1}{x_2^3}Y_{23}).$$

An in general, our tangential operator will be

4.13 $$D_i^r+x_i^r(p_1^1X_1+p_2^1X_2-p_2^1\frac{1}{x_1^2}Y_{12}+p_3^1X_3-p_3^1\frac{1}{x_1^2}(Y_{13}-\frac{x_1^3}{x_2^3}Y_{23})-p_3^1\frac{1}{x_2^3}Y_{23})$$
$$-(x_i^2-x_i^r)(p_2^2\frac{1}{x_1^2}Y_{12}+p_3^2(Y_{13}-\frac{x_1^3}{x_2^3}Y_{23}))-(x_i^3-x_i^r)(p_3^3\frac{1}{x_2^3}Y_{23}).$$

These are then our operators F_i^r. The equations in terms of F_i^r then read

4.14 $$F_i^rf_j^r-x_i^r(-p_2^1\frac{1}{x_1^2}(\delta_{2j}f_i^r-\delta_{1j}f_2^r)-p_3^1\frac{1}{x_1^2}(\delta_{3j}f_1^r-\delta_{1j}f_3^r-\frac{x_1^3}{x_2^3}(\delta_{3j}f_2^r-\delta_{2j}f_3^r))$$
$$-(x_i^2-x_i^r)(p_2^2\frac{1}{x_1^2}(\delta_{2j}f_1^r-\delta_{1j}f_2^r)+p_3^2\frac{1}{x_1^2}(\delta_{3j}f_1^r-\delta_{1j}f_3^r-\frac{x_1^3}{x_2^3}(\delta_{3j}f_2^r-\delta_{2j}f_3^r))$$
$$-(x_i^3-x_i^r)(p_3^3\frac{1}{x_2^3}(\delta_{3j}f_2^r-\delta_{2j}f_3^r))-f_i^rp_j^r-2p_i^rf_j^r=0$$

<u>Proposition</u>. The system of 4.14 with $i = 3$ is quasi-linear symmetric and (locally) hyperbolic off a finite collection of submanifolds of positive codimension.

<u>Proof</u>. Collecting terms in 4.13 we can write

4.15
$$F_i^r = D_i^r + x_i^r(p_1^1 X_1 + p_2^1 X_2 + p_3^1 X_3)$$
$$-[x_i^r(p_2^1 - p_2^2) + x_i^2 p_2^2]\ \frac{1}{x_1^2} Y_{12}$$
$$-[x_i^r(p_3^1 - p_3^2) + x_i^2 p_3^2]\ \frac{1}{x_1^2} Y_{13}$$
$$+ \frac{1}{x_1^2 x_2^2}[x_i^r((p_3^1 - p_3^2)x_1^3 - x_1^2(p_3^1 - p_3^3))$$
$$+ (x_1^3 p_3^2 x_i^2 - x_1^2 p_3^2 x_i^3)]Y_{23}$$

We thus find that

4.16
$$F_3^r x_1^n = (x_3^n - x_3^r)p_1^n + x_3^r(p_1^1) + x_3^r(p_2^1 - p_2^2)\ \frac{1}{x_1^2}\ x_2^n$$
$$+ x_3^r(p_3^1 - p_3^2)\ \frac{1}{x_1^2}\ x_3^n.$$

If the total number of particles (n) is greater than 3, then none of these vanish identically, and so the collection of submanifolds given by $F_3^r x_1^n = 0$ is of positive codimension and finite (algebraic). Calling this set M, we have that if we are at a point z not on M, then there is a neighborhood of z on which the system 4.14 has the form

4.17
$$A\frac{\partial}{\partial x_1^n}\begin{pmatrix} f_1^1 \\ \vdots \\ f_3^n \end{pmatrix} + B\begin{pmatrix} f_1^1 \\ \vdots \\ f_3^n \end{pmatrix} + C\begin{pmatrix} f_1^1 \\ \vdots \\ f_3^n \end{pmatrix} = 0$$

where A is an invertible symmetric matrix of analytic functions, B is a symmetric matrix of differential operators depending analytically on the values of the f_j^r's and the coordinates and not involving $\frac{\partial}{\partial x_1^n}$, and C is a matrix of analytic functions. In our case, the matrices A and B are actually diagonal. This shows the local symmetric hyperbolicity of 4.14 when $n > 3$.

When $n = 3$ $F_3^r x_\ell^3 = 0$ for all s and ℓ and so we can not apply this same argument. Happily, though, in this case F_3^r is a linear differential operator with

4.18 $$F_3^r p_\ell^s = p_3^s p_\ell^s - \delta_{3\ell}.$$

Therefore, for example p_3^3 = constant will define globally noncharacteristic surfaces for the system 4.14, and we can solve 4.14 explicitly with given initial data on one of these systems. Again, we can write 4.14 as a symmetric hyperbolic system using $\frac{\partial}{\partial p_3^3}$ instead of $\frac{\partial}{\partial x_1^n}$. This completes the proof.

We will now show that an equation of the form 4.7 actually holds so that the F_3^r's act as a symmetry for the rest of the equations.

Suppose, then, that we have Euclidean covariant functions f_1^r which, on a transversal to the action of the Euclidean group, satisfy $D_3^r f_j^r = f_3^r p_j^r + 2p_3^r f_3^r$. If we denote by N_i^r the vectorfields made up out of Euclidean generators and which make $D_i^r + N_i^r$ tangent to the transversal, then we would like to compute, on the transversal, the functions

4.19 $$(D_3^r+N_3^r)[(D_j^r+N_j^r)f_m^r - f_j^r p_m^r - 2p_j^r f_m^r - N_j^r f_m^r].$$

Note that we can compute $[D_i^r, D_j^r]$ and $N_i^r f_m^r$ and $[N_i^r, N_j^r]$ and $[N_i^r, N_j^r] f_m^r$ globally. In particular, 4.19 has the same value as

4.20 $$(D_3^r+N_3^r)[D_j^r f_m^r - f_j^r p_m^r - 2p_j^r f_m^r]$$

when this is restricted to the transversal. On the other hand, the equation

4.21 $$\begin{aligned}(D_3^r+N_3^r)[D_j^r f_m^r - f_j^r p_m^r - 2p_j^r f_m^r] &= [D_3^r, D_j^r] f_m^r + [N_3^r, D_j^r] f_m^r + D_j^r(D_3^r f_m^r + N_3^r f_m^r) \\ &\quad - (D_3^r+N_3^r)(f_j^r p_m^r + 2p_j^r f_m^r)\end{aligned}$$

is globally valid.

Now, we know what $D_3^r f_m^r$ is on the transversal and we know what $N_3^r f_m^r$ is globally. Unfortunately, D_j^r is not tangent to the transversal so that, in general, $D_j^r(D_3^r f_m^r)$ on the transversal will not be the restriction of $D_j^r(f_3^r p_j^r + 2p_3^r f_j^r)$. If D_j^r

were tangent, we could use this substitution. Thus we write

4.21
$$(4.20) = [D_3^r, D_j^r]f_m^r + [N_3^r, D_j^r]f_m^r$$
$$+ (D_j^r + N_j^r)(D_3^r f_m^r + N_3^r f_3^r) - N_j^r(D_3^r f_m^r + N_3^r f_m^r)$$
$$- (D_3^r + N_3^r)(f_j^r p_m^r + 2p_j^r f_m^r).$$

Now, N_j^r is not tangent either, however the equation

4.22
$$-N_j^r(D_3^r f_m^r + N_3^r f_m^r) = [D_3^r, N_j^r]f_m^r + [N_3^r, N_j^r]f_m^r$$
$$- (D_3^r + N_3^r)(N_j^r f_m^r)$$

is globally valid. Hence, on the submanifold we have

4.23
$$(4.20) = [D_3^r, D_j^r]f_m^r + ([N_3^r, D_j^r] + [D_3^r, N_j^r])f_m^r$$
$$+ [N_3^r, N_j^r]f_m^r - (D_3^r + N_3^r)(p_m^r f_j^r + 2p_j^r f_m^r + N_j^r f_m^r)$$
$$+ (D_j^r + N_j^r)(p_m^r f_3^r + 2p_3^r f_m^r + N_3^r f_m^r).$$

We have already computed $[D_i^r, D_j^r]$, let us now compute the next term, $[N_i^r, D_j^r] + [D_i^r, N_j^r]$. Using the N_i^r's given above, we find that

