The Distribution Connected with the Dynamical System Generated by a Semi-group of Rotations

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Abstract of the Dissertation

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Marc Kac (in his book "Probability and related topics in physical sciences," Chapter II, §4) gave a method of solution of the following problem. Consider a chain of N links in space, each link having length I. Each link forms a fixed angle (called "valence angle") with the preceding link and no other constraints are put upon the chain.

The problem was to find the distribution of the "size" of the chain (defined as the distance between the initial and final points). The first solution of this problem was given by Moran (4). The new solution given by M. Kac, was based on perturbation theory. We use this method for another problem. Our problem is following: Consider the dynamical system which can be described by the following way. Let us consider in 3-dimensional Euclidean space two axes not situated in one plane, forming the

angle α with each other and at a distance R from each other. Let A^{ϕ} be a random rotation with respect to the first axis, B^{ϕ} a random rotation with respect to the second axis, where the angle ϕ is distributed uniformly on $[0,2\pi]$. We have a problem, which we can call the random walk problem in this context. Consider the transformation:

$$\overline{M}_0 \to \overline{M}_0 A^{\phi_1} B^{\phi_2} A^{\phi_3} B^{\phi_4} \ldots A^{\phi_{n-1}} B^{\phi_n}$$

($\overline{M}_0$ is a starting point) where: $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ are independent and distributed uniformly on $[0,2\pi]$. We wish to find out the limit distribution (as $n \to \infty$) of the three-dimensional random vector:

$$\frac{\overline{M}_n}{\sqrt{n}} = \frac{\overline{M}_0 \cdot A^{\phi_1} \cdot B^{\phi_2} \ldots A^{\phi_{n-1}} B^{\phi_n}}{\sqrt{n}} .$$

Theorem.

The distribution of the three-dimensional random vector $\frac{\overline{M}_n}{\sqrt{n}}$ (as $n \to \infty$) tends to the non-singular normal distribution. (All parameters of N-distribution are calculated.)

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The Distribution Connected with the Dynamical System Generated by a Semi-group of Rotations

Introduction (A) Let Ω be any given set (of total measure 1) on which a completely additive measure μ is given.

We assume that there is given a one-parameter family of transformations T_t of Ω onto itself which preserve the μ-measure.

Definition 1. Let $T_t^{-1}(A)$ be the inverse image of the set $A(\subset\Omega)$. The transformation T_t is said to be measure preserving if

$$(0,1) \qquad \mu\{(T_t^{-1}(A)\} = \mu\{A\}.$$

For one-to-one transformations (0,1) is clearly equivalent to the equal definition of "measure preserving"; i.e.,

$$(0,2) \qquad \mu\{T_t(A)\} = \mu\{A\}.$$

Now let $P_0 \in \Omega$. Denote by $g(P)$ the characteristic function of the measurable set A; i.e.,

$$g(P) = \begin{cases} 1, & P \in A, \\ 0, & P \overline{\in} A. \end{cases}$$

A problem is considered by Poincare and Birkhoff is the existance of the limit:

$$(0,3) \quad \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau g(T_t(P_0))dt \quad \text{for} \quad P_0 \in \Omega.$$

Together with this version, in which the "time" varies continuously, it is convenient to consider a discrete version.

Let T be a measure-preserving transformation; i.e.,

$$(0,4) \qquad \mu\{T^{-1}(A)\} = \mu\{A\},$$

and consider its powers (iterations) $T^2, T^3, \ldots$.

Definition II. We will say that the measurable set Ω, the semi-group of transformations T^k of the set Ω and an invariant measure compose a dynamical system.

The analogue of the limit (0,3) is now

$$(0,5) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(T^k(P_0)).$$

In 1931 G. D. Birkhoff succeeded in proving that the limits (0,3) and (0,5) exist for almost every P_0 (in the sense of μ-measure). A little earlier John von Neumann proved that the limits (0,3) and (0,5) exist in the sense of mean square (see [6]).

Definition III. A transformation is called metrically transitive if the only invariant sets are either of measure zero or one.

It is well-known that if we assume that our transformation T is metrically transitive, then for almost all P_0

$$(0,6) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(T^k(P_0)) = \mu\{A\}.$$

This can be generalized as follows: if $f(P_0)$ is μ-integrable; i.e.,

$$\int_{\Omega} |f(P_0)| d\mu < \infty,$$

and if T is metrically transitive, then for almost all P_0

$$(0,7) \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k(P_0)) = \int_{\Omega} f(P_0) d\mu$$

A generalization of this problem to a non-commutative semi-group of transformaitons, studied by Každan [7], runs as follows.

Let A and B be two rigid motions in the plane. Applying to a point X of the plane the transformations A, B, A^{-1}, B^{-1} in different orders we obtain a sequence of points M(X):

$$X;\ AX, BX, A^{-1}, B^{-1};\ A^2X, ABX, AA^{-1}X,\ \ldots,\ B^{-2}X;\ A^3X,\ \ldots$$

The sequence M(X) turns out to be uniformly distributed in the sense defined below.

We denote by $M^n(X)$ the set of points of the sequence $M(X)$ obtained after n transformations:

$$M^0(X) = X \; ; \; M^1(X) = AX, BX, A^{-1}X, B^{-1}X \; ; \; \ldots$$

Let O be a domain in the plane. We denote by $N^n(O)$ the number of points of $M^n(X)$ falling within O (taking multiplicities into account).

The sequence M(X) is said to be <u>uniformly distributed</u> if

$$\lim_{n\to\infty} \frac{N_X^n(O_1)}{N_X^n(O_2)} = \frac{\mu O_1}{\mu O_2}$$

for an arbitrary pair of Jordan measurable domains O_1 and O_2.

<u>Theorem</u> (D. Každan). If the rigid motions A and B do not commute, and if one of them is a rotation through an angle incommensurable with 2 , then for an arbitrary point X the sequence M(X) is uniformly distributed. (Note that in this case the subgroup generated by A and B is dense in the group of all rigid motions of the plane.) (See [7]).

Kazdan conjectures that an analogous result holds for an arbitrary homogeneous space. The result has been proved for homogeneous spaces of compact Lie groups in [5] and [8].

We will consider in this thesis the analogous <u>continuous</u> problem in 3-dimensional space: consider the dynamical system which can be described by the following way.

Let us consider in 3-dimensional Euclidean space two axes not situated in one plane, forming the angle α with each other and at a distnnce R from each other. Let A^{ϕ} be a random rotation with respect to the first axis, B^{ϕ} a random rotation with respect to the second axis, where the angle ϕ is distributed uniformly on $[0,2\pi]$. We have a problem, which we can call the random walk problem in this context. Consider the transformation:

$$\overline{M}_0 \to \overline{M}_0 \cdot A^{\phi_1} B^{\phi_2} A^{\phi_3} B^{\phi_4} \ldots A^{\phi_{n-1}} B^{\phi_n} \ldots$$

($\overline{M}_0$ is a starting point) where: $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ are independent and distributed uniformly on $[0,2\pi]$.

We wish to find out the limit distribution (as $n \to \infty$) of the 3-dimensional random vector:

$$\frac{\overline{M}_n}{\sqrt{n}} = \frac{M_0 \cdot A^{\phi_1} B^{\phi_2} \ldots A^{\phi_{n-1}} B^{\phi_n}}{\sqrt{n}}$$

We will prove the following result.

<u>Theorem</u>. The distribution of the 3-dimensional random vector $\frac{\overline{M}_n}{\sqrt{n}}$ (as $n \to \infty$) tends to the non-singular normal distribution. (All parameters of N-distribution are calculated.) (Note that if ϕ_2, ϕ_4, ϕ_6, etc. $= 0$ or τ, and ϕ_1, ϕ_3, ϕ_5, etc. $= 0$ or ω,

this gives the positive half of sequence of points studied by Kazdan.)

This problem is related to our studied by Mark Kac. In [1] he gave a method of solution of the following problem. Consider a chain of N links in space, each link having length 1. Each link forms a fixed angle (called "valence angle") with the preceding link and no other constraints are put upon the chain.

The problem was to find the distribution of the "size" of the chain (defined as the distance between the initial and final points).

The first solution of this problem was given by Moran [4]. The new solution given by M. Kac, was based on perturbation theory.

We use this method for our problem.

Let us notice that the most general case (related to our problem) studied by V. Tutubalin. But (in comparison with the solution given by Tutubalin) Kac's method gives us possibility to calculate all parameters of N-distribution.

<u>Introduction</u> (B) M. Kac considered the problem of finding the limiting distribution of a sequence of random variables related to the one we consider here. He introduced a new method which we will use for our problem.

The Kac method for studying the sequence of vector - valued random variables $\frac{R_n}{\sqrt{n}}$ consists in forming

$$g_m(xn^{-\frac{1}{2}}, yn^{-\frac{1}{2}}, zn^{-\frac{1}{2}}) = E\{\exp(xn^{-\frac{1}{2}}R_m\cdot a + y\cdot n^{-\frac{1}{2}}R_m\cdot b + z\cdot n^{-\frac{1}{2}}R_m c)\}$$

and deriving first a single term recursion relation

$$g_m(xn^{-\frac{1}{2}}, y\cdot n^{-\frac{1}{2}}, zn^{-\frac{1}{2}}) =$$
$$= \exp(zn^{-\frac{1}{2}})(2\pi)^{-1}\int_0^{2\pi} g_{m-1}(xn^{-\frac{1}{2}}\sin\theta - yn^{-\frac{1}{2}}\cos\theta, -zn^{-\frac{1}{2}}, xn^{-\frac{1}{2}}\cos\theta + yn^{-\frac{1}{2}}\sin\theta)\,d\theta$$

Kac's main idea is to replace this last relation by another one which enables results from the theory of linear operators on Hilbert space to be used to find the limiting behaviour of $g_m(x)$ as $m \to \infty$

If $g_m((1-z^2)^{\frac{1}{2}}n^{-\frac{1}{2}}, 0, zn^{-\frac{1}{2}}) = f_m(z)$

then the recorsion relation becomes

$$f_m(z) = \exp(zn^{-\frac{1}{2}})\int_{-1}^{1} K(\omega, z) f_{m-1}(\omega)\,d\omega$$

for a certain symmetric kernel $K(\omega, z)$

Therefore $f_{m-1} = T^m f$

for a certain linear operator T.

