CR FUNCTIONS ON TUBE MANIFOLDS

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Abstract of the Dissertation

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In this paper we generalize Bochner's tube theorem on the extendability of holomorphic functions. We prove that if M is a connected locally closed submanifold in $C^n (n > 1)$ which is invariant under translations in the imaginary directions (i.e., M is a tube manifold), then every smooth solution to the tangential Cauchy-Riemann equations on M (i.e., a CR function) uniquely extends to almost all of the convex hull. Moreover, the supremum of the modulus of the extended function equals that of the original function. This is one of the few global CR extension theorems known.

To prove the above theorem, we first define a notion of CR functions on tubes over connected, locally closed,

locally starlike subsets of R^n. Using the techniques of several complex variables, we prove the theorem stated above in this case. We then use the known local CR extension theorems, to reduce the proof in the submanifold case to the locally starlike case.

To all the women that drove me to this.

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§0. Introduction

One of the major differences between holomorphic functions of several complex variables as opposed to one complex variable is the property of holomorphic extendability. On every connected open set Ω in C^n, there exists a holomorphic function f which cannot be extended to a holomorphic function on a larger open set containing Ω. This is no longer true in C^n $(n \geq 1)$. Given a connected open set Ω in C^n $(n > 1)$, is there a largest open set Ω', containing Ω, such that every holomorphic function on Ω extends holomorphically to Ω'? In general Ω' doesn't exist. However, there exists a "largest" complex manifold S, containing Ω, with the property that every holomorphic function on Ω extends to a unique holomorphic function on S.

If we restrict ourselves to special Ω's, there are many results on the extendability of holomorphic functions. A few examples follow:

Theorem (Hartogs): Let Ω be a connected open set in $C^n(n>1)$ and let K be a compact set in C^n such that $\Omega - K$ is connected. Then every holomorphic function on $\Omega - K$ extends to Ω.

<u>Bochner's tube theorem</u>: Let U be a connected open set in R^n and $\tau(U)$ ($= U \times iR^n$), the tube over U, be the set of points in C^n whose real parts belong to U. Then every holomorphic function on $\tau(U)$ extends to the convex hull of $\tau(U)$.

In the 1940's, Bochner and Martinelli among others showed that if Ω is a connected open set in $C^n (n > 1)$ with C^2 boundary then functions satisfying certain differential equations extend from the boundary to the interior. In the 1950's, Hans Lewy gave the first example of functions defined on a lower dimensional subset of C^n, satisfying certain differential equations which extend to holomorphic functions. This work leads us to the following questions:

1) Given a submanifold M of C^n are there differential equations that characterize the smooth functions on M that are boundary values of holomorphic functions near M?

2) If there are such differential equations, do all smooth solutions extend to holomorphic functions on some neighborhood?

The answer to the first question is yes. The differential equations are the M-tangential components of the Cauchy-Riemann equations. They are known as the tangential

Cauchy-Riemann equations to M and their solutions are called CR functions. The answer to the second question is no. For a detailed history of CR function theory see the paper of Wells.

Our major result is a generalization of Bochner's tube theorem, which roughly states:

Theorem: Every C^∞ CR function on $\tau(M)$ ($= M \times iR^n$), where M is a connected locally closed submanifold, extends to a CR function on almost all of the convex hull of $\tau(M)$.

The reason we study tube manifolds (manifolds of the form $\tau(M)$) is to understand how the geometry in this special case influences the extendability of CR functions. It is our hope that the information we get from this special case will enable us to understand what phenomena might occur in the general case, as it did in the classical development of several complex variables.

I would like to mention two known results which are related to the above theorem. The first due to Carmignani states that for M a polygonally connected set in R^n, the germs of holomorphic functions on $\tau(M)$ extend to the convex hull of $\tau(M)$. This can be obtained as a corollary of results in this work. There is some early work of

Rossi on CR extendability on Reinhardt submanifolds and a result by Rossi and Vergne on the extension of CR functions on tube manifolds, where the functions are assumed to be L^2 as well as infinitely differentiable. Using the L^2 assumption and techniques of Fourier Analysis, they extend the functions to the entire convex hull. This is made possible by the growth restrictions on the CR functions at infinity.

We conclude this introduction with an outline of this paper. Section 1 contains the definitions of technical terms and summarizes known results of CR theory we need for reference. Section 2 contains preliminaries on tube manifolds and convexity. Section 3 contains the statement of the lemma of the folding screen, a major tool in this work, and Section 4 contains its proof. Section 5 contains the statement of a CR extension theorem for tubes over a connected, locally closed, locally starlike subset M of R^n. We use the lemma of the folding screen to prove the special case where M is a compact polygonal path. We conclude the proof in Section 6. In Section 7 we state our major theorem and prove some propositions about the excess dimension of tubes over curves. In Section 8 we use the extension theorem of Hunt and Wells to prove a "local" extension for tubes over curves. We then apply our results (Section 5, on tubes over locally starlike subsets) to patch together these local extensions to almost all of the convex hull. In the final section we conclude the proof of the theorem by considering curves through any point on the base manifold M. We use the results of Section 8 to give us extensions from each such curve to its almost convex hull. Finally, we prove that any two such extensions coincide on the intersection of their domains. In the first appendix we prove that the extension theorem gives us an isomorphism of the algebra of CR functions, when they are given an appropriate Frechet structure. In the second appendix we prove a CR extension theorem for Reinhardt submanifolds of C^n.

§1. Definitions and Technical Terms in CR Theory

All manifolds considered here are connected, locally closed, and of class C^{∞}, and all functions will be of class C^{∞} unless otherwise stated.

Definition: Let $p \in C^n$; then $HT_p(C^n)$ denotes the set of all complex tangents vectors V that are complex linear combinations of $\frac{\partial}{\partial z_i} = \frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}$. Let $AT_p(C^n) = \overline{HT_p(C^n)}$. We call $HT_p(C^n)$ the holomorphic tangent space to p in C^n, whereas $AT_p(C^n)$ is the anti-holomorphic tangent space to p in C^n.

Definition: If N is a real submanifold of C^n, then $HT_p(N) = CT_p(N) \cap HT_p(C^n)$ and $AT_p(N) = \overline{HT_p(N)}$ for all $p \in N$. Where $CT_p(N)$ denotes the complex tangent space of N at p.

Definition: A manifold N is a CR submanifold of C^n if the dimension of $HT_p(N)$ is independent of $p \in N$. It is generic if the complex codimension of $HT_p(N)$ as a subset of $HT_p(C^n)$ equals the real codimension of N in C^n. A CR function is a complex valued function f such that $Vf = 0$ for all $V \in AT_p(N)$ and all $p \in N$.

Remark: If N is an open set in C^n, then the CR functions are the holomorphic functions since $Vf = 0$ for all $V \in AT(N)$ are the Cauchy-Riemann equations.

Definition: The Levi Algebra to $p \in N$, $\mathcal{L}_p(N)$, is the stalk at p of the Lie Algebra generated by the germs of the holomorphic and anti-holomorphic vector fields at p to N. The excess dimension at p, $ex_p(N)$, is the complex dimension of $\mathcal{L}_p(N)/[HT_p(N) \oplus AT_p(N)]$.

Local CR extension theorem. (Hunt, Wells, Greenfield, Nirenberg, et. al):

If N is a generic K-dimensional submanifold of C^n $(n > 1)$ and the excess dimension e of N is constant, then N has the property that locally each CR function on N extends to a CR function on a manifold N' of dimension K + e. If the excess dimension is maximal (i.e., the same as the codimension of N in C^n), then N' is an open set, $\bar{N}' \supset N$, and the extension is unique.

We conclude this section with a brief description of the Whitney extension theory. Let $U \subset R^n$ be open, S closed in U, and f an R^n valued function on S. We say that f is of class C^r (resp. smooth or Class C^∞) in the sense of Whitney if for each multi-index $\alpha \in N^n$ such that $|\alpha| \le r$ (resp. for each multi-index α, if f is smooth) there exists a mapping $f_\alpha : S \to R^n$ with $f_o = f$, such that the following conditions are satisfied: if for each $s \le r$ (resp. each integer $s \ge o$) we write

$$f_\alpha(x) = \sum_{|\alpha+\beta| \le s} f_{\alpha+\beta}(\xi) \frac{(x-\xi)^\beta}{\beta!} + R_{\alpha,s}(x,\xi)$$

where $x, \xi \in S$ and $|\alpha| \le s$, then for each $x_o \in S$, and each $\epsilon > 0$, and pair (α,s) with $|\alpha| \le s$, there exists a $\rho > o$ such that $\|R_{\alpha,s}(x,\xi)\| \le \epsilon \|x-\xi\|^{s-|\alpha|}$ for each $x,\xi \in S$ such that $\|x-x_o\| < \rho$ and $\|\xi - x_o\| < \rho$. These conditions imply that the f_α are continuous on S. When S is a submanifold of U, this definition is equivalent to the usual definition of C^r (resp. C^∞).