4.24
$$[N_i^r, D_j^r] + [D_i^r, N_j^r] = \sum_{\ell} p_\ell^1(x_j^r p_i^1 - x_i^r p_j^1)X_\ell + x_i^r X_j - x_j^r X_i$$
$$+ [p_2^1(x_i^r p_j^1 - x_j^r p_i^1) - \frac{p_2^1 p_1^2}{x_1^2}(x_i^r x_j^2 - x_j^r x_i^2) + \delta_{2i}x_j^2 - \delta_{2j}x_i^2]\frac{1}{x_1^2}Y_{12}$$
$$+ [p_3^1(x_i^r p_j^1 - x_j^r p_i^1) - \frac{p_3^1 p_1^2}{x_1^2}(x_i^r x_j^2 - x_j^r x_i^2) + \delta_{3i}x_j^2 - \delta_{3j}x_i^2]\frac{1}{x_1^2}Y_{13}$$

$$+ [(x_j^r x_i^3 - x_i^r x_j^3)(p_1^3 p_3^1 - p_1^3 p_3^2 - p_3^3 p_1^2) + (x_j^r p_2^1 - x_i^r p_j^1) p_3^1 (x_1^3 - x_1^2)$$

$$+ (x_j^2 x_i^3 - x_i^2 x_j^3)(p_3^2 p_1^3 + p_3^3 p_1^2) + (x_j^r x_i^2 - x_i^r x_j^2) p_3^2 p_1^3$$

$$+ \delta_{3i}(x_j^3 x_1^2 - x_j^2 x_1^3) - \delta_{3j}(x_i^3 x_1^2 - x_i^2 x_1^3)$$

$$- (x_j^r (x_1^3 (p_3^1 - p_3^2) - x_1^2 (p_3^1 - p_3^3)) + x_j^2 p_3^2 x_1^3 - x_j^3 p_3^3 x_1^2)((x_i^2 - x_i^r) p_1^2 x_2^3$$

$$+ (x_i^3 - x_i^r) p_2^3 x_1^2)$$

$$+ (x_i^r (x_1^3 (p_3^1 - p_3^2) - x_1^2 (p_3^1 - p_3^3) + x_i^2 p_3^2 x_1^3 - x_i^3 p_3^3 x_1^2 ((x_j^2 - x_j^r) p_1^2 x_2^3$$

$$+ (x_j^3 - x_j^r) p_2^3 x_1^2)]$$

$$\times \frac{1}{x_1^2 x_2^3} Y_{23} - (x_j^r (p_3^1 - p_3^2) + x_j^2 p_3^2) \frac{1}{x_1^2} (\delta_{3i} D_1^r - \delta_{1i} D_3^r)_{(i,j)}$$

$$- (x_j^r (p_2^1 - p_2^2) + x_j^2 p_2^2) \frac{1}{x_1^2} (\delta_{2i} D_1^r - \delta_{1i} D_2^r)_{(i,j)}$$

$$+ [x_j^r (x_1^3 (p_3^1 - p_3^2) + x_1^2 (p_3^1 - p_3^3)) + x_j^2 p_3^2 x_1^3 - x_j^3 p_3^3 x_1^2] \frac{1}{x_1^2 x_2^3}$$

$$\times (\delta_{3i} D_2^r - \delta_{2i} D_3^r)_{(i,j)}$$

In this formula and the ones that follow, we use the subscript (i,j) after a term to indicate that the term should be skew symmetrized in these indices : $p_i p_{j(i,j)} = p_i p_j - p_j p_i$. Using 4.23 in 4.22, we find

$$(4.19) = p^r_m(D^r_j f^r_3 - p^r_3 f^r_j - 2p^r_j f^r_3)$$

$$+ 2p^r_3(D^r_j f^r_m - p^r_m f^r_j - 2p^r_j f^r_m)$$

$$+ \sum_s (x^s_j - x^r_j)(\partial p^s_\ell f^r_m)\cdot[D^s_3 f^s_\ell - p^s_\ell f^s_3 - 2p^s_3 f^s_\ell]_{(j,3)}$$

$$(4.24) \qquad + \sum_{1\le i<\ell\le 3} Q^r_{3i\ell}[\delta_{j\ell}(D^r_i f^r_m - p^r_m f^r_i - 2p^r_i f^r_m)$$

$$- \delta_{ji}(D^r_\ell f^r_m - p^r_m f^r_\ell - 2p^r_\ell f^r_m) + \delta_{im}(D^r_j f^r_i - p^r_i f^r_j - zp^r_j f^r_i)$$

$$- \delta_{im}(D^r_j f^r_\ell - p^r_\ell f^r_j - 2p^r_j f^r_\ell)]\ (3,j)$$

Here $Q^r_{j12} = (x^r_j(p^1_2 - p^2_2) + x^2_j p^2_2)\dfrac{1}{x^2_1}$

$$Q^r_{j13} = (x^r_j(p^1_3 - p^2_3) + x^2_j p^2_3)\frac{1}{x^2_1}$$

$$Q^r_{j23} = (x^r_j(x^3_1(p^1_3 - p^2_3) - x^2_1(p^1_3 - p^3_3))$$

$$+ x^2_j p^2_3 x^3_1 - x^3_j p^3_s x^2_1)\frac{1}{x^2_1 x^3_2}$$

This gives us, then, the required covariance. With this in hand, we may proceed to the next stage.

As we have seen, off a collection of submanifolds, the surfaces x_1^n = constant, in case $n > 3$, and p_3^3 = constant, in case $n = 3$, are transverse to the orbits of $D_3^r + N_3^r$. Therefore, our covariance allows us to pull back the equations to any one of these surfaces. In this case, we find functions v_j^r so that the operators

(4.25) $$D_j^r + N_j^r + v_j^r(D_3^r+N_3^r)$$

are tangent to the surfaces. We will show that the resulting equations with $j = 2$ form a quasi-linear, symmetric hyperbolic system. Further, we will find that, if the $j = 2$ equations are satisfied, then the operators $D_2^r + N_2^r + v_2^r(D_3^r+N_3^r)$ generate a covariance of the remaining equations (appropriate r).

One difficulty is that, whereas we could always extend smoothly by Euclidean invariance, in the present case, the $D_3^r + N_3^r$ covariance equation is quasi-linear and we may not always be able to extend smoothly smooth data on the initial surface and hence the analogue of the argument leading to 4.22 will not be immediately valid. However it is not necessary to extend the f's so that the covariance equation is true in a nbd., but rather only at the surface itself, in order for the analogue of 4.22 to be valid on the surface under consideration. We can certainly make such a smooth extension. Using the analogue of 4.22, we find that if

$$(4.26) \qquad [D_2^r+N_2^r+v_2^r(D_3^r+N_3^r)]f_m^r-p_m^rf_2^r-2p_2^rf_m^r- \Sigma\, Q_{2i\ell}^r(\delta_{\ell m}f^r-\delta_{i\ell n}f_\ell^r)$$

$$-v_2^r(p_m^rf_3^r+2p_3^rf_m^r+\Sigma\, Q_{3i\ell}^r(\delta_{\ell m}f^r-\delta_{im}f_\ell^r)) = 0$$

on the transversal, then we will have

$$[D_2^r+N_2^r+v_2^r(D_3^r+N_3^r)][(D_1^r+N_1^r+v_1^r(D_3^r+N_3^r))f_m^r-p_m^rf_1^r-2p_1^rf_m^r$$

$$-\Sigma\, Q_{1i\ell}^r(\delta_{\ell m}f_i^r-\delta_{im}f_\ell^r)-v_1^r(p_m^rf_3^r+2p_3^rf_m^r-\Sigma\, Q_{3i\ell}^r(\delta_{\ell m}f_i^r-\delta_{im}f_\ell^r)]$$

$$= p_m^r(D_1^rf_2^r-p_2^rf_1^r-2p_1^rf_2^r) + 2p_2^r(D_1^rf_m^r-p_m^rf_1^r-2p_1^rf_m^r)$$

$$+\Sigma\, Q_{2i\ell}^r[\delta_{\ell 1}(D_i^rf_m^r-p_m^rf_i^r-2p_i^rf_m^r)-\delta_{2i}(D_\ell^rf_m^r-p_m^rf_\ell^r-2p_\ell^rf_m^r)$$

$$(4.27) + \delta_{\ell m}(D_1^rf_i^r-p_i^rf_1^r-2p_1^rf_i^r)-\delta_{im}(D_1^rf_\ell^r-p_\ell^rf_1^r-2p_1^rf_\ell^r)]_{(2,1)}$$

$$+ v_2^r\{p_m^r(D_1^rf_3^r-p_3^rf_1^r-2p_1^rf_3^r)+2p_3^r(D_1^rf_m^r-p_m^rf_1^r-2p_1^rf_m^r\}$$

$$+\Sigma\, Q_{3i\ell}^r[\delta_{\ell 1}(D_2^rf_m^r-p_m^rf_i^r-2p_i^rf_m^r)-\delta_{2i}(D_\ell^rf_m^r-p_m^rf_\ell^r-2p_\ell^rf_m^r)$$

$$+\delta_{\ell m}(D_1^rf_i^r-p_i^rf_1^r-2p_1^rf_i^r)-\delta_{im}(D_1^rf_\ell^r-p_\ell^rf_1^r-2p_1^rf_\ell^r)]_{(3,1)}\}_{(2,1)}$$

$$+[\sum_{s,\ell}(x_2^s-x_2^r)(D_1^sf_\ell^s-p_\ell^sf_1^s-2p_1^sf_\ell^s)\partial p_\ell^sf_m^r]_{(2,1)}$$

