CLASSIFICATION OF RELATIVISTIC n-PARTICLE DYNAMICS

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by

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Abstract of the Dissertation

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A relativistic n-particle dynamical law is a collection, $S$ of $n$-tuples of smooth, timelike, one-dimensional submanifolds of Minkowski space, such that the set $S$ is invariant under the natural action of the Poincaré group. A classical $n$-particle dynamics of order $k$ is a smooth one parameter family of diffeomorphisms $H : T^k_f(R^3 \times B^3)^n \times \mathbb{R} \rightarrow T^k_f(R^3 \times B^3)^n$ ($T^k_f = \text{fiber tangent}$) satisfying the usual $k$-th order condition. (The unit open ball is taken for velocity since we do not allow particles to go faster than the speed of light $c = 1$.)

The classification of relativistic $n$-particle dynamics which arise as the orbit-set of a classical $n$-particle dynamics of order $k$ proceeds in two stages. First, the problem is reduced to classifying the solutions of certain systems of first order, non-linear partial differential equations. Although formally overdetermined, these are shown to have some remarkable covariance properties which allow local analytic
solutions to be completely determined. General solutions for various classes are explicitly determined, and for the second order, three particle case it is shown that there are no non-trivial solutions which conserve relativistic momenta. The problem of conservation of generalized momenta is raised.

These results extend the results of Currie, Jordan, Sudarshan: Rev. Mod. Phys., 35, 350 (63), Cannon, Jordan: J. Math. Phys., 5, 299 (64) and Leutwler: Nuov. Cim. 37, 556 (66) when \( k = 2 \) and \( H \) is taken to be Hamiltonian, although completely different techniques are involved, and Arens: Adv. Math. 10, 332 (73) when the generalized momenta are taken to be the standard relativist momenta.
To

Dr. John A. Wheeler
Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>iii</td>
</tr>
<tr>
<td>Dedication</td>
<td>v</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>vi</td>
</tr>
<tr>
<td>Notation</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER I</td>
<td></td>
</tr>
<tr>
<td>§1 Introduction</td>
<td>2</td>
</tr>
<tr>
<td>§2 The Problem as A System of Differential Equations</td>
<td>5</td>
</tr>
<tr>
<td>§3 Specialization to k-th Order Dynamics</td>
<td>10</td>
</tr>
<tr>
<td>§4 Euclidean Invariance</td>
<td>15</td>
</tr>
<tr>
<td>CHAPTER II</td>
<td></td>
</tr>
<tr>
<td>§1 One Particle Dynamics</td>
<td>21</td>
</tr>
<tr>
<td>§2 First Order Dynamics</td>
<td>29</td>
</tr>
<tr>
<td>§3 Second Order Equations</td>
<td>31</td>
</tr>
<tr>
<td>§4 Local Existence of Solutions</td>
<td>47</td>
</tr>
<tr>
<td>§5 Conformal Covariance</td>
<td>63</td>
</tr>
<tr>
<td>§6 Generalized Momentum Conservation</td>
<td>65</td>
</tr>
<tr>
<td>Appendix: Bounce-Time Dynamics</td>
<td>72</td>
</tr>
<tr>
<td>References</td>
<td>86</td>
</tr>
</tbody>
</table>
Notation

To facilitate notation, we have omitted range specifications for subscripts and superscripts. We use the following ranges everywhere unless otherwise noted.

Subscripts

i, j, l, m, i', j', l', m', etc. take all values in \{1, 2, 3\}.

Superscripts

n = number of particles considered.

s, r, s', r', etc. take all values in \{1, \ldots, n\}.

Order

k = order of the dynamics considered.

These allow us to deal with various number of particles and various orders in a uniform manner.

Example

the equation \( p_i p_j = 0 \) actually represents the system of equations \( p_1 p_1 = 0, \ p_1 p_2 = 0, \ p_1 p_3 = 0, \ldots \).
§ 1. In non-relativistic physics, the description of a physical system is constituted in specifying a space, \( M \) (identified as the space of states of the system in question) and a one parameter family, \( \{ \omega_t \mid t \in \mathbb{R} \} \), of automorphisms of the space \( M \). Automorphism here refers to that structure with which we have equipped \( M \).

For example, in classical mechanics, for a single particle \( M = T^* \mathbb{R}^3 \) equipped with the canonical symplectic structure. The \( \omega_t \) are then symplectic diffeomorphisms. In quantum mechanics, \( M = \mathcal{H} \), a separable Hilbert space and \( \omega_t \) are unitary transformations.

The "\( t \)" in \( \omega_t \) refers to time and the one parameter family of \( \omega_t \)'s represents the time development of the system.

There are usually fundamental symmetries of the system present, however. These symmetries are given as a homomorphism \( \rho \), of a group \( G \), into the group \( \Gamma \), of all automorphisms of \( M \). As an example, when \( M = T^* \mathbb{R}^3 \), \( G \) is often \( \mathbb{R}^3 \), \( \text{O}(3) \), or \( \mathbb{R}^3 \times \text{O}(3) \) and \( \rho \) gives the standard action of these on \( T^* \mathbb{R}^3 \).

That these symmetries should remain present as time progresses is expressed in the requirement that \( \forall g \in G, \forall t \in \mathbb{R}, \omega_t \circ \rho(g) = \rho(g) \circ \omega_t \).

In relativistic physics, the symmetries and the dynamics become inextricably intertwined. The usual thing to do in this case is to incorporate the dynamics as "just one more
symmetry". We thus arrive at a homomorphism \( \rho : \text{IL} \rightarrow \Gamma \), where \( \text{IL} \) is the connected component of the group of isometries of Minkowski space, \( \mathbb{M}^4 \) (Poincare group). However, reasonable is this abstraction, it is still a good deal removed from the physical meaning of these isometries. Let us then take a different point of view.

The space-time history of a point-particle in special relativistic physics is described by a time-like curve in Minkowski space (a one-dimensional, time-like submanifold). Correspondingly, the space-time history of an n-particle system would be described by an n-tuple of one-dimensional time-like submanifolds. For the purposes of classical mechanics, it would be completely adequate to describe a set \( S \) of such n-tuples, where each element of this set would correspond to an n-particle space-time history which can occur in nature. We would also need some criterion for distinguishing among elements of this set, especially one which is local in time, for the purposes of prediction. Local in time means, we could design experiments which take a uniformly bounded amount of time to perform and would distinguish trajectories.

Noting that every n-tuple of time-like 1-dimensional submanifold intersects the hyperplane \( t = 0 \) in n points (possibly coincident), it is reasonable to assume that we may parametrize our orbit set \( S \) by \( (\mathbb{R}^3)^n \times \mathbb{P} \), where \( \mathbb{P} \) represents the results of our experiments.

The requirement of relativistic invariance for this
scheme is simply that the set $\mathcal{S}$ be invariant, as a set, under the natural action of the group of isometries of Minkowski space.

Note that if we have any bijective map $\sigma : \mathcal{S} \to \mathcal{S}$ and $\mathcal{S}$ is relativistically invariant, then $\sigma$ induces an action of $\mathfrak{i}L$ on $\mathcal{S}$. In particular, if $\mathcal{S}$ is relativistically invariant and has a parametrization of the type $(\mathbb{R}^3)^n \times P$ given above, and $P$ is a manifold, it is sensible to ask whether the induced action of $\mathfrak{i}L$ on $(\mathbb{R}^3)^n \times P$ is smooth ($C^\infty$) or not. In case it is, we will call $\mathcal{S}$, together with its parametrization, an (n-particle) relativistic dynamical law, or R.D.L. for short.

Among the most interesting, from a physical standpoint, are those R.D.L whose parametrization space corresponds to the phase space of a $k$th order system. That is to say, if $x_1, \ldots, x_{3n} : \mathbb{R}^{3n} \to \mathbb{R}$ are coordinate functions for $\mathbb{R}^{3n}$ and $X_0$ is the vectorfield on $\mathbb{R}^{3n} \times P$ corresponding to the infinitesimal generator of time translations, then the functions $\tilde{x}_1, \ldots, \tilde{x}_{3n}, X_0(\tilde{x}_1), \ldots, X_0(\tilde{x}_{3n}), \ldots, X_0^{k-1}(\tilde{x}_1), \ldots, X_0^{k-1}(\tilde{x}_{3n})$ are a system of coordinates on $\mathbb{R}^{3n} \times P$. ($\tilde{x}_1, \ldots, \tilde{x}_{3n}$ are the natural extensions of $x_1, \ldots, x_{3n}$ to functions on $\mathbb{R}^{3n} \times P$. We will drop the tildas henceforth.)

In this case, any vectorfield is determined by its action on these $k \cdot 3n$ functions. From the fact that each point in $\mathbb{R}^{3n} \times P$ corresponds to an $n$-tuple of time-like, one-dimensional submanifolds of $\mathbb{M}^4$ and the projection $\pi : \mathbb{R}^{3n} \times P \to \mathbb{R}^{3n}$ corresponds to intersecting these submanifolds
with the hyperplane \( t = 0 \), we obtain certain relations which will now be explored.

§ 2. Let us take standard coordinates \((t, x_1, x_2, x_3)\) for \( M^4 \).
Then the following one parameter subgroups of \( iL \) generate

**Time translation**

\[ \tau_\alpha(t, x_1, x_2, x_3) = (t + \alpha, x_1, x_2, x_3) \quad \forall \alpha \in \mathbb{R} \]

corresponding Lie algebra element called \( \mathbb{X}_0 \).

**Space translations**

\[ \mathbb{Y}_\alpha^1(t, x_1, x_2, x_3) = (t, x_1 + \alpha, x_2, x_3) \]
\[ \mathbb{Y}_\alpha^2(t, x_1, x_2, x_3) = (t, x_1, x_2 + \alpha, x_3) \quad \forall \alpha \in \mathbb{R} \]
\[ \mathbb{Y}_\alpha^3(t, x_1, x_2, x_3) = (t, x_1, x_2, x_3 + \alpha) \]
corresponding Lie algebra elements called \( \mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3 \) respectively.

**Space Rotations**

\[ \mathbb{Y}_\alpha^{12}(t, x_1, x_2, x_3) = (t, x_1 \cos(\alpha) - x_2 \sin(\alpha), x_1 \sin(\alpha) + x_2 \cos(\alpha), x_3) \]
\[ \mathbb{Y}_\alpha^{13}(t, x_1, x_2, x_3) = (t, x_1 \cos(\alpha) - x_3 \sin(\alpha), x_2, x_1 \sin(\alpha) + x_3 \cos(\alpha)) \]
\[ \mathbb{Y}_\alpha^{23}(t, x_1, x_2, x_3) = (t, x_1, x_2 \cos(\alpha) - x_3 \sin(\alpha), x_2 \sin(\alpha) + x_3 \cos(\alpha)) \]

Lie algebra elements \( \mathbb{Y}_{12}, \mathbb{Y}_{13}, \mathbb{Y}_{23} \) respectively.

**Lorentz Boosts**

\[ \mathbb{Y}_\alpha^{01}(t, x_1, x_2, x_3) = (\cosh(\alpha)t + \sinh(\alpha)x_1, \sinh(\alpha)t + \cosh(\alpha)x_1, x_2, x_3) \]
\[ \mathbb{Y}_\alpha^{02}(t, x_1, x_2, x_3) = (\cosh(\alpha)t + \sinh(\alpha)x_2, x_1, \sinh(\alpha)t + \cosh(\alpha)x_2, x_3) \]
\[ \mathbb{Y}_\alpha^{03}(t, x_1, x_2, x_3) = (\cosh(\alpha)t + \sinh(\alpha)x_3, x_1, x_2, \sinh(\alpha)t + \cosh(\alpha)x_3) \]
Lie algebra elements respectively \( \gamma_{01}, \gamma_{02}, \gamma_{03} \).

With our choice of coordinates, any one-dimensional time-like submanifold, call it \( \gamma \), can be written uniquely as

\[
\gamma = \{ (\tau, \varphi_1(\tau), \varphi_2(\tau), \varphi_3(\tau)) \mid \tau \in \mathbb{R} \}
\]

where \( \varphi_1, \varphi_2, \varphi_3 \) are smooth functions and

\[
\left( \frac{d\varphi_1}{d\tau} \right)^2 + \left( \frac{d\varphi_2}{d\tau} \right)^2 + \left( \frac{d\varphi_3}{d\tau} \right)^2 < 1
\]

for all \( \tau \in \mathbb{R} \). Hence if \( \tilde{\gamma} \in S \) then it can be written

\[
\tilde{\gamma} = \{ \gamma^i \mid i = 1, \ldots, n \}
\]

where the \( \gamma^i \) are of the above form. In the case (identifying \( S \) with its parametrization) we have

\[
x_j^\gamma(\tilde{\gamma}) = \varphi_j(0)
\]

Clearly we have the following induced actions of the above mentioned one parameter subgroups of \( iL \) on \( R^{3n} \times P \):

\[
x_j^\gamma(\tau_\alpha(\tilde{\gamma})) = \varphi_j^\gamma(-\alpha) \quad \alpha \in \mathbb{R}
\]

\[
x_j^\gamma(\gamma^i_\alpha(\tilde{\gamma})) = \delta_{ij} \alpha + x_j^\gamma(\tilde{\gamma}) \quad \alpha \in \mathbb{R}
\]

\[
x_j^\gamma(\gamma^i_m(\tilde{\gamma})) = \delta_{jm} (x_i^\gamma(\tilde{\gamma}) \sin(\alpha) + x_m^\gamma(\tilde{\gamma}) \cos(\alpha))
\]

\[
+ \delta_{ij} (x_i^\gamma(\tilde{\gamma}) \cos(\alpha) - x_m^\gamma(\tilde{\gamma}) \sin(\alpha)) + (1 - \delta_{ij} - \delta_{jm}) x_j^\gamma(\tilde{\gamma})
\]

If we denote the vectorfields on \( R^{3n} \times P \) induced by these one parameter groups of motions by the unbarred version of the corresponding Lie algebra elements, we thus have

\[
x_\alpha(x_j^\gamma)(\tilde{\gamma}) = \frac{d}{d\tau} \varphi_j^\gamma |_{\tau=0}
\]
\[ X_i(x^r_j)(\gamma) = \delta_{ij} \]

\[ Y_{im}(x^r_j)(\gamma) = \delta_{jm} x^r_j(\gamma) - \delta_{ij} x^r_m(\gamma). \]

Finally, we have

\[ x^r_j(\varphi_\alpha^r(\gamma)) = \delta_{ij}(\sinh(\alpha)t + \cosh(\alpha)\varphi^r_j(t)) + (1 - \delta_{ij})\varphi^r_j(t) \]

where \( t \) satisfies the equation

\[ \cosh(\alpha)t + \sinh(\alpha)\varphi^r_1(t) = 0. \]

Suppose we have a one parameter family of \( t \)'s, say \( t(\alpha) \) satisfying

\[ \cosh(\alpha)t(\alpha) + \sinh(\alpha)\varphi^r_1(t(\alpha)) = 0. \]

Then taking derivative with respect to \( \alpha \) at \( \alpha = 0 \) we have

\[ \frac{d}{d\alpha} t(\alpha) \bigg|_{\alpha=0} + \varphi^r_1(t(\alpha)) \bigg|_{\alpha=0} = 0 \text{ but } t(0) = 0. \]

Thus

\[ \frac{d}{d\alpha} t(\alpha) \bigg|_{\alpha=0} = -\varphi^r_1(0) = -x^r_1(\gamma). \]

We can now write

\[ x^r_j(\varphi_\alpha^r(\gamma)) = \delta_{ij}(\sinh(\alpha)t(\alpha) + \cosh(\alpha)\varphi^r_j(t(\alpha)) + (1 - \delta_{ij})\varphi^r_j(t(\alpha)). \]

Differentiating at \( \alpha = 0 \), we have

\[ Y_{oi}(x^r_j)(\gamma) = -x^r_1(\gamma) \cdot \frac{d}{dt}\varphi^r_j \bigg|_{t=0} = x^r_1(\gamma) \cdot X_o(x^r_j)(\gamma). \]

The vectorfields \( X_o, X_i, Y_{ij} \) and \( Y_{oi} \) inherit bracket relations from the Lie algebra of \( \mathfrak{IL} \). Namely

(2.1) \[ [X_o, X_i] = 0 \]

(2.2) \[ [X_o, Y_{ij}] = 0 \]
(2.3) \[ [X_o, Y_{o1}] = X_1 \]

(2.4) \[ [Y_{ij}, Y_{om}] = \delta_{jm} Y_{oi} - \delta_{im} Y_{oj} \]

(2.5) \[ [Y_{oi}, Y_{oj}] = Y_{ij} \]

(2.6) \[ [X_1, Y_{oj}] = \delta_{ij} X_o \]

From the previous we also must have

(2.7) \[ X_i(x_j^r) = \delta_{ij} \]

(2.8) \[ Y_{ij}(x_m^r) = \delta_{jm} x_i^r - \delta_{im} x_j^r \]

(2.9) \[ Y_{oi}(x_j^r) = x_i^r X_o(x_j^r) \]

(2.10) \[ (X_o(x_1^r))^2 + (X_o(x_2^r))^2 + (X_o(x_3^r))^2 < 1 \]

Let us now combine equations 2.3, 2.7 and 2.9 into the following system:

(2.0)' \[ Y_{oi}(x_j^r) = x_i^r X_o(x_j^r) \]

(2.1)' \[ Y_{oi}X_o(x_j^r) = X_o(x_1^r)X_o(x_j^r) + x_i^r X_o(x_j^r) - \delta_{ij} \]

(2.2)' \[ Y_{oi}X_oX_o(x_j^r) = X_o^2(x_1^r)X_o(x_j^r) + 2X_o(x_1^r)X_0(x_j^r) + x_i^r x_j^r \]

\( \vdots \)

(2.s)' \[ Y_{oi}X_o^{s}(x_j^r) = X_o^{s}(x_1^r)X_o(x_j^r) + sX_o^{s-1}(x_1^r)X_0^{s-1}(x_j^r) + \ldots \]

\[ \vdots \]

\[ \ldots + \left( \frac{s}{l} \right) X_o^{s-l}(x_1^r)X_0^{l+1}(x_j^r) + \ldots + x_i^r x_j^r \]

Lemma. Suppose we have an R.D.L. parametrized by \( R^{3n} \times P \) and a \( k \geq 3 \) and an open subset \( U \subset R^{3n} \times P \) together with a \( j \in (1, 2, 3) \) and an \( r \in (1, \ldots, n) \) such that \( X_o^k(x_j^r) = 0 \) on \( U \).
Then \( x_o^2(x_1^r) = x_o^2(x_2^r) = x_o(x_3^r) = 0 \) on \( U \).

**Proof:** First note that if \( x_o^k(x_j^r) = 0 \), then so is \( x_o^k(x_1^r) \), \( x_o^k(x_2^r) \) and \( x_o^k(x_3^r) \) since they can be obtained by applying the appropriate infinitesimal rotation. We will now proceed by induction on \( k \).

**Case \( k = 3 \).** We have the equation

\[
Y_{0i} x_o^3(x_j^r) = x_o^3(x_1^r) x_o(x_j^r) + 3 x_o^2(x_1^r) x_o^2(x_j^r) + 3 x_o(x_1^r) x_o^3(x_j^r) + x_1^r x_0^4(x_j^r)
\]

which when restricted to \( U \) becomes

\[
0 = 3 x_o^2(x_1^r) x_o^2(x_j^r).
\]

Taking \( i = j \) we find \( x_o^2(x_j^r) = 0 \) on \( U \).

**Reducing Case \( k \) to Case \( k-1 \).** Assume \( x_o^k(x_j^r) = 0 \) on \( U \).

Restricting equation 2.k' to \( U \) we get

\[
0 = k x_o^{k-1}(x_1^r) x_o^2(x_j^r) + \ldots + (k-2) x_o^2(x_1^r) x_o^{k-1}(x_j^r).
\]

Applying \( x_o^{k-3} \) to both sides of this equation on \( U \) and obtain

\[
0 = k x_o^{k-1}(x_1^r) x_o^{k-1}(x_j^r)
\]

on \( U \), where \( K \) is a positive integer. Thus we have \( x_o^{k-1}(x_j^r) = 0 \) on \( U \), completing the induction step.

The meaning of this proposition is that there can be no "constant acceleration" dynamics, other than straight line motion. This rules out the most intuitively plausible possibility for relativistically invariant dynamics.
§ 3. $k$-th Order Dynamics.