T^m is also a symmetric integral operator whose kernel turns out to be defined for non-integer values of m and to be analytic in m in some neighborhood of infinity.

Using the Rellich perturbation theory, Kac shows that the limiting distribution which he seeks depends only on the first eigenfunction of the operator T.

In the course of this work Kac derives some results for a particular kernel

$$K_{(2)}(z,\omega) = \int_{-1}^{1} K(z,\mu)K(\mu,z)\,d\mu$$

with $K(\omega,z) = \begin{cases} \pi^{-1}\left[1-(z^2+\omega^2)\right]^{-\frac{1}{2}}, & \omega^2+z^2<1 \\ 0, & \text{otherwise} \end{cases}$

Namely, he shows that

$$K_{(2)}(z,\omega) \sim \sum_{k=0}^{\infty} \lambda_k^2 \,\frac{2k+1}{2}\, P_k(z) P_k(\omega),$$

$$\lambda_k = \frac{1}{2\pi}\int_0^{2\pi} \sin^k\theta\, d\theta$$

We will use these specific results as well as the general method in treating our problem.

<u>Introduction</u> (C) Now let us return to the eigenvalue problem. Let us formulate some results which are well-known.

<u>Theorem (O,A)</u> (see [2], [10]).

Let $R_0, R_1, \ldots$ be bounded Hermitian operators in an infinite dimensional Hilbert space H. Suppose that $(u, R_0 u) \geqslant 0$ for all u in H and R_0 is completely continuous. Suppose that there exists a constant $k > 0$ such that

$$|(u, R_n u)| \leqslant k^n (u, R_0 u)$$

Then the following statements hold:

$$R(\varepsilon) = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \cdots$$

is a Hermitian, bounded operator in H, regular in

$$-\frac{1}{k} < \varepsilon < \frac{1}{k}$$

In $-\frac{1}{2k} < \varepsilon < \frac{1}{2k}$, the functions

$\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots$ and the elements $\psi_1(\varepsilon), \psi_2(\varepsilon), \ldots$

are all convergent power series in neighborhoods of each ε of this interval such that for $-\frac{1}{2k} < \varepsilon < \frac{1}{2k}$

the following is true:

(o,a) $R(\varepsilon) \cdot \psi_j(\varepsilon) = \lambda_j(\varepsilon) \cdot \psi_j(\varepsilon)$, $j = 1, 2, \ldots$

(o,b) $\psi_1(\varepsilon), \psi_2(\varepsilon), \ldots$ is an orthonormal complete system.

(o,c) $\lim\limits_{j \to \infty} \lambda_j(\varepsilon) = 0$

<u>Remarks.</u> Obviosly the infinitely many conditions

$$|(u, R_n u)| \leqslant k (u, R_0 u)$$

of Theorem (O,A) can be replaced by the conditions:

(o,d) $R(\varepsilon) = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \cdots$

is a bounded regular operator for complex ε with $|\varepsilon| < \rho$,

Hermitian for real ε.

(o,e) For all complex ε with $|\varepsilon| < \rho$ there exists a constant M such that $\|R(\varepsilon)u\| \leq M \|R_0 u\|$

or $|(u, R(\varepsilon)u)| \leq M(u, R_0 u)$

If $R(\varepsilon)$ is an integral operator, $R(\varepsilon)u = \int_{-1}^{1} \mathcal{K}(x,y;\varepsilon)u(y)dy$

and $\mathcal{K}(x,y;\varepsilon)$ is a power series in ε converging uniformly for all x, y of the interval $-1 \leq x, y \leq 1$.

and $\mathcal{K}(x,y;\varepsilon) = \mathcal{K}(y,x;\varepsilon)$

for real ε then our two conditions become

$$\int_{-1}^{1} \left| \int_{-1}^{1} \mathcal{K}(x,y;\varepsilon)u(y)\,dy \right|^2 dx \leq M^2 \int_{-1}^{1} \left| \int_{-1}^{1} \mathcal{K}(x,y;0)u(y)\,dy \right|^2 dx$$

<u>Theorem (O,B)</u> (Dini's Theorem , see [9]).

If the sum of an infinite series of non-negative continuous functions is a continuous function in a closed interval then the series converges uniformly there.

<u>Theorem (O,C)</u> (Mercer's Theorem, see [9]).
If the symmetric L_2-kernel $\mathcal{K}(x,y)$ is continuons and has only positive eigenvalues (or at most a finite number of negative eigenvalues), then the series

$$\sum_j \psi_j(x)\psi_j(y)\cdot\lambda_j$$

converges absolutely and uniformly, and the bilinear formula

$$\mathcal{K}(x,y) = \sum_{j=1}^{\infty} \lambda_j \psi_j(x)\psi_j(y)$$

holds.

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Chapter I

Let us consider in 3-dimensional Euclidean space axes not situated in one plane, forming the angle α with each other and at a distance R from each other.

Let A^{ϕ} be a random rotation with respect to the dirst axis, B^{ϕ} a random rotation with respect to the second axis, where the angle ϕ is distributed uniformly on $[0,2\pi]$. Note that in this case A^{ϕ} and B^{ϕ} do not commute. We have a problem, which we can call the random walk problem in this context. Consider the transformation:

$$\overline{M}_0 \to \overline{M}_0 \cdot A^{\phi_1} B^{\phi_2} A^{\phi_3} B^{\phi_4} \ldots A^{\phi_{n-3}} B^{\phi_{n-2}} A^{n-1} B^{n}$$

($\overline{M}_0$ is a starting point) where: $\phi_1, \phi_2, \ldots, \phi_n$, are independent and distributed uniformly on $[0,2\pi]$.

We wish to find out the limit distribution (as $n \to \infty$) of the three-dimensional random vector:

$$\frac{\overline{M}_n}{\sqrt{n}} = \frac{\overline{M}_0 A^{\phi_1} B^{\phi_2} A^{\phi_3} B^{\phi_n} \ldots A^{\phi_{n-3}} B^{\phi_{n-2}} A^{\phi_{n-1}} B^{\phi_n}}{\sqrt{n}}$$

For the simplicity of calculations, let us assume: $\alpha = \frac{\pi}{2}$; $R = 1$.

For the same reason we will assume that the starting point $\overline{M}_0$ is situated on the straight line crossing the second axis perpendicularly to the first axis.

Let us select Cartesian oordinates such that the random rotation A^{ϕ} has the matrix notation:

$$(X,Y,Z)\begin{Vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{Vmatrix} = (X,Y,Z)A_{\phi}$$

and the random rotation B^{ϕ} has the matrix notation:

$$[(X,Y,Z) - (0,-1,0)]\begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{Vmatrix} + (0,-1,0) = [(X,Y,Z)-e]B_{\phi} + $$

where the starting point is:

$$(X_0,Y_0,Z_0) = (0,-1,0) = e$$

In is easy to see that in our system of coordinates the expression

$$\frac{\bar{M}_n}{\sqrt{n}} = \frac{\bar{M}_0 A^{\phi_1} B^{\phi_2} A^{\phi_3} B^{\phi_4} \cdots A^{\phi_{n-1}} B^{\phi_n}}{\sqrt{n}}$$

becomes

$$(1) \quad \frac{eA_{\phi_1}B_{\phi_2}A_{\phi_3}\cdots A_{\phi_{n-1}}B_{\phi_n} - eB_{\phi_2}A_{\phi_3}\cdots A_{\phi_{n-1}}B_{\phi_n} + eA_{\phi_3}\cdots A_{\phi_{n-1}}B_{\phi_n} - \cdots + eA_{\phi_{n-1}}B_{\phi_n} - eB_{\phi_n} + e}{\sqrt{n}}$$

where I is a unit matrix.

Before we begin to study the limit distribution (1) let us formulate the theorem which will be basic for us.

Theorem I. If for any real ρ:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \exp(\rho a) d\sigma_n(a) = g(\rho) = \int_{-\infty}^{+\infty} \exp(\rho a) d\sigma(a)$$

and if $g(\rho)$ is an entire function (i.e., there is an entire function $g(\rho)$ on the real axis), then $\sigma_n(a) \to \sigma(a)$ at any point of continuity of σ (See [1]).

Let us formulate and let us prove some lemmas which we will need later on.

Lemma 1. Let Xn;Yn;Zn be sequences of one-dimensional random variables and suppose that for any real numbers x,y,z not all zero, the linear combination $xX_n + yY_n + zZ_n$ has the normal distribution as limit as $n \to \infty$. Then the joint distribution of these magnitudes (as $n \to \infty$) tends to the three-dimensional (non-singular) normal distribution. (See [3]).

The last lemma is given without proof. We remark that this statement remains in force under the weaker hypothesis that $x \cdot X_n + y \cdot Y_n + z \cdot Z_n \to$ normal distribution if $x^2 + y^2 + z^2 = 1$.

Lemma 2. Suppose the random vector $\bar{\zeta}$ has a three-dimensional normal distribution. In addition suppose its covariance matrix is diagonal.

Then the random vector $\bar{\zeta}B_\phi$ (where $\bar{\zeta}$ and ϕ are independent and the magnitude ϕ is distributed uniformly on

$[0,2\pi])$ has the three-dimensional normal distribution as well.

The proof is that the rotation of the vector $\overline{\zeta}$ takes place (on the random angle ϕ) around one of the axis of ellipsoid of equal probabilities (for $\overline{\zeta}$) and Jacobian does not depend on ϕ.