This again gives covariance

The functions v_j^r must be

$$(4.28) \qquad v_j^r = -\frac{F_j^r(x_1^n)}{F_3^r(x_1^n)}$$

To complete the reduction on this step, we can choose the local transversals to $D_2^r + N_2^r + v_2^r(D_3^r+N_3^r)$ to be given by x_2^n = constant, when $n > 3$ and x_2^3 = constant when $n = 3$. Then we note that $(D_2^r+N_2^r+v_2^r(D_3^r+N_3^r))x_2^n$ is a rational function, not identically zero and does not involve the functions f_j^r. Thus, as before, off a finite collection of submanifolds of positive codimension we can write the system

$$(D_2^r+N_2^r+v_2^r(D_3^r+N_3^r))f_m^r-p_m^r f_2^r-2p_2^r f_m^r$$

$$-\Sigma\, Q_{2i\ell}(\delta_{\ell m}f_1^r-\delta_{im}f_\ell^r)-v_2^r(p_m^r f_3^r+2p_3^r f_m^r-\Sigma Q_{3i\ell}^r(\delta_{\ell m}f_I^r-\delta_{im}f_\ell^r))= 0$$

as a quasilinear symmetric hyperbolic system with the same analyticity properties as the previous system. The covariance allows us to "factor out" 4.29 from our system, leaving on the local transversals x_2^n = constant, a system of the form

(4.30) $$\{D_1^r+N_1^r+v_1^r(D_3^r+N_3^r)+W_1^r(D_2^r+N_2^r+v_2^r(D_3^r+N_3^r))\}f_m^r-G^r = 0$$

where W_1^r is the appropriate function making this operator tangent to x_2^n = constant, and G is a combination of the f's. This in itself is symmetric hyperbolic off a collection of submanifolds, using the same arguments as above. Noticing that at each stage, the rational functions which must not vanish for hyperbolicity have a denominators a multiple of the previous rational function which could not vanish, (e.g. at the first stage we have x_1^2 and x_2^3 could not vanish : $\frac{1}{x_1^2}$, $\frac{1}{x_2^3}$ and $\frac{1}{x_1^2 x_2^3}$ appear as coef. in the N_j^r's), and repeatedly apply the Cauchy Kowaleska theorem, we have

Proposition. Suppose we have a point q of the n-particle ($n \geq 3$), second order space. By Euclidean covariance we may assume that $x_1^1(q) = x_2^2(q) = x_3^2(q) = x_3^3(q) = 0$. Then there exists $3n + 2$ not-identically-zero polynomials P_j^r, x_1^2, x_2^3, of the coordinates such that $P_j^r(q) \neq 0$, $x_1^2(q) \neq 0$, $x_2^3(q) \neq 0$ implies that there exists coordinate 6n-q plane through q with the property that, given any analytic functions g_j^r defined in a nbd. of q on this plane, there exists a set of analytic functions f_j^r defined in a nbd. of q in the whole space which are solutions of 4.1, 4.2 and agree with the g_j^r on their common domain of definition.

The Two Particle Case.

When $n = 2$, we can do somewhat better. The critical feature being that on $x_i^1 = 0 = x_2^2 = x_3^2$ (section to the Euclidean action) the operator D_2^r and D_3^r loose their non-linear terms and we can solve the corresponding equations exactly. We can factor out these symmetries yielding the following equations on $\{x_i^1=0=x_2^2=x_3^2=p_2^1=p_3^1=p_3^2 \; p_2^2> 0, x_1^2 > 0\}$

$$\begin{aligned} & x_1^2 p_1^2 \partial x_1^2 + [x_1^2 f_2^2 - p_2^2 p_1^1 (p_2^2 p_2^2 - 1)] \partial p_2^2 \\ & + [p_1^2 p_1^2 - 1 - p_2^2 p_1^1 p_1^2 p_2^2 + x_1^2 f_1^2] \partial p_1^2 + (p_1^1 p_1^1 - 1) \partial p_1^1 \} f_j^1 \\ & - p_j^1 f_1^1 - 2 p_1^1 f_j^1 + p_2^2 (\delta_{2j} f_1^1 - \delta_{ij} f_2^1) + p_2^2 p_1^1 (p_j^1 f_2^1) \\ & + x_1^2 f_3^2 \frac{1}{p_2^2} (\delta_{3j} f_2^1 - \delta_{2j} f_3^1) = 0 \end{aligned} \tag{4.31}$$

$$\{x_1^2 p_1^1 \partial x_1^2 + [p_1^1 p_1^1 - 1 - x_1^2 f_1^1]\partial p_1^1 + [p_1^2 p_1^2 - 1 - x_1^2 f_2^1 p_1^2 p_2^2]\partial p^2$$

(4.32)

$$+ [p_1^2 p_2^2 - x_1^2 f_2^1 (p_2^2 p_2^2 - 1)]\partial p_2^2\} f_j^2 - p_j^2 f_2^2 - 2p_2^2 f_j^2$$

$$+ x_1^2 f_2^1 (p_j^2 f_1^2 + 2p_2^2 f_j^2) + x_1^2 f_3^1 (p_j^2 f_3^2 + \frac{1}{p_2^2}(\delta_{3j} f_2^2 - \delta_{zj} f_3^2)) = 0$$

Noticing that this system is symmetric hyperbolic off $x_1^2 = 0$, $p_2^2 = 0$, $p_1^1 = 0$, $p_1^2 = 0$, we apply Kato's theorem (9) to yield

Proposition. Let $\varepsilon > 0$, let $u \subset R^3$ (coordinates p_1^1, p_1^2, p_2^2) be defined by

$U = \{(p_1^1, p_2^1, p_3^1) | \varepsilon < (p_1^1)^2 < 1,\ (p_1^2)^2 + (p_2^2)^2 < 1, \varepsilon < (p_1^2)^2, \varepsilon < (p_2^2)^2$

Let $s > 5/2$ and let D be a bounded open subset of $H^s(U, R^6)$ $D \ni f = (\tilde{f}_1^1, \tilde{f}_2^1, \tilde{f}_3^1, \tilde{f}_2^2, \tilde{f}_2^2, \tilde{f}_3^2)$. Given any $k > 0$ and $f \in D$ there exists a $k^1 > k$ and a solution of 5.1, 5.2, $(f_1^1, \cdots, f_3^2)$ defined on $\{k \le x^2 \le k\} \times U$ such that

(4.33) $\quad (f_1^1, \cdots f_3^2) \in C[k, k^1; D] \cap C^1[k, k^1; H^s(u, R^6)]$

(4.34) $\quad (f_1^1, \cdots, f_3^2)|_{x_1^2 = k} = (\tilde{f}_1^1, \cdots, \tilde{f}_3^2)$

Moreover, such a solution is unique on this interval. (Recall $H^s(U, R^6)$ is the space of functions with square integrable derivatives up to and including order s.)

§5. Conformal covariance

The extension of the Poincaré group by scale changes, $(t,x_1,x_2x_3) \to (ct,cx_1,cx_2,cx_3)$ $c \neq 0$, is of great importance to physics. However, it is often stated that they have relevance only for mass = zero situations. We will show that solutions of the second order R.D.L. equations have a natural action of scale transformations on them. We note that under such a scale transformation, the coordinates $x_i^r \to cx_i^r$ and $p_i^r \to p_i^r$.

Proposition. Let f_j^r's be a solution of :
$D_i^r f_j^r - p_j^r f_i^r - zp_i^r f_j^r = 0$, $X_i f_j^r = 0$ and let $c \neq 0$ then
$\tilde{f}_j^r := cf_j^r(cx_i^s, p_i^s)$ will also be a solution.