<u>Whitney Extension Theorem</u>: Suppose that f, U, and S satisfy the conditions in the above definition, then there exists a C^r (resp. C^∞) function $\tilde{f}$ on U such that

$\frac{\partial^\alpha \tilde{f}}{\partial x^\alpha}(\xi) = f(\xi)$ for all $\xi \in S$. Note that f being of Class C^r does not depend on U.

§2. Convexity and Tubes

Definition: A subset S of R^n or C^n is convex if $x,y \in S$ imply that for all $t \in [0,1]$ $tx + (1-t)y \in S$. The convex hull of S, ch(S), is the smallest convex set containing S. The convex hull of a set always exists.

Proposition: Let S be a subset of R^n; then there exists a unique maximal affine subspace, P(S), containing S.

Corollary: $ch(S) \subset P(S)$, when $S \subset R^n$.

<u>Theorem</u>: Let $S \subset R^n$. Then $y \in ch(S)$ is equivalent to the existence of $y_i \in S$ and $\alpha_i \in [0,1]$ such that

$$y = \sum_{i=1}^{p+1} \alpha_i y_i \quad \text{and} \quad \sum_{i=1}^{p+1} \alpha_i = 1,$$ where p equals the

dimension of P(S).

Definition: The dimension of P(S) is the convex dimension of S. The relative interior of the convex hull of S, rel-int ch(S), is the interior of ch(S) when considered as a subspace of P(S). The almost convex hull of S, ach(S), is the union of S and rel-int ch(S).

Remark: The work of Whitney allows us to use the same definition of smooth functions on sets in R^n of the form ach(S) as we used for closed sets in R^n.

Definition: A subset S of R^n is locally starlike if, for each point $p \in S$, there exists a neighborhood U of p in S such that, for all $p' \in U$, $t(p'-p)+p \in U$, where $t \in [0,1]$.

Proposition: Every connected locally starlike subset S of R^n is polygonally connected.

<u>Theorem</u>: Every locally starlike smooth submanifold M of dimension m in R^n is locally an open subset of some m-dimensional affine subspace of R^n.

Definition: $S \subset R^n$ is locally closed if each point p in S has a neighborhood in S that is the intersection of a closed set in R^n and an open set in R^n. S is locally closed if and only if each point in S has a neighborhood U which is a closed subset of some open subset of R^n. Therefore, we can define the notion of a smooth function on S in the sense of Whitney.

Proposition: Let M be a locally closed submanifold of R^n. Then f is a smooth function on the differentiable manifold M is and only if f is smooth in the sense of Whitney.

Definition: Let $S \subset R^n$. The tube over S is the set $\tau(S) = \{z \in C^n | \text{Re } z \in S\}$, where if $z = (z_1,\ldots,z_n) \in C^n$, $\text{Re } z = (\text{Re } z_1,\ldots,\text{Re } z_n)$.

Note: $\tau(ch(S)) = ch(\tau(S))$ because the convex hull of the cartesian product of a convex set A with a set B is the cartesian product of A with the convex hull of B. Also, $\tau(ach(S)) = ach(\tau(S))$.

Note: If M is a submanifold of R^n, $\tau(M)$ is a real submanifold of C^n.

Definition: Let S be a locally closed, locally star-like subset of R^n. A smooth function f on $\tau(S)$ is a CR' function if for each open ended line segment l in S, $f|\tau(l)$ is a CR function.

Proposition: Let M be a locally closed, locally star-like submanifold of R^n. The notions of CR and CR' on $\tau(M)$ coincide.

Proof: The notions of smoothness coincide. We have to show that CR' implies CR, since CR implies CR' trivially. M is locally an open subset of some affine subspace in R^n. Each line l in M determines an element of a basis for a coordinate chart of that open set, we denote it by l. $\frac{\partial}{\partial l} - i\, J \frac{\partial}{\partial l}$ is an element of

$HT_p(\tau(M))$, where J is the map induced by the complex structure in C^n. Since the dimension of M equals the complex dimension of $HT_p(\tau(M))$, CR' implies CR. We will drop the ' from CR'.

Definition: Let $M \subset R^n$ be a locally closed submanifold or a locally closed and locally starlike set. A CR function f on $\tau(\text{ach}(M))$ is a smooth complex valued function such that both $f|\tau(M)$ and $f|\tau(\text{rel-int ch}(M))$ are CR functions in the usual sense.

If M is any set for which CR functions are defined, let CR(M) be the set of CR functions on M.

Proposition: Let M be a submanifold of R^n. Then $\tau(M)$ is a generic submanifold of C^n.

Proof: Let m be the dimension of M. Then n-m equals the codimension of M in R^n as well as the codimension of $\tau(M)$ in C^n. The space $HT_p(\tau(M))$ $(p \in M)$ is spanned by $\frac{\partial}{\partial s_i} - i\, J \frac{\partial}{\partial s_i}$, where the $\frac{\partial}{\partial s_i}$ form a basis for $T_p(M)$. Therefore, the complex dimension of $HT_p(\tau(M))$ equals m, and the complex codimension of $HT_p(M)$ in $HT_p(C^n)$ equals n-m.

§3. The Lemma of the Folding Screen

This section contains the statement of the lemma of the folding screen. The proofs of certain propositions which are necessary for the proof of the lemma takes up most of this section.

Definition: Let A_1, A_2, and A_3 be three distinct points in R^n and $l_{i,j} = ch(\{A_i,A_j\}) - \{A_j\}$.

<u>Lemma of the folding screen:</u> Let A_1, A_2, and A_3 be distinct points in R^n, $n \geq 2$. Then

$r : CR(\tau(ch(l_{1,2} \cup l_{1,3}))) \rightarrow CR(\tau(l_{1,2} \cup l_{1,3}))$, the

restriction map of functions, is a bijection.

Define F_o^n by

$$F_o^n = \{z \in C^n | \text{Re } z_i = 0 \ i \neq 2 \ \text{ and } 0 \leq \text{Re } z_2 < 1\} \cup$$
$$\{z \in C^n | \text{Re } z_i = 0 \ i \geq 2 \ \text{ and } \ 0 \leq \text{Re } z_1 < 1\}$$

and G_o^n by

$$G_o^n = \{z \in C^n | \text{Re } z_i = 0 \ i \geq 3, \ \text{Re } z_i \geq 0 \ i = 1,2 \ \text{ and}$$
$$\text{Re}(z_1 + z_2) < 1\}$$

We drop the n, when there is no confusion.

There exists an affine isomorphism B of R^n to R^n, such that $B(A_1) = 0$, $B(A_2) = (1,0,\dots,0)$, and $B(A_3) = (0,1,0,\dots,0)$. B equals $G \circ T$ where T is a translation and $G \in GL(n,R)$. Define $T^C(x+iy) = T(x)+iy$ and $G^C(x+iy) = G(x) + iG(y)$. Extend B to a complex affine isomorphism B^C of C^n to C^n by $B^C = G^C \circ T^C$. B^C is biholomorphic and preserves convexity. Since $B^C(\tau(l_{1,2} \cup l_{1,3})) = F_o$ and $B^C(\tau(ch(l_{1,2} \cup l_{1,3}))) = G_o$, it follows that B^{C^*} maps $CR(F_o)$ isomorphically onto $CR(\tau(ch(l_{1,2} \cup l_{1,3})))$.

Let $0 < \epsilon < \frac{1}{2}$, $w_1 = z_1 - z_2$, $w_2 = z_1 + z_2 - \epsilon(z_1^2 + z_2^2)$, and $w_i = z_i$ for $i \geq 3$. Define F_ϵ^n and G_ϵ^n by

$$F_\epsilon^n = \{z \in F_o^n \mid \text{Re } w_2 < 1 - \epsilon\}$$

and

$$G_\epsilon^n = \{z \in G_o^n \mid \text{Re } w_2 < 1 - \epsilon\}.$$

We drop the n when there is no confusion. We will prove that $z_i \to w_i$ is a holomorphic change of variables in a neighborhood of G_ϵ^2. The proof for G_ϵ^n $(n > 2)$ follows easily.

First we prove that $w = (w_1, w_2)$ is one-to-one near G_ϵ. Let $w = (w_1, w_2)$ and $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) = (\tilde{z}_1 - \tilde{z}_2, \tilde{z}_1 + \tilde{z}_2 - \epsilon(\tilde{z}_1^2 + \tilde{z}_2^2))$.