Lemma 3. Let $\overline{\zeta}_0, \overline{\zeta}_1, \ldots, \overline{\zeta}_{2n} \ldots,$ be a sequence of three-dimensional random vectors $\overline{\zeta}_i = (X_i^1, X_i^2, X_i^3)$ and let for $i \neq s$,

$$\left\{\begin{matrix} i; & s = 0,1,2,\ldots,2n,\ldots \\ j; & \mu = 1,2,3. \end{matrix}\right\}; \quad E[(X_i^j - EX_i^j)\cdot(X_s^\mu - EX_s^\mu)] = 0$$

If we know that as $n \to \infty$ the distributions of the random variables

$$\frac{T_n}{\sqrt{2n}} = \left(\frac{\tau_n^1}{\sqrt{2n}}, \frac{\tau_n^2}{\sqrt{2n}}, \frac{\tau_n^3}{\sqrt{2n}}\right) = \frac{\overline{\zeta}_0 + \overline{\zeta}_1 + \ldots + \overline{\zeta}_n}{\sqrt{2n}} \quad \text{and}$$

$$\frac{\Omega_n}{\sqrt{2n}} = \left(\frac{\omega_n^1}{\sqrt{2n}}, \frac{\omega_n^2}{\sqrt{2n}}, \frac{\omega_n^3}{\sqrt{2n}}\right) = \frac{\overline{\zeta}_{n+1} + \overline{\zeta}_{n+2} + \ldots + \overline{\zeta}_{2n}}{\sqrt{2n}} \quad \text{tend to}$$

normal distributions, then we can state, that the limit distribution of the sum $\dfrac{\overline{\zeta}_0 + \ldots + \overline{\zeta}_n + \ldots + \overline{\zeta}_{2n}}{\sqrt{2n}}$ will be normal as well.

Proof: Let us compose $(j,\mu = 1,2,3)$:

$$E\left[\frac{(\tau_n^j - E\tau_n^j)}{\sqrt{2n}} \frac{(\omega_n^\mu - E\omega_n^\mu)}{\sqrt{2n}}\right] = E\left[\frac{\sum_{i=0}^{n}(X_i^j - EX_i^j)}{\sqrt{2n}} \cdot \frac{\sum_{s=n+1}^{2n}(X_s^\mu - EX_s^\mu)}{\sqrt{2n}}\right] =$$

$$E\left[\frac{\sum_{i=0,s=n+1}^{n,2n}(X_i^j - EX_i^j)(X_s^\mu - EX_s^\mu)}{2n}\right] = \frac{1}{2n}\sum_{s,i} E(X_i^j - EX_i^j)(X_s^\mu - EX_s^\mu) = 0.$$

We have to show that:

$$\lim_{n\to\infty} E \frac{\sum_{i=0}^{n}(X_i^j - EX_i^j)}{\sqrt{2n}} \frac{\sum_{s=n+1}^{2n}(X_s^\mu - EX_s^\mu)}{\sqrt{2n}} = 0 =$$

$$E \lim_{n\to\infty} \frac{\sum_{i=0}^{n}(X_i^j - EX_i^j)}{\sqrt{2n}} \lim_{n\to\infty} \frac{\sum_{s=n+1}^{2n}(X_s^\mu - EX_s^\mu)}{\sqrt{2n}}$$

Since coordinates of a vector, which is normally distributed, are normally distributed too, we can interchange the operation of limit and Stieltjes integration using the corresponding theroems of Helly. Then from the non-correlatedness of normally distributed terms follows the normality of the distribution of the sum .(see [II]).

Let us deduce some useful relations, concerning the characteristics of (1). It is known that:

$$A'_\phi = A_\phi^{-1} = A_{-\phi};\ B'_\phi = B_\phi^{-1} = B_{-\phi};\ (A_\phi B_\phi)' = B_{-\phi}A_{-\phi},$$

where prime denotes transposition: $(A')_{ij} = (A)_{ji}$. Let us define $'A$ to be the result of transposing A with respect to the non-standard diagonal: $('A)_{ij} = (A)_{n-j+1,n-i+1}$ if A is

$n \times n$.

It is easy to see that:

1. $'(A_\phi B_\theta) = {'B_\theta}{'A_\phi} = A_\theta B_\phi$, since $'A_\phi = B_\phi$; $'B_\phi = A_\phi$.
2. $'[(A_\phi B_\theta)'] = '[B_\theta' A_\phi'] = '[B_{-\theta} A_{-\phi}] = {'A_{-\phi}}{'B_{-\theta}} = B_{-\phi} A_{-\theta}$
   $= [' (A_\phi B_\theta)]'$
3. $'(e') = '[(0,-1,0)'] = (0,-1,0) = e$.

Let us write (in the expression (1)) $n^{-\frac{1}{2}} T_n = (n^{-\frac{1}{2}}\tau_n^1, n^{-\frac{1}{2}}\tau_n^2, n^{-\frac{1}{2}}\tau_n^3) =$

$$\frac{eA_{\phi_1}B_{\phi_2}\cdots A_{\phi_{n-1}}B_{\phi_n} + eA_{\phi_3}B_{\phi_4}\cdots A_{\phi_{n-1}}B_{\phi_n} + eA_{\phi_5}B_{\phi_6}\cdots A_{\phi_{n-1}}B_{\phi_n} + \cdots + eA_{\phi_{n-1}}B_{\phi_n} + eI}{\sqrt{n}};$$

$n^{-\frac{1}{2}}\check{Q}_n = (n^{-\frac{1}{2}}\check{q}_n^1, n^{-\frac{1}{2}}\check{q}_n^2, n^{-\frac{1}{2}}\check{q}_n^3) =$

$$\frac{eB_{\phi_2}A_{\phi_3}\cdots B_{\phi_n} + eB_{\phi_4}A_{\phi_5}\cdots B_{\phi_n} + eB_{\phi_6}A_{\phi_3}\cdots B_{\phi_n} + eB_{\phi_{n-2}}A_{\phi_{n-1}}B_{\phi_n} + eB_{\phi_n}}{\sqrt{n}}$$

Let us write $Q_n = \check{Q}_n \cdot B_{-\phi_n}$, then $n^{-\frac{1}{2}}Q_n = (n^{-\frac{1}{2}}q_n^1, n^{-\frac{1}{2}}q_n^2, n^{-\frac{1}{2}}q_n^3) =$

$$\frac{eB_{\phi_2}A_{\phi_3}\cdots A_{\phi_{n-1}} + eB_{\phi_4}A_{\phi_5}\cdots A_{\phi_{n-1}} + eB_{\phi_6}A_{\phi_7}\cdots A_{\phi_{n-1}} + eB_{\phi_{n-2}}A_{\phi_{n-1}} + eI}{\sqrt{n}}$$

Then using characteristics (1.,2.,3) we get: $'(n^{-\frac{1}{2}}Q_n') =$

$$(*) \quad \frac{eA_{-\phi_2}B_{-\phi_3}\cdots B_{-\phi_{n-1}} + eA_{-\phi_4}B_{-\phi_5}\cdots B_{-\phi_{n-1}} + eA_{-\phi_6}B_{-\phi_7}\cdots B_{-\phi_{n-1}} + eA_{-\phi_{n-2}}B_{-\phi_{n-1}} + eI}{\sqrt{n}}$$

Let us take an arbitrary linear combination of the components of the vector $n^{-\frac{1}{2}}T_n$:

$$xn^{-\frac{1}{2}}\tau_n^1+yn^{-\frac{1}{2}}\tau_n^2+zn^{-\frac{1}{2}}\tau_n^3 = (\tau_n^1, \tau_n^2,\tau_n^3)\begin{pmatrix} n^{-\frac{1}{2}}x \\ n^{-\frac{1}{2}}y \\ n^{-\frac{1}{2}}z \end{pmatrix} = T_n \cdot \xi', \text{ where } x^2+y^2+z^2 = 1$$

$$\xi = (n^{-\frac{1}{2}}x, n^{-\frac{1}{2}}y, n^{-\frac{1}{2}}z);$$

We now state the next basic lemma.

Lemma 4. The limit distribution of the random variable $xn^{-\frac{1}{2}}\tau_n^1+yn^{-\frac{1}{2}}\tau_n^2+zn^{-\frac{1}{2}}\tau_n^3$ (as $n \to \infty$) will be a one-dimensional normal distribution.

Proof: Let us consider

$$E\{\exp[\rho(xn^{-\frac{1}{2}}\tau_n^1+yn^{-\frac{1}{2}}\tau_n^2+zn^{-\frac{1}{2}}\tau_n^3)]\} = E\{\exp[\rho T_n \cdot \xi']\}.$$

Let us write

$$g_m(\xi) = g_m(xn^{-\frac{1}{2}}, yn^{-\frac{1}{2}}, zn^{-\frac{1}{2}}) = E\{\exp[\rho(xn^{-\frac{1}{2}}\tau_m^1+yn^{-\frac{1}{2}}\tau_m^2+zn^{-\frac{1}{2}}\tau_m^3)]\}.$$

Then we can write down: $g_m(u) =$

$$\left(\frac{1}{2\pi}\right)^m \int_0^{2\pi}\cdots\int_0^{2\pi} \exp\{[eA_{\phi_2}B_{\phi_3}\cdots B_{\phi_m}+eA_{\phi_3}B_{\phi_4}\cdots B_{\phi_m}+\cdots+eA_{\phi_{m-1}}B_{\phi_m}+eI]\cdot\xi'\cdot\rho\} \times d\phi_1\cdots d\phi_m.$$

