ON THE ZEROES OF NONNEGATIVE CURVATURE OPERATORS

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Stanley Michael Zoltek

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Abstract of the Dissertation

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The object of this paper is to study the pointwise behavior of the Riemannian sectional curvature function σ.

Since we are interested in the pointwise behavior of σ, we work in the setting of an arbitrary inner product space V. G is the Grassmann manifold of oriented 2-planes in V. A curvature operator R is a self-adjoint linear transformation of $\Lambda^2(V)$ (e.g. the curvature tensor of a Riemannian manifold M acting on $\Lambda^2(M_m)$, where M_m is the tangent space to M at m). For a curvature operator R, its sectional curvature

$$\sigma_R : G \to \mathbb{R}$$

is given by

$$\sigma_R(P) = \langle RP, P \rangle$$

for P in G.

For dimension $V \leq 4$, Thorpe has shown [1] that the minimum and maximum sets of σ_R are intersections with G of linear subspaces of $\Lambda^2(V)$, and he has given [2] a simple characterization of positive sectional curvature in terms of the curvature tensor. In fact, Thorpe [1], claimed that his description of the minimum and maximum sets of σ_R was true for all dimensions.

We give a new proof of these results for dimension $V \leq 4$ and we show that they do not hold for higher dimensions. More specifically, for dimension $V \geq 5$ we exhibit a family of curvature operators with non-negative sectional curvature each of whose members does not conform to the characterization suggested by Thorpe's results [2] for lower dimensions. Furthermore, it is shown that one member of this family has a zero set which is not the intersection with G of a linear subspace of $\Lambda^2(V)$ and so contradicts Thorpe's result in [1].

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INTRODUCTION

The object of this paper is to study the pointwise behavior of the Riemannian sectional curvature function.

More specifically, the Riemannian sectional curvature of a Riemannian manifold M is a real valued function σ on the Grassmann bundle of tangent 2-planes of M. Let G denote the Grassmann manifold of oriented tangent 2-planes at $m \in M$. G can be made, in a natural way, a submanifold of the vector space Λ^2 of 2-vectors at m. Furthermore, since G is a 2-fold covering space of the manifold of (unoriented) 2-planes at m, we may regard σ as a function on G. We will be interested in the description of the minimum and maximum sets of σ and in the question of characterizing positive sectional curvature in terms of the curvature tensor.

Since we are interested in the pointwise behavior of σ, we shall work in the setting of an arbitrary inner product space V. G is then the Grassmann manifold of oriented 2-planes in V. A curvature operator R is a self-adjoint linear transformation of $\Lambda^2(V)$ (e.g. the curvature tensor R of a Riemannian manifold M acting on $\Lambda^2(M_m)$, where M_m is the tangent space to M at m). For a curvature operator R, its sectional curvature $\sigma_R : G \to R$ is given by

$$\sigma_R(P) = \langle RP, P \rangle$$

for P in G. (For more information regarding the motivation of these definitions see the Preliminaries.)

For dimension $V \leq 4$, Thorpe has shown [1] that the minimum and maximum sets of σ_R are intersections with G of linear subspaces of $\Lambda^2(V)$, and he has given [2] a simple characterization of positive sectional curvature in terms of the curvature tensor. In fact, Thorpe [1] claimed that his description of the minimum and maximum sets of σ_R was true for all dimensions.

In what follows, we shall give a new proof of these results for dimension $V \leq 4$ and we shall show that they do not hold for higher dimensions. More specifically, for dimension $V \geq 5$ we exhibit a family of curvature operators with non-negative sectional curvature each of whose members does not conform to the characterization suggested by Thorpe's result [2] for lower dimensions. Furthermore, it is shown that one member of this family has a zero set which is not the intersection with G of a linear subspace of $\Lambda^2(V)$ and so contradicts Thorpe's result in [1].

BACKGROUND

The Riemannian sectional curvature of a Riemannian manifold M is a real valued function σ on the Grassmann bundle of tangent 2-planes of M. A precise definition in terms of the Riemannian connection of the manifold will be given in the next section.

In this section, it is our goal to explain the importance of sectional curvature and to see why manifolds of strictly positive curvature are so interesting. To this end we shall state several results relating the curvature of a manifold with its topological properties. Finally we shall see that in certain situations, knowing the sectional curvature of a manifold tells us, up to diffeomorphism, that the manifold is a sphere (possibly an exotic sphere).

Perhaps the best known result relating the topology and curvature of a Riemannian manifold is the Gauss-Bonnet theorem [7]. It relates the Euler Characteristic $\chi(M)$ of a Riemannian manifold M with its sectional curvature function σ. In the case when dimension M = 2 it says that if M is a compact connected oriented 2-manifold, then $\int_M \sigma = \chi(M)$. In the case of the torus T, $\chi(T) = 0$ and so the Gauss-Bonnet theorem implies that we can not put a metric on the torus which gives rise to strictly positive (or strictly negative) sectional curvature.

Let δ be a positive number with $0 < \delta \leq 1$. An n-dimensional

Riemannian manifold M is said to be <u>δ-pinched</u> if its sectional curvature function σ satisfies

$$A\delta \le \sigma \le A$$

for some positive number A. If the metric g of a Riemannian manifold is replaced by cg (c a positive number) then the curvature function σ becomes $\frac{1}{c^2}\sigma$ and so we can "normalize" the metric so that $\delta \le \sigma \le 1$.

Hopf [11] proved the following theorem:

<u>Theorem 1</u>. A complete, simply connected 1-pinched Riemannian manifold M is isometric to an ordinary sphere.

J. Wolfe [12] has obtained a complete classification of the complete 1-pinched Riemannian manifolds.

The next theorem due to Myers [13] relates the sectional curvature of a manifold with its fundamental group.

<u>Theorem 2</u>. A complete δ-pinched Riemannian manifold M with $\delta > 0$ is compact and has finite fundamental group.

The following theorem is due to Synge [14].

<u>Theorem 3</u>. A complete δ-pinched Riemannian manifold M (with $\delta > 0$) of even dimension is either (1) simply connected or (2) non-orientable with $\pi_1(M) = Z_2$.

The next two theorems give a partial classification of

δ-pinched complete simply connected manifolds. The first is due to Berger [15] and the second to Klingenberg [16].

Theorem 4. Let M be a complete simply connected Riemannian manifold of even dimension which is δ-pinched. If $\delta > \frac{1}{4}$, then M is homeomorphic to a sphere. If $\delta = \frac{1}{4}$, then M is either homeomorphic to a sphere or isometric to a compact symmetric space of rank 1.

Theorem 5. A complete simply connected Riemannian manifold of odd dimension which is $\frac{1}{4}$-pinched is homeomorphic to a sphere.

In the above theorems (4 and 5) we do not know if M is diffeomorphic with an ordinary sphere. However, Gromoll [17] has proved the following theorem.

Theorem 6. There exists a sequence of numbers $\frac{1}{4} = \delta_1 < \delta_2 < \delta_3 < \ldots, \lim_{\lambda \to \infty} \delta_\lambda = 1$, such that if M is a complete simply connected n-dimensional Riemannian manifold which is δ_{n-2}-pinched, then M is diffeomorphic to an ordinary sphere.

In the previous theorem the pinching constant depended on the dimension of the manifold. Ruh [18] has found a way to avoid this difficulty. He considers the curvature tensor R of a Riemannian manifold M as a self adjoint linear transformation of $\Lambda^2(M_m)$ (the "curvature operator" of M; see the next section for a more detailed explanation). He then calls a Riemannian manifold <u>strongly δ-pinched</u> if the eigenvalues λ_i of R satisfy the condition $\delta \leq \lambda_i \leq 1$ for all i. Phrased in this terminology he gets the following dimension free result.

<u>Theorem 7</u>. There exists a constant $\delta \neq 1$ such that a complete, simply connected, strongly δ-pinched Riemannian manifold is diffeomorphic to the standard sphere of the same dimension.

Considering the wealth of theorems on manifolds of positive curvature it is of great interest to have easy criteria for determining when in fact curvature is positive for a given Riemannian manifold.

Though it is relatively easy to check if a self-adjoint operator is positive definite (e.g. by checking its eigenvalues) it is a formidable task to check positivity of sectional curvature for a non-positive definite curvature operator.

PRELIMINARIES

Let V be an n-dimensional real vector space with inner product $\langle , \rangle$ and for $v \in V$ set $|v| = \sqrt{\langle v,v \rangle}$. For p an integer, $1 \le p \le n$ by $\Lambda^p(V)$ or Λ^p we mean the space of p-vectors of V. If $\{e_1, \ldots, e_n\}$ is a basis for V, then $\{e_{i_1} \wedge \ldots \wedge e_{i_p} | i_1 < \ldots < i_p\}$ is a basis for Λ^p and it follows that Λ^p has dimension $\binom{n}{p}$. (For more detail see [3] or [4].) A p-vector ω is called decomposable if $\omega = v_1 \wedge \ldots \wedge v_p$ where $v_1, \ldots, v_p \in V$. Hence Λ^p has a basis of decomposable vectors. Thus when defining an inner product on Λ^p it suffices to specify its values on decomposable p-vectors. We set $\langle u_1 \wedge \ldots \wedge u_p, v_1 \wedge \ldots \wedge v_p \rangle = \det[\langle u_i, v_j \rangle]$ where $u_i, v_j \in V$. For $\xi \in \Lambda^2$ we set $\|\xi\| = \sqrt{\langle \xi, \xi \rangle}$. It follows that if $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V, then $\{e_{i_1} \wedge \ldots \wedge e_{i_p} | i_1 < \ldots < i_p\}$ is an orthonormal basis for Λ^p. Let $\{v_1, \ldots, v_p\}$ and $\{u_1, \ldots, u_p\}$ be two bases for some p-dimensional subspace P of V. It then follows that $v_1 \wedge \ldots \wedge v_p = \det A \; u_1 \wedge \ldots \wedge u_p$, where A is the change of basis matrix.

The dimension of $\Lambda^n(V) = \binom{n}{n} = 1$. Thus $\Lambda^n(V)$ is isomorphic to R and so $\Lambda^n(V) - \{0\}$ is disconnected and is the union of two connected components. An orientation of V is a choice of one of these components. The orientation determined by an ordered basis $\{v_1, \ldots, v_n\}$ for V is the component of $\Lambda^n(V)$ in which $v_1 \wedge \ldots \wedge v_n$ lies. Let $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_n\}$ be two ordered bases for V and let A be the change of basis matrix. It follows that these bases determine the same orientation of V if and only if det $A > 0$. (See [5].) Given an orientation of V, an ordered basis $\{v_1,\ldots,v_n\}$ is called an oriented basis if it determines the given orientation.

Two decomposable p-vectors α and β will be called equivalent if there exists a real number $\eta > 0$ such that $\alpha = \eta\beta$. The equivalence class of a p-vector $u_1\wedge\ldots\wedge u_p$ will be denoted by $[u_1\wedge\ldots\wedge u_p]$. In this way we can set up a one-to-one correspondence between oriented p-dimensional subspaces of V and the equivalence classes of decomposable p-vectors by:

$$P \leftrightarrow [u_1\wedge\ldots\wedge u_p]$$

where $\{u_1,\ldots,u_p\}$ is any oriented basis for the subspace P. Thus we can identify the Grassmann manifold G of oriented p-dimensional subspaces of V with the submanifold of Λ^p consisting of decomposable p-vectors of length one by $P \leftrightarrow u_1\wedge\ldots\wedge u_p$ where $\{u_1,\ldots,u_p\}$ is an oriented orthonormal basis of P. The elements of G are called p-planes. In what follows G will always be the Grassmann manifold of oriented 2-planes.