Suppose that $w = \tilde{w}$. Then $z_1 - z_2 = \tilde{z}_1 - \tilde{z}_2$ or

(1) $$z_1 - \tilde{z}_1 = z_2 - \tilde{z}_2 \quad \text{and}$$

$$z_1 + z_2 - \varepsilon(z_1^2 + z_2^2) = \tilde{z}_1 + \tilde{z}_2 - \varepsilon(\tilde{z}_1^2 + \tilde{z}_2^2) \quad \text{or}$$

(2) $$z_1 - \tilde{z}_1 - \varepsilon(z_1^2 - \tilde{z}_1^2) + z_2 - \tilde{z}_2 - \varepsilon(z_2^2 - z_2^2) = 0$$

Substituting equation 1, equation 2 becomes

(3) $$(z_1 - \tilde{z}_1)(2 - \varepsilon(z_1 + \tilde{z}_1 + z_2 + \tilde{z}_2)) = 0.$$

$(2 - \varepsilon(z_1 + \tilde{z}_1 + z_2 + \tilde{z}_2)) \neq 0$ near G_ε. Therefore,

$z_1 - \tilde{z}_1 = z_2 - \tilde{z}_2 = 0$ near G_ε.

Note that

$$\frac{\partial(w_1, w_2)}{\partial(z_1, z_2)} = \det \begin{pmatrix} 1 & -1 \\ 1 - 2\varepsilon z_1 & 1 - 2\varepsilon z_2 \end{pmatrix} = 2 - 2\varepsilon(z_1 + z_2).$$

Therefore, the Jacobian in non-zero near G_ε.

Proposition: $\bigcup_{0<\varepsilon<\frac{1}{2}} F_\varepsilon = F_0$ and $\bigcup_{0<\varepsilon<\frac{1}{2}} G_\varepsilon = G_0$

Proof: Suppose without loss of generality that $n = 2$ and $z \in F_0$ is such that $x_2 = 0$. Let $U_\varepsilon(x_1, y_1, y_2) = \operatorname{Re} w_2 = x_1 - \varepsilon x_1^2 + \varepsilon(y_1^2 + y_2^2)$. Let $f(x_1, y_1, y_2)(\varepsilon) = U_\varepsilon(x_1, y_1, y_2) - 1 + \varepsilon$. Then $f(x_1, y_1, y_2)$ is a continuous function of ϵ and $f(x_1,y_1,y_2)(0) = x_1-1$. Therefore, there exists an ϵ' between 0 and $\frac{1}{2}$ such that, for all ϵ between 0 and ϵ', $f(x_1,y_1,y_2)(\epsilon) < 0$ or $z \in F_\epsilon$. The proof is similar for G_0.

Definition. Let u be a smooth complex valued function on F_ϵ, where $0 < \epsilon < \frac{1}{2}$. Then u is a CR function if $u|F_\epsilon - \tau(0)$ is a CR function. If u is a smooth complex valued function on G_ϵ, the u is a CR function if $u|G_\epsilon - F_\epsilon$ is a CR function, where $0 < \epsilon < \frac{1}{2}$. By continuity, $u \in CR(G_\epsilon)$ implies $u|F_\epsilon$ is a CR function.

Definition: Let U be open in R^n or C^n and u be a smooth complex valued function. Then u is flat at $p \in U$ if u vanishes to infinite order at p. If S is a subset of U, u is flat on S if it is flat at all points in S. u is $\bar{\partial}$-flat if u is complex valued function on $S \subset C^n$ and $\frac{\partial u}{\partial \bar{z}_i}$ is flat on S for all i.

<u>Theorem</u>: Let $u \in CR(F_0)$ and U be an open set in C^n containing F_0 such that F_0 is closed in U. Then there exists an extension $\tilde{u}$ of u to U such that $\tilde{u}$ is $\bar{\partial}$-flat on F_0.

Proof: Let $\alpha = (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n)$. Define $u_\alpha(z)$ by

$$u_\alpha(z) = (-i)^{\sum_{i=1}^{n} \xi_i} \frac{\partial^{|\alpha|} u}{\partial y_1^{\xi_1+\eta_1} \dots \partial y_n^{\xi_n+\eta_n}}(z), \quad z \in F_o .$$

Define $R_{(\alpha,s)}$ by

$$u_\alpha(z') = \sum_{|\alpha+\beta| \le s} u_{\alpha+\beta}(z) \frac{(z'-z)^\beta}{\beta!} + R_{(\alpha,s)}(z',z).$$

Note that

$$\sum_{|\alpha+\beta| \le s} u_{\alpha+\beta}(z) \frac{(z'-z)^\beta}{\beta!}$$

is the s-th order Taylor polynomial of U_α. Therefore u_α and $R_{(\alpha,s)}$ satisfy the conditions of the Whitney extension theorem. Therefore there exists a smooth extension $\tilde{u}$ of u such that

$$\frac{\partial^{|\alpha|}\tilde{u}}{\partial x_1^{\xi_1} \dots \partial y_n^{\eta_n}}(z) = u_\alpha(z)$$

for $z \in F_o$. Note that

$$\frac{\partial \tilde{u}}{\partial \bar{z}_i}(z) = \frac{\partial \tilde{u}}{\partial x_i}(z) + i\frac{\partial \tilde{u}}{\partial y_i}(z) = i\frac{\partial \tilde{u}}{\partial y_i}(z) + i\frac{\partial \tilde{u}}{\partial y_i}(z) = 0$$

on F_o.

Since $\frac{\partial}{\partial y_i}$ is tangential to F_o

(1) $$\frac{\partial^k}{\partial y_j^{\,k}} \frac{\partial \tilde{u}}{\partial \bar{z}_i} = 0 \quad \text{on } F_o .$$

Note that

$$\frac{\partial^k}{\partial x_j^{\,k}} \frac{\partial \tilde{u}}{\partial \bar{z}_i} = \frac{\partial}{\partial \bar{z}_i} \frac{\partial^k \tilde{u}}{\partial x_j^{\,k}} .$$

By the Whitney extension theorem and (1), we have

$$\frac{\partial}{\partial \bar{z}_i} \frac{\partial^k \tilde{u}}{\partial x_j^{\,k}} = -i^k \frac{\partial}{\partial \bar{z}} \frac{\partial^k \tilde{u}}{\partial y_j} = 0 \quad \text{on } F_o .$$

Therefore $\frac{\partial \tilde{u}}{\partial \bar{z}_i}$ is flat on F_o .

Note: The above theorem holds with F_o replaced by F_ε since the argument is local.

§4. Proof of the Lemma of the Folding Screen

Before we start the proof of the lemma of the folding screen, we notice that we have reduced the lemma to proving an isomorphism between $CR(F_o)$ and $CR(G_o)$. The following proposition is the special case where n equals 2. The proof (which the general case will not depend on) is nice because it uses certain abstract theorems to give us a short proof. Unfortunately it doesn't generalize to the higher dimensional case. We also give such a proof in the case of F_ε and G_ε. If we view the folding screen, F_o, as foliated by the variables $y_3,\ldots,y_n$, the special case would tell us what the CR extensions should be on each slice . There seems no way using abstract techniques to show those extensions vary smoothly with the extra variables. We will then abandon this abstract approach and reprove the theorems we've used in our special case. We will use standard integral formulas so that we can prove the final extension varies smoothly with respect to all parameters.

Proposition: Let $r : CR(G_o^2) \to CR(F_o^2)$ be the restriction map on functions. Then r is surjective.

Proof: Let V_1 be defined by

$$V_1 = \{z \in C^2 \mid |x_i| < 1 \text{ and } x_1 < 0 \text{ or } x_2 < 0\}$$

and $V_2 = V_1 \cup G_o^2$. Note that V_2 equals $ch(V_1)$. Let u be a CR function on F_o^2 and $\hat{u}$ be a $\bar{\partial}$-flat extension of u to V_2. Define h by

$$h = \begin{cases} 0 \text{ on } V_1 \\ \bar{\partial}\hat{u} \text{ on } G_o^2 \end{cases}$$

Note that h is smooth and that $\bar{\partial}h$ equals zero. Since V_2 is Stein, there exists a smooth function $j : V_2 \to C$ such that $\bar{\partial}j$ equals h. Therefore, j is holomorphic on V_1. By Bochner's tube theorem there exists a unique holomorphic extension k of $j|V_1$ to V_2. Let $\tilde{u} = \hat{u} - j+k$. Then $\tilde{u}$ equals u on F_o^2, since j equals k on F_o^2 by continuity, and $\bar{\partial}\tilde{u}$ equals zero on G_o^2 since k is holomorphic and $\bar{\partial}\hat{u}$ equals ∂j on G_o^2.