Proof. We wish to show that

$$(5.1) \qquad D_i^r \tilde{f}_j^r - p_j^r \tilde{f}_i^r - 2p_i^r \tilde{f}_j^r = 0$$

$$(5.2) \qquad \text{L.H.S.} = \{ \sum_{s,\ell} (x_i^s - x_i^r)(p_\ell^s \partial x_\ell^s + f_\ell^s \partial p_\ell^s) + (p_i^s p_\ell^s - \delta_{i\ell}) \partial p_\ell^s \} f_j^r$$
$$- p_j^r \tilde{f}_i^r - 2p_i^r \tilde{f}_j^r$$

If we plug in our definition of $\tilde{f}_j^r$ and evaluate at a point with coordinates (x_i^s, p_i^s) we find

$$(5.3) \qquad (D^r \tilde{f}^r - p^r \tilde{f}^r - 2p^r \tilde{f}^r)\Big|_{(x_i^s, p_i^s)} = c\{D^r f^r - p^r f^r - 2p^r f^r\}\Big|_{(cx_i^s, p_i^s)}$$

Since the right hand side is assumed to be zero, the left hand side is zero as well. Moreover, it is clear that if $x_i f_j^r = 0$, then $x_i \tilde{f}_j^r = 0$. This concludes the proof.

If in addition, we assume that the solution in which we are interested is actually conformally invariant, i.e. for every $i > 0$, $\tilde{f}_j^r = f_j^r$, this will allow us to factor out yet another symmetry. For example, in the two particle case, we may pull back the final equations to $x_1^2 = 1$, and get the system

$$\begin{aligned}(5.4)\qquad &\{[f_2^2 - p_2^2 p_1^1 (p_2^2 p_2^2 - 1)]\partial p_2^2 + [p_1^2 p_1^2 - 1 - p_2^2 p_1^2 p_1^1 p_2^2 + f_1^2]\partial p_1^2 \\ &+ (p_1^1 p_1^1 - 1)\partial p_1^1\} f_j^1 - p_j^1 f_1^1 - 2p_1^1 f_j^1 + p_2^2(\delta_{2j} f_1^1 - \delta_{ij} f_2^1) + p_1^2 f_j^1 \\ &+ p_2^2 p_1^1 (p_j^1 f_2^1) + \frac{f_3^2}{p_2^2}(\delta_{3j} f_2^1 - \delta_{2j} f_3^1) = 0\end{aligned}$$

$$\begin{aligned}(5.5)\qquad &\{[p_1^1 p_1^1 - 1 - f_1^1]\partial p_1^1 + [p_1^2 p_1^2 - 1 - f_2^1 p_1^2 p_2^2]\partial p_1^2 \\ &+ [p_1^2 p_2^2 - f_2^1 (p_2^2 p_2^2 - 1)]\partial p_2^2\} f_j^2 - p_j^2 f_2^2 - 2p_2^2 f_j^2 \\ &+ p_1^1 f_j^2 + f_2^1 (p_j^2 f_1^2 + 2p_2^2 f_j^2) + f_3^1 (p_j^2 f_3^2 + \frac{1}{p_2^2}(\delta_{3j} f_3^2 - \delta_{2j} f_3^2)) = 0\end{aligned}$$

The most unfortunate feature of this system being that it is distinctly non-hyperbolic.

§6. Generalized Momentum Conservation

In classical particle mechanics, there are a number of "conserved quantities", the standard examples being total momentum and angular momentum. It is already known (8) that if we have the total relativistic momentum conserved in every frame for three particles, then they must (generically) move in a straight line. It is, however, an observed phenomenon that particle interactions in nature have a "center of mass frame", that is, a frame in which a certain linear combination of particle coordinates is unaccelerated by time translation.

We may explore the middle ground between these using the following definition. A generalized mass of the r-th particle will be a smooth, rotationally symmetric function g^r of the coordinates p_1^r, p_2^r, p_3^r. Given a set of such g^r, we may define the total generalized momenta to be the functions $G_i = \sum_r g^r p_i^r$. We would like to know what restriction on a second order R.D.L. is imposed by assuming that some total generalized momenta G_i are conserved under time translation. When $g_r = (1-\sum_\ell p_\ell^r p_\ell^r)^{-\frac{1}{2}}$, this amounts to relativistic momentum conservation; when $g_r \equiv m^r$ = constant, this becomes the condition of non-acceleration of $\sum_r m^r x_i^r$. Our assumption may be simply stated as

$$(6.1) \qquad X_o(G_i) = 0$$

We will concentrate here on the (second order) three particle case. Since we are dealing with an R.D.L., equation 6.6 is Euclidean invariant. Since g^r is assumed to be a rotationally invariant function of the p^r_j we may assume that it is a function of $h^r := \sum_\ell p^r_\ell p^r_\ell$. Let us now compute $X_o(G_i)$

$$(6.2)\qquad \begin{aligned} X_o(G_i) &= \Big(\sum_{s,\ell} p^s_\ell \partial x^s_\ell + f^s_\ell \partial p^s_\ell\Big)\Big(\sum_r g^r(h^r)p^r_i\Big) \\ &= \sum_s g^{s\prime}(h^s)2(\vec{p}^s\cdot\vec{f}^s)p^s_i + g^s(h^s)f^s_i \end{aligned}$$

where we have used the notation $\vec{p}^s,\vec{f}^s : \sum_\ell p^s_\ell f^s_\ell$ and $g^{s\prime} = \frac{dg^s}{dh^s}$.

Since we are dealing with three particles, we may, by Euclidean invariance, assume that they are on the plane $x^s_3 = 0$, in which we have, as before,

$$(6.3)\qquad (Y_{o3} - x^r X_o) = \sum_{s,\ell}(p^s p^s - \delta_{3\ell})\partial p^s_\ell \text{ on } x^s_3 = 0$$

This operator is, of course, tangent to $x^s_3 = 0$. Our equations for an R.D.L. then say

$$(6.4)\qquad \Big(\sum_{s,\ell}(p^s_3 p^s_\ell - \delta_{3\ell})\partial p^s_\ell\Big)f^r_j = p^r_j f^r_3 + 2p^r_3 f^r_j$$

The orbits of the differential operator in 6.4 are given by

$$(6.5)\qquad \begin{aligned} p^s_3(\tau) &= -\tanh(\tau+\eta^s) \\ p^s_i(\tau) &= c^s_i \cosh^{-1}(\tau+\eta^s) \quad i \neq 3 \end{aligned}$$

Where η^s and c_i^s are constants. Along such orbits we have

(6.6)
$$f^s(\tau) = f_3^s \cosh^{-3}(\tau+\eta^3)$$
$$f_i^s(\tau) = \cosh^{-2}(\tau+\eta^s)(\tilde{f}_3^s c_i^s \tanh(\tau+\eta^s) + \tilde{f}_i^s) \quad i \neq 3$$

where the $\tilde{f}_j^s$ are constant along the orbits.

Denoting by "$\cdot$" $\frac{\partial}{\partial\tau}$ we see that along an orbit

(6.7)
$$g^{s\,\prime}(\tau) = \frac{\dot{g}^s(\tau)}{2\tanh(\tau+\eta^s)\cosh^{-2}(\tau+\eta^s)(1-(c_1^s)^2-(c_2^s)^2)}$$

When the denominator is non-zero. Putting this into 6.2 gives us

(6.8)
$$\sum_s \dot{g}^s(\tau)\cosh^{-1}(\tau+\eta^s)\left(-\tilde{f}_3^s + \frac{c_1^s\tilde{f}_1^s + c_2^s\tilde{f}_2^s}{\tanh(\tau+\eta^s)(1-(c_1^s)^2-(c_2^s)^2}\right)p_j^s(\tau)$$
$$+ g^s(\tau)f^s(\tau) = 0$$

If we choose the g^s to be one of the standard examples e.g. $g^s \equiv m^s$, or $g^s = (1-p^s p^s)^{-\frac{1}{2}}$, then for generic c_i^s and η^s the functional coefficients of $\tilde{f}_3^s$ and $(\tilde{f}_1^s+\tilde{f}_2^s)$ will be linearly independent, hence for 6.8 to be satisfied for even a small interval of τ's, these must all vanish. In this case, we may conclude from rotation covariance $(Y_{12}(f_1^s+f_2^s) = -f_2^s+f_1^s)$ and continuity that all the f_j^s are identically zero.