In just the same way we get: $d_m(\xi) = E\{\exp[\rho Q_m \cdot \xi']\} =$

$$\left(\frac{1}{2\pi}\right)^{m-2} \int_0^{2\pi}\cdots\int_0^{2\pi} \exp\{[eB_{\phi_2}A_{\phi_3}\cdots A_{\phi_{n-1}}+eB_{\phi_4}A_{\phi_5}\cdots A_{\phi_{m-1}}+\cdots+eB_{\phi_{m-2}}A_{\phi_{m-1}}+e$$

$$\times \xi'\cdot\rho\}d\phi_1\cdots d\phi_m.$$

From (*) it follows that: $d_m(\xi) = d_m(xn^{-\frac{1}{2}}, yn^{-\frac{1}{2}}, zn^{-\frac{1}{2}}) =$

$$\left(\frac{1}{2\pi}\right)^{m-2}\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{2\pi} \exp\{[eA_{-\phi_2}B_{-\phi_3}\cdots B_{-\phi_{m-1}} + eA_{-\phi_4}B_{-\phi_5}\cdots B_{-\phi_{m-1}} + \cdots + eA_{-\phi_{m-2}}B_{-\phi_{m-1}} + eI]$$

$$\cdot\, {}'\xi'\rho\}\, d\phi_2\cdots d\phi_{m-1}$$

doing the obvious change of variables:

$$= \left(\frac{1}{2\pi}\right)^{m-2}\int_0^{2}\int_0^{2}\cdots\int_0^{2}\int_0^{2} \exp\{[eA_{\phi_1}B_{\phi_2}\cdots A_{\phi_{m-3}}B_{\phi_{m-2}} + eA_{\phi_3}B_{\phi_4}\cdots B_{\phi_{m-2}} + \cdots + eA_{\phi_{m-3}}B_{\phi_{m-2}} + eI]$$

$$\cdot\, {}'\xi\cdot\rho\}\, d\phi_1\cdots d\phi_{m-2} =$$

From here we conclude:

$$(**)\quad d_m(\xi) \equiv g_{m-2}({}'\xi),\ \text{i.e.: } d_m(xn^{-\frac{1}{2}}, yn^{-\frac{1}{2}}, zn^{-\frac{1}{2}})$$

$$\equiv g_{m-2}(zn^{-\frac{1}{2}}, yn^{-\frac{1}{2}}, xn^{-\frac{1}{2}})$$

Then let us note that $g_m(\xi) =$

$$\frac{\exp(e\cdot\xi'\rho)}{(2\pi)^m}\int_0^{2\pi}\cdots\int_0^{2\pi} \exp\{[eA_{\phi_1}B_{\phi_2}\cdots B_{\phi_{m-2}} + eA_{\phi_3}B_{\phi_2}\cdots B_{\phi_{m-2}} + \cdots + eA_{\phi_{m-3}}B_{\phi_{m-2}} + eI]$$

$$(A_{\phi_{m-1}}B_{\phi_m}\cdot\xi'\rho)\, d\phi_1\cdots d\phi_m =$$

$$\frac{\exp(e\cdot\xi'\rho)}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} g_{m-2}(A_{\phi_{m-1}}B_{\phi_m}\xi')\, d\phi_{m-1}\, d\phi_m.$$

Using the fact that $e\cdot\xi' = (0,-1,0)\begin{pmatrix} n^{-\frac{1}{2}}x \\ n^{-\frac{1}{2}}y \\ n^{-\frac{1}{2}}z \end{pmatrix} = -n^{-\frac{1}{2}}y$, we re-write the above in the following way:

$$(2) \quad g_m(\xi) = \{\exp(-\rho n^{-\frac{1}{2}}y)\} \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} g_{m-2}(A_\phi B_\theta \xi')\,d\phi\,d\theta .$$

We may write the expression $A_\phi B_\theta \xi'$ in the following way:

$$\left\| \begin{matrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{matrix} \right\| \begin{pmatrix} n^{-\frac{1}{2}}x \\ n^{-\frac{1}{2}}y \\ n^{-\frac{1}{2}}z \end{pmatrix} = \left\| \begin{matrix} \cos\phi & \sin\phi\cdot\cos\theta & \sin\phi\cdot\sin\theta \\ -\sin\phi & \cos\phi\cdot\cos\theta & \cos\phi\cdot\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{matrix} \right\| \begin{pmatrix} n^{-\frac{1}{2}}x \\ n^{-\frac{1}{2}}y \\ n^{-\frac{1}{2}}z \end{pmatrix}$$

$$= \begin{pmatrix} n^{-\frac{1}{2}}x\cos\phi + n^{-\frac{1}{2}}y\sin\phi\cdot\cos\theta + n^{-\frac{1}{2}}z\sin\phi\cdot\sin\theta \\ -n^{-\frac{1}{2}}x\sin\phi + n^{-\frac{1}{2}}y\cos\phi\cdot\cos\theta + n^{-\frac{1}{2}}z\cos\phi\cdot\sin\theta \\ -n^{-\frac{1}{2}}y\sin\theta + n^{-\frac{1}{2}}z\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} n^{-\frac{1}{2}}x\cos\phi + n^{-\frac{1}{2}}\sqrt{y^2+z^2}\,\sin(\theta+\theta_0)\sin\phi \\ -n^{-\frac{1}{2}}x\sin\phi + n^{-\frac{1}{2}}\sqrt{y^2+z^2}\,\sin(\theta+\theta_0)\cos\phi \\ n^{-\frac{1}{2}}\sqrt{y^2+z^2}\,\cdot\cos(\theta+\theta_0) \end{pmatrix}$$

$$= \begin{pmatrix} n^{-\frac{1}{2}}\cos(\phi-\phi_0)\sqrt{x^2+(y^2+z^2)\sin^2(\theta+\theta_0)} \\ n^{-\frac{1}{2}}\sin(\phi-\phi_0)\sqrt{x^2+(y^2+z^2)\sin^2(\theta+\theta_0)} \\ n^{-\frac{1}{2}}\sqrt{y^2+z^2}\,\cos(\theta+\theta_0) \end{pmatrix}$$

where $\begin{cases} \theta_0 = \arcsin \dfrac{y}{\sqrt{y^2+z^2}} \\ \phi_0 = \arccos \dfrac{x}{\sqrt{x^2+(y^2+z^2)\sin^2(\theta+\theta_0)}} \end{cases}$

Let us substitute the above in (2): $g_m(n^{-\frac{1}{2}}x, n^{-\frac{1}{2}}y, n^{-\frac{1}{2}}z) =$

$$\frac{\exp(-n^{-\frac{1}{2}}y)}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} g_{m-2}\Big[n^{-\frac{1}{2}}\cos(\phi-\phi_0)\sqrt{x^2+(y^2+z^2)\sin^2(\theta+\theta_0)},$$

$$n^{-\frac{1}{2}}\sin(\phi-\phi_0)\sqrt{x^2+(y^2+z^2)\sin^2(\theta+\theta_0)},$$

$$n^{-\frac{1}{2}}\cos(\theta+\theta_0)\sqrt{y^2+z^2}\Big]d\phi d\theta$$

Because of periodicity (after the corresponding change of variables) we get: $g_m(n^{-\frac{1}{2}}x,\ n^{-\frac{1}{2}}y,\ n^{-\frac{1}{2}}z) =$

$$\frac{\exp(-\rho n^{-\frac{1}{2}}y)}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} g_{m-2}\Big[n^{-\frac{1}{2}}\cos\ \phi\sqrt{x^2+(y^2+z^2)\sin^2\ \theta},$$

$$n^{-\frac{1}{2}}\sin\ \phi\sqrt{x^2+(y^2+z^2)\sin^2\ \theta},$$

$$n^{-\frac{1}{2}}\cos\ \theta\sqrt{y^2+z^2}\Big]d\phi d\theta$$

Now let: $g_m(n^{-\frac{1}{2}}x, n^{-\frac{1}{2}}y, n^{-\frac{1}{2}}z)\cdot\exp\left(n^{-\frac{1}{2}}\rho y - \frac{n^{-1}\rho^2x^2}{2}\right) =$ $\tau_m(n^{-\frac{1}{2}}x, n^{-\frac{1}{2}}y, n^{-\frac{1}{2}}z)$ and $x^2+y^2+z^2=1$.

This yields:

(3) $$g_{m-2}\left(n^{-\frac{1}{2}}\cos\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-\frac{1}{2}}\sin\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-\frac{1}{2}}\sqrt{1-x^2}\cos\theta\right)=$$
$$=\tau_{m-2}\left(n^{-\frac{1}{2}}\cos\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-\frac{1}{2}}\sin\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-\frac{1}{2}}\sqrt{1-x^2}\cos\theta\right)\cdot$$
$$\cdot\exp\left(\frac{n^{-1}}{2}\rho^2\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right]-n^{-\frac{1}{2}}\rho\sin\phi\sqrt{1-(1-x^2)\cos^2\theta}\right).$$

We have: $\tau_m(n^{-1/2}x, n^{-1/2}y, n^{-1/2}z) = (2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2 x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi}\tau_{m-2}\Big(n^{-1/2}\cos\phi\sqrt{1-(1-x^2)\cos^2\theta},$
$n^{-1/2}\sin\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-1/2}\sqrt{1-x^2}\cos\theta\Big)\cdot\exp\Big(\frac{n^{-1}}{2}\rho^2\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right] - n^{-1/2}\rho\sin\phi\sqrt{1-(1-x^2)\cos^2\theta}\Big)$
$\cdot\, d\phi\, d\theta$

Set $y = \sqrt{1-x^2}\cos\beta$; $z = \sqrt{1-x^2}\sin\beta$.