Proposition (Maximum Modulus Principle): Suppose that $r : CR(G_\epsilon^n) \to CR(F_\epsilon^n)$ is surjective for $n \geq 2$ and $0 \leq \epsilon < \frac{1}{2}$. If u is a CR function on G_ϵ^n, then

$$\sup_{z \in G_\epsilon^n} |u(z)| = \sup_{z \in F_\epsilon^n} |u(z)| .$$

Proof: If $\sup_{z \in F_\epsilon} |u(z)|$ is infinity there is nothing to prove. Suppose $\sup_{z \in F_\epsilon} |u(z)|$ is finite, and that there exists a $z_o \in G_\epsilon - F_\epsilon$ such that $|u(z_o)| > \sup_{z \in F_\epsilon} |u(z)|$. We will show a contradiction.

Case 1. Let $n = 2$. Let $v(z)$ equal $(u(z)-u(z_o))^{-1}$, so $v(z)$ is smooth in a connected one sided neighborhood Q of F_ϵ, such that $Q \subset G_\epsilon$, and v is holomorphic on $Q - F_\epsilon$. By continuity, $v_1 = v|F_\epsilon$ is a CR function. By assumption there exists a $v_2 \in CR(G_\epsilon)$ such that v_2 equals v on F_ϵ. Define h by

$$h = \begin{cases} v-v_2 & \text{on } Q \\ 0 & \text{on a one-sided neighborhood of } F_\epsilon \text{ which doesn't intersect } Q \end{cases}$$

This h is continuous and $\bar{\partial} h$ equals 0 in the sense of distributions. Therefore, h is a holomorphic function that vanishes on an open set. This implies h is the zero constant and v equals v_2 on Q. Which is impossible because it implies that v_2 equals $(u(z)-u(z_o))^{-1}$ on G_ϵ.

Case 2. Assume $n > 2$, z equals (z_1, z_2, η), and $u_\eta(z_1, z_2)$ equals $u(z_1, z_2, \eta)$. Then $u_\eta \in CR(G_\epsilon^2)$ and by Case 1 we get a contradiction.

Corollary: $CR(G_o^2)$ is isomorphic to $CR(F_o^2)$.

Proof: Let u_1 and u_2 be CR functions on G_o^2 such that $u_1|F_o^2$ equals $u_2|F_o^2$. Then $u_1 - u_2$ equals zero on F_o^2. Therefore, $\sup_{z \in G_o^2} |(u_1 - u_2)(z)|$ equals 0 or u_1 equals u_2.

Lemma: Let $r : CR(G_\epsilon^2) \to CR(F_\epsilon^2)$ be the restriction map where $0 < \epsilon < \frac{1}{2}$. Then r is a bijection.

Proof: Let U be a Stein neighborhood of G_ϵ such that $z \to w$ is biholomorphic and G_ϵ is closed in U. Let $V_1 = \{x \in U | x_1 < 0 \text{ or } x_2 < 0\}$, $V_2 = V_1 \cup G_\epsilon$, and let V_3 be the hull of holomorphy of V_2. Let u be a CR function on F_ϵ. Let $\hat{u}$ be a $\bar{\partial}$-flat extension of u to V_3. Define h by

$$h = \begin{cases} 0 & \text{on } \{z \in V_3 | x_1 < 0 \text{ or } x_2 < 0\} \\ \bar{\partial} u & \text{on } \{z \in V_3 | x_1 \geq 0 \text{ and } x_2 \geq 0\} \end{cases}$$

The function h is smooth on V_3. Since V_3 is Stein there exists a smooth function $j : V_3 \to C$ such that $\bar{\partial} j = h$

and j is holomorphic on V_1. The local version of Bochner's tube theorem (see Komatsu) states that every holomorphic function on a one-sided neighborhood of F_ϵ extends to a neighborhood of G_ϵ. Therefore, there exists a holomorphic function $k : V_2 \to C$ such that k equals j on V_1. Let $\tilde{u} = \hat{u} - j + k$. The proof follows as it did in the case $\epsilon = 0$.

We will now conclude the proof of the lemma of the folding screen with the following proposition:

Proposition: Let $r : CR(G_\epsilon^n) \to CR(F_\epsilon^n)$ be the restriction map on functions, where $n \geq 2$ and $0 \leq \epsilon < \frac{1}{2}$. Then r is a bijection.

Proof: Case 1) Let $0 < \epsilon < \frac{1}{2}$. Choose a neighborhood U' of F_ϵ^n to be $U \times C^{n-2}$ where U is the set defined in the proof of the case $n = 2$. Let $V_i' = V_i \times C^{n-2}$ where the V_i's are the sets defined in the proof of the case $n = 2$. Let u be a CR function on F_ϵ^n and $\hat{u}$ be a $\bar{\partial}$-flat extension of u to U'. Let h_1, h_2, and h be defined by

$$h_1 = \begin{cases} 0 & \text{on } V_1' \\ \dfrac{\bar{\partial}\hat{u}}{\partial \bar{w}_1} & \text{on } G_\epsilon^n \end{cases},$$

$$h_2 = \begin{cases} 0 & \text{on } V_1' \\ \dfrac{\bar{\partial}\hat{u}}{\partial \bar{w}_2} & \text{on } G_\epsilon^n \end{cases},$$

and $h = h_1 d\bar{w}_1 + h_2 d\bar{w}_2$. h, h_1 and h_2 are smooth and have compact support for fixed w_2 and η, where η equals $(y_3,\ldots,y_n)$. Let j be defined by

$$j(w_1,w_2,\eta) = \frac{1}{2\pi i}\int_C \frac{h_1(\zeta,w_2,\eta)}{\zeta - w_1}\, d\zeta \wedge d\bar{\zeta} ,$$

where we define $h_i(-,w_2,\eta)$ to be zero outside of its support. If D is any derivative with respect to $w_1,\bar{w}_1,w_2,\bar{w}_2$, and η we notice that:

$$\int_C \frac{Dh_1(w_1-t,w_2,\eta)}{t}\, dt \wedge d\bar{t} = D\int_C \frac{h_1(w_1-t,w_2,\eta)}{t}\, dt \wedge d\bar{t}$$

$$= D\int_C \frac{h_1(\zeta,w_2,\eta)}{\zeta - w_1}\, d\zeta \wedge d\bar{\zeta}$$

$$= D\, 2\pi i\; j(w_1,w_2,\eta).$$

Therefore j is smooth. Using the generalized Cauchy integral on a curve Γ contained in the unbounded component of $\operatorname{supp} h_1(-,w_2,\eta) \cup \operatorname{supp} h_2(-,w_2,\eta)$, we see that

$$\frac{\partial j}{\partial \bar{w}_1}(w_1,w_2,\eta) = \frac{1}{2\pi i}\int_C \frac{1}{\zeta - w_1} \cdot \frac{\partial}{\partial \bar{w}_1} h_1(\zeta,w_2,\eta) d\zeta \wedge d\bar{\zeta}$$

$$= h_1(w_1,w_2,\eta).$$

If we let $h_\eta(w_1,w_2) = h(w_1,w_2,\eta)$, then $\bar{\partial} h_\eta = 0$. So that

$\frac{\partial h_1}{\partial \bar{w}_2} = \frac{\partial h_2}{\partial \bar{w}_1}$. Therefore:

$$\frac{\partial j}{\partial \bar{w}_2}(w_1,w_2,\eta) = \frac{1}{2\pi i}\int_C \frac{\partial}{\partial \bar{w}_2} \frac{h_1(w_1-t,w_2,\eta)}{t}\, dt \wedge d\bar{t}\ .$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{t}\frac{\partial h_2}{\partial \bar{w}_1}(w_1-t,w_2,\eta)dt \wedge d\bar{t}$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{\zeta - w_1}\frac{\partial h_2}{\partial \bar{w}_1}(\zeta,w_2,\eta)d\bar{\zeta} \wedge d\zeta$$

$$= h_2(w_1,w_2,\eta).$$

For fixed η, j is holomorphic in w_1 and w_2 on V_1' and zero on an open set of V_1' (for $w_1 \notin \operatorname{supp} h_\eta(w_1,w_2)$). Since V_1' is connected, j is zero on V_1' and F^n_ϵ. Therefore, $\tilde{u} = \hat{u} - j$ is a smooth function of G^n_ϵ which equals u on F^n_ϵ. Since $\tilde{u}_\eta(w_1,w_2) = \tilde{u}(w_1,w_2,\eta)$ has the property that $\bar{\partial}\tilde{u}_\eta$ equals zero, $\tilde{u}$ is a CR function. By applying the Maximum Modulus Principle, we conclude that $\tilde{u}$ is the only CR extension of u to G^n_ϵ .