Let M be a Riemannian manifold with Riemannian connection ∇ (see [6], [7]). By M_m we mean the tangent space to M at m. The curvature tensor of a Riemannian connection ∇ is a linear transformation valued tensor R that assigns to each pair of vectors X and Y at m a linear transformation $R(X,Y)$ of M_m into itself. For $Z \in M_m$ we define $R(X,Y)Z$ be extending

X,Y and Z to C^{∞} fields about m and setting
$R(X,Y)Z = (\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z)_m$.

The <u>Riemannian sectional curvature</u> of a Riemannian manifold M is a real valued function σ on the Grassmann bundle of tangent 2-planes at m. For the 2-dimensional subspace P of M_m spanned by the orthonormal vectors X and Y we define its Riemannian sectional curvature by $\sigma(P) = \langle X,R(X,Y)Y\rangle$. It is easily checked that this value does not depend on the choice of orthonormal basis of P. (See [6], page 73.)

R can also be regarded as a self-adjoint linear operator on $\Lambda^2(M_m)$. We set $\langle RX_1\wedge Y_1,X_2\wedge Y_2\rangle = \langle R(X_1,Y_1)Y_2,X_2\rangle$ where $X_i,Y_i \in \Lambda^2(M_m)$. (Since $\Lambda^2(M_m)$ has a basis of decomposable 2-vectors, it suffices to specify the values of R on decomposable 2-vectors.) It is this interpretation of R which leads us to a generalization of the curvature tensor.

Let V be an n-dimensional real inner product space. A <u>curvature operator</u> R is a self-adjoint linear transformation of $\Lambda^2(V)$. The space $\mathfrak{R}$ of all curvature operators has dimension $[\binom{n}{2}^2 + \binom{n}{2}]/2$ and inner product given by:

$$\langle R,T\rangle = \text{trace } R\circ T \text{ where } R,T \in \mathfrak{R}.$$

We also generalize the concept of sectional curvature. Given $R \in \mathfrak{R}$ its <u>sectional curvature</u> is the function $\sigma_R : G \to \mathbb{R}$ defined by $\sigma_R(P) = \langle RP,P\rangle$, $P \in G$. We define the zero set of R by

$$Z(R) = \{P \in G \mid \sigma_R(P) = 0\}.$$

Let $\{e_1,\ldots,e_n\}$ be an oriented orthonormal basis for V. We define the star operator

$$* : \Lambda^p \to \Lambda^{n-p}$$

by

$$\langle *\alpha,\beta\rangle = \langle \alpha\wedge\beta, e_1\wedge\ldots\wedge e_n\rangle$$

where $\alpha \in \Lambda^p$ and $\beta \in \Lambda^{n-p}$. It is easily checked that this definition is independent of the choice of oriented orthonormal basis for V. It is also easily checked that $*^2 = (-1)^{p(n-p)}$ identity and so $*$ is non-singular (see [4]).

If dimension V = 4 and p = 2 then $* : \Lambda^2 \to \Lambda^2$ and since for $\alpha,\beta \in \Lambda^2$, $\alpha\wedge\beta = \beta\wedge\alpha$, it follows that $*$ is symmetric.

By $\mathbb{R}$ we denote the set of all real numbers. For $a,b \in \mathbb{R}$ we set $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$.

Section 1. THE BIANCHI IDENTITY AND THE GRASSMANN QUADRATIC 2-RELATIONS.

In this section we examine the space $\mathcal{S}$ complementary in $\mathcal{R}$ to the subspace $\mathcal{B} = \{R \in \mathcal{R} | R$ satisfies the Bianchi identity$\}$. It is shown that $\mathcal{S}$ is naturally isomorphic to Λ^4 and we establish the relationship between $\mathcal{S}$ and the Grassmann quadratic 2-relations which are necessary and sufficient conditions for decomposability of elements in Λ^2. These results are well known and the proofs we give are found in [1].

Given $R \in \mathcal{R}$ we associate a 2-form on V with values in the vector space of skew symmetric endmorphisms of V by:

$$\langle R(u,v)(w),x\rangle = \langle Ru\wedge v, w\wedge x\rangle, \quad u,v,w,x \in V.$$

It is easily checked that this "association" is a vector spaces isomorphism.

Using this identification we define the Bianchi map $b : \mathcal{R} \to \mathcal{R}$. Given $R \in \mathcal{R}$ we set $[b(R)](u,v)(w) = R(u,v)(w) + R(v,w)(u) + R(w,u)(v)$. Those curvature operators such that $b(R) = 0$ are said to satisfy the first Bianchi identity. It is easily checked that b is a linear map and so its kernel is a linear subspace of $\mathcal{R}$ which we will denote by $\mathcal{B}$.

Let $\mathcal{S} = \mathcal{B}^{\perp}$, the orthogonal compliment of $\mathcal{B}$ in $\mathcal{R}$. For each $\varepsilon \in \Lambda^4$ we associate $S_\varepsilon \in \mathcal{R}$ by $\langle S_\varepsilon \alpha, \beta\rangle = \langle \varepsilon, \alpha \wedge \beta\rangle$, where $\alpha, \beta \in \Lambda^2$. In order to show that $\varepsilon \mapsto S_\varepsilon$ is an isomorphism $\Lambda^4 \to \mathcal{S}$ we will need the following lemma.

<u>Lemma 1.1</u>. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. For $1 \le i < j < k < \ell \le n$, set $S_{ijk\ell} = S_{e_i \wedge e_j \wedge e_k \wedge e_\ell}$. If $R \in \mathcal{R}$, then

$$\langle R, S_{ijk\ell}\rangle = 2[\langle R(e_i \wedge e_j), e_k \wedge e_\ell\rangle + \langle R(e_j \wedge e_k), e_i \wedge e_\ell\rangle + \langle R(e_k \wedge e_i), e_j \wedge e_\ell\rangle].$$

<u>Proof</u>. $$\langle R, S_{ijk\ell}\rangle = \operatorname{tr} R \circ S_{ijk\ell} = \sum_{\alpha<\beta} \langle R \circ S_{ijk\ell}(e_\alpha \wedge e_\beta), e_\alpha \wedge e_\beta\rangle$$

$$= \sum_{\alpha<\beta} \langle S_{ijk\ell}(e_\alpha \wedge e_\beta), R(e_\alpha \wedge e_\beta)\rangle$$

$$= \sum_{\alpha<\beta} \langle S_{ijk\ell}(e_\alpha \wedge e_\beta), \sum_{\gamma<\delta} \langle R(e_\alpha \wedge e_\beta), e_\gamma \wedge e_\delta\rangle e_\gamma \wedge e_\delta\rangle$$

$$= \sum_{\alpha<\beta} \sum_{\gamma<\delta} \langle R(e_\alpha \wedge e_\beta), e_\gamma \wedge e_\delta\rangle \times \langle e_\alpha \wedge e_\beta \wedge e_\gamma \wedge e_\delta, e_i \wedge e_j \wedge e_k \wedge e_\ell\rangle.$$

Collecting terms finishes the proof.

<u>Proposition 1.2</u>. The map $\varepsilon \overset{\mathcal{S}}{\to} S_\varepsilon$ is an isomorphism of Λ^4 onto $\mathcal{S}$. In fact $\varepsilon \to \frac{1}{\sqrt{6}} S_\varepsilon$ is an isometry.

<u>Proof</u>. It is clear that s is a linear map of Λ^4 into $\mathcal{R}$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. The image $S_{ijk\ell}$ of an arbitrary basis element $e_i \wedge e_j \wedge e_k \wedge e_\ell$ $(i < j < k < \ell)$ of

Λ^4 belongs to $\mathcal{S}$. This follows, since by Lemma 1.1, $\langle R, S_{ijk\ell}\rangle = 0$ for all $R \in \mathcal{B}$. Thus s maps Λ^4 into $\mathcal{S}$. In fact given $R \in \mathcal{R}$, Lemma 1.1 implies that $R \in B \Leftrightarrow \langle R, S_{ijk\ell}\rangle = 0$ for all $ijk\ell$. This means that $\{S_{ijk\ell} | 1 \le i < j < k < \ell \le n\}$ spans $\mathcal{S}$ and so s maps Λ^4 onto $\mathcal{S}$. If we take $R = S_{\alpha\beta\gamma\delta}$ in Lemma 1.1 we see that this spanning set is also orthogonal and so a basis of $\mathcal{S}$ and furthermore that $\|S_{ijk\ell}\|^2 = 6$. This proves s is injective and that the map $\varepsilon \to \frac{1}{\sqrt{6}} S_\varepsilon$ is an isometry.

Proposition 1.3. $\alpha \in \Lambda^2$ is decomposable if and only if $\langle S\alpha, \alpha\rangle = 0$ for all $S \in \mathcal{S}$.

Proof (Necessity). By Proposition 1.2 for every $S \in \mathcal{S}$ there exists $\varepsilon \in \Lambda^4$ such that $S_\varepsilon = S$. But then $\langle S\alpha, \alpha\rangle = \langle S_\varepsilon \alpha, \alpha\rangle = \langle \varepsilon, \alpha\wedge\alpha\rangle = 0$.

(Sufficiency). Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. Then $\alpha = \Sigma_{1\le i<j\le n} a_{ij} e_i \wedge e_j$. By Proposition 1.2 $\langle S\alpha, \alpha\rangle = 0$ for all $S \in \mathcal{S}$ if and only if $\langle S_{ijk\ell}\alpha, \alpha\rangle = 0$ $1 \le i < j < k < \ell \le n$. It is easily shown that $\langle S_{ijk\ell}\alpha, \alpha\rangle = 2[a_{ij}a_{k\ell} - a_{ik}a_{j\ell} + a_{i\ell}a_{jk}]$ and it is well-known (see [8], page 309ff) (see also [9]) that α is decomposable if and only if $a_{ij}a_{k\ell} - a_{ik}a_{j\ell} + a_{i\ell}a_{jk} = 0$, $1 \le i < j < k < \ell \le n$.

Corollary 1.4. $\alpha \in \Lambda^2$ is decomposable if and only if $\alpha\wedge\alpha = 0$.

<u>Proof</u>. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. Then

$$\alpha = \Sigma_{1 \le i < j \le n} a_{ij} e_i \wedge e_j$$

and

$$\alpha\wedge\alpha = 2\Sigma_{1 \le i < j < k < \ell \le n} (a_{ij}a_{k\ell} - a_{ik}a_{j\ell} + a_{i\ell}a_{jk}) e_i \wedge e_j \wedge e_k \wedge e_\ell$$

$$= \Sigma_{1 \le i < j < k < \ell \le n} \langle S_{ijk\ell}\alpha, \alpha\rangle e_i \wedge e_j \wedge e_k \wedge e_\ell .$$

Thus $\alpha\wedge\alpha = 0$ if and only if $\langle S_{ijk\ell}\alpha, \alpha\rangle = 0$, $1 \le i < j < k < \ell \le n$, and so by Proposition 1.3 if and only if α is decomposable.

<u>Remark 1</u>. The conditions that $\langle S_{ijk\ell}\alpha, \alpha\rangle = 0$, $1 \le i < j < k < \ell \le n$, are known as the Grassmann quadratic 2-relations.