Then $\tau_m(n^{-1/2}x, \sqrt{1-x^2}\cos\beta, \sqrt{1-x^2}\sin\beta) =$

$$(2\pi)^2\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right)\cdot\int_0^{2\pi}\int_0^{2\pi}\tau_m\Big(n^{-1/2}\cos\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-1/2}\sin\phi\sqrt{1-(1-x^2)\cos^2\theta},$$
$$n^{-1/2}\sqrt{1-x^2}\cos\theta\Big)\cdot\exp\Big(\frac{n^{-1}}{2}\rho^2\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right]-n^{-1/2}\rho\sin\phi\sqrt{1-(1-x^2)\cos^2\theta}\Big)d\phi\, d$$

It thus appears that τ_m is independent of β and hence we can take $\beta = 0$. This gives

$$\tau_m(n^{-1/2}x, n^{-1/2}y, n^{-1/2}z) = \tau_m(n^{-1/2}x, n^{-1/2}\sqrt{1-x^2}, 0)$$

But $\tau_{m-2}\left(n^{-1/2}\cos\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-1/2}\sin\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-1/2}\sqrt{1-x^2}\cos\theta\right) =$

$$\tau_{m-2}\left(n^{-1/2}\cos\phi\sqrt{1-(1-x^2)\cos^2\theta},\ n^{-1/2}\sqrt{1-\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right]},\ 0\right),$$

and setting $\tau_m(n^{-1/2}x, n^{-1/2}\sqrt{1-x^2}, 0) = F_m(x)$

we have: $F_m(x) =$

$$(2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi}F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\Big(\frac{n^{-1}}{2}\rho^2\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right]-n^{-1/2}\rho\sin\phi\sqrt{1-(1-x^2)\cos^2\theta}\Big)$$
$$\cdot\, d\phi\, d$$

Using the above, we will provide some particular results.For example :

$$\left| F_m(x) - (2\pi)^{-2} \exp\left(-\frac{n^{-1}\rho^2 x^2}{2}\right) \int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\varphi\sqrt{1-(1-x^2)\cos^2\theta}\right) \exp\left(\frac{n^{-1}\rho^2}{2}\left[1-(1-x^2)\cos^2\theta\right]\right) d\varphi\, d\theta \right|$$

$$< (2\pi)^{-2} \exp\left(-\frac{n^{-1}\rho^2 x^2}{2}\right) \frac{\Delta \rho^4}{n^2} \int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\varphi\sqrt{1-(1-x^2)\cos^2\theta}\right) \exp\left(\frac{n^{-1}\rho^2}{2}\left[1-(1-x^2)\cos^2\theta\right]\right) d\varphi\, d\theta$$

where Δ is independent of n.

We will use this formula to prove the following integral inequality :

$$\left| f_m - A_{(n)} f_{m-2} \right| < \Delta \rho^4 n^{-2} A_{(n)} f_{m-2}$$

where $F_m(x) \cdot \exp\left(\frac{\rho^2}{4} n^{-1} x^2\right) = f_m(x)$;

$$A_{(n)} f_{m-2}(x) = \int_{-1}^{1} f_{m-2}(p) \left\{ \int_{-1}^{1} \exp\left(-\frac{n^{-1}\rho^2 x^2}{4}\right) K(w,x) \exp\left(\frac{n^{-1}\rho^2}{2}[1-w^2]\right) K(p,w) \exp\left(-\frac{n^{-1}\rho^2 p^2}{4}\right) dw \right\} dp$$

and

$$K(w,x) = \begin{cases} \pi^{-1}\left[1-(x^2+w^2)\right]^{-1/2}, & \text{if } w^2+x^2<1, \\ 0 & \text{otherwise}. \end{cases}$$

Further we will study the first eigenvalue of the integral operator $A_{(n)}$.

Now,

using the expression $\exp\left(-n^{-\frac{1}{2}}\varrho\sin\phi\sqrt{1-(1-x^2)\cos^2\theta}\right) =$

$$\sum_{k=0}^{\infty}\frac{\left[n^{-1/2}\varrho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi\right]^{2k}}{(2k)!}-\sum_{k=1}^{\infty}\frac{\left[n^{-1/2}\varrho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi\right]^{2k-1}}{(2k-1)!}$$

obtain $F_m(x) =$

$$(2\pi)^{-2}\exp\left(-\frac{n^{-1}\varrho^2x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi}F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\varrho^2}{2}\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right]\right)\cdot$$
$$\cdot\sum_{k=0}^{\infty}\frac{\left[n^{-1/2}\varrho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi\right]^{2k}}{(2k)!}\,d\phi\,d\theta-(2\pi)^{-2}\exp\left(-\frac{n^{-1}\varrho^2x^2}{2}\right)\cdot$$
$$\cdot\int_0^{2\pi}\int_0^{2\pi}F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\varrho^2}{2}\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right]\right)\cdot\sum_{k=1}^{\infty}\frac{\left[n^{-1/2}\varrho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi\right]^{2k-1}}{(2k-1)!}\,d\phi\,d\theta$$

The second integral equals zero since

$F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\cdot\exp\left(\frac{n^{-1}\varrho^2}{2}\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right]\right)$ is even

function of ϕ and

$\sum_{k=1}^{\infty}\frac{\left[n^{-1/2}\varrho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi\right]^{2k-1}}{(2k-1)!}$ is odd.

This yields:

$F_m(x) =$

$$(2\pi)^{-2}\exp\left(-\frac{n^{-1}\varrho^2x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi}F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\varrho^2}{2}\cos^2\phi\left[1-(1-x^2)\cos^2\theta\right]\right)\cdot$$
$$\cdot\sum_{k=0}^{\infty}\frac{\left[n^{-1/2}\varrho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi\right]^{2k}}{(2k)!}\,d\phi\,d\theta$$

Setting $\sum_{k=0}^{\infty} \frac{[n^{-1/2}\rho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi]^{2k}}{(2k)!} \equiv \exp\left(\frac{n^{-1}\rho^2}{2}\sin^2\phi[1-(1-x^2)\cos^2\theta]\right) +$

$$+ \sum_{k=0}^{\infty} \frac{[n^{-1/2}\rho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi]^{2k}}{(2k)!} - \exp\left(\frac{n^{-1}\rho^2}{2}\sin^2\phi[1-(1-x^2)\cos^2\theta]\right)$$

we have: $F_m(x) =$

$$(2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\rho^2}{2}\cos^2\phi[1-(1-x^2)\cos^2\theta] + \frac{n^{-1}\rho^2}{2}\sin^2\phi[1-(1-x^2)\cos^2\theta]\right) \cdot d\phi\, d\theta$$

$$+(2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\rho^2}{2}\cos^2\phi[1-(1-x^2)\cos^2\theta]\right) \cdot$$

$$\cdot\left(\sum_{k=0}^{\infty}\frac{[n^{-1/2}\rho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi]^{2k}}{(2k)!} - \exp\left(\frac{n^{-1}\rho^2}{2}\sin^2\phi[1-(1-x^2)\cos^2\theta]\right)\right) d\phi\, d\theta =$$

$$= (2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\rho^2}{2}[1-(1-x^2)\cos^2\theta]\right) d\phi\, d\theta$$

$$+(2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\rho^2}{2}\cos^2\phi[1-(1-x^2)\cos^2\theta]\right) \cdot$$

$$\cdot\left(\sum_{k=0}^{\infty}\frac{[n^{-1/2}\rho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi]^{2k}}{(2k)!} - \exp\left(\frac{n^{-1}\rho^2}{2}\sin^2\phi[1-(1-x^2)\cos^2\theta]\right)\right) d\phi\, d\theta .$$

Let us notice that: $F_m(x) \geq 0$; $\exp\left(\frac{n^{-1}\rho^2}{2}\cos^2\phi[1-(1-x^2)\cos^2\theta]\right) \leq \exp\left(\frac{n^{-1}\rho^2}{2}[1-(1-x^2)\cos^2\theta]\right)$

and $\left|\sum_{k=0}^{\infty}\frac{[n^{-1/2}\rho\sqrt{1-(1-x^2)\cos^2\theta}\,\sin\phi]^{2k}}{(2k)!} - \exp\left(\frac{n^{-1}\rho^2}{2}\sin^2\phi\cdot[1-(1-x^2)\cos^2\theta]\right)\right| < L\rho^4 n^{-2}$,

where L is independent of n.

Hence

$$\left| F_m(x) - (2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right)\int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\rho^2}{2}[1-(1-x^2)\cos^2\theta]\right) d\phi\, d\theta \right|$$

$$< (2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right)\cdot L\rho^4 n^{-2}\int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\rho^2}{2}\cos^2\phi[1-(1-x^2)\cos^2\theta]\right) d\phi\, d\theta$$

$$\leq (2\pi)^{-2}\exp\left(-\frac{n^{-1}\rho^2x^2}{2}\right) L\rho^4 n^{-2}\int_0^{2\pi}\int_0^{2\pi} F_{m-2}\left(\cos\phi\sqrt{1-(1-x^2)\cos^2\theta}\right)\exp\left(\frac{n^{-1}\rho^2}{2}[1-(1-x^2)\cos^2\theta]\right) d\phi\, d\theta$$

Let us substitute $(1-x^2)^{1/2}\cos\theta = w$; $(1-w^2)^{1/2}\cos\phi = \rho$.

Then

$$(4) \quad \left| F_m(x) - \exp\left(-\frac{n^{-1}\varrho^2 x^2}{2}\right) \cdot \int_{-1}^{1} F_{m-2}(\rho)\left[\int_{-1}^{1} K(w,x)\cdot K(\rho,w)\exp\left(\frac{n^{-1}\varrho^2}{2}[1-w^2]\right)dw\right]d\rho \right| <$$

$$< 4\varrho^4 n^{-2}\exp\left(-\frac{n^{-1}\varrho^2 x^2}{2}\right)\int_{-1}^{1} F_{m-2}(\rho)\left[\int_{-1}^{1} K(w,x)\cdot K(\rho,w)\exp\left(\frac{n^{-1}\varrho^2}{2}[1-w^2]\right)dw\right]d\rho ,$$

$$K(w,x) = \begin{cases} \pi^{-1}\left[1-(x^2+w^2)\right]^{-1/2}, & \text{if } w^2+x^2<1 \\ 0, & \text{otherwise} \end{cases}$$

$$K(w,x) = \begin{cases} \pi^{-1}\left[1-(w^2+\rho^2)\right]^{-1/2}, & \text{if } w^2+\rho^2<1 \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that the unperturbed kernel :

$$K_{(2)}(x,\rho) = \int_{-1}^{1} K(w,x)K(\rho,w)\,dw$$

is square integrable, i.e.,

$$\int_{-1}^{1}\int_{-1}^{1} K_{(2)}^2(x,\rho)\,dx\,d\rho < \infty$$

In order to prove this, let us start from the known relation for Legendre polynomials :

$$(!)\quad \frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}_k[(1-x^2)^{\frac{1}{2}}\sin\theta]d\theta = i^k\lambda_k\mathcal{P}_k(x)$$

where $\mathcal{P}_k(x)$ is the k-th Legendre polynomial and

$$\lambda_k = \frac{1}{2\pi}\int_0^{2\pi} \sin^k\theta\, d\theta.$$

Let us rewrite this relation in the equivalent form:

$$(4a)\quad \int_{-1}^{1} K(w,x)\mathcal{P}_k(w)\,dw = i^k\lambda_k\mathcal{P}_k(x).$$