Case 2) Let u be a CR function on F^n_0, let u_ϵ be the restriction of u to F^n_ϵ, where $0 < \epsilon < \frac{1}{2}$, and let $\tilde{u}_\epsilon$ be the unique CR extension of u_ϵ to G^n_ϵ. Choose ϵ and ϵ' such that $0 < \epsilon < \epsilon' < \frac{1}{2}$. Then $G^n_\epsilon \cap G^n_{\epsilon'}$ is connected, has a non-empty relative interior, and contains $F^n_\epsilon \cap F^n_{\epsilon'}$. Consider $\tilde{u}_\epsilon - \tilde{u}_{\epsilon'}$ defined on $G^n_\epsilon \cap G^n_{\epsilon'}$. The function $\tilde{u}_\epsilon - \tilde{u}_{\epsilon'}$ is CR and zero on $F^n_\epsilon \cap F^n_{\epsilon'}$. The proof of the Maximum Modulus Principle shows that $\tilde{u}_\epsilon - \tilde{u}_{\epsilon'}$ is zero everywhere. We can define the CR extension $\tilde{u}$ of u to G^n_o by the values of $\tilde{u}_\epsilon$ on G^n_ϵ. It is unique by the Maximum Modulus Principle.

§5. CR Extension Theorem: The Locally Starlike Case

Theorem: Let M be a connected, locally starlike, locally closed subset of R^n. Then $r : CR(\tau(ach(M))) \to CR(\tau(M))$ is a bijection where r is the restriction map and $n \geq 2$.

In this section we prove the above theorem where M is a compact polygonal path. We conclude the proof for the general case in the next section. First we need the following higher dimensional version of the lemma of the folding screen.

Proposition: Let $\{A_i\}_{i=0}^{K}$ $(K \leq n)$ be a convex linearly independent set of points in R^n, let $l_{i,j}$ equal $ch\{A_i, A_j\} - \{A_j\}$. Then $r : CR(\tau(ch \bigcup_{j=1}^{K} l_{o,j})) \to CR(\tau(\bigcup_{j=1}^{K} l_{o,j}))$ is a bijection, where r is the restriction map and $n \geq 2$.

Proof: If K equals 0 or 1 then there is nothing to prove. If K equals 2, we can apply the lemma of the folding screen. For the inductive step, assume the proposition is true for all positive integers less than or equal to $K \leq n-1$. We prove the proposition for K+1.

By applying a complex affine isomorphism we reduce the proof to the case where $A_o = 0$ and A_i equals e_i (the standard basis vectors in R^n). Let E^i equal $\tau(ch \bigcup_{\substack{j=0 \\ j \neq i}}^{K+1} l_{o,j})$, so E^i is the tube over the face of the simplex spanned by all the A_j's from 0 to K+1 except A_i. If f is a CR function on $\bigcup_{j=0}^{K+1} l_{o,j}$, then by our assumption f has a unique CR extension $\hat{f}$ to $\bigcup_{i=1}^{K+1} E^i$. We now construct the CR extension $\tilde{f}$ of f to $\tau(\mathrm{ch} \bigcup_{j=1}^{K+1} l_{o,j})$ by using $\hat{f}$. If z equals $(z_1,\ldots,z_n)$, let ζ_i equal $(z_2,\ldots,z_{i-1},z_{i+1},\ldots,z_n)$. Let $\hat{f}_{\zeta_i}(z_1,z_i)$ equal $\hat{f}(z)$ for $z \in \bigcup_{j=1}^{K+1} E^j$, $i \neq 1$ and $i \leq K+1$, so that $\hat{f}_{\zeta_i}$ is a CR function on $E^1 \cup E^i$. Let $\tilde{f}_{\zeta_i}$ be the unique CR extension of $\hat{f}_{\zeta_i}$ for fixed ζ_i given by the lemma of the folding screen. Then $\tilde{f}_{\zeta_i}(z_1,z_i)$ equals $\tilde{f}_{\zeta_j}(z_1,z_j)$ because both functions are holomorphic in the interior of the domains of the z_1 variable and their boundary values coincide when x_1 equals zero. Define $\tilde{f}(z)$ to be $\tilde{f}_{\zeta_i}(z_1,z_i)$.

By construction all partial derivatives of $\tilde{f}$ exist. $\frac{\partial \tilde{f}}{\partial \bar{z}_i}$ equals zero for i between 1 and K+1. To conclude the proof that $\tilde{f}$ is CR we must show that $\tilde{f}$ is smooth. We now show that the partial derivatives of $\tilde{f}$ are locally bounded. The smoothness follows. Since $\tilde{f}$ satisfies the tangential Cauchy Riemann equations, $\frac{\partial^p f}{\partial x_i{}^p}$ equals $(-i)^p \frac{\partial^p \tilde{f}}{\partial y_i{}^p}$ for i between 1 and K+1. Therefore the former inherits local boundedness from the latter. The derivative $\left.\frac{\partial^p \tilde{f}}{\partial y_i{}^p}\right|_{\bigcup_{i=1}^{K+1} E^i}$ is a CR function and $\frac{\partial^p \tilde{f}}{\partial y_i{}^p}$ satisfies the tangential Cauchy-

Riemann equations in the relative interior of its domain.

Also $\frac{\partial^p \tilde{f}_{\zeta_2}}{\partial y_i{}^p}$ is a CR function. Let $F_\varepsilon^{\zeta_2}$ (or $G_\varepsilon^{\zeta_2}$) be the pull back by B^C of F_ε (or G_ε) corresponding to the domain of $\frac{\partial^p \tilde{f}_{\zeta_2}}{\partial y_i{}^p}$ with ζ_2 fixed, where B^C is the complex affine isomorphism defined in the lemma of the folding screen. Of course, $F_\varepsilon^{\zeta_2}$ is relatively compact. Choose a point z' in the domain of $\tilde{f}$. Then there exists a ζ_2' and an ε between 0 and $\frac{1}{2}$ such that $z' \in G_\varepsilon^{\zeta_2}$. Let $\delta > 0$ be chosen so that $F_{\varepsilon,\delta}^{\zeta_2'} = \bigcup_{|\zeta_2-\zeta_2'|<\delta} F_\varepsilon^{\zeta_2}$ is relatively compact. Let $G_{\varepsilon,\delta}^{\zeta_2'} = \bigcup_{|\zeta_2-\zeta_2'|<\delta} G_\varepsilon^{\zeta_2'}$. By the Maximum Modulus Principle:

$$\sup_{z\in G_{\varepsilon,\delta}^{\zeta_2'}} \left|\frac{\partial^p \tilde{f}}{\partial y_i{}^p}(z)\right| = \sup_{z\in F_{\varepsilon,\delta}^{\zeta_2}} \left|\frac{\partial^p \tilde{f}(z)}{\partial y_i{}^p}\right|.$$

Since $F_{\varepsilon,\delta}^{\zeta_2'}$ is relatively compact and $\frac{\partial^p \tilde{f}}{\partial y_i{}^p}$ is continuous on $\bigcup_{i=1}^{K+1} E^i$, the right hand side of the equality is finite.

Therefore $\frac{\partial^p \tilde{f}}{\partial y_i{}^p}$ is bounded on $G_{\varepsilon,\delta}^{\zeta_2}$.

To prove that $\tilde{f}$ is the only CR extension of f, suppose g is another such extension, $\tilde{f}$ and g both agree on $\bigcup_{i=1}^{K+1} E^i$. For fixed $y_{K+2},\ldots,y_n$, $\tilde{f}$ and g are holomorphic functions of

$z_1, \ldots, z_{K+1}$ on a connected set and have the same boundary values on the E^i's. Therefore $\tilde{f}$ equals g.

Corollary: Let u be a CR function on $\tau(\text{ch} \bigcup_{j=1}^{K} l_{o,j})$. Then

$$\sup_{z \in \text{dom } u} |u(z)| = \sup_{z \in \tau(\bigcup_{i=1}^{K} l_{o,i})} |u(z)|.$$

Proposition: Let P be a compact polygon path in R^n with vertices $\{v_i\}$

Then $r : CR(\tau(\text{ach}(P))) \to CR(\tau(P))$ is a bijection.

Proof: We order the vertices according to their occurrence, and prove the theorem using mathematical induction on the number m of the vertices. We suppose the proposition is true for all K less than m. We will prove it for m . Note that if m = 0 or 1 there is nothing to prove. If m equals 3 the lemma of the folding screen holds.