<u>Remark 2</u>. By Proposition 1.3 it is clear that each curvature operator $S \in \mathcal{S}$ has sectional curvature identically zero. The converse is also true. We have seen that for $R \in \mathcal{R}$, $R = S+T$ where $S \in \mathcal{S}$ and $T \in \mathcal{B}$. Now $\sigma_R = 0$ implies $\sigma_T = 0$. It is well-known (see [10] page 16) that if a curvature operator has zero sectional curvature and satisfies the Bianchi identity, then it is the zero operator. Thus T is identically zero and $R \in \mathcal{S}$.

Section 2. TWO RESULTS OF THORPE

In this section we restrict ourselves to the case dimension $V = 4$. It is our goal to re-examine two results of Thorpe (see [1], [2]). The first [2] gives a simple characterization of positive sectional curvature in terms of the corresponding curvature operator and the second [1] gives a description of the minimum and maximum sets of σ, the sectional curvature function. The proof we give of the first result will differ from that found in [2] and was suggested by Robert Geroch.

Let $\mathcal{R}^+ = \{R \in \mathcal{R} | \langle RX, X\rangle \geq 0 \ \forall \ X \in \Lambda^2\}$ and $\mathcal{B}^+ = \{R \in \mathcal{B} | \sigma_R \geq 0\}$

By definition of $\mathcal{S}$ and $\mathcal{B}$, $\mathcal{R} = \mathcal{B} \oplus \mathcal{S}$, where $\oplus$ means orthogonal direct sum.

We define π as orthogonal projection from $\mathcal{R}$ into $\mathcal{B}$. Since $\sigma_R = \sigma_{B+S} = \sigma_B$ it follows that $\pi(\mathcal{R}^+) \subseteq \mathcal{B}^+$ and so we can consider π as a map of $\mathcal{R}^+$ into $\mathcal{B}^+$.

<u>Theorem 2.1</u>. If dimension $V = 4$, then the map

$$\pi : \mathcal{R}^+ \to \mathcal{B}^+$$

is onto; i.e. each curvature operator satisfying the Bianchi identity and having non-negative sectional curvature is the orthogonal projection of a positive semi-definite operator

To prove Theorem 2.1 we will need the following lemma.

<u>Lemma 2.2</u>. For $\xi,\eta \in \Lambda^2$ set $\xi\cdot\eta = \langle \overset{*}{}\xi,\eta\rangle$ and $g(\xi) = \xi\cdot\xi$. Let $M = \{\xi \in \Lambda^2 | g(\xi) > 0\}$ and $N = \{\xi \in \Lambda^2 | g(\xi) < 0\}$. For $R \in \mathcal{R}$ set $Q = \{\xi \in \Lambda^2 | \langle R\xi,\xi\rangle < 0\}$. If $\sigma_R \geq 0$, then either $Q \subset M$ or $Q \subset N$.

<u>Proof</u>. Let $S = \{\xi \in \Lambda^2 | \|\xi\| = 1\}$.

Since $\overset{*}{}$ is a non-singular, self-adjoint linear transformation of Λ^2 it follows that $\cdot$ is a non-singular indefinite inner product on Λ^2. By Proposition 1.3 and the fact that Λ^4 is generated by $\overset{*}{} = S_{1234}$ it follows that $\xi \in \Lambda^2$ is decomposable if and only if $g(\xi) = 0$.

If $Q \cap M \neq \emptyset$ then the function $\xi \mapsto \langle R\xi,\xi\rangle$ assumes a minimum on $M \cap S$. In order to show this we will consider the sets $S_i = \{\xi \in S | g(\xi) < \frac{1}{i}\}$. Each S_i^c is a compact subset of S, $S_{i+1}^c \supset S_i^c$ and $\bigcup_{i=1}^{\infty} S_i^c = M \cap S$. Now on the compact set S_i^c the continuous function $\xi \mapsto \langle R\xi,\xi\rangle$ assumes a minimum at some point $\xi_i \in S_i^c$. Since $Q \cap M \neq \emptyset$ there exists $\eta \in M \cap S$ such that $\langle R\eta,\eta\rangle = c < 0$ (select $\tau \in Q \cap M$ and set $\eta = \tau/\|\tau\|$). If we assume the function $\xi \mapsto \langle R\xi,\xi\rangle$ does not attain a minimum on $M \cap S$ then the sequence $\{\xi_i\}_{i=1}^{\infty}$ contains a subsequence $\{\xi_{i_k}\}_{k=1}^{\infty}$ with the properties that $\xi_{i_k} \in S_{i_k}^c - S_{i_{k-1}}^c$ and $\langle R\xi_{i_{k+1}},\xi_{i_{k+1}}\rangle < \langle R\xi_{i_k},\xi_{i_k}\rangle < c$. But then $\lim_{k\to\infty} g(\xi_{i_k}) = 0$ and $\langle R\xi_{i_k},\xi_{i_k}\rangle < c$ for all k which contradicts $\sigma_R \geq 0$.

The minimum value $\mu < 0$ is an eigenvalue (apply the minimization technique of Lagrange multipliers to the function $\xi \to \langle R\xi,\xi\rangle$ on S) and so there exists $x \in M \cap S$ such that $Rx = \mu x$.

Similarly if $Q \cap N \neq \emptyset$, then there exists $y \in N \cap S$ such that $Ry = \varphi y$ where $\varphi < 0$.

Since the eigenvalues μ and φ corresponding to the eigenvectors x and y are both < 0, R is negative definite on the span of $\{x,y\}$. But $g(x) > 0$ and $g(y) < 0$ implies that $g(tx+(1-t)y) = 0$ for some t between 0 and 1. This together with $tx + (1-t)y \neq 0$ (since x and y are linearly independent) contradicts $\sigma_R \geq 0$. (Set $\xi = tx + (1-t)y$ and note $\frac{\xi}{\|\xi\|} \in G$ and $\sigma_R(\xi) < 0$.)

Proof of Theorem 2.1. By Lemma 2.2 $Q = \{\xi \in \Lambda^2 | \langle R\xi,\xi\rangle < 0\}$ is contained (entirely) in M or N.

Consider $R_t = R + t*$, $t \in R$ and set $Q_t = \{\xi \in \Lambda^2 | \langle R_t\xi,\xi\rangle < 0\}$. Applying Lemma 2.2 to R_t it is easily checked that for sufficiently large t, $Q_t \subset N$ and that for sufficiently negative t, $Q_t \subset M$. (In fact for each t, $Q_t \subset M$ or $Q_t \subset N$.) Thus we can define $a = \sup\{t | Q_t \subset M\}$ and $b = \inf\{t | Q_t \subset N\}$. By the definition of b and Lemma 2.2 it follows that $\langle R_b\xi,\xi\rangle \geq 0$ for all $\xi \in M$. Hence it follows by the definition of a that $a \leq b$. Therefore by the definition of a and b, for each $t \in [a,b]$ $Q_t \subset M$ and $Q_t \subset N$, and so by Lemma 2.2 $Q_t = \emptyset$. We thus conclude that for each $t \in [a,b]$, $R + t*$ is positive semi-definite. To complete the proof we set $S = t_o{}^*$, $t_o \in [a,b]$.

Theorem 2.3. Let dimension $V = 4$, and suppose $R \in \mathcal{R}$ is such that $\sigma_R \geq 0$ and $Z(R) \neq \emptyset$. Then there exists a unique $S \in \mathcal{S}$ such that $Z(R) = G \cap \text{kernel } (R+S)$. Recall that $Z(R) = \{\xi \in G | \langle R\xi, \xi\rangle = 0\}$.

Remark. The zero set of R, for $R \in \mathcal{R}$ such that $\sigma_R \geq 0$, $Z(R) \neq \emptyset$ is therefore the intersection with G of a linear subspace of Λ^2, when dimension $V = 4$.

Proof. By Theorem 2.1 there exists $t_o \in R$ such that $R+t_o{}^*$ is a positive semi-definite operator on Λ^2. Since $\langle (R+t_o{}^*)\xi,\xi\rangle \geq 0$ for all $\xi \in \Lambda^2$, it follows that those $\xi \in \Lambda^2$ such that $\langle (R+t_o{}^*)\xi,\xi\rangle = 0$ are eigenvectors of $R+t_o{}^*$ with eigenvalue 0. (Apply the minimization technique of Lagrange multipliers to the function $\xi \mapsto \langle (R+t_o{}^*)\xi,\xi\rangle$ on the unit sphere in Λ^2). Thus if $x \in Z(R)$, then $x \in G$ and $x \in \ker(R+t_o{}^*)$ and so $Z(R) \subset G \cap \ker(R+t_o{}^*)$. Conversely if $x \in G \cap \ker(R+t_o{}^*)$, then $x \in G$ and $\sigma_R(x) = \sigma_{R+t_o{}^*}(x) = \langle (R+t_o{}^*)x,x\rangle = \langle 0,x\rangle = 0$ and so $G \cap \ker(R+t_o{}^*) \subset Z(R)$.

To establish uniqueness, suppose for $t_1 \in R$, $Z(R) = G \cap \ker(R+t_o{}^*) = G \cap \ker(R+t_1{}^*)$. Then for $x \in Z(R)$, $(R+t_o{}^*)x = 0 = (R+t_1{}^*)x$, and so $t_o{}^*x = t_1{}^*x$. Since $*$ is non singular we have $t_o = t_1$.

To complete the proof set $S = t_o{}^*$.

<u>Corollary 2.4</u>. Let dimension $V = 4$ and $R \in \mathcal{R}$, and let λ denote the minimum (or maximum) value of σ_R. Then there exists a unique $S \in \mathcal{S}$ such that $\{P \in G | \sigma_R(P) = \lambda\} = G \cap \ker(R-\lambda I-S)$.

<u>Proof</u>. Follows from Theorem 2.4 by replacing R in that theorem by $R-\lambda I$ (or, when λ is the maximum value of σ_R by $\lambda I-R$).

Section 3. DENSE SUBSETS OF G

In this section dimension V = 5. We describe a collection of dense subsets of the Grassmann manifold G of oriented two dimensional subspaces of V. Specifically, given $P \in G$, we construct a dense subset of G which contains P. In the following sections this tool will greatly simplify our calculations.

Theorem 3.1. Given $P \in G$, let $\{e_1,\ldots,e_5\}$ be an orthonormal basis of V such that $P = e_1 \wedge e_2$. If for $x_1,\ldots,x_5 \in R$ we set $(x_1,x_2,x_3,x_4,x_5) = \sum_{i=1}^{5} x_i e_i$ then

$$Q = \{(1,0,x_3,x_4,x_5)\wedge(0,1,y_3,y_4,y_5)/\|(1,0,x_3,x_4,x_5)\wedge(0,1,y_3,y_4,y_5)\| \mid x_3,x_4,x_5,y_3,y_4,y_5 \in \mathbb{R}\}$$

is a dense subset of G which contains P.

To prove Theorem 3.1 we will need the following lemma.

Lemma 3.2. $G - Q = \{P \in G \mid \langle P, e_1 \wedge e_2 \rangle = 0\}$.