Let us now consider the case of a $k$-th order R.D.L. In this case, the functions

$$x_1^r, x_0(x_1^r), \ldots, x_0^{k-1}(x_1^r)$$

give a coordinatization of $R^{3n} \times P$. In particular any vectorfield is determined by its action on these functions. Thus $X_0$ is completely determined by the functions $x_0^k(x_1^r)$, $r = 1, \ldots, n$; $i = 1, 2, 3$. $Y_{0j}$ is determined by the functions

$$Y_{0j}(x_1^r), Y_{0j}x_0(x_1^r), \ldots, Y_{0j}x_0^{k-1}(x_1^r).$$

Proposition. Suppose $X_0$ is a $k$-th order, $k \geq 2$, vectorfield on $R^{3n} \times P$. Define vectorfields $Y_{0i}$, $i = 1, 2, 3$, by equations $(2.0)', \ldots, (k-1)'$. Then, if $Y_{0i}$ satisfy equations $(2.k')$ and $(k+1)'$, they also satisfy equations $(2.q)'$ for all $q$; moreover the vectorfields

$$Y_{0i}^r = Y_{0i},$$

$$X_0^r = X_0,$$

$$X_i^r = [X_0, Y_{0i}],$$

$$Y_{1j}^r = [Y_{0i}, Y_{0j}]$$

satisfy equations $(2.1), \ldots, (2.9)$.

Proof: Let us first determine $X_1$. Since we are dealing with a $k$-th order system, it is sufficient to operate with $X_1$ on the first $k-1$ $X_0$-derivatives of the $x_j^r$'s.
3.1.0 \[ x_i^r(x_j^r) = [x_o, y_o]_i(x_j^r) = x_o y_o(x_j^r) - y_o x_o(x_j^r) = \delta_{ij} \]
by equations (2.0)' and (2.1)'.

3.1.1 \[ x_1 x_o(x_j^r) = 0 \text{ by equations (2.1)', and (2.2)'.} \]

\[ \vdots \]

3.1.k-1 \[ x_1 x_{(k-1)}(x_j^r) = 0 \text{ by equations (2.(k-1)'; and k'.} \]

Moreover

3.1.k \[ x_k x_o(x_j^r) = 0 \text{ by equations (2.k)', and (k+1)'.} \]

In particular, equation (2.7) is satisfied.

3.2.0 \[ [x_o, x_1] x_j^r = x_o x_1(x_j^r) - x_1 x_o(x_j^r) = x_o(\delta_{ij}) - 0 = 0 \]

3.2.1 \[ [x_o, x_1] x_o(x_j^r) = x_o x_1 x_o(x_j^r) - x_1 x_o x_o(x_j^r) = 0 \]

\[ \vdots \]

3.2.k-1 \[ [x_o, x_1] x_{(k-1)}(x_j^r) = x_o x_1 x_{(k-1)}(x_j^r) - x_{(k-1)} x_o(x_j^r) = 0. \]

Thus, \([x_o, x_1] = 0\) and equation (2.1) is satisfied.

The right hand side of e.g., q' (q > 1) can be obtained from the right hand side of equation (q-1)' by applying \(x_o\) to it. Moreover, applying \(x_o\) to the left hand side of equation (q-1)' gives \(x_o y_o x_{(q-1)}(x_j^r) = y_o x_o x_{(q-1)}(x_j^r) + x_1 x_{(q-1)}(x_j^r)\)

\[ = y_o x_{(q)}(x_j^r), \text{ which is the left hand side of equation q'}. \]

Thus (q-1)' true implies equation q' true, hence they are all satisfied.

We can now compute \([x_1, y_o]_j\).

3.3.0 \[ [x_1, y_o]_j x^r = x_1 y_o(x^r) - y_o x_1(x^r) = \delta_{ij} x^r \]

from equation (2.0)'.

3.3.1 \[ [X_i,Y_{o,j}]X_o(x^r_i) = \delta_{ij}x_o(x^r_i) \]
from equation (2.1)'.

\[ \vdots \]

3.3.k-1 \[ [X_i,Y_{o,j}]x^{k-1}_o(x^r_i) = \delta_{ij}x^{k-1}_o(x^r_i) \]
from equation (2.\(k-1)\)'.

Since the vector fields \([X_i,Y_{o,j}]\) and \(\delta_{ij}x_o\) give the same values on the coordinate functions, we may conclude that they are the same. Thus equation (2.6) is satisfied.

Next note that since \(y^{i,j} = [Y_{o,i},Y_{o,j}],\)
\([X_o,Y_{i,j}] = [X_o,[Y_{o,i},Y_{o,j}]] = -[Y_{o,i},[Y_{o,j},X_o]] - [Y_{o,j},[X_o,Y_{o,i}]]\)
\(= -\delta_{ij}x_o + \delta_{i}{}^{j}x_0 = 0\) and so equation (2.2) is satisfied.

Finally we have the computation of the components of \(y^{i,j}\).

3.4.0 \[ y^{i,j}(x^r_i) = [Y_{o,i},Y_{o,j}]x^r_i = Y_{o,i}Y_{o,j}(x^r_i) - Y_{o,j}Y_{o,i}(x^r_i) \]
\(= -\delta_{i}{}^{j}x^r_j + \delta_{j}{}^{i}x^r_i \)
from equation (2.0)', (2.1)', (2.3), (2.6), (2.7)
(2.3) defines \(x_i\).

3.4.1 \[ y^{i,j}x_o(x^r_i) = -\delta_{i}{}^{j}x_o(x^r_j) + \delta_{j}{}^{i}x^r_i \]
from equation (2.1)', (2.2)', (2.3), (2.6), (2.7).

\[ \vdots \]

3.4.k-1 \[ y^{i,j}x^{k-1}_o(x^r_i) = -\delta_{i}{}^{j}x^{k-1}_o(x^r_j) + \delta_{j}{}^{i}x^r_i \]
from equation (2.\((k-1))', (2.k)', (2.3), (2.6), (2.7).

Note that 3.4.0 yields equation (2.8). We also have
3.5.0 \[ [Y_{ij}, Y_{om}] x^r = Y_{om} Y_{om}(x^r) - Y_{om} Y_{ij}(x^r) \]

\[ = Y_{ij}(x^r x^r) - Y_{om}(\delta^r_{jt} x^r_i - \delta^r_{it} x^r_j) \]

\[ = \delta_{jm} x^r x^r_m x^r_o - \delta_{im} x^r x^r_m x^r_o \]

\[ = \delta_{jm} Y_{oi}(x^r) - \delta_{im} Y_{oj}(x^r) \]

from equations (2.0)', (2.2), (2.8).

3.5.1 \[ [Y_{ij}, Y_{om}] x^r = Y_{ij}(x^r Y_{om} x^r_m) - (x^r Y_{om} x^r_m) Y_{ij}(x^r) \]

(from equation (2.3).)

\[ = x^r Y_{ij} Y_{om}(x^r) - Y_{ij} x^r_m x^r_m - x^r Y_{om} Y_{ij}(x^r) \]

\[ + x^r Y_{ij}(x^r) \]

(from equation (2.2).)

\[ = x^r Y_{ij}(x^r x^r_x(x^r)) - x^r Y_{om}(\delta^r_{jt} x^r_i - \delta^r_{it} x^r_j) \]

\[ + \delta^r_{im} Y_{ij} x^r \]

\[ = \delta_{jm} (x^r x^r_x x^r_o(x^r) + 2x^r x^r_j x^r_o x^r_o(x^r)) - \delta^r_{it} \]

\[ - \delta_{im} (x^r x^r_x x^r_o(x^r) + 2x^r x^r_j x^r_o x^r_o(x^r)) - \delta^r_{jt} \]

\[ = \delta_{jm} Y_{oi}(x^r) - \delta_{im} Y_{oj}(x^r) \]

from equation (2.1)'.

From the components of \( X_m \) and \( Y_{ij} \), and equations (2.1), (2.2) we calculate that \([Y_{ij}, X_m] = \delta_{jm} X_i - \delta_{im} X_j\). We can then find

\[ [X_o, [Y_{ij}, Y_{om}]] = -[Y_{ij}, [Y_{om}, X_o]] - [Y_{om}, [X_o, Y_{ij}]] \]

\[ = \delta_{jm} X_i - \delta_{im} X_j \]

Hence by induction on \( q > 1 \),
3.5.q \[
[Y_{ij}, Y_{om}]x_o^{-q}(x^r_t) = x_o[Y_{ij}, Y_{om}]x_o^{q-1}(x^r_t)
\]
\[- (\delta_{jm} x_i - \delta_{im} x_j)x_o^{q-1}(x^r_t) \]

by induction

\[
x_o(\delta_{jm} y_{oi} x_o^{q-1}(x^r_t) - \delta_{im} y_{oj} x_o^{q-1}(x^r_t))
\]
\[- (\delta_{jm} x_i - \delta_{im} x_j)x_o^{q-1}(x^r_t) \]

(q > 1)

\[
= \delta_{jm} y_{oi} x_i^q(x^r_t) - \delta_{im} y_{oj} x_o^q(x^r_t). \]

Thus we find \([Y_{ij}, Y_{om}] = \delta_{jm} y_{oi} - \delta_{im} y_{oj}\) completing the proof.

With the aid of this proposition, we may begin to study k-th order R.D.L.'s. Clearly, once we specify that we are considering a k-th order R.D.L. the only choice we have left is to specify the k-th order part of the vectorfield \(X_o\), since then everything else is determined. The natural way to specify the k-th order part is to specify the functions \(X^k_o(x^r_j)\). Then equations (2.p)' and (2.(p+1))' define a coupled system of non-linear p.d.e. on the functions \(X^k_o(x^r_j)\) which must be satisfied for \(X_o\) to generate an R.D.L.
§4. **Euclidean Invariance**

We have been led to consider a system of coordinates $x^r_j, a^r_{ij}, a^r_{ij}, \ldots, a^r_{ij}$ for $R^{3n} \times P$. The problem of finding a generator for a $k$-th order ($k > 1$) R.D. L. has been reduced to finding 3$n$ functions, $f^r_j$, of the coordinates so that the vectorfields

$$4.1 \quad X_0 = \Sigma \Sigma a^r_{ij} dx^r_j + \Sigma a^r_{ij} \delta a^r_{ij} + \Sigma a^r_{ij} \delta a^r_{ij} + \ldots + \Sigma a^r_{ij} \delta a^r_{ij} \ldots + \Sigma f^r_j \delta a^r_{ij}$$

$$4.2 \quad Y_{oi} = \Sigma x^r_{iaj} \delta x^r_j + \Sigma (a^r_{ij} - \delta_{ij} + x^r_{iaj}) \delta a^r_{ij} + \Sigma (a^r_{ij} + 2a^r_{ij} a^r_{ij} + x^r_{iaj} a^r_{ij}) \delta a^r_{ij} + \ldots + \Sigma (a^r_{ij} - \delta_{ij} + x^r_{iaj}) \delta a^r_{ij}$$

satisfy

$$4.3.k \quad (Y_{oi} - x^r_{iaj}) f^r_j = f^r_{iaj} + k a^r_{ij} a^r_{ij} + \ldots + k a^r_{ij} f^r_j$$

$$4.3.k+1 \quad (Y_{oi} - x^r_{iaj}) X_0 (f^r_j) = X_0 (f^r_j) a^r_{ij} + (k+1) f^r_{iaj} a^r_{ij} + \ldots + (k+1) a^r_{ij} X_0 (f^r_j)$$

(In these formulae, and throughout, we will use $\delta x^r_j$, etc., instead of $\frac{3}{\delta x^r_j}$, for the coordinate vectorfield tangent to the $x^r_j$.)

We have already shown that if equation 4.k and 4.k+1 are
satisfied, then in particular, we have

4.4.1 \( x_i : = [x_{0i}, y_{0i}] \)

satisfying

4.4.2 \( x_i(x_j^r) = \delta_{ij}, \ x_i(a_j^r) = \ldots = x_i(a_j^{rk-1}) = 0 \)

as well as

4.4.3 \( [x_i, x_0] = 0. \)

We also have

4.5.1 \( y_{ij} : = [y_{0i}, y_{0j}] \)

satisfying

4.5.2 \( y_{ij}(x_i^r) = \delta_{ij}x_i^r - \delta_{it}x_j^r \)

\( y_{ij}(a_i^r) = \delta_{ij}a_i^r - \delta_{it}a_j^r \)

\ldots

\( y_{ij}(a_i^{rk-1}) = \delta_{ij}a_i^{rk-1} - \delta_{it}a_j^{rk-1} \)

as well as

4.5.3 \( [y_{ij}, x_0] = 0. \)

\( x_i \) and \( y_{ij} \) are generators of the euclidean group, the subgroup of \( iL \) taking the hyperplane \( t = 0 \) to itself. The generators have the same form in terms of the adapted coordinates, independent of which \( k \)-th order R.D.L. we are considering. In this sense, the \( [x_i, x_0] = [y_{ij}, x_0] = 0 \) are
quite natural assumptions for the dynamics, even though con-
sequences of 4.3.k and 4.3.k+1. Moreover, these are linear
partial differential equations for the $f^r_j$'s and consequently
easier to handle.

We will next show that 4.3.k and 4.3.k+1 can be replaced
by 4.3.k and $[\dot{X}_i, X_0] = 0$.

**Proposition.** Suppose we have functions $f^r_j$ of the coordinates
$x_j^r, a_j^{rl}, \ldots, a_j^{rk-l}$ so that the vectorfields $X_0$ defined by 4.1
and $Y_{oi}$ defined by 4.2 satisfy equation 4.3.k. If, in addition,
we assume $[X_i, X_0] = 0$, where $X_i$ is defined by 4.4.2, then
4.3.k+1 must be satisfied.

**Proof.** Comparing with equations 3.1.0, \ldots, 1.k-1 we find that
as long as 4.3.k is satisfied, we have $[X_0, Y_{oi}] = X_i$ as we
have defined it in 4.4.2. Assuming $[X_i, X_0] = 0$, then 4.3.k+1
can be obtained from 4.3.k by applying $X_0$ to both sides of
the equation. Hence, 4.3.k+1 is satisfied. This completes
the proof.

This however, does not exhaust all the symmetries of our
system of equations. Equations 4.3.k are euclidean covariant
in a sense we will now explain.

Consider the operator $D$

$$\sum_{r, l, a} f^r_\ell a^{rk-l} = A^D \sum_{i, j} \left[ (Y_{oi} - x_i^r \dot{x}_0) f^r_j - f^r_\ell a_i^j \ldots f^r_\ell a^{rk-1}_j \right]$$

from vectorfields "tangent to the $a^{rk-1}$-directions" to tensor-
fields "tangent to the $a^r_k \cdot l_i$-directions".

Notice that $[X_i, A] = 0 \Rightarrow [X_i, X_o] = 0$ and also $[Y_{ij}, A] = 0 \Rightarrow [Y_{ij}, X_o] = 0$ and finally $D(A) = 0 \Rightarrow$ the $r^j_i$'s satisfy 4.3.3. We already know that if $D(A) = 0$ then $[X_i, X_o] = 0 = [Y_{ij}, X_o]$, however the converse is false. The euclidean covariance of the operator $D$ is expressed in the following

**Proposition.** If $A, D, X_i, Y_{ij}$ as above $(k > 1)$, then

$$\mathcal{L}_{X_i} A = 0 \quad \mathcal{L}_{X_i} D(A) = 0$$

and

$$\mathcal{L}_{Y_{ij}} A = 0 \quad \mathcal{L}_{Y_{ij}} D(A) = 0.$$ 

Here, $\mathcal{L}_{X_i}$ means Lie derivative with respect to $X_i$, etc. (for a vectorfield, $A$, $\mathcal{L}_{X_i} A = [X_i, A]$).

Since we are searching for solutions of $D(A) = 0$ and we know such $A$ must be euclidean invariant, we may as well restrict ourselves to such $A$ from the beginning. This proposition allows us to "factor out" the euclidean invariance of these equations. If $U \subset R^{3n} \times P$ is a subset which has the property that the union of all orbits of the euclidean group which pass through $U$ is all of $R^{3n} \times P$ itself, then any euclidean invariant vector or tensorfield on $R^{3n} \times P$ is completely determined by its restriction to $U$. In particular, if it is zero on $U$, then it is zero everywhere. Thus, if $\mathcal{L}_{X_i} A = 0 = \mathcal{L}_{Y_{ij}} A$ then $D(A) = 0 \Rightarrow D(A)|_U = 0$. If $U$ is an open set, there is not much advantage in this. However, if $U$ is of lower dimension
this allows us to reduce the number of variables in our equations. The procedure is to start with a vectorfield defined over \( U \), extend it to a euclidean invariant vectorfield on \( \mathbb{R}^{3n} \times \mathbb{P} \) (if possible), apply \( D \) to the vectorfield and restrict the result to \( D \). In practice, what we will do is use euclidean invariance to solve for the derivatives "normal" to \( U \) in terms of derivatives along \( U \) and values of the vectorfields themselves.

**Proof of Proposition.** Comparing with equations 3.3.0, ..., 3.k-l we find that \([X_i, Y_{o-j}] = \delta_{i j} X_0\) independent of whether equations 4.3.k are satisfied or not, as long as \([X_i, X_0] = 0\) (hence \([X_i, A] = 0\)). Thus we find

\[
\begin{align*}
\mathcal{L}_{X_i} D(A) &= \sum_{j, \ell, r} \left( [X_i, Y_{o-j}] f_{j \ell}^r + [Y_{o-j}, X_0] X_1(f_{j \ell}^r) \right) \\
&\quad - X_1(f_{j p}^r) \cdots \cdots (k-1) X_1(p_{j p}^r) \delta a_{j}^{r k-l} \circ \delta a_{\ell}^{r k-l} \\
&\quad + \sum_{j, r, \ell} \left( [Y_{o-j}, X_0] f_{j \ell}^r - f_{j p}^r \cdots \cdots (k-1) p_{j p}^r \right) \\
&\quad \left( [X_i, \delta a_{j}^{r k-l}] \circ \delta a_{\ell}^{l k-l} \right) \\
&\quad + \delta a_{j}^{r k-l} \circ [X_i, \delta a_{\ell}^{r k-l}].
\end{align*}
\]

But, since \([X_i, X_0] = 0\) by assumption, we have

\([X_i, \delta a_{j}^{r q}] = 0\) for \( q = 2, \ldots, k-1\)

and also

\[X_i(a_{j}^{r q}) = X_i(p_{j}^r) = X_i(f_{j}^r) = 0.\]
We therefore find

\[ \mathcal{U} \mathcal{X}_1 \mathcal{D}(A) = \sum_{j, r, \ell} \delta_{ij} x_0 - \delta_{ij} x_r \alpha^r_{\ell} \partial_\alpha_{\ell} \otimes \partial_\alpha_{\ell} = 0. \]

This gives the translation part of the proposition.

Similarly, direct computation shows (compare with 3.5) that if \([Y_{ij}, A] = 0\) (\(Y_{ij}\) defined by 4.5.2) then

\[ [Y_{ij}, Y_{0\ell}] = \delta_{jl} Y_{0i} - \delta_{il} Y_{0j} \]

independent of whether 4.3.k is satisfied or not. It is also easily verified that if

\[ [Y_{ij}, A] = 0, \] then

\[ Y_{ij}(f^r_{\ell}) = \delta_{jl} f^r_{iu} - \delta_{il} f^r_{uj}, \] and conversely.