This yields at once $K_{(2)}(x,p) \sim \sum_{k=0}^{\infty} \lambda_k^2\,\frac{1}{2}(2k+1)\mathcal{P}_k(x)\mathcal{P}_k(p)$,

therefore:

$$\int_{-1}^{1}\int_{-1}^{1} K_{(2)}^2(x,p)\,dx\,dp < \infty.$$

Now let:

$$F_m(x)\cdot\exp\left(\frac{\rho^2}{4}n^{-1}x^2\right) = f_m(x).$$

Then (4) will become

$$(5)\quad \left| f_m(x) - \int_{-1}^{1} f_{m-2}(p)\left[\int_{-1}^{1}\exp\left(-\frac{n^{-1}\rho^2x^2}{4}\right)K(w,x)\cdot\exp\left(\frac{n^{-1}\rho^2}{2}[1-w^2]\right)K(p,w)\exp\left(-\frac{n^{-1}\rho^2p^2}{4}\right)dw\right]dp\right.$$

$$< L\rho^4n^{-2}\int_{-1}^{1} f_{m-2}(p)\left[\int_{-1}^{1}\exp\left(-\frac{n^{-1}\rho^2x^2}{4}\right)K(w,x)\exp\left(\frac{n^{-1}\rho^2}{2}[1-w^2]\right)K(p,w)\exp\left(-\frac{n^{-1}\rho^2p^2}{4}\right)dw\right]dp$$

where $f_0(x) = \exp\left(-\frac{n^{-1}\rho^2x^2}{4}\right)$.

So we have an integral Fredholm operator $A_{(n)}$ which is symmetric, positi[ve] and completely continuous. The last remark follows from the square-integrability of the perturbed kernel and from continuity on [-1,1] of the functions $\exp(-\frac{\rho^2}{4}n^{-1}x^2)$, $\exp\left(\frac{n^{-1}\rho^2}{2}[1-W^2]\right)$, $\exp(-\frac{\rho^2}{4}n^{-1}p^2)$. By Hilbert's theorem such an operator has in L_2 a complete orthonormal system of eigenfunctions, with real eigenvalues:

$$\psi_1^{(n)}(x);\ \psi_2^{(n)}(x)\ \ldots;\ \psi_k^{(n)}(x);\ \ldots .$$

$$\lambda_1(n);\ \lambda_2(n);\ \ldots;\ \lambda_k(n);\ \ldots .$$

The function $f_0(x)$ has the series expansion in eigenfunctions of our operator: $f_0(x) = \exp(-\frac{\rho^2}{4}n^{-1}x^2) =$

$$\sum_{j=1}^{\infty}\left[\int_{-1}^{1}\exp(-\frac{\rho^2}{4}n^{-1}x^2)\,\psi_j^{(n)}(x)\,dx\right]\psi_j^{(n)}(x)$$

and in particular /for even n /

$$(6)\quad A_{(n)}^{\frac{n}{2}}f_0(x) = \sum_{j=1}^{\infty}\lambda_j^{\frac{n}{2}}(n)\left[\int_{-1}^{1}\exp(-\frac{\rho^2}{4}n^{-1}x^2)\cdot\psi_j^{(n)}(x)\,dx\right]\psi_j^{(n)}(x).$$

The eigenvalues of the perturbed kernel $K_{(2)}^{(n)}(x,p)$ as $n \to \infty$ tend to the eigenvalues of the non-perturbed kernel $K_{(2)}(x,p)$, and since in the formula (6) all eigenvalues are raised to the $n/2^{th}$ power it is sufficient to consider only the eigenvalue close to 1; i.e., as $n \to \infty$:

$$(7)\quad A_{(n)}^{\frac{n}{2}}f_0(x) \sim \lambda_1^{\frac{n}{2}}(n)\left[\int_{-1}^{1}\exp(-\frac{\rho^2 n^{-1}}{4}x^2)\,\psi_1^{(n)}(x)\,dx\right]\psi_1^{(n)}(x).$$

In order to motivate it, we will prove a special Lemma (see pg. 39)

Now let us study the integral enequality (5) in the following form :

$$(8) \quad \left| f_m - A_{(n)} f_{m-2} \right| < L\rho^4 n^{-2} A_{(n)} f_{m-2},$$

where m and n are even.

Since $f_m - A_{(n)} f_{m-2} = f_m - A_{(n)}^{\frac{m}{2}} f_o - A_{(n)}(f_{m-2} - A_{(n)} f_{m-4}) - A_{(n)}^2 (f_{m-4} - A_{(n)} f_{m-6}) - \dots - A_{(n)}^{\frac{m}{2}-1}(f_2 - A_{(n)} f_o)$

(8) yields

$$(9) \quad \left| f_m - A_{(n)}^{\frac{m}{2}} f_o \right| < L\rho^4 n^{-2} \left(A_{(n)} f_{m-2} + A_{(n)}^2 f_{m-4} + \dots + A_{(n)}^{\frac{m}{2}} f_o \right)$$

Let us show that

$$(10) \quad A_{(n)}^S f_{m-2S} \leq (L\rho^4 n^{-2} + 1)^{\frac{m}{2}-S} \cdot A_{(n)}^{\frac{m}{2}} f_o \qquad \text{for any } 0 \leq S \leq \frac{m}{2}.$$

Using (8) /$\frac{m}{2}-S$ -times/, we get :

$$A_{(n)}^S f_{m-2S} = A_{(n)}^S (f_{m-2S} - A_{(n)} f_{m-2S-2}) + A_{(n)}^{S+1} f_{m-2S-2} < L\rho^4 n^{-2} A_{(n)}^{S+1} f_{m-2S-2} + A_{(n)}^{S+1} f_{m-2S-2} =$$
$$= (L\rho^4 n^{-2} + 1) A_{(n)}^{S+1} f_{m-2S-2} < (L\rho^4 n^{-2} + 1)^2 A_{(n)}^{S+2} f_{m-2S-4} < \dots$$
$$< (L\rho^4 n^{-2} + 1)^{\frac{m}{2}-S} \cdot A_{(n)}^{\frac{m}{2}} f_o$$

Now using the above, we obtain :

$$(8a) \quad \left| f_m - A_{(n)} f_{m-2} \right| < L\rho^4 n^{-2} A_{(n)} f_{m-2} < L\rho^4 n^{-2} (L\rho^4 n^{-2} + 1)^{\frac{m}{2}-1} \cdot A_{(n)}^{\frac{m}{2}} f_o$$

$$(9a) \quad \left| f_m - A_{(n)}^{\frac{m}{2}} f_o \right| < L\rho^4 n^{-2} \left[\sum_{S=1}^{\frac{m}{2}} A_{(n)}^S f_{m-2S} \right] < L\rho^4 n^{-2} \sum_{S=1}^{\frac{m}{2}} (L\rho^4 n^{-2} + 1)^{\frac{m}{2}-S} \cdot A_{(n)}^{\frac{m}{2}} f_o =$$

$$= \frac{L\rho^4 n^{-2} \left[(L\rho^4 n^{-2} + 1)^{\frac{m}{2}} - 1 \right] \cdot A_{(n)}^{\frac{m}{2}} f_o}{L\rho^4 n^2 + 1 - 1} =$$

$$= \left[(L\rho^4 n^{-2} + 1)^{\frac{m}{2}} - 1 \right] \cdot A_{(n)}^{\frac{m}{2}} f_o$$

In particular we have

$$(11)\quad \left| f_n - A_{(n)} f_{n-2} \right| < L\rho^4 n^{-2} \left(L\rho^4 n^{-2} + 1 \right)^{\frac{n}{2}-1} \cdot A_{(n)}^{\frac{n}{2}} f_0 \quad ;$$

$$(12)\quad \left| f_n - A_{(n)}^{\frac{n}{2}} f_0 \right| < \left[\left(L\rho^4 n^{-2} + 1 \right)^{\frac{n}{2}} - 1 \right] A_{(n)}^{\frac{n}{2}} f_0 \quad ,$$

hence :

$$(11a)\quad \left| f_n - A_{(n)} f_{n-2} \right| < \frac{C}{n^2} \quad ;$$

$$(12a)\quad \left| f_n - A_{(n)}^{\frac{n}{2}} f_0 \right| < \frac{D}{n} \quad ,$$

$A_{(n)}^{\frac{n}{2}} f_0$ is bounded uniformly because of Lemma (Pg. 39)

where all constants are independent of $n \, (n \to \infty)$.