Let P_m be the part of the curve from v_o to v_m. Let $\tilde{P}_m$ = rel-int ch(P_m). Let B be defined by

$$B = \{x \in R^n | \exists A_i (i = 0, \ldots, K) \in \tilde{P}_{m-1}, A_o, \ldots, A_K, v_{m-1}, v_m$$
$$\text{are convex linearly independent points,}$$
$$x \in \text{rel-int ch } L\{A_i\}\},$$

where $L\{A_i\} = \bigcup_{i=1} l_{v_{m-1},A_i} \cup l_{v_{m-1},v_m}$ with $l_{x,y} = \text{ch}\{x,y\} - \{y\}$

and K equals the dimension of $\tilde{P}_m$-1 if v_m is in the affine space spanned by $\tilde{P}_{m-1}$ and equals the dimension of $\tilde{P}_m$ otherwise.

The set B is relatively open in the space spanned by B (i.e. rel-open), connected, and $\bar{B}$ contains P_m. Let f be a CR function on $\tau(P_m)$. By assumption f can be extended to a CR function $\hat{f}$ on $\tau(\tilde{P}_{m-1} \cup P_m)$. For $\{A_i\}$ as in the definition of B, one can restrict $\hat{f}$ to $\tau(L\{A\})$ and extend this to a CR function $\tilde{f}\{A_i\}$ on $\tau(\text{achL}\{A_i\})$. Given two sets $\{A_i\}$ and $\{A_i'\}$, let $Q = \tau(\text{achL}\{A_i\}) \cup \tau(\text{achL}\{A_i'\})$. If Q has nonempty relative interior then $\tilde{f}\{A_i\}$ and $\tilde{f}\{A_i'\}$ agree on Q, since Q is connected and the $\tilde{f}$'s agree on $\tau(1_{v_{m-1},v_m})$. Therefore $r : CR(\tau(P_m \cup B)) \to CR(\tau(P_m))$ is a surjection. The injectivity follows by the same argument we used to prove that $\tilde{f}\{A_i\}$ agrees with $\tilde{f}\{A_i'\}$ on Q.

The following argument will prove that every CR function on $\tau(B)$ can be extended to $\tau(\text{ch } B)$. It is an adaptation of the argument Hormander gives in his proof of Bochner's tube theorem.

Assume that B is starlike with respect to the origin. Then there exists a largest starlike (with respect to the origin) rel-open set C containing B such that every CR function g on $\tau(B)$ can be extended to a CR function $\tilde{g}$ on $\tau(C)$. If C isn't convex, it contains two points x^1 and x^2 such that the segment containing these points are not in C. We may choose coordinates so that $x^1 = (1-\delta, 0, \ldots, 0)$ and $x^2 = (0, 1-\delta, 0, \ldots, 0)$, with $\delta \in (0,1)$ and $\lambda e_1, \lambda e_2 \in C$ for $\lambda \in [0,1)$. Since C is starlike with respect to the origin and rel-open, one can find K+1 points d_i in C such that $l_{o,d_i} \subset C$, rel-int$\{ch\{x_1 x_2\} \subset$ rel-int ch $\bigcup_{i=1}^{K+1} l_{o,d_i}$, and $\{o,d_i\}$ is is convex linearly independent. Every CR function g extends to a CR function g' on $\tau(C \cup$ rel-int ch $\bigcup_{i=1}^{K+1} l_{o,d_i})$. The set $C \cup$ rel-int ch $\bigcup_{i=1}^{K+1} l_{o,d_i}$ is rel-open and starlike with respect to the origin. This is a contradiction. Therefore C is convex. Since $C \supset B$ every CR function on B extends to ch B.

Assume B is an arbitrary connected rel-open set such that $0 \in B$. Let C be the largest rel-open set, starlike with respect to the origin such that every $\hat{g} \in CR\tau(B)$ extends to a $\tilde{g} \in CR\tau(C)$. By the above, C is convex. We must prove C contains B.

If not there exists a point $\xi \in B-C$. Join ξ to 0 with a compact polygonal path in B. Let ξ_1 be its last intersection with ∂C. Then ξ_1 is connected to 0 by a polygonal path which apart from ξ_1 belongs to $B \cap C$. Let N be a convex rel-open neighborhood of ξ_1 in B. Then $C \cup N$ is starlike with respect to ξ_1 (C is convex). Let g' be defined to equal g on $\tau(C)$ and $\hat{g}$ on $\tau(B)$. By the above g' extends to a CR function g" on ch$(C \cup N)$. But ch$(C \cup N)$ is starlike with respect to the origin. Therefore $C \supset B$.

Every CR function f on $\tau(P_m)$ extends to $\tilde{f}$ on $\tau(\text{ach } P_m)$, since $\bar{B}$ contains P_m. The extension is unique because the boundary values of any two such extensions agree on P_m.

§6. Conclusion of the CR Extension Theorem: Locally Starlike Case

Assume M is a connected, locally starlike, locally closed subset of R^n. Let P be a compact polygonal path in M such that the convex dimension of P equals the convex hull of M. By the proposition in section 5, every CR function f on $\tau(M)$ extends to a CR function $\tilde{f}_P$ on $\tau(M \cup \text{ach}(P))$. Supoose P' is another such polygonal path. Let D equal $\tau(\text{ach } P' \cap \text{ach } P)$. To prove that $\tilde{f}_P$ agrees with $\tilde{f}_{P'}$ on D if $D \neq \emptyset$, we note the existence of a compact polygonal path P" such that $P'' \supset P \cup P'$. Then $\tilde{f}_P$ agrees with $\tilde{f}_{P''}$ on the intersection of their domains by the uniqueness of CR extensions on tubes over compact polygonal paths. The same is true for $\tilde{f}_{P'}$ and $\tilde{f}_{P''}$. Therefore $\tilde{f}_P$ and $\tilde{f}_{P'}$ agree on D. Let B be defined by

$$B = \{x \in R^n | x \in \text{rel-int ch}(P), \text{ where P is a compact polygonal path with convex dim P = convex dim M}\},$$

so that B is a rel-open, and $\bar{B} \supset M$. Then B is convex since if x_1 and $x_2 \in B$ implies the existence of P_1 and P_2 compact polygonal paths corresponding to x_1 and x_2, and a compact polygonal path P_3 containing P_1 and P_2. Also $\text{ch}\{x_1,x_2\} \subset \text{rel-int ch } P_3$. Therefore r is a surjection.

The injectivity follows as it did in the compact polygonal path case.

Remark: All of these CR extension theorems hold (for tubes over locally starlike subsets of R^n) if one assumes the CR functions to be of class C^s $(s \geq 2)$. This will not hold true when we work on the submanifold case.

§7. The CR Extension Theorem: Manifold Case

Theorem: Let M be a connected locally closed submanifold of R^n. Then $r : CR(\tau(ach(M))) \to CR(\tau(M))$ is a bijection.[1]

The proof of this theorem will occupy sections 8 and 9. In section 8 we prove it when M is a curve. To do this we need detailed information about the CR structures of tubes over curves. That is what this section deals with.

Lemma: Let $\gamma : (-1,1) \to R^n$ be a smooth embedding. Then $ex_{\gamma(0)}(\tau(im\gamma))$ is equal to the dimension of the span of the derivatives of γ at 0 minus one.

Proof: The excess dimension of γ at 0 equals the complex dimension of the Levi Algebra at $\gamma(0)$ modulo the direct sum of $HT_{\gamma(0)}(\tau(im\gamma))$ and $AT_{\gamma(0)}(\tau(im\gamma))$. The Levi Algebra at $\gamma(0)$ is generated by antiholomorphic and holomorphic vector fields to γ near $\gamma(0)$. Since tube manifolds are generic, the complex dimension of the holomorphic tangent space at $\gamma(0)$ is one. The holomorphic vector fields to γ are of the form $c\cdot\sum_{i=1}^{n} \gamma_i'(t) \frac{\partial}{\partial z_i}$ (where c is complex) or

$c\cdot\sum_{i=1}^{n} \gamma_i'(t)(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i})$. The latter way of viewing a holomorphic tangent vector will be more useful at this point. Note that:

[1] There is an extra restriction on the manifolds considered. See the note at the end of this section.