We'd like to solve 6.8 for the various g^s. However, it is clear that if the $f_j^r = 0$ for some r (respectively $g^r = 0$) then 6.8 contains no information about the corresponding g^r(respectively f_j^r). Let us cover these cases first. Suppose, for example, along an orbit $\tilde{f}_j^2 = \tilde{f}_j^3 = 0$ or $g^2 = g^3 = 0$. Then 6.8 reads

$$(6.9)\quad \dot{g}(\tau)\cosh^{-1}(\tau+\eta^1)\left(\tilde{f}_3^1 + \frac{c_2^1\tilde{f}_2^1 + c_1^1\tilde{f}_1^1}{\tanh(\tau+\eta^s(1-(c_1^1)^2-(c_2^1)^2)}\right)\cdot p_j^1 + g^1(\tau)f_j^1(\tau) = 0$$

In particular, we must have

$$(6.10)\quad \frac{g^1(\tau)f_3^1\cosh^{-3}(\tau+\eta^1)}{-\tanh(\tau+\eta^1)} = \frac{\cosh^{-2}(\tau+\eta^1)(f_3^s c_j^1\tanh(\tau+\eta^1)+f_j^1)g^1(\tau)}{c^1\cosh^{-1}(\tau+\eta^1)}$$

Thus, either $g^1 = 0$ or $\tilde{f}_1^1 = \tilde{f}_2^1 = \tilde{f}_3^1 = 0$. Suppose now that only the $\tilde{f}_j^3$ (or g^3) vanishes then the equation becomes

$$(6.11)\quad \begin{aligned} &\dot{g}^1(\tau)\cosh^{-1}(\tau+\eta^1)\left(-\tilde{f}_3^1 + \frac{c_2^1\tilde{f}_2^1 + c_1^1\tilde{f}_1^1}{\tanh(\tau+\eta^1)(1-(c_1^1)^2-(c_2^1)^2}\right)p^1(\tau) \\ &+ g^1(\tau)f^1(\tau) \\ &+ \dot{g}^2(\tau)\cosh^{-1}(\tau+\eta^2)\left(-\tilde{f}_3^2 + \frac{c_1^2\tilde{f}_1^2 + c_2^2\tilde{f}_2^2}{+\tanh(\tau+\eta^2)(1-(c_1^2)^2-(c_2^2)^2)}\right)p_j^2(\tau) \\ &+ g^2(\tau)f^2(\tau) = 0 \end{aligned}$$

Generically, $p^1(\tau)$ and $p^2(\tau)$ will have a common normal vector

$(\vec{p}^1 x \vec{p}^2)(\tau)$. Dotting equation 6.11 with this, we find

$$(6.12) \qquad g^1(\tau)(\vec{f}^1\cdot(\vec{p}^1 x \vec{p}^2)) + g^2(\tau)(\vec{f}^2\cdot(\vec{p}^1 x \vec{p}^2)) = 0$$

We would like to use this to solve for $g^2(\tau)$, which we can do if $(\vec{f}^2\cdot(\vec{p}^1 x \vec{p}^2))(\tau) \neq 0$. Suppose this were zero, then taking the τ derivative, we also have

$$0 = (f_3^2\vec{p}^2 + 2\vec{p}_3^2\vec{f}^2)\cdot(\vec{p}^1 x \vec{p}^2) + \vec{f}^2\cdot((\vec{p}_3^1\vec{p}^1 - \vec{\delta}_3) x \vec{p}^2)$$

$$(6.13) \qquad + \vec{f}^2\cdot(\vec{p}^1 x (p_3^2\vec{p}^1 - \hat{\delta}_3)) = \vec{f}^2\cdot(-\vec{\delta}_3 x \vec{p}^2) + \vec{f}^2\cdot(\vec{p}^1 x - \vec{\delta}_3)$$

$$= p_2^2 f_1^2 - p_1^2 f_2^2 + p_1^1 f_2^2 + p_2^1 f_1^2 = (p_2^2 - p_2^1) f_1^2 - (p_1^2 - p_1^1) f_2^2$$

Thus along our orbit we would have

$$0 = (c_2^2 \cosh^{-1}(\tau+\eta^2) - c_2^1 \cosh^{-1}(\tau+\eta^1))(\cosh^{-2}(t+\eta^2) c^2 \tilde{f}_3^2 \tanh(t+\eta^2)$$

$$(6.14) \qquad + \tilde{f}^2)$$

$$- (c_1^2 \cosh(\tau+\eta^2) - c_1^1 \cosh^{-1}(\tau+\eta^1) \cosh^{-2}(\tau+\eta^2)(c_2^2 \tilde{f}_3^2 \tanh(\tau+\eta^2)$$

$$+ \tilde{f}_2^2)$$

In particular, as long as we have chosen an orbit with

$c_2^2 c_1^1 - c_1^2 c_2^1 \neq 0$, $\tilde{f}_2^2$ must be zero (a generic condition, again)

If, in addition we have $\eta^2 \neq \eta^1$ then $\tilde{f}_1^2 = \tilde{f}_2^2 = 0$, reducing to our previous case. We will thus assume $\vec{f}^2.(\vec{p}^1 x \vec{p}^2) \neq 0$ and write

$$g^2(\tau) = - g^1(\tau) \frac{(\vec{f}^1 \cdot (\vec{p}^1 x \vec{p}^2))}{\vec{f}^2 \cdot (\vec{p}^1 x \vec{p}^2)}$$

$$\dot{g}^2(\tau) = - \dot{g}^1(\tau) \frac{(\vec{f}^1 \cdot (\vec{p}^1 x \vec{p}^2))}{\vec{f}^2 \cdot (\vec{p}^1 x \vec{p}^2)} +$$

$$- g^1(\tau)[(3p^1+p^2)(\vec{f}^1 \cdot (\vec{p}^1 x \vec{p}^2))$$

(6.14) $$+ (p_2^2-p_2^1)f_1^1-(p_1^2-p_1^1)f_2^1](\vec{f}^2 \cdot (\vec{p}^1 x \vec{p}^2))^{-1}$$

$$+ g^1(\tau)(\vec{f}^1 \cdot (\vec{p}^1 x \vec{p}^2))[(3p_3^2+p_3^1)(\vec{f}^2 \cdot (\vec{p}^1 x \vec{p}^2))$$

$$+ (p_2^2-p_2^1)f_1^2-(p_1^2-p_1^1)f_2^2] \ x \ (\vec{f}^2 \cdot (\vec{p}^1 x \vec{p}^2))^{-2}$$

We will use these in 6.11. Let us denote by $\vec{k}^1(\tau)$ and $\vec{k}^2(\tau)$ a dual basis to $p^1(\tau)$ and $p^2(\tau)$, i.e. $\vec{p}^s \cdot \vec{k}^{s'} = \delta_{ss'}$ and $\vec{k}^s \cdot (\vec{p}^1 x \vec{p}^2) = 0$.

Dotting equation 6.11 with these yields the equations

(6.15) $$\dot{g}(\tau)\cosh^{-1}(\tau+\eta^1)(-\tilde{f}_3^1 + \frac{c_2^1\tilde{f}_2^1 + c_1^1\tilde{f}_1^1}{\tanh(\tau+\eta^1)(1-(c_1^1)^2-(c_2^1)^2)})$$

$$+ g^1(\tau)f^1 \cdot k^1 + g^2(\tau)f^2 \cdot k^1 = 0$$

$$\dot{g}^2(\tau)\cos h^{-1}(\tau+\eta^2)(-\tilde{f}_3^2 + \frac{c_1^2\tilde{f}_1^2 + c_2^2\tilde{f}_2^2}{\tanh(\tau+\eta^2)(1-(c_1^2)^2-(c_2^2)^2)})$$

$$+ g^2(\tau)f^2 \cdot k^2 + g^1(\tau)f^1 \cdot k^2 = 0$$

Denoting by L the function $\frac{\vec{f}^1 \cdot (\vec{p}^1 x \vec{p}^2)}{\vec{f}^2 \cdot (\vec{p}^1 x \vec{p}^2)}$ and using 6.14 we may write these as

$$(6.17)\quad \dot{g}^1(\tau)\cosh^{-1}\left(\tau+\eta^1(-\tilde{f}_3^1 + \frac{c_2^1\tilde{f}_2^1 + c_1^1\tilde{f}_1^1}{\tanh(\tau+\eta^1)(1-c_1^1)^2-(c_2^1)^2}\right)$$

$$+ g^1(\tau)\vec{f}^1\cdot\vec{k}^1 + g^1(\tau)L\vec{f}^2\cdot\vec{k}^1 = 0$$

$$(6.18)\quad (\dot{g}^1L+g^1\dot{L})(\cosh^{-1}\left(\tau+\eta^2)(-\tilde{f}_3^2 + \frac{c_1^2\tilde{f}_1^2 + c_2^2\tilde{f}_2^2}{\tanh(\tau+\eta^2)(1+(c_1^2)^2+c_2^2)^2}\right)$$

$$+ g^1(\tau)L\vec{f}^2\cdot\vec{k}^2 + g^1(\tau)\vec{f}^1\cdot\vec{k}^2 = 0$$

On the other hand, if $(\vec{p}^s\cdot\vec{f}^s \neq 0$ we can solve 6.2 along a generic orbit. Extend $\vec{K}_1$, $\vec{K}_2$ with $\vec{K}_3$ so that $\vec{p}^s\cdot\vec{K}^r = \delta sr$. Then

$$(6.19)\qquad \dot{g}^s = -\frac{p_3^s(\vec{p}^s\cdot\vec{p}^s-1)}{2(\vec{p}^s\cdot\vec{f}^s)}\sum_r g^r\vec{f}^r\cdot\vec{K}^s$$