Now consider the integral equation

$$(13)\qquad \lambda f - A_{(n)} f = 0$$

Using the usual methods of perturbation theory (see [2]) we can calculate

$$\lambda_1(n) \quad \text{and} \quad \psi_1^{(n)}(x) .$$

Let us write : $\lambda_1(n) = 1 + \mu_1 n^{-1} + \mu_2 n^{-2} + \mu_3 n^{-3} + \cdots$

$$\psi_1^{(n)}(x) = 2^{-\frac{1}{2}} + n^{-1} \varphi_1(x) + n^{-2} \varphi_2(x) + \cdots$$

and substitute them in the integral equation (13) :

$$\lambda f - A_n f = \lambda f(x) -$$

$$- \int_{-1}^{1} f(p) \left\{ \int_{-1}^{1} \exp\left(-\frac{\rho^2 n^{-1} x^2}{4}\right) \cdot K(w,x) \exp\left(\frac{\rho^2 n^{-1}}{2}[1-w^2]\right) K(p,w) \exp\left(-\frac{\rho^2 n^{-1} p^2}{4}\right) dw \right\} dp$$

$$= 0$$

Comparing the coefficients of n^{-1} on the right and the left sides yields, using the expression:

$$\exp\left(\frac{\rho^2 n^{-1}}{2}[1-w^2]-\frac{\rho^2 n^{-1}}{4}[x^2+\rho^2]\right)=1+\frac{\rho^2 n^{-1}}{2}(1-w^2)-\frac{\rho^2 n^{-1}}{4}(x^2+\rho^2)+\cdots$$

$$(14)\quad \varphi_1(x)+2^{-1/2}\mu_1=\int_{-1}^{1}\left\{2^{-5/2}\left[2\rho^2(1-w^2)-\rho^2x^2-\rho^2\rho^2\right]+\varphi_1(\rho)\right\}\int_{-1}^{1}K(w,x)K(\rho,w)\,dw\,d\rho$$

/Since the function $\varphi_1(x)$ is normalized, then $\varphi_1(x)$ must satisfy the relation:

$$\int_{-1}^{1}1\cdot\varphi_1(x)\,dx=0/.$$

Also we will use the relation:
$$\int_{-1}^{1}\int_{-1}^{1}K(w,x)K(\rho,w)\,dw\,dx==\int_{-1}^{1}\int_{-1}^{1}K(w,x)K(\rho,w)\,dw\,d\rho=(2\pi)^{-2}\int_0^{2\pi}\int_0^{2\pi}d\phi\,d\theta=1.$$

Integrating both sides of the equation (14) in x we get:

$$\int_{-1}^{1}\varphi_1(x)dx+2\cdot2^{-1/2}\mu_1=\int_{-1}^{1}\int_{-1}^{1}2^{-5/2}\left[2\rho^2(1-w^2)-\rho^2x^2-\rho^2\rho^2\right]\int_{-1}^{1}K(w,x)K(\rho,w)\,dw\,d\rho\,dx+$$
$$+\int_{-1}^{1}\varphi_1(\rho)\left\{\int_{-1}^{1}\int_{-1}^{1}K(w,x)\cdot K(\rho,w)\,dw\,dx\right\}d\rho=\int_{-1}^{1}\varphi_1(\rho)\,d\rho+$$
$$+\int_{-1}^{1}2^{-3/2}\rho^2\left\{\int_{-1}^{1}\int_{-1}^{1}(1-w^2)K(w,x)K(\rho,w)\,dw\cdot d\rho\right\}dx-$$
$$-2^{-5/2}\rho^2\int_{-1}^{1}x^2\left\{\int_{-1}^{1}\int_{-1}^{1}K(w,x)K(\rho,w)\,dw\,d\rho\right\}dx-2^{-5/2}\rho^2\int_{-1}^{1}\rho^2\left\{\int_{-1}^{1}\int_{-1}^{1}K(w,x)K(\rho,w)\,dw\,dx\right\}$$
$$=\int_{-1}^{1}\varphi_1(\rho)\,d\rho+2^{-3/2}\rho^2\int_{-1}^{1}(2\pi)^{-2}\left\{\int_0^{2\pi}\int_0^{2\pi}\left[1-(1-x^2)\cos^2\theta\right]d\phi\,d\theta\right\}dx-$$
$$-2^{-5/2}\rho^2\int_{-1}^{1}x^2\,dx-2^{-5/2}\rho^2\int_{-1}^{1}\rho^2\,d\rho.$$

This yields :

$$\mu_1 = \frac{\rho^2}{6}.$$

Let us notice that the integral inequalities $(11a)$ and $(12a)$ become equivalent to the correspondent integral equations (13) and (7) (as $n \to \infty$).

In particular, the difference between $(11a)$ and (13) is at most $\frac{C}{n^2}$.

This estimate gives us possibility to motivate calculation of μ_1 (pg. 32), since we compared the coefficients of n^{-1} and we did not need to compare the coefficients of n^{-2}. (Suppose, the right side of $(11a)$ involves n^{-1}, then it is impossible to compute μ_1).

Now let us substitute the values for $\lambda_1(n)$ and $\psi_1^{(n)}(x)$ in (7) and let us examine the limit as $n \to \infty$, we see that

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} f_n(x) = \exp\left(\frac{\rho^2}{12}\right) = \exp\left(\frac{1}{2}\sigma^2\rho^2\right), \text{ where } \sigma^2 = \frac{1}{6}$$

In addition, the convergence is uniform on every finite interval.

We recall the identities :

$$\exp(\tfrac{1}{2}\sigma^2\rho^2) = \sigma^{-1}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp(\rho a)\exp\left[-\frac{a^2}{2\sigma^2}\right]da$$

or

$$\exp\left[\frac{1}{2}\left(\frac{1}{\sqrt{6}}\right)^2\cdot\rho^2\right] = \sqrt{6}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \exp(\rho a)\exp\left[-3a^2\right]da.$$

Since: $\exp(\frac{1}{2}\sigma^2\rho^2)$ is an entire function, then applying theorem 1,

we finally obtain

$$P(x\tau_n^1 + y\tau_n^2 + z\tau_n^3 < a\sqrt{n}) \to \sqrt{6}(2\pi)^{-1} \int_{-\infty}^{a} \exp\left[-3a^2\right]dt$$

(uniformly with respect to a).

So Lemma 4 is proved.

From the relation (**pg.20) it follows that at the same time we have proved the following result:

$$P(xq_n^1 + yq_n^2 + zq_n^3 < a\sqrt{n}) \to \sqrt{6}\,(2\pi)^{-1} \int_{-\infty}^{a} \exp\left[-3a^2\right]dt$$

$$x^2 + y^2 + z^2 = 1;$$

From Lemma 1 it follows that the limit distribution of the vector

$$n^{-\frac{1}{2}}T_n = (n^{-\frac{1}{2}}\tau_n^1,\ n^{-\frac{1}{2}}\tau_n^2,\ n^{-\frac{1}{2}}\tau_n^3),$$

and also the limit distribution of the vector $n^{-\frac{1}{2}}Q_n = (n^{-\frac{1}{2}}q_n^1,\ n^{-\frac{1}{2}}q_n^2,\ n^{-\frac{1}{2}}q_n^3)$ are non-singular normal distributions. The axes of the ellipsoid of equal probabilities will be the first and second axes of rotation. Then Lemma 2 allows us to write as $n \to \infty$ the distribution of the vector:

$$n^{-\frac{1}{2}}\check{Q}_n = (n^{-\frac{1}{2}}\check{q}_n^1,\ n^{-\frac{1}{2}}\check{q}_n^2,\ n^{-\frac{1}{2}}\check{q}_n^3) = (n^{-\frac{1}{2}}q_n^1,\ n^{-\frac{1}{2}}q_n^2,\ n^{-\frac{1}{2}}q_n^3)\cdot B_{\phi_n}$$

is also a non-singular normal distribution. Let us show that the sum (in fact, any linear combination) of the vectors $n^{-\frac{1}{2}}T_n$ and $n^{-\frac{1}{2}}\check{Q}_n$, that is:

$$\beta\cdot n^{-\frac{1}{2}}T_n + \gamma\cdot n^{-\frac{1}{2}}\check{Q}_n$$

has a non-singular normal distribution as $n \to \infty$. For this, it is enough to check the realization of the conditions of Lemma 3; that is, we have to check for the vectors:

$$n^{-\frac{1}{2}}T_n = (n^{-\frac{1}{2}}\tau_n^1,\ n^{-\frac{1}{2}}\tau_n^2,\ n^{-\frac{1}{2}}\tau_n^3)$$

$$= n^{-\frac{1}{2}}(eA_{\phi_1}B_{\phi_2}\cdots A_{\phi_{n-1}}B_{\phi_n} + eA_{\phi_3}B_{\phi_4}\cdots A_{n-1}B_{\phi_n} + eA_{\phi_5}B_{\phi_6}\cdots B_{\phi_n} + \cdots + eA_{\phi_{n-1}}B_{\phi_n} + eI)$$

$$= n^{-\frac{1}{2}}(\overline{\zeta}_{\frac{n}{2}} + \overline{\zeta}_{\frac{n}{2}-1} + \cdots + \overline{\zeta}_2 + \overline{\zeta}_1 + \overline{\zeta}_0) \text{ where } \overline{\zeta}_0 = eI;$$

$\overline{\zeta}_1 = A_{\phi_{n-1}}B_{\phi_n}$; ... and so on and

$$n^{-\frac{1}{2}}\check{Q}_n = (n^{-\frac{1}{2}}\check{q}_n^1,\ n^{-\frac{1}{2}}\check{q}_n^2,\ n^{-\frac{1}{2}}\check{q}_n^3)$$

$$= n^{-\frac{1}{2}}(eB_{\phi_2}A_{\phi_3}\cdots B_{\phi_n} + eB_{\phi_4}A_{\phi_5}\cdots B_{\phi_n} + eB_{\phi_6}A_{\phi_7}\cdots B_{\phi_n} + \cdots + eB_{\phi_n})$$

$$= n^{-\frac{1}{2}}(\overline{\zeta}_n + \overline{\zeta}_{n-1} + \overline{\zeta}_{n-2} + \cdots + \overline{\zeta}_{\frac{n}{2}+2} + \overline{\zeta}_{\frac{n}{2}+1}) \text{ where}$$

$$\overline{\zeta}_{\frac{n}{2}+1} = eB_{\phi_n};\ \overline{\zeta}_{\frac{n}{2}+2} = eB_{\phi_{n-2}}A_{\phi_{n-1}}B_{\phi_n},\ \ldots$$

the realization of condition: (for $i \neq s$).