(1) $$[\sum_{i=1}^{n} \gamma_i'(t)\frac{\partial}{\partial z_i}, \sum_{i=1}^{n} \gamma_i'(t)\frac{\partial}{\partial \bar{z}_i}] = 2i[\sum_{i=1}^{n} \gamma_i'(t)\frac{\partial}{\partial x_i}, \sum_{i=1}^{n} \gamma_i'(t)\frac{\partial}{\partial y_i}].$$

Since $\gamma_i'(t)\frac{\partial}{\partial y_i}$ is tangent to imγ, we may write the right hand side of the equality as

(2) $$\sum_{j=1}^{n} 2i[\sum_{i=1}^{n} \gamma_i'(t)\frac{\partial}{\partial x_i}, \gamma'(t)\frac{\partial}{\partial y_j}].$$

Using Lie derivatives we see that (2) is just

(3) $$\sum_{j=1}^{n} 2i\ \gamma_j''(t)\frac{\partial}{\partial y_j}.$$

Following the above procedure, we get the following formulas:

(4) $$[\sum_{i=1}^{n} \gamma_i'(t)\frac{\partial}{\partial z_i}, \sum_{i=1}^{n} \gamma_i^{(K)}(t)\frac{\partial}{\partial y_j}] = \sum_{j=1}^{n} \gamma_j^{(K+1)}(t)\frac{\partial}{\partial y_j} \qquad K \geq 2$$

(5) $$[\sum_{i=1}^{n} \gamma_i'(t)\frac{\partial}{\partial \bar{z}_i}, \sum_{i=1}^{n} \gamma_i^{(K)}\frac{\partial}{\partial y_i}] = \sum_{i=1}^{n} \gamma_i^{(K+1)}(t)\frac{\partial}{\partial y_i}$$

(6) $$[\sum_{i=1}^{n} \gamma_i^{(K)}(t)\frac{\partial}{\partial y_i}, \sum_{i=1}^{n} \gamma_{i(t)}^{(\ell)}\frac{\partial}{\partial y_i}] = 0.$$

Therefore

$$\sum_{i=1}^{n} \gamma_i'(0)\frac{\partial}{\partial z_i}, \sum_{i=1}^{n} \gamma_i'(0)\frac{\partial}{\partial \bar{z}_i}, \text{ and } \sum_{i=1}^{n} \gamma_i^{(K)}(0)\frac{\partial}{\partial y_j}\ (K \geq 2)$$

span the Levi Algebra of τ(imγ) at $\gamma(0)$.

<u>Theorem</u>: Let $\gamma : I \to R^n$ be a smooth embedding, where I is a closed interval in R. Assume that $\{\gamma^{(i)}(t)\}_{i=1}^{j}$(i-th derivative of γ) is linearly independent for all $t \in I$, $j \leq n$, and

that $\{\gamma^{(i)}(t)\}_{i=1}^{j+1}$ is dependent for all $t \in I$. Then the image of γ is contained in a j-dimensional affine subspace of R^n.

Proof: Without loss of generality assume that $I = [-1,1]$, $\gamma(0) = 0$, and that V equals the span of the $\gamma^{(i)}(0)$ where $i = 1,\ldots,j$. By assumption, $\gamma^{(j+1)}(t) = \sum_{i=1}^{j} c_i(t)\gamma^{(i)}(t)$, where the c_i are smooth, since the Wronskian of the $\gamma^{(i)}(t)$ isn't zero. Let $\pi : R^n \to R^n/V$ be the quotient map and $\tilde{\gamma} = \pi\circ\gamma$. Then $\tilde{\gamma}^{(j+1)}$ equals $\sum_{i=1}^{j} c_i\tilde{\gamma}^i$ and $\tilde{\gamma}^i(0) = 0$ for $i = 0,\ldots,j$. Therefore $\tilde{\gamma}$ satisfies an ordinary linear differential equation. There exists a unique solution satisfying the initial conditions above, and $\tilde{\gamma} \equiv 0$ satisfies both conditions. Therefore $\gamma(t) \in V$ for all $t \in V$ and the dimension of V equals j.

Lemma: Let M be a one dimensional embedded submanifold of R^n. Then M can be decomposed into two disjoint sets R and I. The set I is closed and nowhere dense in M. The set R satisfies $R = \cup R^j$ such that $x \in R^j$ implies the existence of an arbitrarily small neighborhoods U of x in M such that U is a subset of a j-dimensional affine subspace Γ_x and Γ_x is the smallest affine subspace containing U.

Proof: Without loss of generality consider M as the image of a smooth embedding $\gamma : (-1,1) \to R^n$. Define the sets R^i and J^i as follows:

$$R^n = \{m \in M \mid m = \gamma(t) \text{ and } \Lambda^n\gamma(t) \neq 0\} \text{ where}$$
$$\Lambda^i\gamma(t) = \gamma'(t)\wedge\ldots\wedge\gamma^{(i)}(t), \text{ and}$$
$$J^n = M - R^n.$$

Let R^{n-1} be the set of points, $m = \gamma(t)$, in the interior of J^n such that $\Lambda^{n-1}\gamma(t)$ is non-zero. Define J^{n-1} to be $M - (R^n \cup R^{n-1})$. Define the sets R^i and J^i recursively. Let R equal $\bigcup_{i=1}^{n} R^i$ and $I = J^1$. By construction the R^i are open and I is closed and nowhere dense. The R^i's have the desired property.

Note: Henceforth, we shall assume that all submanifolds M of R^n have the property that every pair of points in M are contained in a curve γ such that the set I (corresponding to the previous lemma) is finite. It would suffice to limit ourselves to the case where I is countably infinite; for simplicity we avoid this, for the proofs only involve induction. This hypothesis is satisfied when M is an analytic manifold.

§8. The Proof of the CR Extension Theorem: The Curve Case.

<u>Theorem</u>: Let M be a 1-dimensional, locally closed, embedded submanifold of R^n. Then $r : CR(\tau(ach(M))) \to CR(\tau(M))$ the restriction map on functions is a bijection.

Proof: Let M = RUI as constructed in the above lemma. Then for $x \in \tau(R^j)$ the excess dimension of M at x equals j-1. Near x, $\tau(M)$ is contained in $\tau(\Gamma_x)$. By the local CR extension theorem quoted in section 1, every CR function on a sufficiently small neighborhood U of x in $\tau(M)$ can be extended to a CR function on a connected manifold $\tilde{M}$ of dimension n+j whose closure contains that neighborhood. Since Γ_x has dimension j, $\tilde{M}$ is an open set in $\tau(\Gamma_x)$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, where the ε_i's are non-negative real numbers. Let f be a CR function on $U \cup (U+i\varepsilon)$, where $U \cap (U+i\varepsilon) \neq \emptyset$. The extension of $f|U$ to $\tilde{M}$ must agree on $\tilde{M} \cap (\tilde{M}+i\varepsilon)$ with the extension of $f|U+i\varepsilon$ to $\tilde{M}+i\varepsilon$ by the uniqueness of the extension of $f|U\cap(U+i\varepsilon)$. Therefore for any $x \in \tau(R^j)$ there exists a sufficiently small tubular neighborhood U_x of x in M such that every CR function extends uniquely to a tubular rel-open set $\tilde{M}_x$, whose closure contains U_x. Note that the real part of $\tilde{M}_x$ is locally starlike and locally closed. By our CR extension theorem for tubes over connected, locally starlike, locally connected subsets of R^n, $r : CR(ach\ U_x) \to CR(U_x)$ is a bijection.

Consider $\widetilde{M} = \tau(I) \cup \{z \in C^n | z \in \text{ach } U_x$ for some $x \in \tau(R^j)$ $j = 1,\ldots,n\}$, this set is locally starlike. The set $\widetilde{M}$ might not be locally closed. It seems that a CR function f on $\tau(M)$ might not extend to a well defined function on $\widetilde{M}$ by the method described in the previous paragraph. Since M is locally closed we can extend f to a CR function on a tube M' over a locally closed, locally starlike subset of R^n, where $M' \subset \widetilde{M}$ and $M' \supset \tau(M)$. We do this by extending $f|U_x$ to a locally closed, locally starlike tubular subset M_x' of $\widetilde{M}_x$ such that the convex dimension of $M_x' = \dim \widetilde{M}_x$, $M_x' \supset U_x$, the closure of the interior of M_x' with respect to $\widetilde{M}_x$ contains U_x, and the extension of f to M_x' is well defined (without worrying what point x we chose). Let M' equal the union of the M_x' and I. Let f' be the CR extension of f to M'. By our CR extension theorem on tubes over connected, locally closed, locally starlike subsets, there exists a unique extension of f' to $\tau(\text{achM})$.

What we have just done is consider the local tubular extensions we constructed earlier and restricted them to tubular sets near $\tau(M)$ so that the extension is well defined. We then applied our CR extension theorem.

§9. Conclusion of the CR Extension Theorem: Manifold Case.

Let M be a connected, locally closed submanifold of R^n of arbitrary dimension. Let f be a CR function on $\tau(M)$. Let C_1 and C_2 be two compact locally closed submanifolds of M of dimension one such that their convex dimension equals that of M. Using the results of the previous section, we know that there exists CR functions $\tilde{f}_{C_i}$ which are the extensions of $f|\tau(C_i)$ to τ(ach C_i). We will show that $\tilde{f}_{C_1}$ equals $\tilde{f}_{C_2}$ on the intersection of their domains.