Thus we find that if \( Y_{ij} ^2 (A) = 0 \) then

\[ \mathcal{U} \mathcal{Y}_{ij} \mathcal{D}(A) = \sum_{i, m, r} \left\{ [Y_{0\ell} - x_i^r X_0] f^r_{lm} f^r_{im} - (k-l) p^r_{lm} \right\} \partial_\alpha_{\ell} \otimes \partial_\alpha_{m} \]

\[ = \sum_{\ell, m, r} \left\{ \delta_{jl} \left( [Y_{0i} - x_i^r X_0] f^r_{lm} f^r_{im} - (k-l) p^r_{lm} \right) \right. \]

\[ - \delta_{il} \left( [Y_{0j} - x_j^r X_0] f^r_{lm} f^r_{im} - (k-l) p^r_{lm} \right) \]

\[ + \delta_{jm} \left( [Y_{0l} - x_l^r X_0] f^r_{im} f^r_{lm} - (k-l) p^r_{im} \right) \]

\[ - \delta_{im} \left( [Y_{0l} - x_l^r X_0] f^r_{jm} f^r_{jl} - (k-l) p^r_{jm} \right) \]

\[ x \partial_\alpha_{\ell} \otimes \partial_\alpha_{m} \]

\[ + \sum_{\ell, m, r} \left\{ [Y_{0\ell} - x_i^r X_0] f^r_{lm} f^r_{im} - (k-l) p^r_{lm} \right\} \]

\[ \times \left\{ (\delta_{jl} \partial_\alpha_{i} - \delta_{il} \partial_\alpha_{j}) \otimes \partial_\alpha_{m} \right. \]

\[ + \partial_\alpha_{\ell} \otimes (\delta_{jm} \partial_\alpha_{i} - \delta_{im} \partial_\alpha_{j}) \}

\[ = 0. \]

This completes the proof.
CHAPTER II

§1. We will begin this chapter by considering the simplest case of these equations we have derived for generators of k-th order R.D.L.'s. This is the one particle case. Since there is only one particle involved, we may drop the superscript "r" in all the equations. With the notation of 1.4, the relevant equations become, when $k > 1$

1.1 $$(Y_{oi}-x_iX_o)f_j = f_i f_j + ka_i^{k-1}a_j 2 + \ldots + kp_i f_j$$

1.2 $X_i(f_j) = 0$

and if $k = 1$

1.3 $$(Y_{oi}-x_iX_o)f_j = f_i f_j - \delta_{ij}$$

1.4 $$(Y_{oi}-x_iX_o)X_o(f_j) = X_o(f_i)f_j + 2f_i X_o(f_j)$$

The most important thing about the one particle equations is that the operator $(Y_{oi}-x_iX_o)$ is completely independent of the $f_j$'s and these (except 1.4) equations are therefore linear first order partial differential equations.

Since we have $Y_{oi}(x_j) = x_i X_o(x_j)$, we find that in the first order case ($k = 1$) the operator $(Y_{oi}-x_iX_o)$, which is determined here by its action on the coordinate functions $x_j$, must be identically zero. We may then state

**Proposition.** There are no one particle, first order R.D.L.'s.

**Proof.** If there were, then 1.3 would be satisfied. However,
\((Y_{oi} - x_i x_o) = 0\) here so that 1.3 reads

\[0 = f_i f_j - \delta_{ij}.
\]

Taking \(i = j\) we have \((f_i)^2 = 1\) but taking \(i \neq j\) we have \(f_i f_j = 0\). Squaring this equation, we have \((f_i)^2(f_j)^2 = 0 = 1 \cdot 1 = 1\) a contradiction.

The next case to be considered is that of a one particle second order R.D.L.; the classical case. Computing the operator \(Y_{oi} - x_i x_o\) from definition we find that it may be written

\[D_i = Y_{oi} - x_i x_o = \sum_j (p_i p_j - \delta_{ij}) \phi_j.
\]

In particular, \(D_i(x_j) = 0\), \(D_i(p_j) = p_i p_j - \delta_{ij}\) and \([D_i, D_j] x_\ell = 0\) and also

\[\begin{align*}
[D_i, D_j] p_\ell &= D_i(p_j p_\ell - \delta_{j\ell}) - D_j(p_i p_\ell - \delta_{i\ell}) \\
&= p_\ell(p_i p_j - \delta_{ij}) + p_j(p_i p_\ell - \delta_{i\ell}) \\
&\quad - p_\ell(p_j p_i - \delta_{ji}) - p_i(p_j p_\ell - \delta_{j\ell}) \\
&= p_j D_i(p_\ell) - p_i D_j(p_\ell).
\end{align*}
\]

Thus \([D_i, D_j] = p_j D_i - p_i D_j\).

**Proposition.** There is one and only one one particle, second order R.D.L.

**Proof.** Equation 1.1 may be written
1.5 \( D_1 f_j = f_1 p_j + 2p_1 f_j \).

Using 1.5 and the values of \( D_1(p_j) \) computed above, we find

1.6 \[ [D_1, D_j] f_\ell = -\delta_{i\ell} f_j + \delta_{j\ell} f_i. \]

On the other hand, we have found that the operator \([D_1, D_j]\) may be expressed as a linear combination of \( D_i \) and \( D_j \),

\[ [D_1, D_j] = p_j D_1 - p_1 D_j. \]

Consequently, we must also have

1.7 \[ [D_1, D_j] f_\ell = (p_j D_1 - p_1 D_j) f_\ell \]

\[ = p_j (p_\ell f_i + 2p_i f_\ell) - p_1 (p_\ell f_j + 2p_j f_\ell) \]

\[ = p_j p_\ell f_i - p_1 p_\ell f_j. \]

Combining equations 1.6 and 1.7 we obtain

1.8 \[ (p_j p_\ell - \delta_{j\ell}) f_i - (p_1 p_\ell - \delta_{i\ell}) f_j = 0. \]

We can write the \( i = \ell \) component of this system as the matrix equation

\[
\begin{pmatrix}
1-(p_2)^2 & p_1 p_2 & 0 \\
0 & 1-(p_3)^2 & p_2 p_3 \\
p_1 p_3 & 0 & 1-(p_1)^2
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2 \\
f_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

In order to have a non-zero solution to this system, we must have \( \det( ) = 0 \) which is the same as

1.9 \[ (1-(p_1)^2-(p_2)^2-(p_3)^2) + (p_1 p_2)^2 + (p_1 p_3)^2 + (p_2 p_3)^2 = 0. \]
Since we want speed less than 1, the left hand side is greater than zero. Thus \( f_1 = f_2 = f_3 = 0 \). Moreover \( f_1 = f_2 = f_3 = 0 \) is clearly a solution, corresponding to straight line motion of our particle. This completes the proof.

There are many one particle third, or higher, order solutions. We will now construct a quite general class of third order solutions. The higher order construction is analogous but involves more variables.

The third order equations may be written

\[
1.10 \quad \left\{ \sum_{\ell} (p_{i\ell} - \delta_{i\ell}) \partial p_{\ell} + (a_{i\ell} + 2p_{i\ell}) \partial a_{\ell} \right\} f_j = f_1 p_j + 3a_i a_j + 3p_i f_j = D_i f_j.
\]

The vectorfields \( D_i \) generate an action of the Lorentz group on the phase space. The domain of our functions \( f_j \) must be invariant under this action as well as that induced by the time translation generator \( X_o = \sum_j p_j \partial x_j + a_j \partial p_j + f_j \partial a_j \) and the euclidean group.

**Proposition.** Let \( \Gamma^1, \Gamma^2, \Gamma^3 \) be smooth functions defined on the closed ball of radius \( K > 0 \) in \( \mathbb{R}^3 \) such that

\[
\mathcal{Y}_{ij}(\Gamma_k) = \delta_{jk} \Gamma^i - \delta_{ik} \Gamma^j,
\]

where \( \mathcal{Y}_{ij} \)'s are the standard generators of the action of the rotation group on \( \mathbb{R}^3 \).

Finally, assume that the \( \Gamma \)'s vanish on the boundary of the ball. Then there exists a domain \( U \) in the third order, one particle phase space which is invariant under the action of the euclidean group and the action of the Lorentz group.
generated by the $D_{\perp}$'s and functions $f_1, f_2, f_3$ on $U$ so that the $f$'s are translation invariant and satisfy 1.10. Moreover, $U$ is invariant under $X_0$,

$$U \cap \{p_1 = p_2 = p_3 = x_1 = x_2 = x_3 = 0\} = U \cap V \cap \{a_1^2 + a_2^2 + a_3^2 < K\} \cap V$$

and

$$f_i \mid_{U \cap V} = f_i.$$

**Proof.** First note that the set $V = \{p_2 = p_3 = x_1 = x_2 = x_3 = 0, -1 < p < 0\}$ is of the type mentioned in the proposition on euclidean invariance. Namely, the union of the orbits of the euclidean group passing through $V$ is the whole phase space. Thus, if we can find $f$'s which satisfy the "pullback" of 5.10 on $V$ and which can be extended by euclidean invariance, then the extended functions will satisfy 1.10 everywhere.

In this case, however, $D_{\perp}$ is tangent to $V$. Moreover, since $[Y_{i, j}, D_{\perp}] = \delta_{j i} D_i - \delta_{i j} D_j$ and $[X_i, D_j] = 0$ the pullback system is the equation

$$1.11 \quad D_{i} f_{j} = f_{i} p_{j} + 3 a_{1 j} a_{i} + 3 p_{i} f_{j}$$

itself. In order for $U$ to satisfy the invariance requirements, $U \cap V$ must, at least, contain all the orbits of $D_{\perp}$ through $U \cap V$, and we will find that this will be enough to generate $U$.

Let us first find the orbits of $D_{\perp}$ in $V$. If we let
\[ p_1(t) = -\tanh(t) \quad (t \geq 0) \] then, taking everything on \( V \) unless otherwise stated, we have

1.12 i) \[ p_1(t) = -\tanh(t) \quad p_2(t) = p_2(0) = 0 \quad p_3(t) = p_3(0) = 0 \]

ii) \[ a_1(t) = a_1(0) \cosh^{-3}(t) \]
\[ a_2(t) = a_2(0) \cosh^{-2}(t) \]
\[ a_3(t) = a_3(0) \cosh^{-2}(t) \]

as a complete orbit set of \( D_1 \) with initial conditions
\[ p_1(0) = 0 = p_2(0) = p_3(0), \] others as stated.

We now take \( U \cap V \) to be the union of all the orbits of \( D_1 \) through \( U \cap V \), namely the subset of \( V \) given by

1.13 \[
(\left[1-p_1p_1\right]^{-3/2}a_1)^2 + (\left[1-p_1p_1\right]^{-1}a_2)^2 + (\left[1-p_1p_1\right]^{-1}a_3)^2 < K.
\]

We can now solve equation 1.11 in parametrized form in a quite straightforward manner.

1.14 \[ f_1 = \cosh^{-4}(t)[3(a_1(0))^2 \tanh(t) + \Phi_1(a_1(0), a_2(0), a_3(0))] \]
\[ f_2 = \cosh^{-3}(t)[3a_1(0)a_2(0) \tanh(t) + \Phi_2(a_1(0), a_2(0), a_3(0))] \]
\[ f_3 = \cosh^{-3}(t)[3a_1(0)a_2(0) \tanh(t) + \Phi_3(a_1(0), a_2(0), a_3(0))] \]

In particular the \( f_j = \Phi_j \) on \( U \cap V \) and \( f_j \to 0 \) along every orbit of \( D_1 \).
If we set $G(x, p, a)$ to be the euclidean invariant extension of the function given by the left hand side of 1.13, the $U$ defined by $G < K$ and, of course, $p^2_1 + p^2_2 + p^2_3 < 1$, clearly satisfies all the invariance requirements, and the euclidean invariant extension of $\Sigma f_j a_j$ is then defined on $U$. (The possibility of such an extension is assured by the symmetry requirements of our initial functions $F_j$).

All we have left to check is that the resulting vector-field defines a global flow on our domain, i.e., $U$ is invariant under $X_0$. There are three things to check under this heading. First, that the spacial coordinates of an integral curve of $X_0$ don't run off to infinity in finite time, which is clear since $(X_0(x_1))^2 + (X_0(x_2))^2 + (X_0(x_3))^2 < 1$. Second, we must check that $X_0$ is tangent to the boundary of $U$ defined by $G = K$. Since $F_j$ were taken to be defined and smooth on $a_1(0)^2 + a_2(0)^2 + a_3(0)^2 = K$, 1.14 actually defines smooth solutions up to and including $G = K$. We must only check that $X_0(G)|_{G=K} = 0$ to be assured of this tangency. Since both $X_0$ and $G$ are euclidean invariant, it suffices to check this on $U \cap \overline{V}$. There, taking into account the euclidean invariance of $G$, the pullback equation reads

$$1.15 \quad [a_1 \delta p_1 + {\frac{a_2}{p_1}}(a_2 \delta a_1 - a_1 \delta a_2) + {\frac{a_3}{p_1}}(a_3 \delta a_1 - a_1 \delta a_3)] + f_1 \delta a_1 + f_2 \delta a_2 + f_3 \delta a_3][[1-p_1 p_1]^{-3} a_1^2 + [1-p_1 p_1]^{-2} a_2^2 + [1-p_1 p_1]^{-2} a_3^2] = 0$$
when 

\[(1-p_1p_1)^{-3}a_1^2 + [1-p_1p_1]^{-2}(a_2^2+a_3^2) = K \text{ and } p_1 \neq 0.\]

When \( p_1 = 0 \), we have \(-D_1 = \partial p_1\), etc., and the equation may be written

\[1.16 \quad [-a_1D_1-a_2D_2-a_3D_3+F_1\partial a_1+F_2\partial a_2+F_3\partial a_3]G = 0\]

when \( G = K \) and \( 0 = p_1 (= p_2 = p_3) \).

Taking 1.16 first, \( G \) was constructed so that \( D_jG = 0 \) always, on the other hand \( F_1,F_2,F_3 \) were required to vanish on \( G = K \), thus 1.16 is satisfied.

Converting 1.14 to unparametrized form we have

\[1.17 \quad f_1 = -3p_1a_1^2(1-p_1p_1)^{-1} + (1-p_1p_1)^2F_1([1-p_1p_1]^{-3/2})a_1,\]

\[\quad [1-p_1p_1]^{-1/2}a_2, [1-p_1p_1]^{-1/2}a_3\]

\[f_2 = -3p_1a_1a_2(1-p_1p_1)^{-1} + (1-p_1p_1)^3/2F_2([1-p_1p_1]^{-3/2})a_1,\]

\[\quad [1-p_1p_1]^{-1/2}a_2, [1-p_1p_1]^{-1/2}a_3\]

\[f_3 = -3p_1a_1a_3(1-p_1p_1)^{-1} + (1-p_1p_1)^3/2F_3([1-p_1p_1]^{-3/2})a_1,\]

\[\quad [1-p_1p_1]^{-1/2}a_2, [1-p_1p_1]^{-1/2}a_3.\]

Plugging these into 1.15, we find that everything but the contributions of the \( F_j \)'s cancel identically and then the \( F_j \)'s themselves vanish on \( G = K \) by assumption.

Finally, we must check that the integral curves of \( X_0 \) do not reach \( p_1^2 + p_2^2 + p_3^2 = 1 \) in finite time. On the other hand

\[(X_0(p_1))^2 + (X_0(p_2))^2 + (X_0(p_3))^2 < (1-p_1p_1-p_2p_2-p_3p_3)^2K.\]
Thus the speed goes up like \( \tanh(t) \), in particular, never reaches \( 1 \) in finite time. This completes the proof.

We might note, at this point that the Dirac solutions of the one body electrodynamics problem (1) are of this type. This is the best evidence that the "true" \( n \)-body electrodynamics is at least third order. This procedure may be mimicked for any higher order dynamics we wish, giving us a large, essentially exhaustive class of solutions for the single particle dynamics.

§2. Let us now consider two and three particle, first order dynamics. For first order dynamics, we may begin by considering equation 4.3.1 which reads

\[
(Y_{0i} - x^r_{1i} X_0) f^r_j = f^r_i f^r_j - s^r_{ij}
\]

In this case \( X_0 = \sum_{s,j} f^s_j d x^r_j \) and \( Y_{0i} = \sum_{s,j} x^s_i f^s_j d x^s_j \) reflecting the fact that we are dealing with a first order dynamics. We thus have

\[
(Y_{0i} - x^r_{1i} X_0) = \sum_{s,j} (x^s_i - x^r_i) f^r_j d x^r_j = D^r_i.
\]

**Proposition.** There are no two particle, first order dynamics.

**Proof.** First note that on the submanifold given by \( x^1_2 = x^1_3 = x^2_2 = x^2_3 = 0 \), the operators \( D^r_2 \) and \( D^r_3 \) vanish and are thus tangent to this submanifold. If we had \( f^r_j \)'s which defined a first order R.D.L. they would have to satisfy
2.3 \[ 0 = f_2^r \delta_{2j} - f_2^r \delta_{2j} \] and \[ 0 = f_3^r \delta_{3j}. \]

On this submanifold. Taking \( j = 2 \) and \( j = 3 \) in these equation we find \( f_2^r \delta_2 = 1 = f_3^r \delta_3 \) but \( f_2^r \delta_3 = 0 \), an impossible situation. Moreover, this submanifold generates the entire space under the action of the euclidean group so that not even local solutions can exist.

**Proposition.** There are no (subluminal) three particle, first order R.D.L.

**Proof.** We again consider equations 2.2. This time we restrict to the submanifold \( x_1^1 = x_2^2 = x_3^3 = 0 \). Again, \( D_2^r \) vanishes there and the \( f_2^r \) must satisfy

\[ 2.4 \quad 0 = f_2^r \delta_{2j}. \]

Hence \( (f_2^r)^2 = 1 \) and \( f_2^r = f_1^r = 0 \). However, since we want speeds less than 1, this rules this out as a possible solution, completing the proof.

It is interesting to note that if we are willing to include speed = 1 solutions, then this actually gives us one. Namely, if we exclude from \( R^3 \times R^3 \times R^3 \) the various diagonals, then we can give the prescription that three particles travel at the speed of light in the direction (euclidean) normal to the plane they span with the sense given by the right hand rule applied to their ordering. This defines a three particle, first order R.D.L.
§3. The second order equations.

The second order case is the most important, physically, for in most physical systems, second order is assumed, and Hamiltonian (hence quantum) mechanics is intrinsically second order.

We will first construct a class of solutions which, in a sense, skirts the non-linearity of the equations. With our previous notation, the relevant equations are

\[ 3.1 \quad [\Sigma \left( x_i^s - x_i^r \right) (p_i^s \delta x_i^l + p_i^r \delta p_i^l) + \left( p_i^s p_i^l - \delta_{i,l} \right) \delta p_i^r] f_j^r = f_i^r p_j^r + 2 p_i^r f_j^r. \]

\[ 3.2 \quad x_i(f_j^r) = 0 = [\Sigma \delta x_i^r] f_j^r. \]

For the purpose of our construction, we will define the linearized operators \( F_i^r \).

\[ 3.3 \quad F_i^r = \Sigma (x_i^s - x_i^r) p_i^s \delta x_i^l + (p_i^s p_i^l - \delta_{i,l}) \delta p_i^l. \]

**Proposition.** Suppose we have \( 3n \) smooth functions \( f_j^r \) satisfying the linear equations \( F_i^r(f_j^r) = f_i^r p_j^r + 2 p_i^r f_j^r \) and \( x_i(f_j^r) = 0. \) Suppose, moreover, that we have \( n \) smooth functions \( g^r \) satisfying \( g^r g^s = 0 \) for \( r \neq s \) and \( F_i^r(g^r) = 0 \) and \( x_i(g^r) = 0. \) Then the functions \( f_j^r : g^r f_j^r \) will satisfy equations 3.1 and 3.2.

**Proof.** With our assumptions 3.2 reads
3.4 \( x'_1 f'_j = x'_1 (g^r y^r) = x'_1 (g^S y^r) + g^S x'_1 (f^r) = 0. \)

Thus 3.2 is satisfied.

On the other hand 3.1 reads

3.5 \[
\begin{align*}
[ \sum_{s, t} (x'_1 - x'_1) (p^s_{l'} \delta p_{l'} + f_{l'} \delta p_{l'}) + (p^s_{l'} \delta p_{l'}) & \delta p_{l'} ] f^r_j \\
= F^r_1 (f^r_j) + [ \sum_{s, t} (x'_1 - x'_1) f_{l'} \delta p_{l'} ] f^r_j & = F^r_1 (g^r y^r) \\
+ [ \sum_{s, t} (x'_1 - x'_1) g^s f_{l'} \delta p_{l'} ] (g^r y^r) & = g^r (f^r_1 p^r_j + 2 p^r y^r + [ \sum_{s, t} (x'_1 - x'_1) g^s f_{l'} \delta p_{l'} ] (g^r y^r) \\
= f^r_1 p^r_j + 2 p^r y^r & + [ \sum_{s, t} (x'_1 - x'_1) g^s f_{l'} \delta p_{l'} ] (g^r y^r). \\
\end{align*}
\]

Thus, as long as the last term is zero, 3.1 will be satisfied.