Proof. (Using the notation of Theorem 3.1)

$$P \in G \Rightarrow P = \frac{(x_1,x_2,x_3,x_4,x_5)\wedge(y_1,y_2,y_3,y_4,y_5)}{\|(x_1,x_2,x_3,x_4,x_5)\wedge(y_1,y_2,y_3,y_4,y_5)\|}$$

Now if

$$\langle P, e_1 \wedge e_2 \rangle = \frac{x_1 y_2 - x_2 y_1}{\|(x_1,x_2,x_3,x_4,x_5)\wedge(y_1,y_2,y_3,y_4,y_5)\|} \neq 0.$$

Then either $x_1 \neq 0$ or $y_1 \neq 0$. We can assume $x_1 \neq 0$ (by interchanging x's and y's if necessary). Dividing each x_i by x_1 we get

$$P = \frac{(1,x_2,x_3,x_4,x_5)\wedge(y_1,y_2,y_3,y_4,y_5)}{\|(1,x_2,x_3,x_4,x_5)\wedge(y_1,y_2,y_3,y_4,y_5)\|},$$

where for $i = 2,\ldots,5$ we abusively denote $\frac{x_i}{x_1}$ by x_i. Replacing y_i by $y_i - x_i y_1$ we get

$$P = \frac{(1,x_2,x_3,x_4,x_5)\wedge(0,y_2,y_3,y_4,y_5)}{\|(1,x_2,x_3,x_4,x_5)\wedge(0,y_2,y_3,y_4,y_5)\|}$$

where for $i = 2,\ldots,5$ we abusively denote $y_i - x_i y_1$ by y_i. Since $0 \neq \langle P, e_1 \wedge e_2 \rangle = y_2$ (the new y_2) we can divide each y_i by y_2 to get

$$P = \frac{(1,x_2,x_3,x_4,x_5)\wedge(0,1,y_3,y_4,y_5)}{\|(1,x_2,x_3,x_4,x_5)\wedge(0,1,y_3,y_4,y_5)\|}$$

where for $i = 2,\ldots,5$ we abusively denote $\frac{y_i}{y_2}$ by y_i. Finally by replacing x_i by $x_i - y_i x_2$ we get

$$P = \frac{(1,0,x_3,x_4,x_5)\wedge(0,1,y_3,y_4,y_5)}{\|(1,0,x_3,x_4,x_5)\wedge(0,1,y_3,y_4,y_5)\|}$$

where for $i = 3,4,5$ we abusively denote $x_i - y_i x_2$ by x_i. Hence we have shown $G - Q \subset \{P \in G \mid \langle P, e_1 \wedge e_2 \rangle = 0\}$.

It is clear that if $P \in Q \Rightarrow \langle P, e_1 \wedge e_2 \rangle \neq 0$ and so

$$\{P \in G \mid \langle P, e_1 \wedge e_2 \rangle = 0\} \subset G\text{-}Q.$$

Proof of Theorem 3.1. Lemma 3.2 shows that the complement of Q in G, $G\text{-}Q = \{P \in G \mid \langle P, e_1 \wedge e_2 \rangle = 0\}$. Since the function $P \to \langle P, e_1 \wedge e_2 \rangle$ is a smooth function on G it follows by the implicit function theorem that G-Q has co-dimension one in G and therefore Q is dense in G.

Section 4. THE CURVATURE OPERATOR R_k

In this section we discuss the possibility of extending Theorem 2.1 and 2.4 to the case dimension $V \geq 5$. Two claims are made and an example is presented. It will be the analysis of this example which occupies most of the remaining sections and results in a verification of these claims.

<u>Claim 4.1</u>. When the dimension $V \geq 5$, the zero set of a curvature operator with non-negative sectional curvature need not be the intersection with G of a linear subspace of Λ^2.

<u>Claim 4.2</u>. The map π, defined in Section 3, need not be onto. Indeed for dimension $V \geq 5$, there exist curvature operators with non-negative sectional curvature which can not be made positive semi-definite by adding an element of Λ^4.

Until further notice dimension $V = 5$. Let $\{e_1,\ldots,e_5\}$ be an orthonormal basis for V and k a real number. Set $e_{ij} = e_i \wedge e_j$ and consider the following example.

Let $R_k : \Lambda^2 \to \Lambda^2$ be defined by

$$R_k e_{12} = e_{12} - e_{15} - e_{34}$$
$$R_k e_{15} = e_{15} - e_{12} - e_{34}$$
$$R_k e_{34} = e_{34} - e_{12} - e_{15}$$
$$R_k e_{24} = R_k e_{35} = 0$$
$$R_k e_{13} = k e_{13}$$

$$R_k e_{14} = k e_{14}$$

$$R_k e_{23} = k e_{23}$$

$$R_k e_{25} = k e_{25}$$

$$R_k e_{45} = k e_{45}$$

It is easily checked that R_k is self-adjoint.

Let $\alpha = e_{12} + e_{15} + e_{34}$. Then

$$R_k \alpha = e_{12} - e_{15} - e_{34} + e_{15} - e_{12} - e_{34} + e_{34} - e_{12} - e_{15} = -\alpha.$$

In the next section it will be shown that R_k has non-negative sectional curvature.

Section 5. THE SECTIONAL CURVATURE OF R_k

In this section we will analyse sectional curvature on a dense subset of G containing the zero e_{24} of R. The sectional curvature of R will be shown to be non-negative on this subset, and so on all of G.

By Theorem 3.1

$$Q = \{[(\alpha,1,\beta,0,\gamma)\wedge(\delta,0,\varepsilon,1,\theta)]/\|[\alpha,1,\beta,0,\gamma)\wedge(\delta,0,\varepsilon,1,\theta)]\|$$

$$: \alpha,\beta,\gamma,\delta,\varepsilon,\theta \in \mathbb{R}\}$$

is a dense subset of G containing e_{24}.

Let ζ be a typical element of Q. Our goal being to show $\sigma_{R_k} \geq 0$, we can disregard the normalization factor, since $\langle R_k\|\zeta\|\zeta,\|\zeta\|\zeta\rangle = \|\zeta\|^2\langle R_k\zeta,\zeta\rangle$. Set $\xi = \|\zeta\|\zeta$. Then

$$\begin{aligned}
\xi &= [\alpha e_1+e_2+\beta e_3+\gamma e_5]\wedge[\delta e_1+\varepsilon e_3+e_4+\theta e_5] \\
&= \alpha\varepsilon e_{13}+\alpha e_{14}+\alpha\theta e_{15}-\delta e_{12}+\varepsilon e_{23} \\
&\quad +e_{24}+\theta e_{25}-\beta\delta e_{13}+\beta e_{34}+\beta\theta e_{35}-\gamma\delta e_{15} \\
&\quad -\gamma\varepsilon e_{35}-\gamma e_{45} \\
&= -\delta e_{12}+(\alpha\varepsilon-\beta\delta)e_{13}+\alpha e_{14}+(\alpha\theta-\gamma\delta)e_{15}+\beta e_{34} \\
&\quad +\varepsilon e_{23}+\theta e_{25}+e_{24}+(\beta\theta-\gamma\varepsilon)e_{35}-\gamma e_{45}
\end{aligned}$$

$$\begin{aligned}
R_k\xi &= -\delta[e_{12}-e_{15}-e_{34}] + (\alpha\theta-\gamma\delta)[e_{15}-e_{12}-e_{34}] \\
&\quad +\beta[e_{34}-e_{12}-e_{15}]+k[(\alpha\varepsilon-\beta\delta)e_{13}+\alpha e_{14}+\varepsilon e_{23}+\theta e_{25}-\gamma e_{45}] \\
&= (-\delta-\alpha\theta+\gamma\delta-\beta)e_{12}+(\delta+\alpha\theta-\gamma\delta-\beta)e_{15}+(\delta-\alpha\theta+\gamma\delta+\beta)e_{34} \\
&\quad +k[(\alpha\varepsilon-\beta\delta)e_{13}+\alpha e_{14}+\varepsilon e_{23}+\theta e_{25}-\gamma e_{45}]
\end{aligned}$$

$$\langle R_k\xi,\xi\rangle = (-\delta)(-\delta-\alpha\theta+\gamma\delta-\beta)+(\alpha\theta-\gamma\delta)(\delta+\alpha\theta-\gamma\delta-\beta)$$
$$+\beta(\delta-\alpha\theta+\gamma\delta+\beta)+k[(\alpha\varepsilon-\beta\delta)^2+\alpha^2+\varepsilon^2+\theta^2+\gamma^2]$$
$$= \delta^2+\delta\alpha\theta-\gamma\delta^2+\delta\beta+\alpha\theta\delta-\gamma\delta^2+\alpha^2\theta^2-2\alpha\theta\gamma\delta$$
$$-\alpha\beta\theta+\beta\gamma\delta+\gamma^2\delta^2+\beta\delta-\alpha\theta\beta+\beta\gamma\delta+\beta^2$$
$$+k[(\alpha\varepsilon-\beta\delta)^2+\alpha^2+\varepsilon^2+\theta^2+\gamma^2]$$
$$= (\delta+\beta)^2-2\gamma\delta^2+2\delta\alpha\theta-2\alpha\theta\beta+2\beta\gamma\delta-2\alpha\theta\gamma\delta$$
$$+\gamma^2\delta^2+\alpha^2\theta^2+k[(\alpha\varepsilon-\beta\delta)^2+\alpha^2+\varepsilon^2+\theta^2+\gamma^2] = (*).$$

For $k \geq 2$, we will write $(*)$ as the sum of squares of rational functions and hence conclude it is non-negative.

<u>Theorem 5.1</u>.

$$\langle R_k\xi,\xi\rangle = (1+\delta^2)\left[\left(\gamma + \frac{-\delta^2+\alpha\varepsilon-\alpha\theta\delta}{1+\delta^2}\right)^2 + \left(\beta + \frac{\delta-\alpha\varepsilon\delta-\alpha\theta}{1+\delta^2}\right)^2\right]$$
$$+ \frac{2(\alpha+\theta\delta)^2}{1+\delta^2} + \frac{2\varepsilon^2}{1+\delta^2} + \frac{2(\alpha+\varepsilon)^2\delta^2}{1+\delta^2} + \frac{2\theta^2}{1+\delta^2} + (\alpha\varepsilon-\beta\delta-\gamma)^2$$
$$+ (k-2)[(\alpha\varepsilon-\beta\delta)^2 + \alpha^2 + \varepsilon^2 + \theta^2 + \gamma^2].$$

<u>Proof</u>. It suffices to check this for $k = 2$, since $\langle R_k\xi,\xi\rangle = \langle R_2\xi,\xi\rangle + (k-2)[(\alpha\varepsilon+\beta\delta)^2 + \alpha^2 + \varepsilon^2 + \theta^2 + \gamma^2]$. Expanding the right side (RHS) we get

$$\text{RHS} = (1+\delta^2)[\gamma^2 + \frac{-2\gamma\delta^2+2\alpha\gamma\varepsilon-2\alpha\theta\gamma\delta}{1+\delta^2}$$
$$+ \frac{\delta^4-2\delta^2\alpha\varepsilon+2\alpha\theta\delta^3+\alpha^2\varepsilon^2-2\alpha^2\theta\delta\varepsilon+\alpha^2\theta^2\delta^2}{(1+\delta^2)^2}$$
$$+ \beta^2 + \frac{2\beta\delta-2\alpha\varepsilon\beta\delta-2\alpha\theta\beta}{1+\delta^2} + \frac{\delta^2-2\alpha\varepsilon\delta^2-2\alpha\theta\delta+\alpha^2\varepsilon^2\delta^2+2\alpha^2\theta\delta\varepsilon+\alpha^2\theta}{(1+\delta^2)^2}$$

$$+ \frac{2(\varepsilon^2+\alpha^2+\theta^2)(1+\delta^2)}{1+\delta^2} + \frac{(4\alpha\theta\delta+4\alpha\varepsilon\delta^2)}{1+\delta^2} + (\alpha\varepsilon-\beta\delta-\gamma)^2$$

$$= \gamma^2+\gamma^2\delta^2-2\gamma\delta^2+2\alpha\gamma\varepsilon-2\alpha\theta\gamma\delta + \frac{\delta^4}{1+\delta^2} - \frac{2\delta^2\alpha\varepsilon}{1+\delta^2}$$

$$+ \frac{2\alpha\theta\delta^3}{1+\delta^2} + \frac{\alpha^2\varepsilon^2}{1+\delta^2} + \frac{\alpha^2\theta^2\delta^2}{1+\delta^2} + \beta^2 + \beta^2\delta^2 + 2\beta\delta$$

$$- 2\alpha\varepsilon\beta\delta - 2\alpha\theta\beta + \frac{\delta^2}{1+\delta^2} - \frac{2\delta^2\alpha\varepsilon}{1+\delta^2} - \frac{2\alpha\theta\delta}{1+\delta^2}$$

$$+ \frac{\alpha^2\varepsilon^2\delta^2}{1+\delta^2} + \frac{\alpha^2\theta^2}{1+\delta^2} + 2\varepsilon^2 + 2\alpha^2 + 2\theta^2 + \frac{4\alpha\theta\delta}{1+\delta^2}$$

$$+ \frac{4\alpha\varepsilon\delta^2}{1+\delta^2} + (\alpha\varepsilon-\beta\delta-\gamma)^2$$

NOTE: $\frac{-2\alpha^2\theta\delta\varepsilon}{1+\delta^2}$ and $\frac{2\alpha^2\theta\delta\varepsilon}{1+\delta^2}$ have been combined.