Thus, whereas conservation of relativistic momentum limits the interaction kinematically (we didn't use the dynamical equations at all) the conservation of generalized momentum does not. We can certainly solve 6.19 if it is thought of as an o.d.e. along orbits of L_3 for the g^s. The question of whether we can solve the equation for all orbits in a way consistent with the dependence requirements of the g^s and the dynamical equations for the f^s's together still remains.

Appendix I

In this section, we will show the restrictiveness of the k-th order assumption on an R.D.L. by exhibiting a smooth R.D.L., together with a parametrization, which is completely geometric and yet cannot be k-th order for any k. The naturalness of the R.D.L. indicates, perhaps the unnaturality of the k-th order assumption.

We will work for the moment in two dimensions: coordinates (x,t) and metric $ds^2 = dt^2 - dx^2$. If p_1, p_2 are two points in this space-time we will write $p_1 \longrightarrow p_2$ when p_2 lies on the forward light cone of p_1 and we will write $p_1 < p_2$ when p_2 lies within the forward light cone of p_1. If γ is a time-like path in Minkowski space containing p_1 and p_2, we write $d_\gamma(p_1,p_2)$ for the arclength of the part of γ between p_1 and p_2.

Now let (γ_1,γ_2) be a pair of one-dimensional time-like submanifolds. We define functions $\beta_1 : \gamma_1 \to R \cup \{\infty\}$ and $\beta_2 : \gamma_2 \to R \cup \{\infty\}$ by

$$\beta_1(p) = \begin{cases} d_{\gamma_1}(p,p') \text{ when } \exists\, p' \in \gamma_1 \text{ such that} \\ \qquad\qquad \exists\, p'' \in \gamma_2 \text{ with } p \longrightarrow p'' \longrightarrow p', \\ 0 \text{ otherwise} \end{cases}$$

and β_2 is defined analogously.

Given two numbers $0 < M_1 < \infty$, $0 < M_2 < \infty$ we can define $S_{M_1,M_2} = \{\text{all pairs } (\gamma_1,\gamma_2) \text{ such that } \beta_1 \equiv M, \beta_2 \equiv M_2\}$.

Thus, we may say a pair (γ_1,γ_2) is a solution of the bounce time dynamics if the particles stay a constant distance apart, in each others reckoning.

Clearly $\mathcal{S}_{M_1,M_2}$ is an invariant set under the action of isometries. Moreover, the initial value data for this problem may be rather nicely circumscribed.

Initial Value Data

Let $0 < M_1 < \infty$, $0 < M_2 < \infty$ and $\begin{matrix} 0 \le \varepsilon_1 \\ 0 \le \varepsilon_2 \end{matrix}$. Define an $(M_1,M_2,\varepsilon_1,\varepsilon_2)$ light-trapezoid as a quartuple of points (p_1,p_2,p_3,p_4) in space time such that $p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_4$ and $p_1 < p_3$ and $p_2 < p_4$ and finally $d(p_1,p_3) = M_1 + \varepsilon_1$, $d(p_2,p_4) = M_2 + \varepsilon_2$ where $d(p_1,p_3)$ is the straight-line (Minkowski) arclength. (Figure 1; see page 84.) Given any pair (γ_1,γ_2) of time-like paths between p_1 and p_3 (respectively p_2,p_4) such that $d_{\gamma_1}(p_1,p_3) = M_1$ and $d_{\gamma_2}(p_2,p_4) = M_2$ we can extend these in both directions uniquely to get piecewise smooth solutions in $\mathcal{S}_{M_1,M_2}$. This is accomplished in the following way (see Figure 2; page 84). The paths γ_1 and γ_2 define a smooth tangent line-field in the regions

$\Gamma_1 = \{p \mid p_3 < p \text{ or } p_3 \rightarrow p\} - \{p \mid p_4 < p\}$ and

$\Gamma_2 = \{p \mid p < p_2 \text{ or } p \rightarrow p_3\} - \{p \mid p < p_1\}$ where the slope (in (x,t) coordinates) of the tangent line element at a point $z \in \Gamma_1$ is given by

$$S(z) = \frac{1-(1-k)^2\left(\frac{1+\omega}{1-\omega}\right)}{1+(1-k)^2\left(\frac{1+\omega}{1-\omega}\right)} \qquad \text{if } x(p_1) < x(p_2)$$

$$S(z) = \frac{(1+k)^2(\frac{1-\omega}{1+\omega})-1}{1+(1+k)^2(\frac{1-\omega}{1+\omega})} \qquad \text{if } x(p_1) > x(p_2)$$

where k is the slope of γ_2 at that point p such that $d(p,z) = 0$ and ω is the slope of γ_1 at that point p' such that $d(p,p') = 0$ (see Figure 3, page 84).

The extension of γ_1 is then that integral curve of this tangent line field which begins at the point p_3. Notice that, since γ_1 and γ_2 are time-like up to and including the endpoints, the extension of γ_1 is also time-like and intersects the forward time boundary of Γ_1 at some finite time. Call this point p_5, and the extension of γ_1, call $\tilde{\gamma}_1$. Then (p_2,p_3,p_4,p_5) form a new light-trapezoid with constants $(M_2,M_1,\varepsilon_2,\tilde{\varepsilon}_1)$ where $\tilde{\varepsilon}_1 = d(p_3,p_5)$. We can thus iterate this procedure with paths γ_2 and $\tilde{\gamma}_1$, to arrive at a point p_6 with an extension $\tilde{\gamma}_2$ of γ_2, etc.

Clearly, by imposing C^∞ boundary condition on γ_1 and γ_2 we can make these extensions C^∞ instead of piecewise smooth.

Asymptotics

The next question to consider is whether this procedure gives complete solutions, i.e. it we iterate this process infinitely many times into the past and future, will the curves thus defined intersect every t = constant slice? To see that this is actually the case, let us call p_{2n+1} the n-th iteration of the point p_1 and p_{2n+2} the n-th iteration of p_2. Then we know $d_{\gamma_1}(p_{2n+1},p_{2(n+1)+1}) = M_1$ so that

$d(p_{2n+1}, p_{2(n+1)+1}) \geq M_1$.

On the other hand

$|t(p_{2n+1}) - t(p_{2(n+1)+1})| \geq d(p_{2n+1}, p_{2(n+1)+1}) \geq M_1$. Thus, succeeding iterations of p_1 are at least M_1 units of time apart, and so after infinitely many iterations, every time value will be covered.

We may therefore conclude that these extensions exist for all time. Are all these solutions "nicely behaved as time goes to infinity? Certainly, if γ_1 and γ_2 are null translations of one another, this is the case. Here, the extension process reproduces copies of the curves, translated to the appropriate endpoints (see Figure 4, page 84) On the other hand, this is essentially the only "nicely behaved" case. We may see this by considering merely a skelton of one of these general solutions.

Knowing the velocities (in standard coordinates) of the curves at points p_1 and p_2, we may use formulae * to determine the velocities at all the other "bounce points".

It may be seen that if the velocities at p_1 and p_2 are not identical, then the velocities at the bounce point proceed in absolute value to the speed of light ($c = 1$ in our coordinates). Moreover, the velocities of the two particles at adjacent bound points eventually have the same sign. In the situation illustrated in Figure 5, (see page 84) (velocity at p_1) > (velocity at p_2) (respectively <)

implies the velocities at bounce points in future direction go to -1 (respectively +1) as time goes to infinity. Thus, if the velocity of γ_1 is greater than that of γ_2, then the extensions will depict a pair of particles accelerating toward the left of the picture.