(11) $0 = E[(X_i^j - EX_i^j)(X_s^\mu - EX_s^\mu)]$ where $\overline{\zeta}_i = (X_i^1, X_i^2, X_i^3)$
$(i; s=0,1,2,\ldots,n$
$j,\mu=1,2,3.)$

If we introduce the notations:

$$\{e_j\}_{j=1,2,3} = \begin{cases} e_1 = (1,0,0) \\ e_2 = (0,1,0) = -e, \\ e_3 = (0,0,1) \end{cases}$$

then writing the coordinates of vectors as scalar products of these vectors with e_1, e_2, e_3 we may write the condition (11) of the Lemma (3) in the following way:

$$E\{[e_j \cdot (\overline{\zeta}_i - E\overline{\zeta}_i)']\cdot[(\overline{\zeta}_s - E\overline{\zeta}_s)\cdot e_\mu']\} = 0 \qquad (i \neq s)$$

Since products of independent random matrices correspond to vectors $\overline{\zeta}_i$ we may write

$$E\overline{\zeta}_i = -e_2 \cdot E(A_\phi B_\theta A_\omega B_\delta \cdots B_\tau) = -e_2 \cdot EA_\phi \cdot EB_\theta EA_\omega \cdots EB_\tau$$

$$= -(0,1,0)\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \cdots \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} = (0,0,0).$$

The mathematical expectation will be a non-zero vector only for $\overline{\zeta}_0$:

$$E\overline{\zeta}_0 = Ee = E(0,-1,0) = e = (0,-1,0) = -e_2.$$

Let us take two arbitrary vectors $\overline{\zeta}_i$, $\overline{\zeta}_s$ of the

$$\overline{\zeta}_i = -e_2 \cdot A_{\phi_{i_1}} B_{\phi_{i_2}} A_{\phi_{i_3}} \cdots B_{\phi_n}; \quad \overline{\zeta}_s = -e_2 B_{\phi_{i_s}} \cdots A_{\phi_{i_1}} B_{\phi_{i_2}} \cdots B_{\phi_n}$$

then we may write:

$$E\{\cdot[e_j \cdot (e_2 \cdot A_{\phi_{i_1}} B_{\phi_{i_2}} \cdots B_{\phi_n})' \quad e_2 \cdot B_{\phi_{i_s}} A_{\phi_{i_{s+1}}} \cdots A_{\phi_{i_1}} B_{\phi_{i_2}} \cdots B_{\phi_n}) \cdot e']\}$$

(using the independence of the matrices)

$$= E\{[e_j \cdot (e_2 A_{\phi_{i_1}} B_{\phi_{i_2}} \cdots B_{\phi_n})'][e_2 \cdot EB_{\phi_{i_s}} \; EA_{\phi_{i_{s+1}}} \cdots A_{\phi_{i_1}} B_{\phi_{i_2}} \cdots B_{\phi_n}) \cdot e'_\mu]\}$$

$$= E\ [e_j \ (e_2 A_{\phi_{i_1}} B_{\phi_{i_2}} \cdots B_{\phi_n})'][((0,0,0) A_{\phi_{i_1}} B_{\phi_{i_2}} \cdots B_{\phi_n}) \ e']\} = 0$$

which is what we wanted to prove. So, as a result, we have that the limit distribution of the random vector $\beta n^{-\frac{1}{2}} T_n + \gamma n^{-\frac{1}{2}} \check{Q}_n$ is a non-singular normal distribution. Assuming $\beta = 1$, $\gamma = -1$, we get the solution of our original problem: the distribution of vector

$$\frac{\overline{M}_n}{\sqrt{n}} = \frac{\overline{M}_0 A^{\phi_1} B^{\phi_2} A^{\phi_3} \ldots A^{\phi_{n-1}} B^{\phi_n}}{\sqrt{n}} =$$

$$\frac{eA_{\phi_1} B_{\phi_2} A_{\phi_3} \ldots A_{\phi_{n-1}} B_{\phi_n} - eB_{\phi_2} A_{\phi_3} \ldots A_{\phi_{n-1}} B_{\phi_n} + eA_{\phi_3} \ldots A_{\phi_{n-1}} B_{\phi} - \ldots - eB_{\phi_n} + eI}{\sqrt{n}}$$

as $n \to \infty$ tends to the non-singular normal distribution.

Now let us calculate the variance.

The variance for $n^{-1/2} \overline{T}_n - n^{-1/2} \check{Q}_n$

is equal to the sum of variances (because of statistical independency) : $\Sigma^2 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Since any one-dimentional N-distribution is independent on X, Y, Z , we can notice that N -distribution of $\frac{\overline{M}_n}{\sqrt{n}}$ is symmetric and Σ^2 is equal $\frac{1}{3}$.

Using another notation, we get :

$$\| \Sigma^2 \| = \left\| \begin{matrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{matrix} \right\| .$$

Chapter II

We wish to give a rigorous proof of equation (7).

For this purpose we will prove the following result:

Lemma.

$$f_n(x) = \sum_{j=1}^{\infty} \lambda_j^{n/2}(n)\left[\int_{-1}^{1} \exp\left(-\frac{\varrho^2}{4} n^{-1} x^2\right) \psi_j^{(n)}(x)\,dx\right] \psi_j^{(n)}(x)$$

converges uniformly for $n > N$, a constant independent of x.

This lemma enables us to assert that

$$\lim_{n\to\infty} f_n(x) = \sum_{j=1}^{\infty} \lim_{n\to\infty} \lambda_j^{n/2}(n)\left[\int_{-1}^{1} \exp\left(-\frac{\varrho^2}{4} n^{-1} x^2\right) \psi_j^{(n)}(x)\,dx\right] \psi_j^{(n)}(x)$$

but $\psi_j^{(n)}(x)$ are bounded uniformly in x, and

$$\lim_{n\to\infty} \lambda_j^{n/2}(n) = 0 \quad , \quad j = 2, 3, \ldots$$

Now remember that in our case

$$K_{(m)}(x, \rho; 0) = \sum_{k=0}^{\infty} \lambda_k^m \cdot \frac{2k+1}{2} P_k(x) \cdot P_k(\rho),$$

$$\lambda_k = \frac{1}{2\pi} \int_0^{2\pi} \sin^k\theta \, d\theta .$$

We can see that our iterated kernel is continuous and positive. We note that for our perturbed kernel all conditions of Theorem (0,A) are satisfied. Further we have to remember that eigenfunctions of a continuous kernel are also continuous (see [9]).

Let $n^{-1} = \varepsilon$

Theorem (0,A) yields that all eigenfunctions and eigenvalues of our perturbed kernel are continuous functions of ε in a closed interval (in neighborhood of zero).

We obtain $0<\lambda_j(\varepsilon)<1$ where $\frac{1}{\varepsilon}=n>N$; $j=2,3,\ldots$

$$\lambda_j(\varepsilon) = \frac{1}{2\pi}\int_0^{2\pi} \sin^j\theta\, d\theta + \varepsilon\lambda_1^{(j)} + \varepsilon^2\lambda_2^{(j)} + \cdots$$

In view of Theorem (O,C) we see thus that

$$\sum_j \lambda_j^m(\varepsilon)\,\psi_j(x,\varepsilon)\cdot\psi_j(y,\varepsilon)$$

converges uniformly in $-1 \leq x, y \leq 1$

and the bilinear formula

$$\mathcal{K}_{(m)}(x,y;\varepsilon) = \sum_j \lambda_j^m(\varepsilon)\,\psi_j(x,\varepsilon)\,\psi_j(y,\varepsilon)$$

holds.

Let us consider the diagonal kernel

$$\mathcal{K}_{(m)}(x,x;\varepsilon) = \sum_{j=0}^{\infty} \lambda_j^m(\varepsilon)\,\psi_j^2(x,\varepsilon)$$

The functions

$$\lambda_j(\varepsilon)\,\psi_j^2(x,\varepsilon)$$

are positive and continuous in closed domain $-1\leq x\leq 1$; $|\varepsilon| \leq \rho < \frac{1}{2k}$

Theorem (O,A) yields that

$$\mathcal{K}_{(m)}(x,y;\varepsilon)$$

is continuous function of ε in neighborhood of zero $(|\varepsilon|\leq\rho)$

Since the convergence of the series $\sum_{j=0}^{\infty}\lambda_j^m(\varepsilon)\,\psi_j^2(x,\varepsilon)$

is uniform in ε by virtue of Dini's Theorem, we see that

$$\sum_{j=0}^{\infty}\lambda_j^m(\varepsilon)\,\psi_j(x,\varepsilon)\,\psi_j(y,\varepsilon)$$

converges uniformly (in $\mathcal{E}$).

Therefore we can see, that for any integer ℓ, q the Schwarz inequality for sums gives:

$$\left[\sum_{h=\ell+1}^{\ell+q} \left|\lambda_h^m(\varepsilon)\psi_h(x,\varepsilon)\psi_h(y,\varepsilon)\right|\right]^2 \leq \sum_{h=\ell+1}^{\ell+q}\lambda_h^m(\varepsilon)\psi_h^2(x,\varepsilon)\cdot\sum_{h=\ell+1}^{\ell+q}\lambda_h^m(\varepsilon)\psi_h^2(y,\varepsilon)$$

So Lemma is proved.

BIBLIOGRAPHY

(* denotes books)

*[1] KAC, MARK. Probability and related topics in physical sciences, Chapter II, §4.

[2] RELLICH. Storungstheorie der Spektralzerlegung I, II, Math. Ann., 113 (1937), 600-619, 677-685.

*[3] ANDERSON, T. An introduction to multivariate statistical analysis, Chapter 2 (1957).

[4] MORAN. The statistical distribution of the length of a rubber molecule. Proc. Cambridge Phil Soc. 44 (1942), pp. 342-344.

[5] ARNOL'D, V. and KRYLOV, A. Uniform distributions of points on a sphere and some ergodic properties of solutions of ordinaty linear differential equations in a complex domain. Dokl, Acad. Nauk SSSR 148 (1963), 9-12 = Soviet Math. Dokl. 4 (1963), 1-5, MR27 #375.

[6] HALMOS, P. Lectures on ergodic theory, Chelsea, New York, 1956, Russian transl., Moscow 1959. MR22 #2677.

[7] KAZDAN, D. Uniform distribution in the plane. Transactions of the Moscow Math. Society for the year 1965 (14) published by the American Math. Society (1967).

[8] HELMBERG, G. A theorem on equidistribution on compact groups. Pacific J. Math. 8 (1958), 227-241.

*[9] TRICOMI, Integral equations , 3-12 , 124-127

*[10] RELLICH, F. Perturbation theory of eigenvalues problem published by N.Y.U., (1953), 154-155.

*[11] FELLER, W. An introduction to probability theory and its applications. Volume I, 1966.