However, since we have assumed \( g^s = 0 \) for \( s \neq r \)

3.6 \[
[ \sum_{s, t} (x'_1 - x'_1) g^s f_{l'} \delta p_{l'} ] (g^r y^r) \\
= [ \sum_{s, t} (x'_1 - x'_1) g^s f_{l'} \delta p_{l'} ] (g^r y^r) \\
+ T^r_j [ \sum_{s, t} (x'_1 - x'_1) g^s f_{l'} \delta p_{l'} ] g^r.
\]

Now at any point we have one of the following situations

i) \( g^s = 0 \) all \( s \), in which case the last term of 3.6 also vanishes.

ii) \( g^r \neq 0 \), in which case \( g^s = 0 \) for \( s \neq r \) and again the last term is zero.
iii) $g^s \neq 0$ some $s \neq r$, in which case $g^s \neq 0$ in a neighborhood of the point in question. This would then imply that $g^r = 0$ in that same neighborhood, thus all derivatives of $g^r$ would vanish at the point in question and again the last term vanishes.

This concludes the proof.

Let us consider a procedure for constructing solutions of this type. We will take $n = 3$ for specificity, although this type of construction works for any number of particles but it will differ in details.

To find the $\tilde{\tau}_j^r$'s, we first note that, like the non-linear equations, the linearized equations

$$3.7 \quad \tilde{X}_i^r \tilde{\tau}_j^r = \tilde{X}_i^r (\tilde{p}_j^r + 2\tilde{p}_i^r)$$

have the Euclidean covariance property outlined in Proposition of Section I. Moreover, if $\tilde{\tau}_j^r$ satisfy $3.7$ and $X_i(\tilde{\tau}_j^r) = 0$ then they must also be Euclidean covariant, i.e.

$$3.8 \quad \tilde{Y}_{ij} \tilde{\tau}_j^r = \delta_{ij} \tilde{\tau}_i^r - \delta_{ij} \tilde{\tau}_j^r.$$

These can be derived from the corresponding statements for the nonlinear equation by setting $g^r = 1$ and $g^s = 0$ for $s \neq r$ and applying the above proposition.

Consider the submanifold given by

$$3.9: \quad x_1^1 = x_2^1 = x_3^1 = x_2^2 = x_3^2 = x_3^3 = 0.$$
Certainly, the union of the orbits of the Euclidean group through this submanifold is all of the three particle, second order space. We will then pull back 3.7 to this submanifold, solve there and extend by Euclidean covariance.

The only derivatives in the $f_i$'s which are not tangential to this submanifold are, of course, $\partial x^1_1, \partial x^1_2, \partial x^1_3, \partial x^2_2, \partial x^2_3, \partial x^3_3$. On this submanifold, we can solve for these in terms of the $Y_{ij}, X_i$ and tangential derivatives.

On the "transverse strata" given by $x^3_2 > 0, x^2_1 > 0$ (we will consider the lower strata separately since there are subgroups of the Euclidean group leaving them invariant).

We can write

\[ \partial x^3_2 j = \frac{1}{x^2_2} (Y_{2j} - \Sigma_s p^s_2 \delta p^s_3 - p^s_3 \delta p^s_2) Y^r_j \]
\[ = \frac{1}{x^2_2} (\delta^r_j - \delta^r_2 - \delta^r_3 - \Sigma_s (p^s_2 \delta p^s_3 - p^s_3 \delta p^s_2) Y^r_j) \]

\[ \partial x^2_3 j = \frac{1}{x^1_1} (Y_{13} - x^3_1 \partial x^3_2 - \Sigma_s p^s_1 \delta p^s_3 - p^s_3 \delta p^s_1) Y^r_j \]
\[ = \frac{1}{x^1_1} (\delta^r_j - \delta^r_1 - \delta^r_3 - \Sigma_s (p^s_2 \delta p^s_3 - p^s_3 \delta p^s_2) Y^r_j) \]
\[ - \Sigma_s (p^3_1 \delta p^s_3 - p^s_3 \delta p^s_1) Y^r_j) \]

\[ \partial x^1_3 j = (x_3 - \partial x^2_3 - \partial x^3_3) Y^r_j = -(\partial x^2_3 + \partial x^3_3) Y^r_j \]
3.10.4 \[ \partial x^2_{2r} = \frac{1}{x^1_2}(y^1_{12} - x^3_1 \partial x^3_{2r} - \Sigma \quad p^s_1 \partial p^s_2 - p^s_2 \partial p^s_1) x^2_{1r} \]

= \frac{1}{x^1_2} \left( \delta_{2j} x^2_{1r} - \delta_{1j} x^2_{2r} - x^3_1 \partial x^3_{2r} x^2_{1j} + x^3_2 \partial x^3_{1r} x^2_{2j} \right) - \Sigma \left( p^s_1 \partial p^s_2 - p^s_2 \partial p^s_1 \right) x^2_{1r} \]

3.10.5 \[ \partial x^{1r} = (x^2_2 - \partial x^2_2 - \partial x^3_2) x^2_{1r} = - (\partial x^2_2 + \partial x^3_2) x^2_{1r} \]

3.10.6 \[ \partial x^{1r} = (x^1_2 - \partial x^2_1 - \partial x^3_1) x^2_{1r} = - (\partial x^2_1 + \partial x^3_1) x^2_{1r} \]

Restricted to the manifold, equation 3.7 becomes

3.11.1 \[ \left[ (x^2_1 \Sigma p^2_\ell \partial x^3_\ell + x^3_1 \Sigma p^2_\ell \partial x^3_\ell + \Sigma \left( p^s_1 p^s_\ell - \delta_{1\ell} \right) \partial p^s_\ell x^1_1 \right] \]

= \frac{1}{1_1} p^1_1 + 2p^1_2. \]

3.11.2 \[ \left[ x^3_2 \Sigma p^3_\ell \partial x^3_\ell + \Sigma \left( p^s_2 p^s_\ell - \delta_{2\ell} \right) \partial p^s_\ell x^2_2 \right] \]

= \frac{1}{2_2} p^2_1 + 2p^1_2. \]

3.11.3 \[ \left[ \Sigma \left( p^s_3 p^s_\ell - \delta_{3\ell} \right) \partial p^s_\ell x^1_3 \right] = \frac{1}{2_3} p^1_1 + 2p^1_2. \]

3.12.1 \[ \left[ -x^2_1 \Sigma p^1_\ell \partial x^1_\ell + (x^3_1 - x^2_1) \Sigma p^3_\ell \partial x^3_\ell + \Sigma \left( p^s_1 p^s_\ell - \delta_{1\ell} \right) \partial p^s_\ell x^2_2 \right] \]

= \frac{1}{1_1} p^2_1 + 2p^1_2.
3.12.2 \[ \sum_{s, l} \left( p_{2s}^s p_{2l}^s - \delta_{2l}^s \right) \delta p_{l}^s \gamma_j^2 \]
\[ = \frac{\gamma_j^2}{2} + 2p_{2j}^3. \]

3.12.3 \[ \sum_{s, l} \left( p_{2s}^s p_{2l}^s - \delta_{2l}^s \right) \delta p_{l}^s \gamma_j^2 \]
\[ = \frac{\gamma_j^2}{2} + 2p_{2j}^3. \]

3.13.1 \[ -x_1^2 \sum_{l} p_{l}^1 x_1^l + \left( x_1^2 - x_1^3 \right) \sum_{l} p_{l}^2 x_2^l + \sum_{s, l} \left( p_{2s}^s p_{2l}^s - \delta_{2l}^s \right) \delta p_{l}^s \gamma_j^3 \]
\[ = p_{j}^3 + 2p_{2j}^3. \]

3.13.2 \[ -x_2^3 \sum_{l} p_{l}^1 x_1^l - x_2^2 \sum_{l} p_{l}^2 x_2^l + \sum_{s, l} \left( p_{2s}^s p_{2l}^s - \delta_{2l}^s \right) \delta p_{l}^s \gamma_j^3 \]
\[ = \frac{\gamma_j^3}{2} + 2p_{2j}^3. \]

3.13.3 \[ \sum_{s, l} \left( p_{2s}^s p_{2l}^s - \delta_{2l}^s \right) \delta p_{l}^s \gamma_j^3 \]
\[ = \frac{\gamma_j^3}{2} + 2p_{2j}^3. \]

Note that equation 3.11.3, 3.12.3, 3.13.3 all involve the same operator \( L_3 := \sum_{s, l} \left( p_{2s}^s p_{2l}^s - \delta_{2l}^s \right) \delta p_{l}^s. \) Also note that this operator is tangent to the submanifold we are considering. We will show that \( L_3 \) is an additional "symmetry" of the system. In particular, if \( \gamma^r_j \) satisfy \( L_3 \gamma^r_j = \chi^r_j \gamma^r_j + 2p_{2j}^3 \gamma^r_j \) (equation 3.11.3, 3.12.3, 3.13.3) then the functions given by the tangential versions of the other equations transform under \( L_3 \) in such a way that if they vanish at a point, then they vanish along the entire orbit of \( L_3 \) through the point.
This will allow us to factor out one more dimension.

To make the calculation easier, let us adopt the notation

3.14.1 \( \psi_{13} := \sum_s p_s^3 p_3^s - p_3^3 p_2^s \)

3.14.2 \( \psi_{13} := \sum_s p_1^s p_3^s - p_3^s p_1^s \)

3.14.3 \( \psi_{12} := x_1^3 x_2^3 - x_2^3 x_1^3 + \sum_s p_1^s p_2^s - p_2^s p_1^s \).

Each of these is tangent to the submanifold. With this notation, equation 3.11, 3.12, 3.13 may be written on the transverse strata as

3.15.1 \[ \{ x_1^2[p_1^2 p_2^2 \frac{1}{x_1^2}(x_{12} - \psi_{12}) + p_3^2 \frac{1}{x_1^2}(x_{13} - \frac{x_1^3}{x_2^3}(y_{23} - \psi_{23}) - \psi_{13})] \]

\[ = x_1^3[p_1^3 x_1^3 + p_2^3 x_2^3 + p_3^3 \frac{1}{x_2^3}(y_{23} - \psi_{23})] \]

\[ + \sum_{s, t} (p_{1t} p_{s}^t - \delta_{1t} \delta_{st}) p_{st}^1 \psi_{1j} - \frac{1}{2} p_{1j} - 2 p_{1}^1 \psi_{1j} = 0 \]

3.15.2 \[ \{ x_2^3[p_1^3 x_1^3 + p_2^3 x_2^3 + p_3^3 \frac{1}{x_2^3}(y_{23} - \psi_{23})] \]

\[ + \sum_{s, t} (p_{2t} p_{s}^t - \delta_{2t} \delta_{st}) p_{st}^1 \psi_{1j} - \frac{1}{2} p_{2j} - 2 p_{2}^1 \psi_{1j} = 0. \]

3.16.1 \[ \{ -x_1^2[-p_1^{1}(\delta x_1^2 + \delta x_1^3)] - p_2^1(\frac{1}{x_1^2}(y_{12} - \psi_{12}) + \delta x_2^3) \]

\[ - p_3^1(\frac{1}{x_1^2}(y_{13} - \frac{x_1^3}{x_2^3}(y_{23} - \psi_{23}) - \psi_{13}) + \frac{1}{x_2^3}(y_{23} - \psi_{23})) \]
\[ + (x_1^2 - x_2^2)[p_1^3 \delta x_1^3 + p_2^3 \delta x_2^3 + p_3^3 \frac{1}{x_2^3}(y_{23} - y_{23})] \\
+ \sum_{s, \ell} (p_1^s p_2^s \delta s - \delta_{s, \ell}) \delta p_4^s \delta x_4^2 \eta_{12} j - \eta_{22} p_j - 2p_2^2 \eta_{12} j = 0. \]

3.16.2 \[ x_2^3[p_1^3 \delta x_1^3 + p_2^3 \delta x_2^3 + p_3^3 \frac{1}{x_2^3}(y_{23} - y_{23})] \\
+ \sum_{s, \ell} (p_2^s p_2^s \delta s - \delta_{s, \ell}) \delta p_4^s \eta_{22}^2 j - \eta_{22}^2 p_j - 2p_2^2 \eta_{12}^2 j = 0. \]

3.17.1 \[ -x_1^3[-p_1^1(\delta x_1^2 + \delta x_1^2) - p_2^2(\frac{1}{x_1^2} y_{12} - y_{12}) + \delta x_2^3] \\
- p_3^3(\frac{1}{x_2^3}(y_{13} - \frac{x_1^3}{x_2^3}(y_{23} - y_{23}) - y_{23}) + \frac{1}{x_2^3}(y_{23} - y_{23})) \]
\[ + (x_1^2 - x_2^2)[p_1^2 \delta x_1^2 + p_2^2 \frac{1}{x_1^2}(y_{12} - y_{12}) + p_3^2 \frac{1}{x_1^2}(y_{13} - \frac{x_1^3}{x_2^3} (y_{23} - y_{23} - y_{13})) \]
\[ + \sum_{s, \ell} (p_1^s p_2^s \delta s - \delta_{s, \ell}) \delta p_4^s \eta_{12}^3 j - \eta_{22}^3 p_j - 2p_2^3 \eta_{12}^3 j = 0. \]

3.17.2 \[ -x_2^3[-p_1^1(\delta x_1^2 + \delta x_1^2) - p_2^2(\frac{1}{x_2^2} y_{12} - y_{12}) + \delta x_2^3] \\
- p_3^3(\frac{1}{x_2^3}(y_{13} - \frac{x_1^3}{x_2^3}(y_{23} - y_{23}) - y_{13}) + \frac{1}{x_2^3}(y_{23} - y_{23})) \]
\[ + p_1^2 \delta x_1^2 + p_2^2 \frac{1}{x_1^2}(y_{12} - y_{12}) + p_3^2 \frac{1}{x_1^2}(y_{13} - \frac{x_1^3}{x_2^3} (y_{23} - y_{23} - y_{13})) \]
\[ + \sum_{s, \ell} (p_2^s p_2^s \delta s - \delta_{s, \ell}) \delta p_4^s \eta_{12}^3 j - \eta_{22}^3 p_j - 2p_2^3 \eta_{12}^3 j = 0. \]
It is straightforward to check that if we define functions $G_1, G_2, G_1^2, G_2^2, G_1^3, G_2^3$ by the left hand sides of 3.15.1, 3.15.2, 3.16.1, 3.16.2, 3.17.1, 3.17.2, where we have used $Y_{i j}^{r r} = \delta_{j i} Y_{i 1}^{r r} - \delta_{i j} Y_{j 1}^{r r}$, and if the $Y_{i j}^{r r}$ satisfy
$L_3^{r r}_{i j} = Y_{3 r}^{r r}_{i j} + 2p_3^{r r}_{i j}$, then

3.18.1 $L_3(G_2^1) = p_3^3 G_2^1$

3.18.2 $L_3(G_2^2) = p_3^3 G_2^2$.

In particular, since $L_3(p_3^3) \neq 0$ everywhere, we see that if $G_2^1 = 0$ at some point (i.e., 3.15.2 is satisfied), then $G_2^1 = 0$ along the whole orbit of $L_3$ through that point, and similarly for $G_2^2$.

We also find that if we let $G_1^2 = G_1 - \frac{x_1^3}{x_2} G_2$ and

$G_1^2 = G_1 - \frac{x_1^3 - x_1^2}{x_2} G_2$ then we have, with the above assumptions

3.19.1 $L_3(G_1^1) = p_3^2 G_1^1$

3.19.2 $L_3(G_1^2) = p_3^1 G_1^2$.

In particular, if $G_1^1 = 0$ at some point, then it is zero along the whole orbit of $L_3$ through that point ($L_3 p_3^2 \neq 0$, $L_3 p_3^1 \neq 0$ everywhere). Similarly for $G_1^2$.

Finally, if we define $G_1^3 = G_1 - \frac{x_1^3}{x_2} G_2$ and $G_2^3 = G_2 + \frac{x_1^2 - x_1^3}{x_2} G_2^3$ we find
3.20.1 \( L_3(q^3_1) = p^2q^3_2 \)

3.20.2 \( L_3(q^3_2) = p^3q^3_2 \).

We, therefore, have

**Proposition.** If \( L_3^{\mu r} = f^{\mu r}_j p^r_j + 2p^{\mu r}_j \) and equation 3.15.1, 3.15.2, 3.16.1, 3.16.2, 3.17.1, 3.17.2 are satisfied at a point, then they are satisfied on the whole orbit of \( L_3 \) through that point.

Since they have nicer covariance properties under \( L_3 \), we will use, instead of equation 3.15, 3.16, 3.17 the equivalent system

3.21.1 \( q^1_1 = 0 \)

3.21.2 \( q^1_2 = 0 \)

3.22.1 \( q^2_1 = 0 \)

3.22.2 \( q^2_2 = 0 \)

3.23.1 \( q^3_1 = 0 \)

3.23.2 \( q^3_2 = 0 \)

(assuming, \( L_3^{\mu r} = f^{\mu r}_j p^r_j + 2p^{\mu r}_j \)).

Because of the above proposition, we may pull back these equations to a subset of one lower dimension. We will consider in detail only the system 3.21.1 and 3.21.2, the others are similar. In fact, since the systems are completely decoupled
we could have chosen different sections for the Euclidean action on each one so that the resulting equations would look exactly like 21.1 and 21.2 for the other systems as well. This nice feature will not appear in the full nonlinear equations.

Writing out 3.21.1 and 3.21.2 in full, we have

\[
3.24.1 \quad [x_1^2 p_1^2 x_1^2 + p_2 x_2 x_1^3 + p_2 x_1 x_2^3 - p_2 x_1^2 x_2^2 + \frac{p_2}{p_1^2} + p_3 x_3 + p_1 p_1 - \frac{x_1^3}{x_2^3} \rho_1^3]
\]

\[
\rho_1^2
\]

\[
+ \left[ p_2^2 - p_3 p_1^2 + \frac{1}{x_2} \right] \rho_1^2
\]

\[
\rho_1^3
\]

\[
+ \left[ p_3^2 - p_1^2 + \frac{1}{x_2} \right] \rho_1^3
\]

\[
\rho_1^2
\]

\[
+ \left[ p_3^2 - p_1^2 + \frac{1}{x_2} \right] \rho_1^2
\]

\[
\rho_1^3
\]

\[
+ \left[ p_3^2 - p_1^2 + \frac{1}{x_2} \right] \rho_1^2
\]

\[
\rho_1^3
\]

\[
+ \left[ p_3^2 - p_1^2 + \frac{1}{x_2} \right] \rho_1^2
\]

\[
\rho_1^3
\]

\[
- \frac{x_1^2}{x_2^2} \rho_1^3
\]

\[
- \frac{x_1^3}{x_2^3} \rho_1^3
\]

\[
= 0.
\]
3.24.2 \[ x_{2}p_{1}^{3}x_{1}^{3} + x_{2}p_{2}^{3}x_{2} + p_{2}p_{1}^{2}p_{1}^{2} + \left[p_{3}^{2}p_{3}^{3}p_{2}^{2} - 1\right]p_{2}^{1} \\
+ \left[-p_{3}^{2}p_{2}^{1}p_{3}^{3}\right]p_{3}^{1} + p_{1}^{2}p_{1}^{2} + \left[p_{3}^{2}p_{3}^{2}p_{2}^{2} - 1\right]p_{2}^{2} \\
+ \left[-p_{3}^{2}p_{2}^{1}p_{3}^{3}\right]p_{3}^{2} + p_{3}^{3}p_{1}^{3} + \left[p_{3}^{2}p_{3}^{3}p_{2}^{3} - 1\right]p_{2}^{3} \delta_{j}^{1} \\
- \delta_{2}^{2}p_{j}^{1} - \delta_{2}^{1}p_{j}^{2} + p_{3}^{3} \left[\delta_{3}^{1} \delta_{j}^{1} - \delta_{3}^{1} \delta_{j}^{1}\right] = 0.\]

Notice that 3.24.1 is tangent to the submanifolds given by \( p_{3}^{2} = \) constant and 3.24.2 is tangent to the submanifolds \( p_{3}^{3} = \) constant. Also note that the orbits of \( L_{3} \) through either of these fills up the whole space. We can thus choose \( p_{3}^{3} = 0 \) as defining the reduced space.

