THE PHRAGMEN-LINDELÖF TYPE THEOREM FOR THE QUASILINEAR EQUATION

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ABSTRACT OF THE DISSERTATION

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The Phragmen-Lindelöf type theorems were studied from different points of view by various mathemeticians. Results of this type are used to obtain the uniqueness and approximation theorems by means of the maximum principle on unbounded domains by imposing some restrictions on the growth of the function at infinity. At the same time, these results are helpful in establishing the maximum principle for solutions in bounded domains when there are gaps in the prescription of boundary data or singular points on the boundary.

Here we introduce a result of the Phragmen-Lindelöf type

for a quasilinear equation of the form $\operatorname{div}\left(|u_x|^{p-2} u_x\right)=0$ which may be considered as a nonlinear analog of the Laplace operator $(p=2)$.

For this purpose we complete our previous investigation of the behavior of particular solutions of the above equation in the spherical cone in R^n which are to be used as barrier functions for the maximum principle.

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INTRODUCTION

The purpose of this paper is to establish a Phragmen-Lindelöf type theorem for the quasilinear equation

(0.1) $$\operatorname{div}\left(|u_x|^{p-2} u_x\right) = 0$$

where $u_x = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$, $1 < p < \infty$.

Equation (0.1) is a special case of an equation of the form

(0.2) $$\operatorname{div} a(x, u_x) = 0$$

where x lies in a bounded domain $\Omega \subset R^n$ and $a(x, \xi)$ is an n-dimensional vector-valued function defined on $\Omega \times R^n$ and satisfying the following conditions:

1) The function $\xi \longrightarrow a(x, \xi)$ is continuous on R^n for almost all $x \in \Omega$; the function $x \longrightarrow a(x, \xi)$ is measurable for every $\xi \in R^n$;
2) $|a(x, \xi)| \leq \gamma |\xi|^{p-1}$, $\gamma > 1$ for almost all $x \in \Omega$;
3) $a(x, \xi) \cdot \xi \geq |\xi|^p$ for almost all $x \in \Omega$;
4) $[a(x, \xi) - a(x, \zeta)](\xi - \zeta) > 0$, $\xi \neq \zeta$ for almost all $x \in \Omega$.

Let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω. Let $L_p^1(\Omega)$

be the space of locally summable functions on Ω with gradients (in the sense of distributions) belonging to $L_p(\Omega)$, $1<p<\infty$. The completion of $C_0^\infty(\Omega)$ in the norm $\|v_x\|_{L_p(\Omega)}$ will be written $\overset{\circ}{L}{}_p^1(\Omega)$. We denote by C various positive constants and by D_ρ an open ball centered at 0. Moreover, $C\Omega$ and $\overline{\Omega}$ are the complement and the closure of Ω, and $\Omega_\rho = \Omega \cap D_\rho$, $C_\rho = C\Omega \cap D_\rho$. Let h be a function in $L_p^1(\Omega)$. We say that a function $u \in L_p^1(\Omega)$ is a solution of the Dirichlet problem for the equation (0.2), if for all $\varphi \in \overset{\circ}{L}{}_p^1(\Omega)$

$$(0.3) \quad \int_\Omega a(x, u_x)\varphi_x \, dx = 0 \quad , \quad (u-\varphi) \in \overset{\circ}{L}{}_p^1(\Omega)$$

Equation (0.2) and more general elliptic equations of second order were studied in the works of O. A. Ladyzenskaya and N. N. Ural'ceva (see [1]), J. Serrin ([2]-[4]) and other authors. It was established in [1], as a particular case, that every generalized solution of (0.2) which belongs to $L_p^1(\Omega)$ satisfies a Hölder condition in the interior of Ω. This property also holds in the neighborhood of a boundary point provided that the lower (volume) density of $R^n \setminus \Omega$ at this point is positive.

J. Serrin, in his works [2]-[4], established a maximum principle, a Harnack inequality for a broad class of quasilinear elliptic equations, and obtained estimates of the solution in the neighborhood of the isolated singularity.

The question of the existence of generalized solutions of general quasilinear elliptic equations was taken up in the papers [5]-[8]. From these results it can be shown without much difficulty that a unique solution of the problem (0.3) does exist.

Different forms of the Phragmen-Lindelöf principle for linear elliptic equations of second order have been given by Gilbarg and Serrin [9], Hopf [10], Gilbarg [11], Serrin [12], Meyers and Serrin [13], Landis [14], and Blochina [15].