Next note the following terms simplify:

$$\left(\frac{-2\delta^2\alpha\varepsilon}{1+\delta^2} + \frac{-2\delta^2\alpha\varepsilon}{1+\delta^2} + \frac{4\delta^2\alpha\varepsilon}{1+\delta^2}\right) = 0$$

$$\left(\frac{2\alpha\theta\delta^3}{1+\delta^2} + \frac{-2\alpha\theta\delta}{1+\delta^2} + \frac{4\alpha\theta\delta}{1+\delta^2}\right) = 2\alpha\theta\delta$$

$$\left(\frac{\alpha^2\theta^2}{1+\delta^2} + \frac{\alpha^2\theta^2\delta^2}{1+\delta^2}\right) = \alpha^2\theta^2$$

$$\left(\frac{\alpha^2\varepsilon^2}{1+\delta^2} + \frac{\alpha^2\varepsilon^2\delta^2}{1+\delta^2}\right) = \alpha^2\varepsilon^2$$

$$\beta^2 + 2\beta\delta - \delta^2 + \delta^2 + \frac{\delta^4}{1+\delta^2} + \frac{\delta^2}{1+\delta^2} = (\beta+\delta)^2.$$

Substitution yields:

$$\text{RHS} = \gamma^2+\gamma^2\delta^2-2\gamma\delta^2+2\alpha\gamma\varepsilon-2\alpha\theta\gamma\delta+\beta^2\delta^2-2\alpha\varepsilon\beta\delta-2\alpha\theta\beta$$

$$+ 2(\varepsilon^2+\theta^2+\alpha^2) + 2\alpha\theta\delta + \alpha^2\theta^2 + \alpha^2\varepsilon^2 + (\beta+\delta)^2 + (\alpha\varepsilon-\beta\delta-\gamma)^2.$$

Now

$$(\alpha\varepsilon-\beta\delta-\gamma)^2 + \gamma^2 + \alpha^2\varepsilon^2 + 2\alpha\gamma\varepsilon + \beta^2\delta^2 - 2\alpha\varepsilon\beta\delta$$
$$= 2(\alpha\varepsilon-\beta\delta)^2 + 2\beta\gamma\delta + 2\gamma^2.$$

Substitution yields:

$$\text{RHS} = \gamma^2\delta^2 - 2\gamma\delta^2 - 2\alpha\theta\gamma\delta - 2\alpha\theta\beta + 2(\varepsilon^2+\theta^2+\alpha^2)$$
$$+ 2\alpha\theta\delta + \alpha^2\theta^2 + 2\gamma^2 + 2\beta\gamma\delta + 2(\alpha\varepsilon-\beta\delta)^2 + (\beta+\delta)^2.$$

Remark. From the above expression of $\langle R_k\xi,\xi\rangle$ as the sum of squares of rational functions it follows that $\langle R_2\xi,\xi\rangle = 0$ if and only if $\alpha = \varepsilon = \theta = 0$ and $\gamma = \frac{\delta^2}{1+\delta^2}$, $\beta = \frac{-\delta}{1+\delta^2}$.

Normalizing, this gives a curve of zeroes, parametrized by δ, through e_{24}.

Section 6. SOME ZEROES OF R_2

In this section it is our goal to find two curves of zeroes of R_2 through the zero $(e_{12}+e_{15})/\sqrt{2}$. We will begin by examining a subset Q of G and finding a polynomial expression for $\langle R_2\xi,\xi\rangle$ for $\xi \in \Lambda^2$ such that $\xi/\|\xi\| \in Q$.

Let

$$Q = \{\zeta \in \Lambda^2 | \zeta = (1,\gamma,\alpha,\beta,-\gamma)\wedge(0,1+\theta,\delta,\varepsilon,1-\theta),$$
$$\alpha,\beta,\gamma,\delta,\varepsilon,\theta \in R\}$$

<u>Remark</u>. It can be shown that normalizing makes Q into a dense subset of G. However, for what follows we only need to know that it contains $(e_{12}+e_{15})$, which is obvious.

Let

$$\begin{aligned}
\xi &= (1,\gamma,\alpha,\beta,-\gamma)\wedge(0,1+\theta,\delta,\varepsilon,1-\theta)\\
&= [e_1+\gamma e_2+\alpha e_3+\beta e_4-\gamma e_5]\wedge[(1+\theta)e_2+\delta e_3+\varepsilon e_4+(1-\theta)e_5]\\
&= [e_1+\alpha e_3+\beta e_4+\gamma(e_2-e_5)]\wedge[e_2+\delta e_3+\varepsilon e_4+e_5+\theta(e_2-e_5)]\\
&= e_{12}+\delta e_{13}+\varepsilon e_{14}+e_{15}+\theta e_{12}-\theta e_{15}-\alpha e_{23}+\alpha\varepsilon_{34}\\
&\quad +\alpha e_{35}-\alpha\theta e_{23}-\alpha\theta e_{35}-\beta e_{24}-\beta\delta e_{34}+\beta e_{45}-\beta\theta e_{24}-\beta\theta e_{45}\\
&\quad +\gamma\delta e_{23}+\gamma\varepsilon e_{24}+\gamma e_{25}-\gamma\theta e_{25}+\gamma e_{25}+\gamma\delta e_{35}+\gamma\varepsilon e_{45}+\gamma\theta e_{25}\\
&= (1+\theta)e_{12}+\delta e_{13}+\varepsilon e_{14}+(1-\theta)e_{15}+(-\alpha-\alpha\theta+\gamma\delta)e_{23}\\
&\quad +(-\beta-\beta\theta+\gamma\varepsilon)e_{24}+2\gamma e_{25}+(\alpha\varepsilon-\beta\delta)e_{34}+(\alpha-\alpha\theta+\gamma\delta)e_{35}\\
&\quad +(\beta-\beta\theta+\gamma\varepsilon)e_{45}.
\end{aligned}$$

$$R_k\xi = (1+\theta)[e_{12}-e_{15}-e_{34}]+k\delta e_{13}+k\varepsilon e_{14}+(1-\theta)[e_{15}-e_{12}-e_{34}]$$
$$+k[-\alpha-\alpha\theta+\gamma\delta]e_{23}+k(2\gamma)e_{25}+(\alpha\varepsilon-\beta\delta)[e_{34}-e_{12}-e_{15}]$$
$$+k[\beta-\beta\theta+\gamma\varepsilon]e_{45}$$
$$= (1+\theta-1+\theta-\alpha\varepsilon+\beta\delta)e_{12}+(-1-\theta+1-\theta-\alpha\varepsilon+\beta\delta)e_{15}$$
$$+(-1-\theta-1+\theta+\alpha\varepsilon-\beta\delta)e_{34}+k[-\alpha-\alpha\theta+\gamma\delta]e_{23}$$
$$+k(2\gamma)e_{25}+k[\beta-\beta\theta+\gamma\varepsilon]e_{45}+k\delta e_{13}+k\varepsilon e_{14}$$

$$\langle R_k\xi,\xi\rangle = (1+\theta)(2\theta-(\alpha\varepsilon-\beta\delta))+(1-\theta)(-2\theta-(\alpha\varepsilon-\beta\delta))$$
$$+(\alpha\varepsilon-\beta\delta)(-2+\alpha\varepsilon-\beta\delta)+k[\delta^2+\varepsilon^2+(-\alpha-\alpha\theta+\gamma\delta)^2$$
$$+4\gamma^2+(\beta-\beta\theta+\gamma\varepsilon)^2]$$
$$= 2\theta^2+2\theta-(\alpha\varepsilon-\beta\delta)-\theta(\alpha\varepsilon-\beta\delta)+2\theta^2-2\theta-(\alpha\varepsilon-\beta\delta)$$
$$+\theta(\alpha\varepsilon-\beta\delta)-2(\alpha\varepsilon-\beta\delta)+(\alpha\varepsilon-\beta\delta)^2$$
$$+k[(-\alpha-\alpha\theta+\gamma\delta)^2\ (\beta-\beta\theta+\gamma\varepsilon)^2+\varepsilon^2+\delta^2+4\gamma^2]$$
$$= 4\theta^2-4(\alpha\varepsilon-\beta\delta)+(\alpha\varepsilon-\beta\delta)^2+k[(-\alpha-\alpha\theta+\gamma\delta)^2$$
$$+(\beta-\beta\theta+\gamma\varepsilon)^2+\delta^2+\varepsilon^2+4\gamma^2] = (*)$$

Set $\alpha = \gamma = \varepsilon = 0$ and $k = 2$. Then

$$\langle R_2\xi,\xi\rangle = 4\theta^2 + 4\beta\delta + \beta^2\delta^2 + 2\beta^2(1-\theta)^2 + 2\delta^2.$$

For fixed β set

$$f(\theta,\delta) = 4\theta^2 + 4\beta\delta + \beta^2\delta^2 + 2\beta^2(1-\theta)^2 + 2\delta^2.$$

Now $\sigma_{R_2} \geq 0 \Rightarrow \langle R_2\xi,\xi\rangle \geq 0 \Rightarrow f(\theta,\delta) \geq 0$. Thus a zero of f is minimum of f. But, at a minimum of f,

$$0 = \frac{\partial f}{\partial\theta} = 8\theta - 4\beta^2(1-\theta) = 4(2+\beta^2)\theta - 4\beta^2$$

and

$$0 = \frac{\partial f}{\partial\delta} = 4\beta + 2\beta^2\delta + 4\delta = 2(\beta^2+2)\delta + 4\beta.$$

Hence $\theta = \frac{\beta^2}{\beta^2+2}$ and $\delta = \frac{-2\beta}{\beta^2+2}$.