Four Dimensional Space-Time

We may attempt to repeat this whole scheme of two particle dynamics in the four dimensional Minkowski space.

If p_1 and p_2 are two points in M^4 we will write $p_1 \rightarrow p_2$ when p_2 lies on the forward light-cone of p_1 and $p_1 < p_2$ when p_2 lies within the forward light-cone of p_1. If γ is a time-like path containing p_1 and p_2 we write $d_\gamma(p_1,p_2)$ for the Minkowski arclength of the part of γ between p_1 and p_2. If (γ_1,γ_2) are a pair of time-like, one-dimensional submanifolds, $\beta_1 : \gamma_1 \to R \cup \{\infty\}$ and $\beta_2 : \gamma_2 \to R \cup \{\infty\}$ are defined as in the two dimensional case.

Again, given two numbers $0 < M_1 < \infty$ and $0 < M_2 < \infty$ we can define $\mathcal{S}_{M_1,M_2} = \{$all pairs (γ_1,γ_2) such that $\beta_1 \equiv M_1$, $\beta_2 \equiv M_2\}$. This is again an invariant set under the natural action of the Poincaré group in four dimensions.

In this case, however, the analogous light-trapezoid initial value data does not select out a unique solution. This problem arises because when we repeat the extension procedure used in the two dimensional case, we do not get a smooth line element field which we can then integrate.

In fact, what is obtained is a paraboloid of tangent directions at each point. This may be remedied by suitably refining our dynamical law. It is consistent to require of our paths γ_1 and γ_2 not only that $\beta_1 \equiv M_1$ and $\beta_2 \equiv M_2$, but also that if p_1 and p_3 are points of γ_1 (respectively γ_2) and p_2 is on γ_2 (respectively γ_1) with $p_1 \longrightarrow p_2 \longrightarrow p_3$, then the difference of the normalized tangent vectors $\dot{\gamma}(p_3) - \dot{\gamma}_1(p_1)$ (respectively, $\dot{\gamma}_2(p_3) - \dot{\gamma}_2(p_1)$) is parallel to the plane generated by the null line through p_1 and p_2 and the null line through p_2 and p_3. We will call the set of (γ_1, γ_2) satisfying these conditions $\mathscr{S}_{M_1,M_2}$.

Clearly, $\mathscr{S}_{M_1,M_2}$ is invariant under the natural action of the Poincaré group. Moreover, on any two dimensional sub-space-time this dynamical law reduces to the one considered above.

The light trapezoid gives good initial value data for $\mathscr{S}_{M_1,M_2}$ in the following sense. Given points p_1, p_2, p_3, p_4 in space-time with $p_1 \longrightarrow p_2 \longrightarrow p_3 \quad p_4$ and $p_1 < p_3$, $p_2 < p_4$ and smooth time-like paths γ_1, γ_2 from p_1 to p_3 respectively p_2 to p_4 such that $d_{\gamma_1}(p_1, p_3) = M_1$ and $d_{\gamma_2}(p_2, p_4) = M_2$ there exists unique piecewise smooth extension $\tilde{\gamma}_1, \tilde{\gamma}_2$ defined for all time of γ_1 and γ_2 which satisfy $\beta_1 \equiv M_1$, $\beta_2 \equiv M_2$ everywhere and the parallelism requirement everywhere except possibly at the bounce iterates of p_1 and p_2.

Asymptotics

In this system we find many nicely behaved solutions. If $R(t)$ is any one parameter group of space rotations and p,q are distinct points on the $t = 0$ hyperplane, then $\gamma_1 = \{T(t)\ R(a\cdot t)[p] \mid t \in R\}$ and $\gamma_2 = \{T(t)\ R(a\cdot t)[q] \mid t \in R\}$, where $T(t)$ is forward time translation by t, and $a > 0$ gives us a pair of curves which are time-like if a is sufficiently small and which satisfy $\beta_1 \equiv$ some constant and $\beta_2 \equiv$ some constant. The parallelism requirement, in this case, translates to the requirement that p,q and the center-line of rotation be co-planar (see Figure 6, p.85).

Unfortunately, this system differs from the planar one in that the velocities at the bounce points depend on the positions as well as the various velocities and are thus not directly computable as a "skeleton".

We note that the light-trapezoid initial data is also sufficient for the Fokker-Tetrode formulation of two particle electrodynamics.[1,2] However, global existence theorems are not as readily obtainable here as in the case of integrating line-element fields. However, we may again expect that there will be infinitely many degrees of freedom in the two particle electrodynamics. Moreover, it is hoped that techniques developed for computer solutions of the two particle bounce time dynamics will be of assistance in the corresponding electrodynamics problem.

Sub-Appendix

If we parametrize γ_1 by arc-length, $\gamma_1(\tau)$ and parametrize $\gamma_2(\tau)$ so that $\|\gamma_2(\tau) - \gamma_1(\tau)\|^2 = 0$ and finally parametrize the extension $\tilde{\gamma}_1$ of γ_1 by arc-length $\tilde{\gamma}_1(\tau)$ we have $\gamma_1(0) = p_1$, $\gamma_1(M) = p_3$, $\gamma_2(0) = p_2$, $\gamma_2(M) = p_4$ and $\tilde{\gamma}_1(0) = p_3$ and $\|\tilde{\gamma}_1(\tau) - \gamma_2(\tau)\|^2 = 0$. Taking first derivatives of these

$$\begin{aligned} 0 &= \langle\gamma_2(\tau)-\gamma_1(\tau),\gamma_2(\tau)-\gamma_1(\tau)\rangle = \langle\tilde{\gamma}_1(\tau)-\gamma_2(\tau),\tilde{\gamma}_1(\tau)-\gamma_1(\tau)\rangle \\ &= 2\langle\dot{\gamma}_2(\tau)-\gamma_1(\tau),\gamma_2(\tau)-\gamma_1(\tau)\rangle \\ &= 2\langle\tilde{\gamma}_1(\tau)-\dot{\gamma}_2(\tau),\tilde{\gamma}_1(\tau)-\gamma_2(\tau)\rangle. \end{aligned}$$

Letting $V_1(\tau)$ be the null vector $\gamma_2(\tau)-\gamma_1(\tau)$

$V_2(\tau)$ be the null vector $\tilde{\gamma}_1(\tau)-\gamma_2(\tau)$

$W(\tau) = \dot{\gamma}_1(\tau) \qquad \|W(\tau)\|^2 = 1$

$X(\tau) = \dot{\gamma}_2(\tau)$

$W'(\tau) = \dot{\tilde{\gamma}}_1(\tau) \qquad \|W'(\tau)\|^2 = 1$

We find $\langle X-W,V_1\rangle = 0$, $\langle W'-X,V_2\rangle = 0$, $\|W\|^2 = \|W'\|^2 = 1$, $\langle V_1,V_1\rangle = 0$, $\langle V_2,V_2\rangle = 0$, $(V_1,V_2 \neq 0)$ and X is tangent to γ_2, and non-zero.

We can translate all of these to a single tangent space.

If we know the velocity, in standard reference frame, of $\gamma_1(\tau)$ and $\gamma_2(\tau)$ we can compute the velocity of $\tilde{\gamma}_1(\tau)$ from the above as follows.

W in standard coordinates is given by (t_1, x_1) with $t_1^2 - x_1^2 = 1$. If the velocity (slope) is ω, then $(t_1 > 0)$ $t_1 = (1-\omega)^{-\frac{1}{2}}$ and $x_1 = \omega\cdot(1-\omega)^{-\frac{1}{2}}$. V_1 is represented by a pair (ℓ,ℓ) or $(\ell,-\ell)$ depending on the initial configuration of our particles and V_2 will be represented by a pair $(n,-n)$ or (n,n) respectively.

If the velocity of X is k, then X is represented by a pair $(\alpha, k\alpha)$, $\alpha > 0$ which by $\langle X-W, V_1\rangle = 0$ satisfies

$$(t_1-\alpha)\ell - (x_1-k\alpha)\ell = 0 \quad \text{(taking } V_1 = (\ell,\ell) \text{ for specificity)}$$

or $$(t_1-\alpha) = x_1 - k\alpha$$

or $$\alpha = \frac{t_1-x_1}{1-k}.$$

Next, if W' is represented by a pair t_2, x_2 from $\langle W'-X, V_2\rangle = 0$ we have

$$(t_2-\alpha)n + (x_2-k\alpha)n = 0$$

or $$t_2 = \alpha(1+k) - x_2.$$

From $\langle W', W'\rangle = 1$ we find

$$(\alpha(1+k) - x_2)^2 - x_2^2 = 1$$

or

$$\alpha^2(1+k)^2 - 2\alpha(1+k)x_2 = 1$$

or

$$x_2 = \frac{\alpha^2(1+k)^2-1}{2\alpha(1+k)}$$

Thus

$$\frac{x_2}{t_2} = \frac{\alpha^2(1+k)^2-1}{2\alpha(1+k)} \cdot (\alpha(1+k) - \frac{\alpha^2(1+k)^2-1}{2\alpha(1+k)})^{-1}.$$

Substituting above quantities, we get formulae *.