On \( p_{3}^{3} = 0 \) we have

3.25 \[ L_{3} = -\delta_{3}^{3} + \sum_{j \neq 3} \left(p_{j}^{3}p_{j}^{3} - \delta_{3}^{3}\right)\delta_{j}^{3}.\]

On \( p_{3}^{3} = 0 \), 3.24.2 simplifies to

3.26 \[ x_{2}p_{1}^{3}x_{1}^{3} + x_{2}p_{2}^{3}x_{2} + p_{2}p_{1}^{1}p_{1}^{1} + \left(p_{2}^{1}p_{2}^{1} - 1\right)p_{2}^{1} \\
+ p_{2}^{1}p_{2}^{1}p_{2}^{1} + p_{2}^{1}p_{2}^{1} + \left(p_{2}^{1}p_{2}^{1} - 1\right)p_{2}^{2} + p_{2}^{2}p_{2}^{2}p_{2}^{2} \\
+ p_{2}^{2}p_{2}^{1} + \left(p_{2}^{3}p_{2}^{3} - 1\right)p_{2}^{3} \delta_{j}^{1} + \delta_{j}^{1} \delta_{2}^{1} + \delta_{j}^{1} - 2p_{2}^{1} \delta_{j}^{1} = 0.\]

On the other hand, writing \( L_{3} = \sum_{j \neq 3} \left(p_{j}^{3}p_{j}^{3} - \delta_{3}^{3}\right)\delta_{j}^{3} \) on \( p_{3}^{3} = 0 \), 3.24.1 becomes
If we define $F$ to be the differential operator in 3.26 we may write 3.26 as $F\tilde{f}_j - \delta_{2j}p_j - 2\alpha^{1\mu}_j = 0$. If we let $a_1$ be the function defined by the left hand side of 3.27 then if $F\tilde{a}_1 - \delta_{2j}p_j - 2\alpha^{1\mu}_j = 0$ we have

\[ a_1 = p_2a_1. \]
These equations 3.18.1, 3.18.2 and 3.28 are the complete analogues of \([D_i, D_j] = p_j D_i - p_i D_j\) shown in the first section of this chapter. (See Section 4 for general calculation of 3.28.)

Equation 3.28 allows us to factor out one further dimension. For example, the orbits of \(F\) through \(p_2^2 = 0\) fill out the whole space at hand. On \(p_2^2 = 0\), \(F\) may be written as

\[
3.29 \quad F = -\partial p_2^2 + \Sigma (p_2 p_1^2 - \delta_{2\ell}) \partial p_1^1 + p_2 p_1^3 \partial p_1^3 + (p_2 p_2^1 - 1) \partial p_2^3
\]

\[
+ x_2^2 p_1^3 \partial x_1^1 + x_2^3 p_2^3 \partial x_2^3.
\]

Thus, using 3.29 and \(F^j \partial_{p_j} = -(p_2 p_2^1 - 1) = 0\) we may pull back 3.27 to \(p_2^2 = 0\) and get the final equation.

\[
3.30 \quad \{x_1^2 p_1^2 \partial x_1^2 + [p_3 p_3^2 + p_1 p_1^1 - 1] x_2^3 \partial x_2^3
\]

\[
+ [-p_3^2 x_1^2 + p_1 p_2^1 - x_2^3 (p_2 p_2^1 - 1)] \partial p_2^3
\]

\[
+ [p_3 x_1^1 + x_1^2 p_3^1 + p_1 p_3 - x_2^3 p_2 p_3^1] \partial x_2^3
\]

\[
+ [p_3 p_3^1 + p_1 p_1^1 - 1] \partial p_1^1
\]

\[
+ [-p_3^2 x_1^3 + x_1^3 x_2^2] x_2^3 \partial x_2^3 + (p_2 p_2^1 - 1) \partial p_2^1
\]

\[
+ p_2 x_2^3 \partial x_1^1 + x_2^3 p_2^3 \partial x_2^3 + \Sigma (p_2 p_2^1 - \delta_{2\ell}) \partial x_2^3
\]

\[
+ p_2 p_1^3 \partial x_1 + (p_2 p_2^1 - 1) \partial p_2^3
\]
\[ + \left[ p_1^3 \cdot x_1^3 \cdot p_2^3 - \frac{x_1^3}{3} p_2^3 p_1^3 \right] \partial p_1^3 \]
\[ + \left[ p_1^3 \cdot x_1^3 \cdot p_2^3 - \frac{x_1^3}{3} (p_2^3 p_2^3 - 1) \right] \partial p_2^3 \]
\[ + \left[ \frac{2x_1^3}{x_2} - \frac{x_1^3}{3} p_2^3 p_1^3 \right] L_3 \]
\[ \frac{1}{x_2} \left( 1 - \frac{1}{x_1} \right) x_1^3 \cdot p_1^3 \cdot p_2^3 \cdot \frac{1}{x_2} \]
\[ - \frac{2x_1^3}{x_2} (x_2 \cdot x_2^3 - x_2^3) \]
\[ + \frac{x_1^3}{3} (x_2 \cdot x_2^3) \]
\[ - \left[ \frac{2x_1^3}{x_2} - \frac{x_1^3}{3} \right] (x_2 \cdot x_2^3) \]
\[ - \left[ \frac{2x_1^3}{x_2} - \frac{x_1^3}{3} \right] \frac{1}{x_2} \]
\[ \left[ \frac{1}{x_2} \right] \]
\[ = 0. \]
We can solve 3.29 by parametrization of the orbits of the vectorfield involved, e.g. noting that \( p_3^2 \frac{x_1^2}{x_2} \) : = k must be constant along the orbits, \( p_1^2(\tau) = \sqrt{1+k} \tanh (\sqrt{1+k} \tau + \eta_1^2) \) etc. We can propagate functions defined on \( p_1^2 = 0 \) to a solution of 3.29.

We may complete this section by noting that the functions

\[
\begin{align*}
    h_{12} &= (M_1 M_2)^{-1}(1 - p_1^1 p_1^2 - p_2^1 p_2^2 - p_3^1 p_3^2) \\
    h_{13} &= (M_1 M_3)^{-1}(1 - p_1^3 p_2^3 - p_2^3 p_3^3 - p_3^3 p_3^3) \\
    h_{23} &= (M_2 M_3)^{-1}(1 - p_1^3 p_1^3 - p_2^3 p_2^3 - p_3^3 p_3^3)
\end{align*}
\]

where \( M_i = (\sum_{\xi} 1 - p_\xi^i p_\xi^i)^{\frac{1}{2}} \)

(the Lorentz inner products in our coordinate system) are invariant under the \( X_1 \) and \( F_2^r \) of the proposition. Thus, if \( v^s \) are functions on \( \mathbb{R}^3 \) with mutually disjoint supports, then \( g^s : = v^s(h_{12}, h_{13}, h_{23}) \) give us functions of the type required for the proposition.
§ 4. Local Existence of Solutions.

We will now return to the full non-linear equations.

\[
\sum_{s, \ell} \left( x^s_i - x^s_j \right) \left( p^{\alpha}_s \delta x^\alpha_{s, \ell} + r^{\alpha}_s \delta p^{\alpha}_s \right) + \left( p^{\ell}_i - p^{\ell}_j \right) p^{s}_{s, \ell} f^r_{j} = f^r_{j} p^{r}_j + 2 p^{r}_j f^r_{j}.
\]

\[4.2 \quad x^r_i f^r_{j} = 0.\]

As with the linearized versions, we will factor out more and more of the symmetries. The difference in this case being that, after the initial Euclidean invariance, the symmetries are generated by quasi-linear rather than linear operators and thus we can get only local extensions under these "symmetries". Because, in the general case, we are not interested in the exact details of the equation, but rather only that at each stage they are symmetric hyperbolic and analytic, we will deal with them rather more abstractly than in the linearized case.

If we denote by \( D^r_{i,j} \), the differential operator given by the left hand side of 4.1, we may write 4.1 as

\[ D^r_{i,j} f^r_{j} - f^r_{j} p^{r}_{j} = 2 p^{r}_{j} f^r_{j} = 0. \]

We will find functions \( Q^r_{i,lm} \) and \( S^r_{i,\ell} \) so that the operators

\[4.3 \quad D^r_{i,j} - \sum_{\ell, m} Q^r_{i,lm} y^m_{\ell} - \sum_{\ell} S^r_{i,\ell} x^\ell_{\ell} \]

are tangent to a submanifold transverse to the orbits of the Euclidean group.
This is the "pullback" of $D_{i}^{r}$ to this submanifold. We then write $4.1$ as

$$4.4\quad (D_{i}^{r} - \Sigma Q_{i\ell m}^{r} Y_{\ell m} - \Sigma S_{i\ell}^{r} X_{\ell}) f_{j}^{r} + (\Sigma Q_{i\ell m}^{r} Y_{\ell m} + \Sigma S_{i\ell}^{r} X_{\ell}) f_{j}^{r} - f_{i}^{r} p_{j}^{r} - 2p_{i}^{r} f_{j}^{r} = 0.$$ 

Then use Euclidean covariance to write this as

$$4.5\quad (D_{i}^{r} - \Sigma Q_{i\ell m}^{r} Y_{\ell m} - \Sigma S_{i\ell}^{r} X_{\ell}) f_{j}^{r} + \Sigma Q_{i\ell m}^{r} (\delta_{m j}^{r} - \delta_{j m}^{r}) f_{j}^{r} - f_{i}^{r} p_{j}^{r} - 2p_{i}^{r} f_{j}^{r} = 0.$$ 

$4.5$ is then tangent to the submanifold.

We then pick out a particular subset of the equations of $4.5$, e.g. $i = 3$, and think of this set as generating a symmetry of the others. This can be done by showing that (writing $F_{i}^{r} = (D_{i}^{r} - \Sigma Q_{i\ell m}^{r} Y_{\ell m} - \Sigma S_{i\ell}^{r} X_{\ell})$) Euclidean covariance of the $f_{j}^{r}$'s together with

$$4.6\quad F_{3}^{r} f_{j}^{r} + \Sigma Q_{3\ell m}^{r} (\delta_{m j}^{r} - \delta_{j m}^{r}) f_{j}^{r} - f_{3}^{r} p_{j}^{r} - 2p_{3}^{r} f_{j}^{r} = 0$$

assumed on the transverse submanifold implies that

$$4.7\quad F_{3}^{r} (F_{i}^{r} f_{j}^{r} + \Sigma Q_{i\ell m}^{r} (\delta_{m j}^{r} - \delta_{j m}^{r}) - f_{i}^{r} p_{j}^{r} - 2p_{i}^{r} f_{j}^{r}) = \Sigma T_{i \ell}^{r} (F_{\ell}^{r} f_{j}^{r} + \Sigma Q_{\ell \ell m}^{r} (\delta_{m j}^{r} - \delta_{j m}^{r}) - f_{\ell}^{r} p_{j}^{r} - 2p_{\ell}^{r} f_{j}^{r})$$

on the submanifold. Here $T_{i \ell}^{r}$ are functions. This will show...
that if \( F^r_j \sum_{\ell,m} Q^r_{3 \ell m} (\delta^r_{mj} F^r_\ell - \delta^r_{lj} F^r_m) - f^r_p_j - 2p^{r^r}_j = 0 \) on the submanifold and \( F^r_j \sum_{\ell,m} Q^r_{1 \ell m} (\delta^r_{mj} F^r_\ell - \delta^r_{lj} F^r_m) - f^r_p_j - 2p^{r^r}_j = 0 \) at a point of the submanifold, then these are zero along the whole orbit of \( F^r_j \) (appropriate \( r \) through this point. Thus if we can then find a sub-submanifold transverse to the orbits of all the \( F^r_j \) and can solve 4.7 starting with data on this submanifold (as we can with the Euclidean action) then we can reduce the problem to one on this sub-submanifold and repeat the procedure with, say, the pull-back of \( F^r_2 \).

Let us first collect some facts about the operators \( D^r_j \).

Assuming only the Euclidean covariance of the \( f^r_j \)'s
\[
(X^r_j f^r_j = 0, \ Y^r_{ij} f^r_j = \delta^r_{ji} f^r_i - \delta^r_{ij} f^r_j)
\]
we find

\[
4.8 \quad [X^r_i, D^r_j] = 0, \quad [Y^r_{ij}, D^r_k] = \delta^r_{jk} D^r_i - \delta^r_{ik} D^r_j.
\]

Assuming nothing about the functions \( f^r_j \), direct computation yields

\[
4.9 \quad [D^r_i, D^r_j] x^s_\ell = \delta^r_{ji} (x^s_i - x^r_i) - \delta^r_{ij} (x^s_j - x^r_j)
\]

\[
4.10 \quad [D^r_i, D^r_j] p^s_\ell = (\delta^r_{ji} p^s_i - \delta^r_{ij} p^s_j) + (x^s_j - x^r_j) [D^r_i (f^s_\ell) - f^s_i p^s_\ell - 2f^s_p^s_\ell]
\]

\[
- (x^s_i - x^r_i) [D^r_j (f^s_\ell) - f^s_j p^s_\ell - 2f^s_p^s_\ell].
\]

Noting also that \( D^r_i = D^s_i + (x^s_i - x^r_i) \sum_{s', \ell} p^s_\ell \delta x^{s'} \ell + f^s_i \delta p^s_\ell \)
we may write
4.11 \[ [D_i^r, D_j^r] = Y_{i,j} - x_1^r x_{j,1} + x_{j,1} x_1^r + \Sigma_{\ell} (x_{s,\ell}^r - x_{s,1}^r) [D_i^s(f_\ell^s) - f_\ell^s p_{\ell}^s - 2 p_{\ell}^s f_\ell^s] \]
\[ - (x_{s,1}^r - x_{1}^r) [D_j^s(f_\ell^s) - f_\ell^s p_{\ell}^s - 2 p_{\ell}^s f_\ell^s] p_{\ell}^s. \]

We may now begin to carry out this program. Given a point \( z \) in the second order, \( n \)-particle phase space, by Euclidean covariance we may assume that it lies on the submanifold given by \( x_1^2 = x_2^2 = x_3^2 = x_4^2 = x_5^2 = 0 \). (For notational convenience we will consider the case \( n = 2 \) separately. Since \( n = 1 \) has already been considered, we will take \( n \geq 3 \).)

As we go along, we will exclude certain submanifolds from consideration. The first of these is embodied in the assumption \( x_1^2(z) > 0, x_2^3(z) > 0 \). If these are non-zero, we can make them positive by an action of the Euclidean group.

We find that the following operators are tangential.

4.12.1 \[ D_1^2 - p_2^2 Y_{12} - p_3^2 (Y_{13} - \frac{x_3^2}{x_2^2} Y_{23}) - \frac{x_1^2}{x_2^2} p_3^2 Y_{23} \]

4.12.2 \[ D_1^2 + p_1^2 x_1^2 + p_2^1 x_1 x_2 + p_2^1 Y_{12} + x_1^2 p_3^1 x_3 - p_3^1 (Y_{13} - \frac{x_3^2}{x_2^2} Y_{23}) - \frac{x_1^2}{x_2^2} p_3^1 Y_{23} \]
\[ - (x_{1}^2 - x_{1}^2) \frac{p_3^1}{x_2^2} Y_{23} \]

4.12.3 \[ D_1^3 + p_1^3 x_1^3 + x_1^3 p_2^2 x_2 - \frac{x_1^3}{x_2^2} Y_{12} + x_1^3 p_3^1 x_3 - \frac{x_1^3}{x_2^2} (Y_{13} - \frac{x_3^2}{x_2^2} Y_{23}) - \frac{x_1^3}{x_2^2} p_3^1 Y_{23} \]
\[ - (x_{1}^2 - x_{1}^2) \frac{p_2^1}{x_2^2} Y_{23} - (x_{1}^2 - x_{1}^2) \frac{p_3^1}{x_1^2} (Y_{23} - \frac{x_3^2}{x_2^2} Y_{23}) \]
4.12.4 \[ D_1^4 + x_1^4 - x_1^4 + x_1^4 p_2 x_2 - p_2^4 x_2^4 + x_1^4 p_3 x_3 - x_1^4 p_3 x_3 (y_{13} - \frac{x_2^3}{x_2^2 y_{23}}) \]

\[ - x_1^4 \frac{x_1}{x_2^2} y_{23} - (x_1 - x_1^4) p_2^2 \frac{x_2^2 y_{12}}{} \]

\[ -(x_1^2 - x_1^4) p_3^2 \frac{x_2^2 y_{12}}{} -(x_1^3 - x_1^4) p_3^2 \frac{x_2^2 y_{12}}{} \]

\[ \vdots \]

4.12.\(n\) \[ D_1^{n+1} + x_1^n \left( p_1 x_1 + p_2 x_2 - p_2 x_2 y_{12} + \frac{x_1}{x_2} \frac{x_3}{x_2} (y_{13} - \frac{x_2^3}{x_2^2 y_{23}}) - p_2 \frac{x_2}{x_2} y_{23} \right) \]

\[ - (x_1^2 - x_1^n) \left( p_2^2 \frac{x_2 y_{12}}{} + p_3^2 \frac{x_2 y_{12}}{} (y_{13} - \frac{x_2^3}{x_2^2 y_{23}}) \right) \]

\[ - (x_1^3 - x_1^n) \left( p_3^3 \frac{x_2 y_{12}}{} \right) \]

An in general, our tangential operator will be

4.13 \[ D_1^r + x_1^r \left( p_1 x_1 + p_2 x_2 - p_2^r y_{12} + \frac{x_1}{x_2} \frac{x_3}{x_2} (y_{13} - \frac{x_2^3}{x_2^2 y_{23}}) - p_2^r \frac{x_2}{x_2} y_{23} \right) \]

\[ - (x_1^2 - x_1^r) \left( p_2 \frac{x_2 y_{12}}{} + p_3 \frac{x_2 y_{12}}{} (y_{13} - \frac{x_2^3}{x_2^2 y_{23}}) \right) -(x_1^3 - x_1^r) \left( p_3 \frac{x_2 y_{12}}{} \right) \]

These are then our operators \( F_i^r \). The equations in terms of \( F_i^r \) then read

4.14 \[ F_{ij}^r - x_1 \left( - p_2^r \frac{1}{x_1} (\delta_{2j} f^r_{i2} - \delta_{ij} f^r_{22}) \right) - p_3^r \frac{1}{x_1} (\delta_{3j} f^r_{i3} - \delta_{ij} f^r_{33}) - \frac{x_3^3}{x_2^3} (\delta_{3j} f^r_{23} - \delta_{ij} f^r_{23}) \]

\[ - (x_1^2 - x_1^r) \left( p_2^r \frac{1}{x_1} (\delta_{2j} f^r_{i2} - \delta_{ij} f^r_{22}) + p_3^r \frac{1}{x_1} (\delta_{3j} f^r_{i3} - \delta_{ij} f^r_{33}) - \frac{x_3^3}{x_2^3} (\delta_{3j} f^r_{23} - \delta_{ij} f^r_{23}) \right) \]

\[ - (x_1^3 - x_1^r) \left( p_3^r \frac{1}{x_2} (\delta_{3j} f^r_{i3} - \delta_{ij} f^r_{33}) + p_3^r \frac{1}{x_2} (\delta_{3j} f^r_{23} - \delta_{ij} f^r_{23}) \right) \]

\[ r^i_{ij} - 2 p^i_{ij} \right) = 0 \]
Proposition. The system of 4.14 with $1 = 3$ is quasi-linear symmetric and (locally) hyperbolic off a finite collection of submanifolds of positive codimension.

Proof. Collecting terms in 4.13 we can write

$$4.15 \quad \mathfrak{F}^r_1 = D^r_1 + x^r_1 (p^1_1 x^1_1 + p^1_2 x^1_2 + p^1_3 x^1_3)$$

$$- [x^r_1 (p^2_1 - p^2_2) + x^2_1 p^2_2] \frac{1}{x^2_1} y^1_2$$

$$- [x^r_1 (p^2_1 - p^2_3) + x^2_1 p^2_3] \frac{1}{x^2_1} y^1_3$$

$$+ \frac{1}{2 x^2_1} [x^r_1 ((p^3_3 - p^3_1) x^3_1 - x^3_1 (p^3_1 - p^3_3))$$

$$+ (x^3_1 p^3_3 x^3_1 - x^3_1 p^3_1 x^3_1)] y^2_3$$

We thus find that

$$4.16 \quad \mathfrak{F}^r_3 x^n_1 = (x^n_3 - x^n_1) p^n_1 + x^n_3 (p^n_1 - p^n_2) \frac{1}{x^n_1} x^n_2$$

$$+ x^n_3 (p^n_3 - p^n_2) \frac{1}{x^n_1} x^n_3.$$  

If the total number of particles ($n$) is greater than 3, then none of these vanish identically, and so the collection of submanifolds given by $\mathfrak{F}^r_3 x^n_1 = 0$ is of positive codimension and finite (algebraic). Calling this set $M$, we have that if we are at a point $z$ not on $M$, then there is a neighborhood of $z$ on which the system 4.14 has the form
4.17  
\[ A \frac{\partial}{\partial x_1} \left( \begin{array}{c} f_1^1 \\ \vdots \\ f_n^1 \\ f_3^1 \end{array} \right) + B \left( \begin{array}{c} f_1^1 \\ \vdots \\ f_n^1 \\ f_3^1 \end{array} \right) + C \left( \begin{array}{c} f_1^1 \\ \vdots \\ f_n^1 \\ f_3^1 \end{array} \right) = 0 \]

where \( A \) is an invertible symmetric matrix of analytic functions, \( B \) is a symmetric matrix of differential operators depending analytically on the values of the \( r_j^r \)'s and the coordinates and not involving \( \frac{\partial}{\partial x_1} \), and \( C \) is a matrix of analytic functions.