It remains to verify that for these values of θ and δ, $f(\theta,\delta) = 0$.

$$f\left(\frac{\beta^2}{\beta^2+2},\frac{-2\beta}{\beta^2+2}\right) = 4\left(\frac{\beta^2}{\beta^2+2}\right)^2 + 4\beta\left(\frac{-2\beta}{\beta^2+2}\right) + \beta^2\left(\frac{-2\beta}{\beta^2+2}\right)^2$$

$$+ 2\beta^2\left(1-\frac{\beta^2}{\beta^2+2}\right)^2 + 2\left(\frac{-2\beta}{\beta^2+2}\right)^2$$

$$= \frac{4\beta^4}{(\beta^2+2)^2} - \frac{8\beta^2}{\beta^2+2} + \frac{4\beta^4}{(\beta^2+2)^2} + 2\beta^2\left(\frac{2}{\beta^2+2}\right)^2 + \frac{8\beta^2}{(\beta^2+2)^2}$$

$$= \frac{4\beta^4}{(\beta^2+2)^2} - \frac{8\beta^2(\beta^2+2)}{(\beta^2+2)^2} + \frac{4\beta^4}{(\beta^2+2)^2} + \frac{8\beta^2}{(\beta^2+2)^2} + \frac{8\beta^2}{(\beta^2+2)^2}$$

$$= \frac{4\beta^4-8\beta^4-16\beta^2+4\beta^4+16\beta^2}{(\beta^2+2)^2} = \frac{0}{(\beta^2+2)^2} = 0.$$

Thus $\langle R_2\xi,\xi\rangle = 0$ if

$$\xi = \left(1 + \frac{\beta^2}{\beta^2+2}\right)e_{12} + \left(\frac{-2\beta}{\beta^2+2}\right)e_{13} + \left(1 - \frac{\beta^2}{\beta^2+2}\right)e_{15}$$

$$+ (-\beta)\left(1 + \frac{\beta^2}{\beta^2+2}\right)e_{24} + \left(\frac{2\beta^2}{\beta^2+2}\right)e_{34} + (\beta)\left(1 - \frac{\beta^2}{\beta^2+2}\right)e_{45}.$$

Set $\xi^1 = (\beta^2+2)\xi = (2\beta^2+2)e_{12} + (-2\beta)e_{13} + 2e_{15}$

$$+ (-\beta)(2\beta^2+2)e_{24} + (2\beta^2)e_{34} + (2\beta)e_{45}$$

then

$$\langle R_2\xi^1,\xi^1\rangle = 0.$$

We then have $\beta \mapsto \frac{\xi^1(\beta)}{\|\xi^1(\beta)\|}$ is a curve of zeroes through $(e_{12}+e_{15})/\sqrt{2}$.

If in (*) we set $k = 2$ and $\delta = \beta = \gamma = 0$ we get $\langle R_2\xi,\xi\rangle = 4\theta^2 - 4\alpha\varepsilon + \alpha^2\varepsilon^2 + 2(\alpha+\alpha\theta)^2 + \varepsilon^2$. Following an approach identical to that above gives

$$\xi^2 = 2e_{12} + (2\alpha)e_{14} + (2\alpha^2+2)e_{15} - 2\alpha e_{23} + 2\alpha^2 e_{34} + (\alpha)(2\alpha^2+2)e_{35}.$$

For the sake of brevity we will merely check that $\frac{\xi^2}{\|\xi^2\|}$ is decomposable and that $\sigma_{R_2}(\frac{\xi^2}{\|\xi^2\|}) = 0$. It suffices to show $\xi^2\wedge\xi^2 = 0$ and $\langle R_2\xi^2,\xi^2\rangle = 0$.

$$\begin{aligned}\tfrac{1}{2}\xi^2\wedge\xi^2 &= [2e_{12} + 2\alpha e_{14} + (2\alpha^2+2)e_{15} - 2\alpha e_{23} + 2\alpha^2 e_{34} \\ &\quad + (\alpha)(2\alpha^2+2)e_{35}]\wedge[2e_{12} + 2\alpha e_{14} + (2\alpha^2+2)e_{15} \\ &\quad - 2\alpha e_{23} + 2\alpha^2 e_{34} + (\alpha)(2\alpha^2+2)e_{35}] \\ &= (4\alpha^2-4\alpha^2)e_{1234} + (2\alpha^3+4\alpha-2\alpha^3-4\alpha)e_{1235} \\ &\quad + (0)e_{1245} + (4\alpha^4+4\alpha^2-4\alpha^4-4\alpha^2)e_{1345} + (0)e_{2345} \\ &= 0.\end{aligned}$$

$$\begin{aligned}R_2\xi^2 &= 2[e_{12}-e_{15}-e_{34}] + 4\alpha e_{14} + (2\alpha^2+2)[e_{15}-e_{12}-e_{34}] - 4\alpha e_{23} \\ &\quad + 2\alpha^2[e_{34}-e_{12}-e_{15}] \\ &= [2-2\alpha^2-2-2\alpha^2]e_{12} + 4\alpha e_{14} + [-2+2\alpha^2+2-2\alpha^2]e_{15} \\ &\quad + [-2-2\alpha^2-2+2\alpha^2]e_{34} - 4\alpha e_{23}\end{aligned}$$

$$\begin{aligned}\langle R_2\xi^2,\xi^2\rangle &= \langle -4\alpha^2 e_{12} + 4\alpha e_{14} - 4e_{34} - 4\alpha e_{23},\xi^2\rangle \\ &= -8\alpha^2 + 8\alpha^2 - 8\alpha^2 + 8\alpha^2 = 0.\end{aligned}$$

Thus $\beta \to \frac{\xi^2(\beta)}{\|\xi^2(\beta)\|}$ is another curve of zeroes through $(e_{12}+e_{15})/\sqrt{2}$.

Section 7. THE ZERO SET OF R_k

In this section we prove Claims 4.1 and 4.2 and for each $k > 2$ we explicitly describe the zero set of R_k.

Until further notice we set $k = 2$.

Consider the following five vectors.

$$\alpha_1 = \xi^1(1) = 4e_{12}-2e_{13}+2e_{15}-4e_{24}+2e_{34}+2e_{45}$$

$$\alpha_2 = \xi^1(-1) = 4e_{12}+2e_{13}+2e_{15}+4e_{24}+2e_{34}-2e_{45}$$

$$\alpha_3 = \xi^2(1) = 2e_{12}+2e_{14}+4e_{15}-2e_{23}+2e_{34}+4e_{35}$$

$$\alpha_4 = \xi^2(-1) = 2e_{12}-2e_{14}+4e_{15}+2e_{23}+2e_{34}-4e_{35}$$

$$\alpha_5 = -12e_{12}-12e_{15}$$

It is clear from the above construction of ξ^1 and ξ^2 that $\langle R_2\alpha_i,\alpha_i\rangle = 0$, $i = 1,\ldots,5$ and thus $\beta_i = \frac{\alpha_i}{\|\alpha_i\|} \in Z(R_2)$ for $i = 1,\ldots,5$.

Let

$$\beta = \|\alpha_1\|\beta_1+\|\alpha_2\|\beta_2+\|\alpha_3\|\beta_3+\|\alpha_4\|\beta_4+\|\alpha_5\|\beta_5 = \sum_{i=1}^{5} \alpha_i .$$

It is easily checked that $\beta = 8e_{34}$ and so $\frac{\beta}{8} \in G$. Now $\langle R_2\frac{\beta}{8},\frac{\beta}{8}\rangle = \langle e_{34}-e_{12}-e_{15},e_{34}\rangle = 1$.

We have found five zeroes of R_2 whose linear span contains a 2-plane in G with non-zero sectional curvature. Let $L_2 = \pi(R_2)$ (To verify Claim 4.1 we need an example which satisfies the Bianchi identity.) Now by the remark at the end of Section 1, $\sigma_{L_2} = \sigma_{R_2}$ and so Claim 4.1 of Section 4 is verified.

Claim 4.2 is now easily verified. If there existed $S \in \Lambda^4$ such that L_2+S were positive semi-definite, then each $x \in Z(L_2)$ would be a minimum of $\langle (L+S)\xi,\xi \rangle$ on the unit sphere in Λ^2, and so would be an eigenvector of L+S with zero eigenvalue. It would then follow by Theorem 2.3 that $Z(L_2)$ was the intersection with G of a linear subspace of Λ^2, namely the null space of L_2+S. However, we have shown that this is not the case.

Lemma 7.1. If

$$Q = \{P \in G | P = (\alpha,1,\beta,0,\gamma)\wedge(\delta,0,\varepsilon,1,\theta)/\|(\alpha,1,\beta,0,\gamma)\wedge(\delta,0,\varepsilon,1,\theta)\| : \alpha,\beta,\gamma,\delta,\varepsilon,\theta \in R\}$$

then

$$G-Q = \{P \in G | P = (\alpha,0,\beta,0,\gamma)\wedge(\delta,\mu,\varepsilon,\eta,\theta); \alpha,\beta,\gamma,\delta,\mu,\varepsilon,\eta,\theta \in R\}.$$

Proof. Replacing $e_1\wedge e_2$ by $e_2\wedge e_4$ in Lemma 3.2 shows that $G-Q = \{P \in G | \langle P, e_2\wedge e_4\rangle = 0\}$. Now

$$0 = \langle P,e_2\wedge e_4\rangle = -\langle P,*(e_1\wedge e_3\wedge e_5)\rangle \Rightarrow P\wedge e_1\wedge e_3\wedge e_5 = 0.$$

Hence (considering P as a 2-dimensional subspace of V and $e_1\wedge e_3\wedge e_5$ as a 3-dimensional subspace of V) it follows that $P \cap (e_1\wedge e_3\wedge e_5) \neq (0)$ and so there exists $v \in P$ such that $|v| = 1$ and $v = (\alpha,0,\beta,0,\gamma)$. Choosing $w \in P$ such that

$|w| = 1$ and $\langle w,v\rangle = 0$ we have that

$$P = v\wedge w = (\alpha,0,\beta,0,\gamma)\wedge(\delta,\mu,\varepsilon,\eta,\theta);\ \alpha,\beta,\gamma,\delta,\mu,\varepsilon,\eta,\theta \in R.$$

Next we analyse the sectional curvature of R_k $(k \geq 2)$ on G-Q. Our goal being to explicitly describe $Z(R_k)(k > 2)$ we can disregard the normalization factor.