A) If the distance of C_1 from C_2 is greater than 0, then there exists a one dimensional locally closed manifold C_3 in M containing C_1 and C_2. As in the polygonal case $\tilde{f}_{C_3}$ equals $\tilde{f}_{C_i}$ $(i = 1,2)$ on the intersection of their domains. Therefore $\tilde{f}_{C_1}$ and $\tilde{f}_{C_2}$ are equal on the intersection of their domains.

B) Suppose that $C_1 \cap C_2 = \emptyset$, and the distance between the C_i's is zero. Let x be an element of the intersection of relative interior of the convex hull of the C_i's. There exists a compact curve Γ_i in each C_i such that x is in the relative interior of the convex hull of the image of Γ_i. This reduces the question of $\tilde{f}_{C_1}$ agreeing with $\tilde{f}_{C_2}$ near x to the previous case.

c) Suppose that $C_1 \cap C_2 \neq \emptyset$. Let $\xi \in C_1 \cap C_2$. Let x be an element of rel-int ch $C_1 \cap$ rel-int ch C_2. Let f'_{x,C_i} be the restriction of $\tilde{f}_{C_i}$ to the tube over line segment between x and ξ. f'_{x,C_i} $(i = 1,2)$ is a CR function. The functions f'_{x,C_1} and f'_{x,C_2} have the same boundary value on $\tau(\xi)$. Therefore they are equal.

Define B by

$$B = \{x \in R^n | x \in \text{rel-int ch } C, \text{ where } C \text{ is a 1-dimensional manifold whose convex dimension equals that of } M\}$$

We can extend any $f \in CR(\tau(M))$ to an $\hat{f} \in CR(\tau(M \cup B))$. Since B is locally starlike and locally closed $\hat{f}$ extends to a $\tilde{f} \in CR(\tau(M \cup \text{ch } B))$. The closure of B contains M, therefore ch B equals rel-int ch(M). This extension is unique because of the unique extension to B.

Corollary (The general Maximum Modulus Theorem): Let M be a connected locally closed subset of R^n (or a locally closed submanifold of R^n). Let u be a CR function on $\tau(\text{ach}(M))$. Then

$$\sup_{z \in \text{dom} u} |u(z)| = \sup_{z \in \tau(M)} |u(z)|.$$

The proof is the same as it was in the lemma of the folding screen.

Appendix 1

Let N be the tube over a locally closed submanifold M of R^n. The CR functions on N are a Frechet algebra when given the induced topology from the Frechet algebra of smooth complex valued functions on N. The only property of Frechet algebras that CR(N) doesn't trivially satisfy is that of completeness. Since the topology on CR(N) is that of convergence on compact subsets of the function and its partial derivatives, we only need to show that locally any Cauchy sequence of CR functions converges to a CR function. For any small neighborhood U in N, the CR functions on U are elements of the Ker X, where X is an anti-holomorphic vector field. The vector field X is a continuous operator on the Frechet space of smooth complex valued functions. Therefore the Kernel of X is closed, which implies CR(U) is closed in $C^\infty(U,C)$.

The topology on CR(N) can be induced by the following set of seminorms. Let K be a compact set in N. Let $\{\tilde{U}_i\}$ be a locally finite coordinate covering of M and $U_i = \tau(\tilde{U}_i)$. Define $\|f\|_{\alpha,K}$ where α is a multi-index by

$$\|f\|_{\alpha,K} = \sup_{|m|\leq\alpha} \sup_{U_i} \sup_{z\in K\cap U_i} |D^m f(z)|$$

where $m = (m_1, m_2)$ and

$D^m = \frac{\partial^{m_1}}{\partial \tilde{x}^{m_1}} \frac{\partial^{m_2}}{\partial y^{m_2}}$. The symbol $\frac{\partial^{m_2}}{\partial y^{m_2}}$ stands for the usual derivative in the y directions. The symbol $\frac{\partial^{m_1}}{\partial \tilde{x}^{m_1}}$ stands for derivatives tangent to M.

We will now put a Frechet algebra structure on the CR functions on $\tau(\text{ach } M) = \text{ach } N$. Let $\{\tilde{W}_i\}$ be a locally finite cover of ach M, such that $W_i \cap M$ is a coordinate patch for M. Let K be a compact set and α a multi-index. Define $p_{K,\alpha}$ by

$$p_{K,\alpha}(f) = \max(\|f\|_{\alpha,K}, \sup_{|m| \leq \alpha} \sup_{\{W_i\}} \sup_{z \in W_i \cap \text{rel-int ch } N \cap K} |D'^m f(z)|),$$

where D'^m are the usual derivatives with respect to standard coordinates. The $p_{K,\alpha}$ are seminorms. The topology is Hausdorff and locally convex. It is complete since if f_n is a Cauchy sequence then $\lim_{n\to\infty} f_n$ exists and is continuous. By the above, $(\lim_{n\to\infty} f_n)|M$ equals $\lim_{n\to\infty}(f_n|N)$ and $(\lim_{n\to\infty} f_n)|\text{rel-int ch } N$ equals $\lim_{n\to\infty}(f_n|\text{rel-int ch } N)$. Both functions are CR. Since $\lim_{n\to\infty} f_n$ is continuous it is CR as well.

<u>Theorem:</u> Let $r : CR(\text{ach } N) \to CR(N)$ be the restriction map. Then r is a Frechet isomorphism.

Proof: The restriction map is obviously continuous. That r is one to one was proved earlier. By the closed graph theorem, the inverse map is continuous.

Appendix 2

Definition: Let M be a subset of C^n. Define exp(M) and log(M) by:

$$\exp(M) = \{z \in C^n | z = (e^{w_1}, \ldots, e^{w_n}), (w_1, \ldots, w_n) \in M\}$$

and

$$\log(M) = \{z \in C^n | \exp\{z\} \subset M\}.$$

Let K be an integer between a and n. Define $\exp^K : C^n \to C$ by

$$\exp^K(z) = (z_1, \ldots, z_K, e^{z_{K+1}}, \ldots, e^{z_n}).$$

Definition: Let K be an integer between 0 and 1. A set $M \subset C^n$ is a part-tubular part circular submanifold if $M = T \oplus C$, where T is a tube manifold in C^K and C is the image under exp of a tube manifold in C^{n-K}.

Definition: Let $M = T \oplus C$, as above, be connected and locally closed. Then a continuous function on $\exp^K(\text{ach}(T \oplus \log C))$ is a CR-function if $u|T \oplus C$ and $u|\exp^K(\text{rel-int ch}(T \oplus ,\text{og } C))$ are both CR functions.

Note: The relative interior of ch($T \oplus \log C$) equals rel-int ch(T) $\oplus$ rel-int ch(log C), since both are convex, rel-open, and have the same closure.

<u>Theorem</u>: Let M be a connected, locally closed, part tubular part circular submanifold of C^n. Then $r : CR(\exp^K(\mathrm{ach}(T \oplus \log C)) \to CR(T \oplus C)$ where $M = T \oplus C$ as above.

Proof: The map $\exp^K$ is a locally injective holomorphic map such that $\exp^K(T \oplus \log C) = T \oplus C$. Therefore $\widetilde{\exp} = \exp^K | T \oplus \log C$ is a CR map. Let $u \in CR(T \oplus C)$. Let $\hat{u}$ be the CR extension to $\mathrm{ach}(T \oplus \log C)$ of the pullback of u by $\widetilde{\exp}$ to $T \oplus \log C$. The map $\hat{u}$ is periodic, because $\hat{u}(z_1, \ldots, z_K, z_{K+1} + 2\pi p_{K+1} i, \ldots, z_n + 2\pi i p_n)$ has the same boundary values on $T \oplus \log C$ as $\hat{u}$. Therefore $\hat{u}$ projects to a CR map on $\exp^K(\mathrm{ach}(T \oplus \log C))$. This is the only CR extension of u because any CR extension u_1 of u could be pulled back by exp to a CR function u_1'. But $u_1' = \hat{u}$ by the uniqueness of the extension of CR functions on $T \oplus \log C$.

Corollary: Let $M = T \oplus C$ be as above. If $u \in CR(\exp^K(\mathrm{ach}$
$u \in CR(\exp^K(\mathrm{ach}(T \oplus \log C)))$ then

$$\sup_{z \in \mathrm{dom}\, u} |u(z)| = \sup_{z \in T \oplus C} |u(z)|.$$

Proof: Let $(\exp^K)^* u$ be the pullback of u, then

$$\sup_{z \in \mathrm{dom}\, u} |u(z)| = \sup_{z \in \mathrm{ach}(T \oplus \log C)} |(\exp^K)^* u(z)|$$

$$= \sup_{z \in T \oplus \log C} |(\exp^K)^* u(z)|$$

$$= \sup_{z \in T \oplus C} |u(z)|.$$

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