In our case, the matrices \( A \) and \( B \) are actually diagonal. This shows the local symmetric hyperbolicity of 4.14 when \( n > 3 \).

When \( n = 3 \) \( F_{3n}^{r} x_{L}^{n} = 0 \) for all \( s \) and \( t \) and so we can not apply this same argument. Happily, though, in this case \( F_{3}^{r} \) is a linear differential operator with

4.18  
\[ F_{3n}^{r} x_{L}^{n} = p_{3}^{s} x_{L}^{n} - \delta_{3L} . \]

Therefore, for example \( p_{3}^{3} \) = constant will define globally noncharacteristic surfaces for the system 4.14, and we can solve 4.14 explicitly with given initial data on one of these systems. Again, we can write 4.14 as a symmetric hyperbolic system using \( \frac{\partial}{\partial p_{3}} \) instead of \( \frac{\partial}{\partial x_{1}} \). This completes the proof.

We will now show that an equation of the form 4.7 actually holds so that the \( F_{3}^{r} \)'s act as a symmetry for the rest of the equations.
Suppose, then, that we have Euclidean covariant functions \( f^r_1 \) which, on a transversal to the action of the Euclidean group, satisfy \( D^r_3 f^r_j = f^r_3 p^r_3 + 2 p^r_3 f^r_3 \). If we denote by \( N^r_1 \) the vector fields made up out of Euclidean generators and which make \( D^r_1 + N^r_1 \) tangent to the transversal, then we would like to compute, on the transversal, the functions

\[
4.19 \quad (D^r_3 + N^r_3)[(D^r_j + N^r_j) f^r_m - f^r_j p^r_m - 2 p^r_j f^r_m - N^r_j f^r_m].
\]

Note that we can compute \([D^r_1, D^r_j]\) and \(N^r_1 f^r_m\) and \([N^r_1, N^r_j]\) and \([N^r_1, N^r_j] f^r_m\) globally. In particular, 4.19 has the same value as

\[
4.20 \quad (D^r_3 + N^r_3)[D^r_j f^r_m - f^r_j p^r_m - 2 p^r_j f^r_m]
\]

when this is restricted to the transversal. On the other hand, the equation

\[
4.21 \quad (D^r_3 + N^r_3)[D^r_j f^r_m - f^r_j p^r_m - 2 p^r_j f^r_m]
\]

\[
= [D^r_3, D^r_j] f^r_m + [N^r_3, D^r_j] f^r_m + D^r_j (D^r_3 f^r_m + N^r_3 f^r_m)
\]

\[
- (D^r_3 + N^r_3)(f^r_j p^r_m + 2 p^r_j f^r_m)
\]

is globally valid.

Now, we know what \( D^r_3 f^r_m \) is on the transversal and we know what \( N^r_3 f^r_m \) is globally. Unfortunately, \( D^r_j \) is not tangent to the transversal so that, in general, \( D^r_j (D^r_3 f^r_m) \) on the transversal will not be the restriction of \( D^r_j (f^r_3 p^r_j + 2 p^r_3 f^r_j) \). If \( D^r_j \)
were tangent, we could use this substitution. Thus we write

\begin{equation}
4.21 \quad (4.20) = [D_3^r, D_j^r] f_m^r + [N_3^r, D_j^r] f_m^r \\
+ (D_j^r + N_j^r) (D_3^r f_m^r + N_3^r f_m^r) - N_j^r (D_3^r f_m^r + N_3^r f_m^r) \\
- (D_3^r + N_3^r) (f_m^r p_{j m} + 2 p_{j m} f_m^r).
\end{equation}

Now, \( N_j^r \) is not tangent either, however the equation

\begin{equation}
4.22 \quad -N_j^r (D_3^r f_m^r + N_3^r f_m^r) = [D_3^r, N_j^r] f_m^r + [N_3^r, N_j^r] f_m^r \\
- (D_3^r + N_3^r) (N_j^r f_m^r)
\end{equation}

is globally valid. Hence, on the submanifold we have

\begin{equation}
4.23 \quad (4.20) = [D_3^r, D_j^r] f_m^r + ([N_3^r, D_j^r] + [D_3^r, N_j^r]) f_m^r \\
+ [N_3^r, N_j^r] f_m^r - (D_3^r + N_3^r) (p_m^r f_j^r + 2 p_j^r f_m^r + N_j^r f_m^r) \\
+ (D_j^r + N_j^r) (p_m^r f_j^r + 2 p_j^r f_m^r + N_j^r f_m^r).
\end{equation}

We have already computed \([D_3^r, D_j^r]\), let us now compute the next term, \([N_3^r, D_j^r] + [D_3^r, N_j^r]\). Using the \( N_i^r \)'s given above, we find that

\begin{equation}
4.24 \quad [N_3^r, D_j^r] + [D_3^r, N_j^r] = \sum_l p_l^1 (x_j^r p_1^1 - x_1^r p_j^1) x_l + x_1^r x_j - x_j^r x_1 \\
+ [p_2^1 (x_1^r p_j^1 - x_j^r p_1^1) - \frac{p_3^1 p_1^2}{x_1^2} (x_1^r x_j^2 - x_j^r x_1^2)] \frac{1}{2} x_{12}^2 \\
+ [p_3^1 (x_1^r p_j^1 - x_j^r p_1^1) - \frac{p_3^1 p_1^2}{x_1^2} (x_1^r x_j^2 - x_j^r x_1^2)] \frac{1}{2} x_{13}^2.
\end{equation}
\[ + [ (x_j^r x_1^3 - x_1^r x_j^3) (p_1^3 p_2^3 - p_1^3 p_2^2 - p_3^3 p_1^2) + (x_j^r p_2^1 - x_1^r p_1^1) p_3^1 (x_1^3 - x_1^2) \\
+ (x_j^r x_1^2 - x_1^r x_j^2) (p_3 p_1^3 p_2^3) + (x_j^r x_1^2 - x_1^r x_j^2) p_3^2 p_1^2] \\
+ \delta_{31} (x_j^3 x_1^3 - x_j^3 x_1^3) - \delta_{3J} (x_1^3 x_1^3 - x_1^3 x_1^3) \\
- (x_j^r (x_1^3 (p_3^1 - p_3^2) - x_1^2 (p_1^3 - p_3^3)) + x_j^r p_3^3 x_1^3 - x_j^3 p_3^3 x_1^3) ((x_1^2 - x_1^2) p_1^2 x_2^3 \\
\quad + (x_1^3 - x_1^3) p_2^3 x_1^2) \\
+ (x_1^r (x_1^3 (p_3^1 - p_3^2) - x_1^2 (p_1^3 - p_3^3)) + x_1^r p_3^3 x_1^3 - x_1^3 p_3^3 x_1^3) ((x_1^2 - x_1^2) p_1^2 x_2^3 \\
\quad + (x_1^3 - x_1^3) p_2^3 x_1^2)] \]

\[ \times \frac{1}{x_1^2 x_2} \gamma_{23} - (x_j^r (p_3^3 - p_3^2) + x_j^2 p_3^2) \frac{1}{x_1^2} (\delta_{31} D_1^r - \delta_{11} D_3^r) (i, j) \]

\[ - (x_j^r (p_3^3 - p_3^2) + x_j^2 p_3^2) \frac{1}{x_1^2} (\delta_{21} D_1^r - \delta_{11} D_2^r) (i, j) \]

\[ + [ x_j^r (x_1^3 (p_3^3 - p_3^2) + x_1^2 (p_3^3 - p_3^2)) + x_j^2 p_3^3 x_1^3 - x_j^3 p_3^3 x_1^3] \frac{1}{x_1^2 x_2} \]

\[ \times (\delta_{31} D_2^r - \delta_{21} D_3^r) (i, j) \]
In this formula and the ones that follow, we use the subscript \((1,j)\) after a term to indicate that the term should be skew symmetrized in these indices:

\[ p_1 p_j(1, j) = p_1 p_j - p_j p_1. \]

Using 4.23 in 4.22, we find

\[
(4.19) = p_m^r (D_j p_j p_j - p_j p_j p_j) + 2p_j^r (D_j p_j p_j - p_j p_j p_j) + \sum_{s} (x_j^s - x_j^r) (\partial_{p_j} p_j^s) \cdot [D_j p_j^s - p_j p_j p_j] - 2p_j^r (D_j p_j p_j - p_j p_j p_j)
\]

\[
(4.24) + \sum_{1 \leq i < j \leq 3} Q_{3ij}^r \{ \delta_{j1} (D_j p_j p_j - p_j p_j p_j) + \delta_{im} (D_j p_j p_j - p_j p_j p_j) - \delta_{im} (D_j p_j p_j - p_j p_j p_j) \} (j, 3)
\]

Here \( Q_{j12}^r = (x_j^r (p_2^1 - p_2^2) + x_j^2 p_2^1) \frac{1}{x_1^2} \)

\[ Q_{j13}^r = (x_j^r (p_3^1 - p_3^2) + x_j^2 p_3^1) \frac{1}{x_1^2} \]

\[ Q_{j23}^r = (x_j^r (x_j^3 (p_3^1 - p_3^2) - x_j^2 (p_3^1 - p_3^2))) + x_j^2 p_3 x_1^2 - x_j^3 p_3 x_1^2 \]

This gives us, then, the required covariance. With this in hand, we may proceed to the next stage.

As we have seen, off a collection of submanifolds, the
surfaces \( x_1^n = \text{constant} \), in case \( n > 3 \), and \( p_3^3 = \text{constant} \), in case \( n = 3 \), are transverse to the orbits of \( D_3^r + N_3^r \). Therefore, our covariance allows us to pull back the equations to any one of these surfaces. In this case, we find functions \( v_j^r \) so that the operators

\[(4.25) \quad D_j^r + N_j^r + v_j^r(D_3^r + N_3^r)\]

are tangent to the surfaces. We will show that the resulting equations with \( j = 2 \) form a quasi-linear, symmetric hyperbolic system. Further, we will find that, if the \( j = 2 \) equations are satisfied, then the operators \( D_2^r + N_2^r + v_2^r(D_3^r + N_3^r) \) generate a covariance of the remaining equations (appropriate \( r \)).

One difficulty is that, whereas we could always extend smoothly by Euclidean invariance, in the present case, the \( D_3^r + N_3^r \) covariance equation is quasi-linear and we may not always be able to extend smoothly smooth data on the initial surface and hence the analogue of the argument leading to 4.22 will not be immediately valid. However it is not necessary to extend the \( f \)'s so that the covariance equation is true in a nbd., but rather only at the surface itself, in order for the analogue of 4.22 to be valid on the surface under consideration. We can certainly make such a smooth extension. Using the analogue of 4.22, we find that if
on the transversal, then we will have

\[
[D_{2}^{N_{2}} + v_{2}(D_{3}^{N_{3}})](D_{1}^{f_{1}} + v_{1}(D_{3}^{f_{3}})) f_{m}^{r} p_{m}^{r_{1}} - 2 p_{1}^{r_{1}} f_{m}^{r} \\
\quad - \Sigma Q_{21l}^{r}(\delta_{l_{1}^{r_{1}}} - \delta_{l_{1}^{m_{1}}} f_{l}^{r}) - v_{2}(p_{m}^{r_{2}} + 2 p_{3}^{r_{2}}, f_{m}^{r} - \Sigma Q_{3i}^{r}(\delta_{l_{1}^{r_{1}}} - \delta_{l_{1}^{m_{1}}} f_{l}^{r})) = 0
\]

\[
[D_{2}^{N_{2}} + v_{2}(D_{3}^{N_{3}})](D_{1}^{f_{1}} + v_{1}(D_{3}^{f_{3}})) f_{m}^{r} f_{m}^{r_{1}} - 2 p_{1}^{r_{1}} f_{m}^{r} \\
\quad - \Sigma Q_{1l}^{r}(\delta_{l_{1}^{r_{1}}} - \delta_{l_{1}^{m_{1}}} f_{l}^{r}) - v_{2}(p_{m}^{r_{2}} + 2 p_{3}^{r_{2}}, f_{m}^{r} - \Sigma Q_{3i}^{r}(\delta_{l_{1}^{r_{1}}} - \delta_{l_{1}^{m_{1}}} f_{l}^{r})) = 0
\]

\[
= p_{m}^{r_{2}}(D_{1}^{f_{1}} - 2 p_{2}^{f_{1}} f_{m}^{r}) + 2 p_{2}^{r_{2}}(D_{1}^{f_{1}} - 2 p_{2}^{f_{1}} f_{m}^{r}) \\
\quad + \Sigma Q_{21l}^{r}(\delta_{l_{1}^{r_{1}}} - \delta_{l_{1}^{m_{1}}} f_{l}^{r}) - \delta_{21}(D_{m}^{f_{m}} - 2 p_{1}^{f_{m}} f_{m}^{r})
\]

\[
(4.27) + \delta_{l_{1}^{r_{1}}} - \delta_{l_{1}^{m_{1}}} f_{l}^{r} - \delta_{l_{1}^{m_{1}}} f_{l}^{r} - \delta_{m_{1}}^{r_{1}} f_{m_{1}}^{r} - \delta_{m_{1}}^{r_{1}} f_{m_{1}}^{r}
\]

\[
(2, 1)
\]

\[
+ v_{2}^{r_{2}} \left[ p_{m}^{r_{2}}(D_{1}^{f_{1}} - 2 p_{2}^{f_{1}} f_{m}^{r}) + 2 p_{2}^{r_{2}}(D_{1}^{f_{1}} - 2 p_{2}^{f_{1}} f_{m}^{r}) \\
\quad + \Sigma Q_{3i}^{r}(\delta_{l_{1}^{r_{1}}} - \delta_{l_{1}^{m_{1}}} f_{l}^{r}) - \delta_{21}(D_{m}^{f_{m}} - 2 p_{1}^{f_{m}} f_{m}^{r})
\]

\[
+ \delta_{l_{1}^{r_{1}}} - \delta_{l_{1}^{m_{1}}} f_{l}^{r} - \delta_{m_{1}}^{r_{1}} f_{m_{1}}^{r})
\]

\[
(3, 1)
\]

\[
(2, 1)
\]

\[
+ \left[ \Sigma \left( x_{2}^{r_{2}}(x_{2}^{r_{2}}) - p_{2}^{r_{2}}(x_{2}^{r_{2}})ight)
\]

\[
(4.28) v_{j}^{r} = - \frac{F_{j}^{r}(x_{1}^{r_{1}})}{F_{j}^{r}(x_{1}^{r_{1}})}
\]

To complete the reduction on this step, we can choose
the local transversals to $D_2^r + N_2^r + v_2^r(D_3^r + N_3^r)$ to be given
by $x_2^n = \text{constant}$, when $n > 3$ and $x_2^3 = \text{constant}$ when $n = 3$.
Then we note that $(D_2^r + N_2^r + v_2^r(D_3^r + N_3^r))x_2^n$ is a rational function,
not identically zero and does not involve the functions $f_j^r$.
Thus, as before, off a finite collection of submanifolds of
positive codimension we can write the system

$$(D_2^r + N_2^r + v_2^r(D_3^r + N_3^r))f_m^r - p_m^r f_2^r - 2p_2^r f_m^r$$

$- \sum \theta_{214}(\delta_m^l f_1^r - \delta_m^l f_4^r) - v_2^r(p_m^r f_3^r + 2p_3^r f_m^r) - \sum \theta_{314}(\delta_m^l f_3^r - \delta_m^l f_4^r) = 0$$

as a quasilinear symmetric hyperbolic system with the same
analyticity properties as the previous system. The covariance
allows us to "factor out" 4.29 from our system, leaving on
the local transversals $x_2^n = \text{constant}$, a system of the form

$$(4.30) \quad [D_1^r + N_1^r + v_1^r(D_3^r + N_3^r) + W_1^r(D_2^r + N_2^r + v_2^r(D_3^r + N_3^r))]f_m^r - G_m^r = 0$$

where $W_1^r$ is the appropriate function making this operator
tangent to $x_2^n = \text{constant}$, and $G$ is a combination of the $f_i$'s.
This in itself is symmetric hyperbolic off a collection of
submanifolds, using the same arguments as above. Noticing
that at each stage, the rational functions which must not
vanish for hyperbolicity have a denominators a multiple of
the previous rational function which could not vanish, (e.g.
at the first stage we have $x_1^2$ and $x_2^3$ could not vanish:
$\frac{1}{x_1^2}, \frac{1}{x_2^3}$ and $\frac{1}{x_1 x_2}$ appear as coeff. in the $N_j^r$'s), and repeatedly
apply the Cauchy-Kowaleska theorem, we have

**Proposition.** Suppose we have a point \( q \) of the \( n \)-particle \((n \geq 3)\), second order space. By Euclidean covariance we may assume that \( x_1^1(q) = x_2^2(q) = x_3^3(q) = x_3^3(q) = 0 \). Then there exists \( 3n + 2 \) not-identically-zero polynomials \( P_j^r, x_1^2, x_2^3, \) of the coordinates such that if \( P_j^r(q) \neq 0, x_1^2(q) \neq 0, x_2^3(q) \neq 0 \) implies that there exists coordinate \( 6n-q \) plane through \( q \) with the property that, given any analytic functions \( g_j^r \) defined in a nbd. of \( q \) on this plane, there exists a set of analytic functions \( f_j^r \) defined in a nbd. of \( q \) in the whole space which are solutions of 4.1, 4.2 and agree with the \( g_j^r \) on their common domain of definition.

**The Two Particle Case.**

When \( n = 2 \), we can do somewhat better. The critical feature being that on \( x_1^1 = 0 = x_2^2 = x_3^3 \) (section to the Euclidean action) the operator \( D_2^r \) and \( D_3^r \) loose their non-linear terms and we can solve the corresponding equations exactly. We can factor out these symmetries yielding the following equations on \((x_1^1 = 0 = x_2^2 = x_3^3 = p_2^2 = p_3^2 \geq 0, x_1^2 > 0)\):

\[
\begin{align*}
\xi_1^2 p_1^2 \partial x_1^2 + [x_1^2 p_2^2, p_1^1 (p_2^2 p_2^2 - 1)] \partial p_2^2 \\
+ [p_2^2 p_2^2 - p_2^3 p_2^2 + x_2^2 f_1^2] \partial p_1^1 (p_2^2 p_2^2 - 1) \partial p_1^1 f_j^1
\end{align*}
\]

\[(4.31)\]

\[
\begin{align*}
-p_j^r f_1^1 - 2 p_j^r f_1^1 + p_2^3 (\delta_j^f_1 - \delta_j^f_2) + p_2^3 p_1^1 (p_2^f_1 p_2^f_2) \\
x_1^2 f_3^2 \left( \delta_j^f_2 - \delta_j^f_3 \right) = 0
\end{align*}
\]
\[
\begin{align*}
(x_1^2 p_1^1 x_1^1 + [p_1^1 p_1^1 1 - x_1^2 f_1^1]) p_1^1 + & [p_1^2 p_1^2 1 - x_1^2 f_2^1 p_1^2] p_1^2 \\
+ [p_1^2 p_2^2 - x_1^2 f_1^1 (p_2^2 p_2^2 1)] p_2^2 f_1^2 p_1^2 - & 2 p_1^2 p_2^2 f_2^2 j \\
+ x_1^2 f_1^1 (p_1^2 p_2^2 + 2 p_2^1 p_2^2) + x_1^2 f_1^1 (p_2^2 f_1^2 + \frac{1}{p_2^1} (5 f_2^2 - 5 f_3^2)) = 0
\end{align*}
\]

Noticing that this system is symmetric hyperbolic off
\(x_1^2 = 0, p_2^2 = 0, p_1^1 = 0, p_1^2 = 0\), we apply Kato's theorem
(9) to yield

**Proposition.** Let \(\epsilon > 0\), let \(u \subset \mathbb{R}^3\) (coordinates \(p_1^1, p_1^2, p_2^2\))
be defined by
\[
u = \left\{ (p_1^1, p_2^1, p_3^1) \mid \epsilon < (p_1^1)^2 < 1, (p_1^2)^2 + (p_2^2)^2 < 1, \epsilon < (p_1^2)^2, \epsilon < (p_2^2)^2 \right\}
\]

Let \(s > 5/2\) and let \(D\) be a bounded open subset of
\(H^s(U, \mathbb{R}^6) \ni f = (\hat{f}_1^1, \hat{f}_1^2, \hat{f}_1^3, \hat{f}_2^1, \hat{f}_2^2, \hat{f}_2^3)\). Given any \(k > 0\) and
\(f \in D\) there exists a \(k^1 > k\) and a solution of 5.1, 5.2,
\((f_1^1, \ldots, f_3^2)\) defined on \(k \leq x^2 \leq k\) \(x \in U\) such that
\[
(4.33) \quad (f_1^1, \ldots, f_3^2) \in C[k, k^1; D] \cap C^1[k, k^1; H^s(U, \mathbb{R}^6)]
\]

\[
(4.34) \quad (f_1^1, \ldots, f_3^2) \mid _{x_1^2 = k} = (\hat{f}_1^1, \ldots, \hat{f}_3^2)
\]

Moreover, such a solution is unique on this interval.