Let

$$\begin{aligned}
\xi &= (\alpha,0,\beta,0,\gamma)\wedge(\delta,\mu,\varepsilon,\eta,\theta)\\
&= (\alpha e_1+\beta e_3+\gamma e_5)\wedge(\delta e_1+\mu e_2+\varepsilon e_3+\eta e_4+\theta e_5)\\
&= \alpha\mu e_{12}+\alpha\varepsilon e_{13}+\alpha\eta e_{14}+\alpha\theta e_{15}-\beta\delta e_{13}-\beta\mu e_{23}+\beta\eta e_{34}+\beta\theta e_{35}\\
&\quad -\gamma\delta e_{15}-\gamma\mu e\ _{5}-\gamma\varepsilon e_{35}-\gamma\eta e_{45}\\
&= \alpha\mu e_{12}+(\alpha\varepsilon-\beta\delta)e_{13}+\alpha\eta e_{14}+(\alpha\theta-\gamma\delta)e_{15}-\beta\mu e_{23}\\
&\quad -\gamma\mu e_{25}+\beta\eta e_{34}+(\beta\theta-\gamma\varepsilon)e_{35}-\gamma\eta e_{45}.
\end{aligned}$$

$$\begin{aligned}
R_k\xi &= \alpha\mu(e_{12}-e_{15}-e_{34})+(\alpha\theta-\gamma\delta)(e_{15}-e_{12}-e_{34})+\beta\eta(e_{34}-e_{12}-e_{15})\\
&\quad +k[(\alpha\varepsilon-\beta\delta)e_{13}+\alpha\eta e_{14}-\beta\mu e_{23}-\gamma\mu e_{25}-\gamma\eta e_{45}]\\
&= (\alpha\mu-\alpha\theta+\gamma\delta-\beta\eta)e_{12}+(\alpha\theta-\gamma\delta-\alpha\mu-\beta\eta)e_{15}\\
&\quad +(\beta\eta-\alpha\theta+\gamma\delta-\alpha\mu)e_{34}+k[(\alpha\varepsilon-\beta\delta)e_{13}+\alpha\eta e_{14}-\beta\mu e_{23}-\gamma\mu e_{25}-\gamma\eta e_{45}].
\end{aligned}$$

$$\begin{aligned}
\langle R_k,\xi,\xi\rangle &= \alpha\mu(\alpha\mu-\alpha\theta+\gamma\delta-\beta\eta)+(\alpha\theta-\gamma\delta)(\alpha\theta-\gamma\delta-\alpha\mu-\beta\eta)\\
&\quad +\beta\eta(\beta\eta-\alpha\theta+\gamma\delta-\alpha\mu)+k[(\alpha\varepsilon-\beta\delta)^2+\alpha^2\eta^2+\beta^2\mu^2+\gamma^2\mu^2+\gamma^2\eta^2]\\
&= \alpha^2\mu^2-\alpha^2\mu\theta+\alpha\mu\gamma\delta-\alpha\mu\beta\eta+(\alpha\theta-\gamma\delta)^2-\alpha^2\mu\theta-\alpha\theta\beta\eta\\
&\quad +\gamma\delta\alpha\mu+\gamma\delta\beta\eta+\beta^2\eta^2-\beta\eta\alpha\theta+\beta\eta\gamma\delta-\beta\eta\alpha\mu\\
&\quad +k[(\alpha\varepsilon-\beta\delta)^2+\alpha^2\eta^2+\beta^2\mu^2+\gamma^2\mu^2+\gamma^2\eta^2]\\
&= \alpha^2\mu^2-2\alpha^2\mu\theta+2\alpha\mu\gamma\delta-2\alpha\mu\beta\eta+(\alpha\theta-\gamma\delta)^2-2\alpha\theta\beta\eta+2\gamma\delta\beta\eta\\
&\quad +\beta^2\eta^2+k[(\alpha\varepsilon-\beta\delta)^2+\alpha^2\eta^2+\beta^2\mu^2+\gamma^2\mu^2+\gamma^2\eta^2] = (*).
\end{aligned}$$

For $k \geq 2$ we will write (*) as the sum of squares of polynomial functions.

Theorem 7.2. For $k \geq 2$ and $\xi \in G-Q$,

$$\begin{aligned}\langle R_k\xi,\xi\rangle &= (-\beta\eta+\alpha\theta-\gamma\delta-\alpha\mu)^2+2(\beta\mu-\alpha\eta)^2\\ &\quad +k[(\alpha\varepsilon-\beta\delta)^2+\gamma^2\mu^2+\gamma^2\eta^2]\\ &\quad +(k-2)(\alpha^2\eta^2+\beta^2\mu^2).\end{aligned}$$

Proof. Expanding the right hand side (RHS) we get

$$\begin{aligned}\text{RHS} &= \beta^2\eta^2-2\alpha\theta\beta\eta+2\gamma\delta\beta\eta+2\alpha\mu\beta\eta+\alpha^2\theta^2-2\alpha\theta\gamma\delta-2\alpha^2\mu\theta\\ &\quad +\gamma^2\delta^2+2\alpha\mu\gamma\delta+\alpha^2\mu^2+2\beta^2\mu^2-4\alpha\mu\beta\eta+2\alpha^2\eta^2\\ &\quad +k[(\alpha\varepsilon-\beta\delta)^2+\gamma^2\mu^2+\gamma^2\eta^2+\alpha^2\eta^2+\beta^2\mu^2]\\ &\quad -2\alpha^2\eta^2-2\beta^2\mu^2.\end{aligned}$$

Note the following terms simplify as follows:

$$\begin{aligned}2\alpha\mu\beta\eta-4\alpha\mu\beta\eta &= -2\alpha\mu\beta\eta\\ 2\alpha^2\eta^2-2\alpha^2\eta^2 &= 0\\ 2\beta^2\mu^2-2\beta^2\mu^2 &= 0\\ \alpha^2\theta^2-2\alpha\theta\gamma\delta+\gamma^2\delta^2 &= (\alpha\theta-\gamma\delta)^2.\end{aligned}$$

Substitution yields:

$$\begin{aligned}\text{RHS} &= \alpha^2\mu^2-2\alpha^2\mu\theta+2\alpha\mu\gamma\delta-2\alpha\mu\beta\eta+(\alpha\theta-\gamma\delta)^2\\ &\quad -2\alpha\theta\beta\eta+2\gamma\delta\beta\eta+\beta^2\eta^2+k[(\alpha\varepsilon-\beta\delta)^2+\alpha^2\eta^2+\beta^2\mu^2+\gamma^2\mu^2+\gamma^2\eta^2].\end{aligned}$$

Theorem 7.3. For $k > 2$, $Z(R_k) = \{\pm(\frac{e_{12}+e_{15}}{\sqrt{2}}),\pm e_{24},\pm e_{35}\}$.

<u>Proof</u>. For $k > 2$ Theorem 5.1 implies that the only zeroes of R_k in Q are $\pm e_{24}$. For $k > 2$ and $\xi \in G\text{-}Q$ an analysis of the polynomial expression for $\langle R_k\xi,\xi\rangle$ given by Theorem 7.2 will show that the only zeroes of R_k in $G\text{-}Q$ are $\pm(e_{12}+e_{15})/\sqrt{2}$ and $\pm e_{35}$.

For $k > 2$ and $\xi \in G\text{-}Q$ it follows by Theorem 7.2 that

$$\begin{aligned}\langle R_k\xi,\xi\rangle = &(-\beta\eta+\alpha\theta-\gamma\delta-\alpha\mu)^2 + 2(\beta\mu-\alpha\eta)^2 \\ &+k[(\alpha\varepsilon-\beta\delta)^2 + \gamma^2\mu^2 + \gamma^2\eta^2] \\ &+(k-2)(\alpha^2\eta^2+\beta^2\mu^2).\end{aligned}$$

Now for $k > 2$ ($\xi \in G\text{-}Q$), $\langle R_k\xi,\xi\rangle = 0$ implies that $\alpha^2\eta^2 + \beta^2\mu^2 = 0$.

<u>Case 1</u>. $\alpha = \beta = 0$.

Then $\langle R_k\xi,\xi\rangle = \gamma^2\delta^2 + k(\gamma^2\mu^2+\gamma^2\eta^2) = 0$ implies that $\mu = \eta = \delta = 0$. ($\gamma \neq 0$ since $\alpha = \beta = \gamma = 0$ would imply that $\xi = 0$ which is not possible since $\|\xi\| = 1$.) Then $\xi = (0,0,0,0,\gamma)\wedge(0,0,\varepsilon,0,\theta) = -\gamma\varepsilon e_{35} = \pm e_{35}$ (since $\|\xi\| = 1$).

<u>Case 2</u>. $\alpha = \mu = 0$.

Then $\langle R_k\xi,\xi\rangle = (\beta\eta+\gamma\delta)^2 + k(\beta^2\delta^2+\gamma^2\eta^2) = 0$ implies that $\beta = \eta = 0$ or $\delta = \gamma = 0$ or $\delta = \eta = 0$. ($\alpha = \beta = \gamma = 0$ would imply that $\xi = 0$).

If $\beta = \eta = 0$ then $\delta = 0$ and so

$$\xi = (0,0,0,0,\gamma)\wedge(0,0,\varepsilon,0,\theta) = -\gamma\varepsilon e_{35} = \pm e_{35}.$$

If $\delta = \gamma = 0$ then $\eta = 0$ ($\beta \neq 0$ since $\alpha = \beta = \gamma = 0$ would imply that $\xi = 0$) and so

$$\xi = (0,0,\beta,0,0)\wedge(0,0,\varepsilon,0,\theta) = \beta\theta e_{35} = \pm e_{35}.$$

If $\delta = \eta = 0$ then

$$\xi = (0,0,\beta,0,\gamma)\wedge(0,0,\varepsilon,0,\theta) = (\beta\theta-\gamma\varepsilon)e_{35} = \pm e_{35}$$

Case 3. $\eta = \beta = 0$.

Then $\langle R_k\xi,\xi\rangle = (\alpha\theta-\gamma\delta-\alpha\mu)^2 + k(\alpha^2\varepsilon^2+\gamma^2\mu^2) = 0$ implies that $\alpha = \mu = 0$ or $\varepsilon = \gamma = 0$ or $\varepsilon = \mu = 0$. ($\alpha = \gamma = 0$ would imply that $\xi = 0$.)

If $\alpha = \mu = 0$ then $\delta = 0$ and so

$$\xi = (0,0,0,0,\gamma)\wedge(0,0,\varepsilon,0,\theta) = -\gamma\varepsilon e_{35} = \pm e_{35}.$$

If $\varepsilon = \gamma = 0$ then $\alpha^2(\theta-\mu)^2 = 0$ implying that $\theta = \mu$. ($\alpha \neq 0$ since $\alpha = \beta = \gamma = 0$ would imply that $\xi = 0$.) Thus

$$\xi = (\alpha,0,0,0,0)\wedge(\delta,\mu,0,0,\mu) = \alpha\mu(e_{12}+e_{15}) = \pm(\frac{e_{12}+e_{15}}{\sqrt{2}}).$$

If $\varepsilon = \mu = 0$ then $(\alpha\theta-\gamma\delta)^2 = 0$ and thus

$$\xi = (\alpha,0,0,0,\gamma)\wedge(\delta,0,0,0,\theta) = (\alpha\theta-\gamma\delta)e_{15} = 0.$$

Case 4. $\eta = \mu = 0$.

Then $\langle R_k\xi,\xi\rangle = (\alpha\theta-\gamma\delta)^2 + k(\alpha\varepsilon-\beta\delta)^2 = 0$ implies that $\alpha\theta-\gamma\delta = 0$ and $\alpha\varepsilon-\beta\delta = 0$. Thus

$$\xi = (\alpha,0,\beta,0,\gamma)\wedge(\delta,0,\varepsilon,0,\theta)$$
$$= (\alpha\varepsilon-\beta\delta)e_{13} + (\alpha\theta-\gamma\delta)e_{15} + (\beta\theta-\gamma\varepsilon)e_{35}$$
$$= (\beta\theta-\gamma\varepsilon)e_{35} = \pm e_{35}.$$

Proposition 7.4. For $k \geq 2$ L_k is not the projection under π of a positive semi-definite operator on Λ^2.

Proof. Suppose it is. Then for some $S \in \Lambda^4$ R_k+S is a positive semi-definite operator on Λ^2.

Let $\alpha = e_{12}+e_{15}+e_{34}$. Then $R_k\alpha = -\alpha$ and $\langle R_k\alpha,\alpha\rangle = -3$. Thus $\langle(R_k+S)\alpha,\alpha\rangle \geq 0$ imples that $\langle S\alpha,\alpha\rangle \geq 3$.