We can make an analogous study in four space-time dimensions. If γ_1 is parametrized by arclength $\gamma_1(\tau)$ and γ_2 parametrized so that $\|\gamma_2(\tau) - \gamma_1(\tau)\|^2 = 0$ and $\tilde{\gamma}_1$ parametrized by arclength, $\tilde{\gamma}_1(\tau)$. Again, letting $V_1 = \gamma_2(\tau) - \gamma_1(\tau)$ and $V_2 = \tilde{\gamma}_1(\tau) - \gamma_2(\tau)$, $W = \mathring{\gamma}_1$, $X = \dot{\gamma}_2$, $W' = \dot{\tilde{\gamma}}_1$.

We have

(i) V_1, V_2 null, (non-zero and non-collinear)

(ii) $\langle X-W, V_1\rangle = 0$

(iii) $\langle W'-X, V_2\rangle = 0$

(iv) $\langle W, W\rangle = 1$

(v) $\langle W', W'\rangle = 1$

(vi) $W'-W = aV_1 + bV_2$.

We can then solve for W' in terms of W and the lines corresponding to X, V_1, and V_2, in the following way:

From (vi) and (iii)

$$0 = \langle W-X + aV_1 + bV_2, V_2\rangle = \langle W'X, V_2\rangle + a\langle V_1, V_2\rangle$$

thus

$$a = \frac{\langle X-W, V_2\rangle}{\langle V_1, V_2\rangle}.$$

From (v) and (vi) and (iv)

$$1 = \langle W', W'\rangle = \langle W + aV_1 + bV_2, W + aV_1 + bV_2\rangle$$

$$= \langle W, W\rangle + 2a\langle W, V_1\rangle + 2b\langle W, V_2\rangle + 2ab\langle V_1, V_2\rangle.$$

Thus

$$b = \frac{-a\langle W, V_1\rangle}{\langle W, V_2\rangle + a\langle V_1, V_2\rangle}$$

Combining these, we have

$$** \quad W' = W + \frac{\langle X-W, V_2\rangle}{\langle V_1, V_2\rangle}\left(V_1 - \frac{\langle W, V_1\rangle}{\langle W, V_2\rangle + \langle X-W, V_2\rangle} V_2\right)$$

$$= W + \frac{\langle X-W, V_2\rangle}{\langle V_1, V_2\rangle}\left(V_1 - \frac{\langle X, V_1\rangle}{\langle X, V_2\rangle} V_2\right).$$

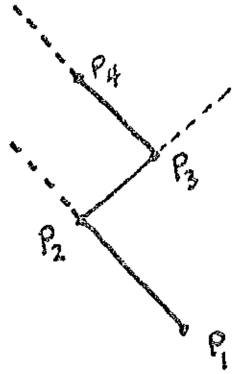
Notice that this is independent of the "lengths" of V_1 and V_2, and given the ray corresponding to X, (ii) fixes it uniquely. Moreover

$$\langle W'-W, X\rangle = \frac{\langle X-W, V_2\rangle}{\langle V_1, V_2\rangle}\left(\langle V_1, X\rangle - \frac{\langle X, V_1\rangle}{\langle X, V_2\rangle}\langle V_2, X\rangle\right) = 0.$$

Thus, since the velocities of γ_1 and γ_2 are bounded (continuous and defined on closed interval) the velocity of $\tilde{\gamma}_1$ is also bounded, $\tilde{\gamma}_1(M)$ has finite coordinate time. The same argument as in the two dimensional space-time case shows that coordinate time of $\tilde{\gamma}_1(M)$ is bounded away from zero.

The formula ** then defines a smooth unit time-like vector field of bounded velocity in the region bounded by the forward light cones of p_2 and p_4, minus those null lines through γ_1 and γ_2, and this can be extended smoothly through this surface as well. Hence there exists a unique curve $\tilde{\gamma}_1(\tau)$ which is an integral curve of this vector field with $\tilde{\gamma}_1(0) = p_3$. The analogous procedure can be used in the backwards time direction.

Figure 1.



A light-trapezoid. The indicated lines are null.

Figure 2

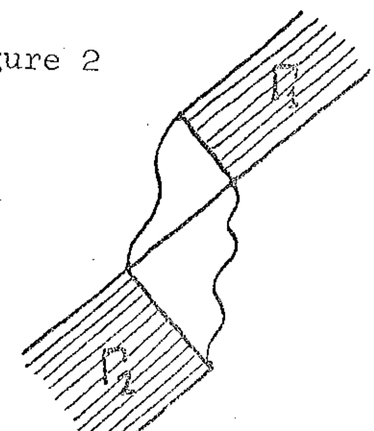
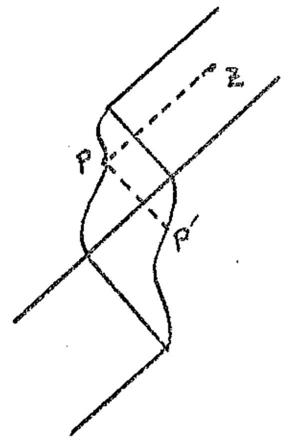


Figure 3



A light-trapezoid with initial curves. The regions on which the line-element field is defined are shaded (Γ_1 and Γ_2).

Figure 4

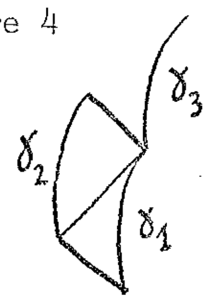
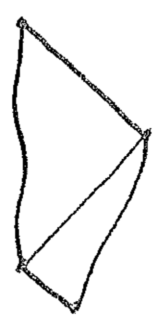
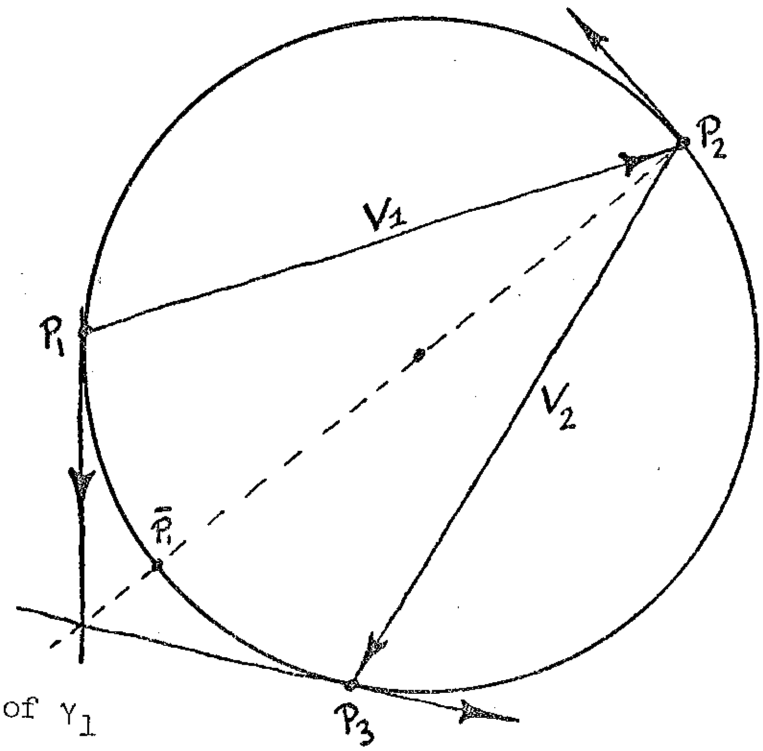


Figure 5



γ_2 is a null translate of γ_1. γ_3 is a time-like translate of γ_1.

Figure 6



p_1 point of γ_1 simultaneous to p_2.

In order that $\dot{\gamma}_1(p_3) - \dot{\gamma}_1(p_1)$ be parallel to the plane generated by V_1 and V_2 (all shown here in projection onto a space-like slice tangent to the rotation) p_2 and $\bar{p}_1$ and centerline of rotation (out of the page) must be coplanar.

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