(Recall \(H^s(U, \mathbb{R}^6)\) is the space of functions with square
integrable derivatives up to and including order \(s\)).
§ 5. Conformal covariance

The extension of the Poincaré group by scale changes, \((t, x_1, x_2, x_3) \rightarrow (ct, cx_1, cx_2, cx_3)\) \(c \neq 0\), is of great importance to physics. However, it is often stated that they have relevance only for mass = zero situations. We will show that solutions of the second order R.D.L. equations have a natural action of scale transformations on them. We note that under such a scale transformation, the coordinates \(x_i^r \rightarrow cx_i^r\) and \(p_i^r \rightarrow p_i^r\).

Proposition. Let \(r_j^r\)'s be a solution of:

\[
D_{ij}^r r_j^r - p_{j1}^r r_j^r - z p_{i1}^r r_j^r = 0, \quad x_{1j}^r = 0\]

and let \(c \neq 0\) then \(\tilde{r}_j^r = c r_j^r (cx_i^s, p_i^s)\) will also be a solution.

Proof. We wish to show that

\[
(5.1) \quad D_{1j}^r r_j^r - p_{j1}^r r_j^r - 2p_{i1}^r r_j^r = 0
\]

\[
(5.2) \quad L.H.S. = \sum_{s, t} (x_i^s - x_i^r)(p^s_{k1} x_i^s + f^s_{k1} \partial x_i^s + p^s_{k1} \partial p_i^s) r_j^r
\]

\[- p_{j1}^r r_j^r - 2p_{i1}^r r_j^r
\]

If we plug in our definition of \(r_j^r\) and evaluate at a point with coordinates \((x_i^s, p_i^s)\) we find

\[
(5.3) \quad \left. \frac{D_{1j}^r r_j^r - p_{j1}^r r_j^r - 2p_{i1}^r r_j^r}{(x_i^s, p_i^s)} \right|_{\{(c x_i^s, p_i^s)\}} = c \left. \frac{D_{1j}^r r_j^r - p_{j1}^r r_j^r - 2p_{i1}^r r_j^r}{(x_i^s, p_i^s)} \right|_{\{(x_i^s, p_i^s)\}}
\]
Since the right hand side is assumed to be zero, the left hand side is zero as well. Moreover, it is clear that if \( x_1^j f_j^r = 0 \), then \( x_1^j f_j^r = 0 \). This concludes the proof.

If in addition, we assume that the solution in which we are interested is actually conformally invariant, i.e., for every \( i > 0 \), \( f_j^r = f_j^r \), this will allow us to factor out yet another symmetry. For example, in the two particle case, we may pull back the final equations to \( x_1^2 = 1 \), and get the system

\[
\begin{aligned}
\{ [f_2^r - p_2 p_1 (p_2^2 - 1)] \partial p_2^2 &+ [p_1^2 p_1^2 - p_2^2 p_1^2 p_2 + f_1^2] \partial p_1^2 \\
+ (p_1^2 p_1^2 - 1) \partial p_1^1 f_1^1 &- p_1^2 f_1^1 - 2 p_1^2 f_1^1 + p_2^2 (x_2 f_1^1 - x_1 f_2^1) + p_1^2 f_2^1 \\
+ p_2^2 p_1^2 f_1^2 &+ \frac{f_3^2}{p_2^2} (x_3 f_2^1 - x_2 f_3^1) = 0
\end{aligned}
\]

\[
\begin{aligned}
\{ [p_1 p_1^2 - f_1^1] \partial p_1^1 &+ [p_1^2 p_1^2 - f_2^1 p_1^2] \partial p_1^2 \\
+ [p_2^2 p_2^2 - f_2^1 (p_2^2 - 1)] \partial p_2^2 &- p_2^2 f_2^2 - 2 p_2^2 f_2^2 \\
+ p_1^2 f_1^2 &+ f_2^1 (p_2^2 f_1^2 - p_2^2 f_1^2) + f_3^1 (p_2^2 f_3^2 + \frac{1}{p_2^2} (x_3 f_3^2 - x_2 f_3^2)) = 0
\end{aligned}
\]

The most unfortunate feature of this system being that it is distinctly non-hyperbolic.
§6. Generalized Momentum Conservation

In classical particle mechanics, there are a number of "conserved quantities", the standard examples being total momentum and angular momentum. It is already known (8) that if we have the total relativistic momentum conserved in every frame for three particles, then they must (generically) move in a straight line. It is, however, an observed phenomenon that particle interactions in nature have a "center of mass frame", that is, a frame in which a certain linear combination of particle coordinates is unaccelerated by time translation.

We may explore the middle ground between these using the following definition. A generalized mass of the r-th particle will be a smooth, rotationally symmetric function \( g^r \) of the coordinates \( p^r_1, p^r_2, p^r_3 \). Given a set of such \( g^r \), we may define the total generalized momenta to be the functions \( G^1_i = \sum_r g^r p^r_{i1} \). We would like to know what restriction on a second order R.D.L. is imposed by assuming that some total generalized momenta \( G^1_i \) are conserved under time translation. When \( g^r = (1-\sum_r p^r_{i1})^{-\frac{1}{2}} \), this amounts to relativistic momentum conservation; when \( g^r = m^r = \text{constant} \), this becomes the condition of non-acceleration of \( \sum_r m^r x^r_{i1} \). Our assumption may be simply stated as

\[
(6.1) \quad x^0_0(G^1_i) = 0
\]
We will concentrate here on the (second order) three particle case. Since we are dealing with an R.D.L., equation 6.6 is Euclidean invariant. Since $g^r$ is assumed to be a rotationally invariant function of the $p^r_j$ we may assume that it is a function of $h^r = \sum_i p^r_i p^r_i$. Let us now compute $X_0(G_1)$

$$X_0(G_1) = \left( \sum_{s,t} p^s_t \partial x^s_t + i \sum_t \partial p^s_t \right) \left( \sum_r g^r \frac{d}{dh^r} p^r_1 \right)$$

(6.2) 

$$= \sum_s g^{s'}(h^s)2(p^{s'}_s f^s_1)p^s_1 + g^s(h^s)f^s_1$$

where we have used the notation $p^{s', t}_s : \sum_t p^s_t f^s_t$ and $g^{s'} = \frac{dg^s}{dh^s}$.

Since we are dealing with three particles, we may, by Euclidean invariance, assume that they are on the plane $x^s_3 = 0$, in which we have, as before,

(6.3) 

$$\left( Y_0 - x^r_3 X_0 \right) = \sum_{s,t} (p^s_{3} p^s_{-3} \delta_{3t} \partial p^s_t) \text{ on } x^s_3 = 0$$

This operator is, of course, tangent to $x^s_3 = 0$. Our equations for an R.D.L. then say

(6.4) 

$$\left( \sum_{s,t} (p^s_{3} p^s_{-3} \delta_{3t} \partial p^s_t) \right) f^r_j = p^r_j f^r_3 + 2p^r_{3j}$$

The orbits of the differential operator in 6.4 are given by

(6.5) 

$$p^s_3(\tau) = - \tanh(\tau + \eta^s)$$

$$p^s_i(\tau) = c^s_i \cosh^{-1} (\tau + \eta^s) \quad i \neq 3$$
Where \( \eta^S \) and \( c_1^S \) are constants. Along such orbits we have

\[
\dot{r}^S(\tau) = r_3^S \cosh^{-3} (\tau \eta^3)
\]

\[
\dot{f}_1^S(\tau) = \cosh^{-2} (\tau \eta^S) (\dot{r}_3^S c_1^S \tanh (\tau \eta^S) + \dot{r}_1^S) \quad i \neq 3
\]

where the \( \dot{r}_j^S \) are constant along the orbits.

Denoting by \( \frac{\partial}{\partial \tau} \) we see that along an orbit

\[
g^S(\tau) = \frac{\dot{g}^S(\tau)}{2 \tanh (\tau \eta^S) \cosh^{-2} (\tau \eta^S) \left[ (1- (c_1^S)^2) - (c_2^S)^2 \right]}
\]

When the denominator is non-zero. Putting this into

6.2 gives us

\[
\sum_{s} g^S(\tau) \cosh^{-1} (\tau \eta^S) (-\dot{r}_3^S + c_1^S \dot{r}_1^S + \dot{r}_2^S) p_j^S(\tau)
\]

\[
\frac{\tanh (\tau \eta^S) \left[ (1- (c_1^S)^2) - (c_2^S)^2 \right]}{\tanh (\tau \eta^S) \left[ (1- (c_1^S)^2) - (c_2^S)^2 \right]}
\]

\[+ g^S(\tau) f^S(\tau) = 0\]

If we choose the \( g^S \) to be one of the standard examples e.g. \( g^S = m^S \), or \( g^S = (1 - p^S p^S)^{-\frac{1}{2}} \), then for generic \( c_1^S \) and \( \eta^S \) the functional coefficients of \( r_3^S \) and \( (f_1^S + f_2^S) \) will be linearly independent, hence for 6.8 to be satisfied for even a small interval of \( \tau \)'s, these must all vanish. In this case, we may conclude from rotation covariance

\[
(Y_{12}(f_1^S + f_2^S) = -f_2^S + f_1^S)
\]

and continuity that all the \( f_j^S \) are identically zero.

We'd like to solve 6.8 for the various \( g^S \). However,
it is clear that if the $f^r_j = 0$ for some $r$ (respectively $g^r = 0$) then 6.8 contains no information about the corresponding $g^r$ (respectively $f^r_j$). Let us cover these cases first. Suppose, for example, along an orbit $\tilde{f}^2_j = \tilde{f}^3_j = 0$ or $g^2 = g^3 = 0$. Then 6.8 reads

$$g^3 (\tau) \cosh^{-1} (\tau + \eta^1) \left( f^1_3 + \frac{c_2^1 f^1_2 + c_1^1 f^1_1}{\tanh (\tau + \eta^1) (1 - (c_1^1)^2 - (c_2^1)^2)} \right) \cdot p^1_f_j \tag{6.9}$$

$$+ g^1 (\tau) f^1_j (\tau) = 0$$

In particular, we must have

$$g^1 (\tau) f^1_3 \cosh^{-3} (\tau + \eta^1) = \frac{\cosh^{-2} (\tau + \eta^1) (f^1_3 \cosh (\tau + \eta^1) + f^1_j) g^1 (\tau)}{\cosh^{-1} (\tau + \eta^1)} \tag{6.10}$$

Thus, either $g^1 = 0$ or $f^1_1 = f^1_2 = f^1_3 = 0$. Suppose now that only the $\tilde{f}^3_j$ (or $g^3$) vanishes then the equation becomes

$$g^1 (\tau) \cosh^{-1} (\tau + \eta^1)(-f^3_3 + \frac{c_1^1 f^1_2 + c_1^1 f^1_1}{\tanh (\tau + \eta^1) (1 - (c_1^1)^2 - (c_2^1)^2)} \cdot p^1 (\tau)$$

$$+ g^1 (\tau) f^1 (\tau) \tag{6.11}$$

$$+ g^2 (\tau) \cosh^{-1} (\tau + \eta^2)(-f^2_3 + \frac{c_2^2 f^2_2 + c_2^2 f^2_1}{\tanh (\tau + \eta^2) (1 - (c_1^2)^2 - (c_2^2)^2)} \cdot p^2 (\tau)$$

$$+ g^2 (\tau) f^2 (\tau) = 0$$

Generically, $p^1 (\tau)$ and $p^2 (\tau)$ will have a common normal vector
Dotting equation 6.11 with this, we find

\[(6.12) \quad g^1(\tau)(\vec{r} \cdot (p^1xp^2)) + g^2(\tau)(\vec{r} \cdot (p^1xp^2)) = 0\]

We would like to use this to solve for \(g^2(\tau)\), which we can do if \((\vec{r} \cdot (p^1xp^2))(\tau) \neq 0\). Suppose this were zero, then taking the \(\tau\) derivative, we also have

\begin{align*}
0 &= (p_3^2 + 2p_3^2p_2^2)(\vec{r} \cdot (p^1xp^2)) + \vec{r}^2 \cdot ((p_3^1 - \vec{b}_3)x \cdot p^2) \\
&\quad + \vec{r}^2 \cdot (p^1x(p_3^2p_1 - \vec{b}_3)) = \vec{r}^2 \cdot (-\vec{b}_3xp^2) + \vec{r}^2 \cdot (p^1x - \vec{b}_3) \\
&\quad = p_2^2p_1^2 - p_1f_2^2 + p_1^2f_2^2 + p_2p_1 = (p_2^2 - p_1)f_1^2 - (p_2p_1^2)f_2^2
\end{align*}

Thus along our orbit we would have

\[(6.13) \quad 0 = (c_2^2 \cosh^{-1} (\tau + \eta^2) - c_2^1 \cosh^{-1} (\tau + \eta^1))(\cosh^{-2} (\tau + \eta^2)c_2^2p_3^2 \tanh (\tau + \eta^2) + \vec{r}^2) \\
- (c_1^2 \cosh (\tau + \eta^2) - c_1^1 \cosh^{-1} (\tau + \eta^1))(\cosh^{-2} (\tau + \eta^2)(c_2^2p_3^2 \tanh (\tau + \eta^2) + \vec{r}^2))
\]

In particular, as long as we have chosen an orbit with

\[c_2^2c_1^1 - c_1^2c_2^1 \neq 0, \quad \vec{r}_2^2 \text{ must be zero (a generic condition, again)}\]

If, in addition we have \(\eta^2 \neq \eta^1\), then \(\vec{r}_1^2 = \vec{r}_2^2 = 0\), reducing to our previous case. We will thus assume \(\vec{r}^2 \cdot (p^1xp^2) \neq 0\) and write
\[ g^2(\tau) = -g^1(\tau) \frac{\mathbf{f}^1 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)}{f^2 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)} \]

\[ \dot{g}^2(\tau) = -g^1(\tau) \frac{\mathbf{f}^1 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)}{f^2 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)} + \]

\[ -g^1(\tau)(3\mathbf{p}^1 \cdot \mathbf{p}^2)(\mathbf{f}^1 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)) \]

\[ + (p_2^2-p_1^2)f^1_{12}-(p_1^2-p_2^1)f^1_{21}][\mathbf{f}^2 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)]^{-1} \]

\[ + g^1(\tau)(\mathbf{f}^1 \cdot (\mathbf{p}^1 \times \mathbf{p}^2))[(3p_3^2+p_2^1)(\mathbf{f}^2 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)) \]

\[ + (p_2^2-p_1^2)f^2_{12}-(p_1^2-p_2^1)f^2_{21}][\mathbf{f}^2 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)]^{-2} \]

(6.14)

We will use these in 6.11. Let us denote by \( \mathbf{k}^1(\tau) \) and \( \mathbf{k}^2(\tau) \) a dual basis to \( \mathbf{p}^1(\tau) \) and \( \mathbf{p}^2(\tau) \), i.e. \( \mathbf{p}^s \cdot \mathbf{k}^s' = \delta_{ss'} \) and \( \mathbf{k}^s \cdot (\mathbf{p}^1 \times \mathbf{p}^2) = 0 \).

Dotting equation 6.11 with these yields the equations

\[ \dot{g}(\tau) \cosh^{-1}(\tau+\eta^1)(-\mathbf{f}^1 + c_2^1 f^1_{22} + c_1^1 f^1_{11}) \]

\[ \tanh(\tau+\eta^1)(1-(c_2^1)^2-(c_2^2)^2) \]

\[ + g^1(\tau)f^1 \cdot k^1 + g^2(\tau)f^2 \cdot k^1 = 0 \]

\[ \dot{g}^2(\tau) \cosh^{-1}(\tau+\eta^2)(-\mathbf{f}^2 + c_2^2 f^2_{22} + c_2^2 f^2_{11}) \]

\[ \tanh(\tau+\eta^2)(1-(c_2^2)^2-(c_2^2)^2) \]

\[ + g^2(\tau)f^2 \cdot k^2 + g^1(\tau)f^1 \cdot k^2 = 0 \]

Denoting by \( L \) the function \( \frac{\mathbf{f}^1 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)}{f^2 \cdot (\mathbf{p}^1 \times \mathbf{p}^2)} \) and using 6.14 we may
write these as

\[ g^1(\tau) \cosh^{-1} \left( \tau + \eta^1 \left( -\tau_3^1 + \frac{c_1 f_1}{c_2 f_2} \right) \right) \frac{1}{\tanh (\tau + \eta^1)(1 - c_1^2 - c_2^2)} \]

\[ + \frac{g^1(\tau) \tau^1 k^1 + g^1(\tau) L^2 \tau^2 k^1}{(\tau^2 - \tau_3^1 + \frac{c_1^2 \tau_1^1 + c_2^2 \tau_2^1}{1 + \eta^2_1(\tau + \eta^2)(1 + (c_1^2 + c_2^2)^2)} \right) \frac{1}{\tanh (\tau + \eta^2)(1 + (c_1^2 + c_2^2)^2)} \]

\[ + \frac{g^1(\tau) \tau^2 k^2 + g^1(\tau) \tau^1 k^2}{0} \]

(6.17)

(6.18)

On the other hand, if \( \tau^1 \tau^1 \neq 0 \) we can solve 6.2 along a generic orbit. Extend \( K_1, K_2 \) with \( K_3 \) so that \( \tau^1 \tau = \delta sr. \) Then

\[ g^s = - \frac{\tau^s}{2(\tau^s \tau^s)} \Sigma g^r \tau^r \tau^s \]

(6.19)

Thus, whereas conservation of relativistic momentum limits the interaction kinematically (we didn't use the dynamical equations at all) the conservation of generalized momentum does not. We can certainly solve 6.19 if it is thought of as an o.d.e. along orbits of \( L_3 \) for the \( g^s \). The question of whether we can solve the equation for all orbits in a way consistent with the dependence requirements of the \( g^s \) and the dynamical equations for the \( f^s \)'s together still remains.
Appendix I

In this section, we will show the restrictiveness of the k-th order assumption on an R.D.L. by exhibiting a smooth R.D.L., together with a parametrization, which is completely geometric and yet cannot be k-th order for any k. The naturalness of the R.D.L. indicates, perhaps the unnaturality of the k-th order assumption.

We will work for the moment in two dimensions: coordinates \((x, t)\) and metric \(ds^2 = dt^2 - dx^2\). If \(p_1, p_2\) are two points in this space-time we will write \(p_1 \rightarrow p_2\) when \(p_2\) lies on the forward light cone of \(p_1\) and we will write \(p_1 < p_2\) when \(p_2\) lies within the forward light cone of \(p_1\). If \(\gamma\) is a time-like path in Minkowski space containing \(p_1\) and \(p_2\), we write \(d_\gamma(p_1, p_2)\) for the arclength of the part of \(\gamma\) between \(p_1\) and \(p_2\).

Now let \((\gamma_1, \gamma_2)\) be a pair of one-dimensional time-like submanifolds. We define functions \(\beta_1 : \gamma_1 \rightarrow \mathbb{R} \cup \{\infty\}\) and \(\beta_2 : \gamma_2 \rightarrow \mathbb{R} \cup \{\infty\}\) by

\[
\beta_1(p) = \begin{cases} 
  d_{\gamma_1}(p, p') & \text{when } \exists p' \in \gamma_1 \text{ such that } \\
  \exists p'' \in \gamma_2 \text{ with } p \rightarrow p'' \rightarrow p', \\
  0 & \text{otherwise}
\end{cases}
\]

and \(\beta_2\) is defined analogously.