Now since $S \in \Lambda^4$ it follows by Proposition 1.2 that $S = \Sigma_{1\leq i<j<k<\ell\leq 5}\lambda_{ijk\ell}S_{ijk\ell}$, $\lambda_{ijk\ell} \in \mathbb{R}$. Thus

$$\langle S\alpha,\alpha\rangle = \Sigma_{1\leq i<j<k<\ell\leq 5}\lambda_{ijk\ell}\langle S_{ijk\ell}\alpha,\alpha\rangle$$
$$= \lambda_{1234}\langle e_{34}+e_{12},e_{12}+e_{15}+e_{34}\rangle + \lambda_{1235}\langle e_{35}+e_{23},e_{12}+e_{15}+e_{34}\rangle$$
$$+\lambda_{1245}\langle e_{45}+e_{24},e_{12}+e_{15}+e_{34}\rangle + \lambda_{1345}\langle e_{34}+e_{15},e_{12}+e_{15}+e_{34}\rangle$$
$$+\lambda_{2345}\langle e_{25},e_{12}+e_{15}+e_{34}\rangle$$
$$= 2\lambda_{1234} + 2\lambda_{1345}.$$

Since $\langle S\alpha,\alpha\rangle \geq 3$ it follows that $\lambda_{1234} + \lambda_{1345} \geq \frac{3}{2}$.

Setting $w_1 = e_{13}+ke_{24}$ and $w_2 = e_{14}+ke_{35}$ we get the following:

$$\langle R_k w_1, w_1\rangle = \langle ke_{13}, e_{13}+ke_{24}\rangle = k$$

$$\langle R_k w_2, w_2\rangle = \langle ke_{14}, e_{14}+ke_{35}\rangle = k$$

$$\begin{aligned}\langle Sw_1, w_1\rangle &= \lambda_{1234}\langle -e_{24}-ke_{13}, e_{13}+ke_{24}\rangle \\ &\quad +\lambda_{1235}\langle -e_{25}, e_{13}+ke_{24}\rangle \\ &\quad +\lambda_{1245}\langle ke_{15}, e_{13}+ke_{24}\rangle \\ &\quad +\lambda_{1345}\langle e_{45}, e_{13}+ke_{24}\rangle \\ &\quad +\lambda_{2345}\langle -ke_{35}, e_{13}+ke_{24}\rangle \\ &= -2k\lambda_{1234}\end{aligned}$$

$$\begin{aligned}\langle Sw_2, w_2\rangle &= \lambda_{1234}\langle e_{23}, e_{14}+ke_{35}\rangle \\ &\quad +\lambda_{1235}\langle ke_{12}, e_{14}+ke_{35}\rangle \\ &\quad +\lambda_{1245}\langle -e_{25}, e_{14}+ke_{35}\rangle \\ &\quad +\lambda_{1345}\langle -e_{35}-ke_{14}, e_{14}+ke_{35}\rangle \\ &= -2k\lambda_{1345}.\end{aligned}$$

Thus $\langle (R_k+S)w_1, w_1\rangle = k(1-2)\lambda_{1234}$ and $\langle (R_k+S)w_2, w_2\rangle = k(1-2)\lambda_{1345}$. But $\lambda_{1234} + \lambda_{1345} \geq \frac{3}{2}$ implies that $\lambda_{1234} \geq \frac{3}{4}$ or $\lambda_{1345} \geq \frac{3}{4}$ implying that $\langle (R_k+S)w_1, w_1\rangle < 0$ or $\langle (R_k+S)w_2, w_2\rangle < 0$, thus contradicting the assumption that R_k+S is positive semi-definite.

Theorem 7.5. There exist curvature operators which satisfy the Bianchi identity, have nonnegative sectional curvature, and each of whose zero sets is the intersection with G of a linear subspace of Λ^2; but which are not the projection under π of a positive semi-definite operator on Λ^2.

Proof. We claim that for k > 2, each curvature operator L_k is of this type.

It follows by Theorem 7.4 that L_k is not the projection of a positive semi-definite operator and by Theorem 5.1 that $\sigma(L_k) \geq 0$.

By Theorem 7.3, for k > 2 $Z(L_k) = \{\frac{\pm(e_{12}+e_{15})}{\sqrt{2}}, \pm e_{24}, \pm e_{35}\}$. To complete the proof we verify that (for k > 2) $Z(L_k) = \mathrm{span}Z(L_k) \cap G$.

That $Z(L_k) \subset \mathrm{span}Z(L_k) \cap G$ is clear. If $\xi \in \mathrm{span}Z(L_k)$, then $\xi = a(e_{12}+e_{15})/\sqrt{2} + be_{24}+ce_{35}$, $a,b,c \in \mathbb{R}$. By Corollary 1.4, ξ is decomposable

$$\begin{aligned}\Leftrightarrow 0 = \xi\wedge\xi &= [a(e_{12}+e_{15})/\sqrt{2} + be_{24} + ce_{35}] \\ &\quad \wedge[a(e_{12}+e_{15})/\sqrt{2} + be_{24} + ce_{35}] \\ &= \frac{2ac}{\sqrt{2}} e_1\wedge e_2\wedge e_3\wedge e_5 + \frac{2ab}{\sqrt{2}} e_1\wedge e_2\wedge e_4\wedge e_5 + 2bce_2\wedge e_4\wedge e_3\wedge e_5\end{aligned}$$

$$\Leftrightarrow ab = ac = bc = 0 \Leftrightarrow a = b = 0 \text{ or } b = c = 0 \text{ or } a = c = 0$$

$$\Leftrightarrow \xi = ce_{35} \text{ or } \xi = a(e_{12}+e_{15})/\sqrt{2} \text{ or } \xi = be_{24}.$$

Theorem 7.6. If dimension V = n ≥ 5, then there exist curvature operators L_k^n which satisfy the Bianchi identity and have the following properties:

1. For $k \geq 2$, $\sigma_{L_k^n} \geq 0$

2. For $k = 2$, $Z(L_k^n)$ is not the intersection with G of a linear subspace of Λ^2.

3. For $k \geq 2$, L_k^n is not the projection under π of a positive semi-definite operator on Λ^2.

Proof. For $n \geq 5$ let $\{e_1,\ldots,e_n\}$ be an orthonormal basis for V and let $w = \text{span}\{e_1,e_2,e_3,e_4,e_5\}$. Since $W \subset V$, $\Lambda^2(W) \subset \Lambda^2(V)$.

We define the linear maps $\pi_1 \Lambda^2(V) \to \Lambda^2(W)$ as follows: for

$$\xi = \Sigma_{1\leq i<j\leq n}\, a_{ij}e_{ij} \in \Lambda^2(V)$$

$$\pi_1(\xi) = \Sigma_{1\leq i<j\leq 5}\, a_{ij}e_{ij}.$$

Note that if $\xi = (\sum_{i=1}^{n} a_i e_i)\wedge(\sum_{j=1}^{n} b_j e_j)$ then $\pi_1(\xi) = (\sum_{i=1}^{5} a_i e_i) \wedge(\sum_{j=1}^{5} b_j e_j)$. Thus if ξ is decomposable then $\pi_1(\xi)$ is decomposable.

For k a real number and dimension $V = n \geq 5$, consider the following example:

Let $R_k^n : \Lambda^2(V) \to \Lambda^2(V)$ be defined by

$$R_k^n e_{12} = e_{12}-e_{15}-e_{34}$$

$$R_k^n e_{15} = e_{15}-e_{12}-e_{34}$$

$$R_k^n e_{34} = e_{34} - e_{12} - e_{15}$$

$$R_k^n e_{24} = R_k^n e_{35} = 0.$$

$$R_k^n e_{ij} = k e_{ij} \text{ for remaining } e_{ij}.$$

Note that for $k > 0$ $\langle R_k^n \xi, \xi \rangle \geq \langle R_k \pi_1(\xi), \pi_1(\xi) \rangle$ for all $\xi \in \Lambda^2(V)$.

Let $L_k^n = \pi(R_k^n)$. Then L_k^n satisfies the Bianchi identity and for $k \geq 2$

$$\sigma_{L_k^n}(\xi) = \sigma_{R_k^n}(\xi) = \langle R_k^n \xi, \xi \rangle \geq \langle R_k \pi_1(\xi), \pi_1(\xi) \rangle \geq 0.$$

Thus L_k^n has property 1.

To see that L_k^n has property 2, let β_i $(i = 1, \ldots, 5)$ and β be defined as above. By the natural inclusion of $\Lambda^2(W)$ in $\Lambda^2(V)$ we can consider β and β_i as elements of $\Lambda^2(V)$. Then $\sigma_{L_2^n}(\beta_i) = \sigma_{R_2^n}(\beta_i) = \sigma_{R_2}(\beta_i) = 0$ and $\sigma_{L_2^n}(\beta/8) = \sigma_{R_2^n}(\beta/8)$ $= \sigma_{R_2}(\beta/8) = 1$. Thus we have found five zeroes of L_2^n whose linear span contains a 2-plane in G with non-zero sectional curvature and so $Z(L_2^n)$ is not the intersection with G of a linear subspace of $\Lambda^2(V)$.

To see that L_k^n has property 3, let's suppose it is the projection of a positive semi-definite operator on $\Lambda^2(V)$. Then for some $S \in \Lambda^4(V)$, $R_k^n + S$ is a positive semi-definite operator on $\Lambda^2(V)$.

Let $\alpha = e_{12} + e_{15} + e_{34}$. Then $R_k^n \alpha = -\alpha$ and $\langle R_k^n \alpha, \alpha \rangle = -3$. Thus $\langle (R_k^n + S)\alpha, \alpha \rangle \geq 0$ implies that $\langle S\alpha, \alpha \rangle \geq 3$.

Now since $S \in \Lambda^4$, it follows by Proposition 1.2 that $S = \Sigma_{1 \le i<j<k<\ell \le n} \lambda_{ijk\ell} S_{ijk\ell}$, $\lambda_{ijk\ell} \in \mathbb{R}$.

A direct compution (similar to that in the proof of Proposition 7.4) shows that $\langle S\alpha,\alpha\rangle \ge 3$ only if $\lambda_{1234} + \lambda_{1345} \ge \frac{3}{2}$.

Set $w_1 = e_{13}+ke_{24}$ and $w_2 = e_{14}+ke_{35}$. A direct computation shows that

$$\langle (R^n_k+S)w_1,w_1\rangle = k(1-2)\lambda_{1234}$$

and

$$\langle (R^n_k+S)w_2,w_2\rangle = k(1-2)\lambda_{1345}.$$

Now $\lambda_{1234} + \lambda_{1345} \ge \frac{3}{2}$ implies that $\lambda_{1234} \ge \frac{3}{4}$ or $\lambda_{1345} \ge \frac{3}{4}$ implying that $\langle (R^n_k+S)w_1,w_1\rangle < 0$ or $\langle (R^n_k+S)w_2,w_2\rangle < 0$, thus contradicting the assumption that R^n_k+S is positive semi-definite.

CONCLUSION

In this section we summarize what has been accomplished and we point out what still needs to be done.

Our main contribution has been the construction of a family of curvature operators possessing quite unexpected properties.

More specifically, for dimension $V \geq 5$, we have constructed curvature operators with nonnegative sectional curvature which are not the projection of positive semi-definite operators. We have also constructed curvature operators with nonnegative sectional curvature each of whose zero zet is not the intersection with G of a linear subspace of Λ^2. These latter examples show that Thorpe's description of the zero set of such an operator [1] is only valid in dimensions ≤ 4. The first examples show that Thorpe's result of [2] can not be generalized to dimension ≥ 5.

What remains to be done, for dimensions ≥ 5, is to find a simple characterization of positive sectional curvature and to find a description of the zero sets of curvature operators which have nonnegative sectional curvature.

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