Given two numbers \(0 < M_1 < \infty, 0 < M_2 < \infty\) we can define \(\mathcal{S}_{M_1, M_2} = \{\text{all pairs } (\gamma_1, \gamma_2) \text{ such that } \beta_1 = M, \beta_2 = M_2\}\).
Thus, we may say a pair \((\gamma_1, \gamma_2)\) is a solution of the bounce time dynamics if the particles stay a constant distance apart, in each others reckoning.

Clearly \(S_{M_1, M_2}\) is an invariant set under the action of isometries. Moreover, the initial value data for this problem may be rather nicely circumscribed.

**Initial Value Data**

Let \(0 < M_1 < \infty, 0 < M_2 < \infty\) and \(0 \leq \epsilon_1, \epsilon_2\). Define an \((M_1, M_2, \epsilon_1, \epsilon_2)\) light-trapezoid as a quartuple of points \((p_1, p_2, p_3, p_4)\) in space time such that \(p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_4\) and \(p_1 < p_3\) and \(p_2 < p_4\) and finally \(d(p_1, p_3) = M_1 + \epsilon_1\), \(d(p_2, p_4) = M_2 + \epsilon_2\) where \(d(p_1, p_3)\) is the straight-line (Minkowski) arclength. (Figure 1; see page 84.) Given any pair \((\gamma_1, \gamma_2)\) of time-like paths between \(p_1\) and \(p_3\) (respectively \(p_2, p_4\)) such that \(d_{\gamma_1}(p_1, p_3) = M_1\) and \(d_{\gamma_2}(p_2, p_4) = M_2\) we can extend these in both directions uniquely to get piecewise smooth solutions in \(S_{M_1, M_2}\). This is accomplished in the following way (see Figure 2; page 84). The paths \(\gamma_1\) and \(\gamma_2\) define a smooth tangent line-field in the regions

\[ \Gamma_1 = \{ p | p_3 < p \text{ or } p_3 \rightarrow p \} - \{ p | p_4 < p \} \text{ and} \]

\[ \Gamma_2 = \{ p | p < p_2 \text{ or } p \rightarrow p_3 \} - \{ p | p < p_1 \} \]

where the slope (in \((x, t)\) coordinates) of the tangent line element at a point \(z \in \Gamma_1\) is given by

\[ S(z) = \frac{(1-(1-k)^2/(1-w))}{1+(1-k)^2/(1-w)} \quad \text{if } x(p_1) < x(p_2) \]
\[ S(z) = \frac{(1+k)^2(\frac{1-w}{1+w})-1}{1+(1+k)^2(\frac{1-w}{1+w})} \quad \text{if } x(p_1) > x(p_2) \]

where \( k \) is the slope of \( \gamma_2 \) at that point \( p \) such that \( d(p,z) = 0 \) and \( w \) is the slope of \( \gamma_1 \) at that point \( p' \) such that \( d(p,p') = 0 \) (see Figure 3; page 84).

The extension of \( \gamma_1 \) is then that integral curve of this tangent line field which begins at the point \( p_3 \). Notice that, since \( \gamma_1 \) and \( \gamma_2 \) are time-like up to and including the endpoints, the extension of \( \gamma_1 \) is also time-like and intersects the forward time boundary of \( \Gamma_1 \) at some finite time. Call this point \( p_5 \), and the extension of \( \gamma_1 \), call \( \tilde{\gamma}_1 \). Then \( (p_2,p_3,p_4,p_5) \) form a new light-trapezoid with constants \( (M_2,M_1,\varepsilon_2,\tilde{\varepsilon}_1) \) where \( \tilde{\varepsilon}_1 = d(p_3,p_5) \). We can thus iterate this procedure with paths \( \gamma_2 \) and \( \tilde{\gamma}_1 \), to arrive at a point \( p_6 \) with an extension \( \tilde{\gamma}_2 \) of \( \gamma_2 \), etc.

Clearly, by imposing \( C^\infty \) boundary condition on \( \gamma_1 \) and \( \gamma_2 \) we can make these extensions \( C^\infty \) instead of piecewise smooth.

**Asymptotics**

The next question to consider is whether this procedure gives complete solutions, i.e. if we iterate this process infinitely many times into the past and future, will the curves thus defined intersect every \( t \) = constant slice? To see that this is actually the case, let us call \( p_{2n+1} \) the \( n \)-th iteration of the point \( p_1 \) and \( p_{2n+2} \) the \( n \)-th iteration of \( p_2 \). Then we know \( d_{\gamma_1}(p_{2n+1},p_{2(n+1)+1}) = M_1 \) so that
On the other hand

\[ |t(p_{2n+1}) - t(p_{2(n+1)+1})| \geq d(p_{2n+1}, p_{2(n+1)+1}) \geq M_1. \]

Thus, succeeding iterations of \( p_1 \) are at least \( M_1 \) units of time apart, and so after infinitely many iterations, every time value will be covered.

We may therefore conclude that these extensions exist for all time. Are all these solutions "nicely behaved as time goes to infinity? Certainly, if \( \gamma_1 \) and \( \gamma_2 \) are null translations of one another, this is the case. Here, the extension process reproduces copies of the curves, translated to the appropriate endpoints (see Figure 4, page 84 ) On the other hand, this is essentially the only "nicely behaved" case. We may see this by considering merely a skeleton of one of these general solutions.

Knowing the velocities (in standard coordinates) of the curves at points \( p_1 \) and \( p_2 \), we may use formulae * to determine the velocities at all the other "bounce points".

It may be seen that if the velocities at \( p_1 \) and \( p_2 \) are not identical, then the velocities at the bounce point proceed in absolute value to the speed of light \( (c = 1 \text{ in our coordinates}) \). Moreover, the velocities of the two particles at adjacent bound points eventually have the same sign. In the situation illustrated in Figure 5, (see page 84 ) (velocity at \( p_1 \)) > (velocity at \( p_2 \)) (respectively <)
implies the velocities at bounce points in future direction go to -1 (respectively +1) as time goes to infinity. Thus, if the velocity of $\gamma_1$ is greater than that of $\gamma_2$, then the extensions will depict a pair of particles accelerating toward the left of the picture.

Four Dimensional Space-Time

We may attempt to repeat this whole scheme of two particle dynamics in the four dimensional Minkowski space.

If $p_1$ and $p_2$ are two points in $M^4$ we will write $p_1 \rightarrow p_2$ when $p_2$ lies on the forward light-cone of $p_1$ and $p_1 < p_2$ when $p_2$ lies within the forward light-cone of $p_1$. If $\gamma$ is a time-like path containing $p_1$ and $p_2$ we write $d_\gamma(p_1, p_2)$ for the Minkowski arclength of the part of $\gamma$ between $p_1$ and $p_2$. If $(\gamma_1, \gamma_2)$ are a pair of time-like, one-dimensional submanifolds, $\beta_1 : \gamma_1 \rightarrow \mathbb{R} \cup \{\infty\}$ and $\beta_2 : \gamma_2 \rightarrow \mathbb{R} \cup \{\infty\}$ are defined as in the two dimensional case.

Again, given two numbers $0 < M_1 < \infty$ and $0 < M_2 < \infty$ we can define $S_{M_1, M_2} = \{\text{all pairs } (\gamma_1, \gamma_2) \text{ such that } \beta_1 = M_1, \beta_2 = M_2\}$. This is again an invariant set under the natural action of the Poincaré group in four dimensions.

In this case, however, the analogous light-trapezoid initial value data does not select out a unique solution. This problem arises because when we repeat the extension procedure used in the two dimensional case, we do not get
a smooth line element field which we can then integrate.

In fact, what is obtained is a paraboloid of tangent directions at each point. This may be remedied by suitably refining our dynamical law. It is consistent to require of our paths $\gamma_1$ and $\gamma_2$ not only that $\beta_1 = M_1$ and $\beta_2 = M_2$, but also that if $p_1$ and $p_3$ are points of $\gamma_1$ (respectively $\gamma_2$) and $p_2$ is on $\gamma_2$ (respectively $\gamma_1$) with $p_1 \rightarrow p_2 \rightarrow p_3$, then the difference of the normalized tangent vectors $\dot{\gamma}(p_3) - \dot{\gamma}(p_1)$ (respectively, $\dot{\gamma}_2(p_3) - \dot{\gamma}_2(p_1)$) is parallel to the plane generated by the null line through $p_1$ and $p_2$ and the null line through $p_2$ and $p_3$. We will call the set of $(\gamma_1, \gamma_2)$ satisfying these conditions $\mathcal{E}_{M_1, M_2}$.

Clearly, $\mathcal{E}_{M_1, M_2}$ is invariant under the natural action of the Poincaré group. Moreover, on any two dimensional subspace-time this dynamical law reduces to the one considered above.

The light trapezoid gives good initial value data for $\mathcal{E}_{M_1, M_2}$ in the following sense. Given points $p_1, p_2, p_3, p_4$ in space-time with $p_1 \rightarrow p_2 \rightarrow p_3 \rightarrow p_4$ and $p_1 < p_3$, $p_2 < p_4$ and smooth time-like paths $\gamma_1, \gamma_2$ from $p_1$ to $p_3$ respectively $p_2$ to $p_4$ such that $d_{\gamma_1}(p_1, p_3) = M_1$ and $d_{\gamma_2}(p_2, p_4) = M_2$ there exists unique piecewise smooth extension $\tilde{\gamma}_1, \tilde{\gamma}_2$ defined for all time of $\gamma_1$ and $\gamma_2$ which satisfy $\beta_1 = M_1$, $\beta_2 = M_2$ everywhere and the parallelism requirement everywhere except
possibly at the bounce iterates of $p_1$ and $p_2$.

Asymptotics

In this system we find many nicely behaved solutions. If $R(t)$ is any one parameter group of space rotations and $p,q$ are distinct points on the $t=0$ hyperplane, then

$$\gamma_1 = \{ T(t) \ R(a \cdot t) [p] | t \in \mathbb{R} \} \text{ and } \gamma_2 = \{ T(t) \ R(a \cdot t) [q] | t \in \mathbb{R} \},$$

where $T(t)$ is forward time translation by $t$, and $a > 0$ gives us a pair of curves which are time-like if $a$ is sufficiently small and which satisfy $\beta_1 = \text{some constant}$ and $\beta_2 = \text{some constant}$. The parallelism requirement, in this case, translates to the requirement that $p,q$ and the center-line of rotation be co-planar (see Figure 6, p.85).

Unfortunately, this system differs from the planar one in that the velocities at the bounce points depend on the positions as well as the various velocities and are thus not directly computable as a "skeleton".

We note that the light-trapezoid initial data is also sufficient for the Fokker-Tetrode formulation of two particle electrodynamics. However, global existence theorems are not as readily obtainable here as in the case of integrating line-element fields. However, we may again expect that there will be infinitely many degrees of freedom in the two particle electrodynamics. Moreover, it is hoped that techniques developed for computer solutions of the two particle bounce
time dynamics will be of assistance in the corresponding electrodynamics problem.
Sub-Appendix

If we parametrize $\gamma_1$ by arc-length, $\gamma_1(\tau)$ and parametrize
$\gamma_2(\tau)$ so that $\|\gamma_2(\tau) - \gamma_1(\tau)\|^2 = 0$ and finally parametrize the
extension $\tilde{\gamma}_1$ of $\gamma_1$ by arc-length $\tilde{\gamma}_1(\tau)$ we have $\gamma_1(0) = p_1,$
$\gamma_1(M) = p_3,$ $\gamma_2(0) = p_2,$ $\gamma_2(M) = p_4$ and $\tilde{\gamma}_1(0) = p_3$ and
$\|\tilde{\gamma}_1(\tau) - \gamma_2(\tau)\|^2 = 0$. Taking first derivatives of these

$$0 = \langle \gamma_2(\tau) - \gamma_1(\tau), \gamma_2(\tau) - \gamma_1(\tau) \rangle = \langle \tilde{\gamma}_1(\tau) - \gamma_2(\tau), \tilde{\gamma}_1(\tau) - \gamma_1(\tau) \rangle$$
$$= 2 \langle \dot{\gamma}_2(\tau) - \dot{\gamma}_1(\tau), \gamma_2(\tau) - \gamma_1(\tau) \rangle$$
$$= 2 \langle \tilde{\gamma}_1(\tau) - \dot{\gamma}_2(\tau), \tilde{\gamma}_1(\tau) - \gamma_2(\tau) \rangle.$$ 

Letting $V_1(\tau)$ be the null vector $\gamma_2(\tau) - \gamma_1(\tau)$
$V_2(\tau)$ be the null vector $\tilde{\gamma}_1(\tau) - \gamma_2(\tau)$
$W(\tau) = \dot{\gamma}_1(\tau)$ \hspace{1cm} $\|W(\tau)\|^2 = 1$
$X(\tau) = \dot{\gamma}_2(\tau)$
$W'(\tau) = \tilde{\gamma}_1(\tau)$ \hspace{1cm} $\|W'(\tau)\|^2 = 1$

We find $\langle X - W, V_1 \rangle = 0,$ $\langle W' - X, V_2 \rangle = 0,$ $\|W\|^2 = \|W'\|^2 = 1,$
$\langle V_1, V_1 \rangle = 0,$ $\langle V_2, V_2 \rangle = 0,$ $(V_1, V_2 \neq 0)$ and $X$ is tangent to
$\gamma_2,$ and non-zero.

We can translate all of these to a single tangent space.
If we know the velocity, in standard reference frame, of \( \gamma_1(\tau) \) and \( \gamma_2(\tau) \) we can compute the velocity of \( \tilde{\gamma}_1(\tau) \) from the above as follows.

With standard coordinates is given by \((t_1, x_1)\) with
t_1^2 - x_1^2 = 1. If the velocity (slope) is \( \omega \), then \( t_1 > 0 \)
t_1 = (1-\omega)^{-\frac{1}{2}} \) and \( x_1 = \omega(1-\omega)^{-\frac{1}{2}} \). \( V_1 \) is represented by a
pair \((\ell, \ell)\) or \((\ell, -\ell)\) depending on the initial configuration
of our particles and \( V_2 \) will be represented by a pair \((n, -n)\)
or \((n, n)\) respectively.

If the velocity of \( X \) is \( k \), then \( X \) is represented by a
pair \((\alpha, k\alpha)\), \( \alpha > 0 \) which by \( \langle X - \tilde{W}, V_1 \rangle = 0 \) satisfies

\[(t_1 - \alpha)\ell - (x_1 - k\alpha)\ell = 0 \quad \text{(taking } V_1 = (\ell, \ell) \text{ for specificity)}\]
or
\[(t_1 - \alpha) = x_1 - k\alpha\]
or
\[\alpha = \frac{t_1 - x_1}{1 - k}.\]

Next, if \( W' \) is represented by a pair \( t_2, x_2 \) from
\[\langle W' - X, V_2 \rangle = 0 \] we have

\[(t_2 - \alpha)n + (x_2 - k\alpha)n = 0\]
or
\[t_2 = \alpha(l+k) - x_2.\]

From \( \langle W', W' \rangle = 1 \) we find
\[(\alpha(l+k) - x_2)^2 - x_2^2 = 1\]
or
\[\alpha^2(l+k)^2 - 2\alpha(l+k)x_2 = 1\]
or
\[x_2 = \frac{\alpha^2(l+k)^2 - 1}{2\alpha(l+k)}\]
Thus

\[ \frac{x_2}{t_2} = \frac{a^2(1+k)^2-1}{2a(1+k)} \cdot (a(1+k) - \frac{a^2(1+k)^2-1}{2a(1+k)})^{-1}. \]

Substituting above quantities, we get formulae *.

We can make an analogous study in four space-time dimensions. If \( \gamma_1 \) is parametrized by arclength \( \gamma_1(\tau) \) and \( \gamma_2 \) parametrized so that \( \|\gamma_2(\tau) - \gamma_1(\tau)\|^2 = 0 \) and \( \gamma_1 \) parameterized by arclength, \( \gamma_1(\tau) \). Again, letting \( V_1 = \gamma_2(\tau) - \gamma_1(\tau) \) and \( V_2 = \gamma_1(\tau) - \gamma_2(\tau) \), \( W = \dot{\gamma}_1, X = \dot{\gamma}_2, W' = \dot{\gamma}_1 \).

We have

(i) \( V_1, V_2 \) null, (non-zero and non-collinear)
(ii) \( \langle X-W, V_1 \rangle = 0 \)
(iii) \( \langle W'-X, V_2 \rangle = 0 \)
(iv) \( \langle W, W \rangle = 1 \)
(v) \( \langle W', W' \rangle = 1 \)
(vi) \( W'-W = aV_1 + bV_2 \).

We can then solve for \( W' \) in terms of \( W \) and the lines corresponding to \( X, V_1, \) and \( V_2, \) in the following way:

From (vi) and (iii)

\[ 0 = \langle W-X + aV_1 + bV_2, V_2 \rangle = \langle W', V_2 \rangle + a\langle V_1, V_2 \rangle \]

thus

\[ a = \frac{\langle X-W, V_2 \rangle}{\langle V_1, V_2 \rangle}. \]

From (v) and (vi) and (iv)

\[ 1 = \langle W', W' \rangle = \langle W + aV_1 + bV_2, W + aV_1 + bV_2 \rangle \]

\[ = \langle W, W \rangle + 2a\langle W, V_1 \rangle + 2b\langle W, V_2 \rangle + 2ab\langle V_1, V_2 \rangle. \]
Thus
\[
b = \frac{-a\langle W, V_1 \rangle}{\langle W, V_2 \rangle + a\langle V_1, V_2 \rangle}
\]

Combining these, we have
\[
\star\star\quad W' = W + \frac{\langle x - W, V_2 \rangle}{\langle V_1, V_2 \rangle} (V_1 \quad - \quad \frac{\langle W, V_1 \rangle}{\langle W, V_2 \rangle + \langle x - W, V_2 \rangle} V_2)
\]
\[
= W + \frac{\langle x - W, V_2 \rangle}{\langle V_1, V_2 \rangle} (V_1 \quad - \quad \frac{\langle x, V_1 \rangle}{\langle x, V_2 \rangle} V_2).
\]

Notice that this is independent of the "lengths" of \( V_1 \) and \( V_2 \), and given the ray corresponding to \( x \), (ii) fixes it uniquely. Moreover
\[
\langle W' - W, X \rangle = \frac{\langle x - W, V_2 \rangle}{\langle V_1, V_2 \rangle} (\langle V_1, X \rangle \quad - \quad \frac{\langle x, V_1 \rangle}{\langle x, V_2 \rangle} \langle V_2, X \rangle) = 0.
\]

Thus, since the velocities of \( \gamma_1 \) and \( \gamma_2 \) are bounded (continuous and defined on closed interval) the velocity of \( \tilde{\gamma}_1 \) is also bounded, \( \tilde{\gamma}_1(M) \) has finite coordinate time. The same argument as in the two dimensional space-time case shows that coordinate time of \( \tilde{\gamma}_1(M) \) is bounded away from zero.

The formula \( \star\star \) then defines a smooth unit time-like vector field of bounded velocity in the region bounded by the forward light cones of \( p_2 \) and \( p_4 \), minus those null lines through \( \gamma_1 \) and \( \gamma_2 \), and this can be extended smoothly through this surface as well. Hence there exists a unique curve \( \tilde{\gamma}_1(\tau) \) which is an integral curve of this vector field with \( \tilde{\gamma}_1(0) = p_3 \).

The analogous procedure can be used in the backwards time direction.
A light-trapezoid. The indicated lines are null.

A light-trapezoid with initial curves. The regions on which the line-element field is defined are shaded ($\Gamma_1$ and $\Gamma_2$).

$\gamma_2$ is a null translate of $\gamma_1$. $\gamma_3$ is a time-like translate of $\gamma_1$. 
Figure 6

$P_1$ point of $\gamma_1$
simultaneous to $P_2$.
In order that $\dot{\gamma}_1(P_3) - \dot{\gamma}_1(P_1)$ be parallel to the plane generated by $V_1$ and $V_2$ (all shown here in projection onto a spacelike slice tangent to the rotation) $P_2$ and $\bar{p}_1$ and centerline of rotation (out of the page) must be coplanar.
References


*All these may be found as reprints in Kerner, Theory of Action at a Distance in Relativistic Particle Mechanics, Gordon and Breach, 1972.