Further bibliographical references may be found in these papers.

To illustrate the Phragmen-Lindelöf principle, we recall a classical result concerning the growth of subharmonic functions in an unbounded sector in the plane.

Let K be the sector, defined by the inequalities $-cx < y < cx$, $x > 0$. In polar coordinates (ρ, θ) the equation of the boundary ∂K will be $\theta = \pm \frac{\pi}{2\alpha}$ where $c = \tan(\pi/2\alpha)$. The function

$W = \rho^{\alpha} \cos \alpha\theta$ is harmonic in the sector K and vanishes on the boundary ∂K. This function approaches infinity like ρ^{α} as $\rho \to \infty$ on every ray θ =constant.

<u>Theorem</u> (Phragmen-Lindelöf)

Let u satisfy the inequality

$$\Delta u \geqslant 0$$

in a sector K of angle $\frac{\pi}{\alpha}$. Assume that $u \leq M$ on the boundary $\theta = \frac{\pi}{2\alpha}$ and suppose that

$$\liminf_{R \to \infty} \left\{ R^{\alpha} \max_{\rho = R} u(\rho, \theta) \right\} \leq 0 .$$

Then $u \leq M$ in K.

The Phragmen-Lindelöf theorem asserts that the growth of the above function W as $\rho \to \infty$ is characteristic of harmonic functions which are unbounded in a sector. That is, any harmonic function which vanishes on the boundary and is not identically zero must grow as fast as ρ^{α}. Moreover, if a harmonic (or subharmonic) function is bounded along the entire boundary of a sector K of angle $\frac{\pi}{\alpha}$ and if it grows more slowly than ρ^{α} as $\rho \to \infty$ then it does not grow at all (i.e., it is bounded).

The main device of the proof consists in the determination of a harmonic function in $K_R = K \cap D_R$ which remains bounded away from zero and has an appropriate growth as $R \to \infty$. This function is to be used as a comparison function or a barrier in the maximum principle.

In the linear case a function having these properties is

$$W_R(\rho, \theta) = 1 + \frac{2}{\pi} R^{\alpha} \tan^{-1} \frac{2 R^{\alpha} \rho^{\alpha} \cos \alpha\theta}{R^{2\alpha} - \rho^{2\alpha}}$$

It is easily verified that W_R is harmonic since it is the imaginary part of the analytic function of $z = \rho e^{i\theta}$

$$f(z) = i + \frac{2}{\pi} R^{\alpha} \log \frac{R^{\alpha} + iz^{\alpha}}{R^{\alpha} - iz^{\alpha}} .$$

The basic method applies to much more general situations, and the theorem can be extended to include not only more general linear elliptic operators but also bounded as well as unbounded domains.

Our main goal is to establish a result of Phragmen-Lindelöf type for the quasilinear equation (0.1). The proof will be carried out by a scheme analogous to that used in the example above. Since the maximum principle for the equation (0.1) has been established by J. Serrin [2], we ought to find a solution of the equation (0.1) with properties similar to those of the function $W_R(\rho, \theta)$ in the linear case.

Let $K(\ell)$ denote the cone in R^n

$$K(\ell) = \{ x : 0 \le \theta \le \ell \}$$

where $\cos\theta = x_n |x|^{-1}$ and $0 < \ell < \pi$. We shall show that equation (0.1) has a non-negative solution of the form

$$u(\rho,\theta) = \rho^{\lambda} f_{\lambda}(\theta) \tag{1.1}$$

which vanishes on $\partial K(\ell)$. Here $\lambda = \lambda(\ell)$ is a number and $f_{\lambda}(\theta) \in C^2[0,\pi)$. For the function equation (0.1) takes the form

$$\sin^{n-2}\theta \left\{ [u_{\rho}'^2 + \rho^{-2} u_{\theta}'^2]^{\frac{p-2}{2}} \rho^{n-1} u_{\rho}' \right\}_{\rho}' +$$

$$+ \rho^{n-3} \left\{ [u_{\rho}'^2 + \rho^{-2} u_{\theta}'^2]^{\frac{p-2}{2}} u_{\theta} \sin^{n-2}\theta \right\}_{\theta}' = 0$$

Thus, $f_{\lambda}(\theta)$, $\theta \in [0,\ell]$ must satisfy the ordinary differential equation

$$\left\{ [\lambda^2 f^2(\theta) + f'^2(\theta)]^{\frac{p-2}{2}} f'(\theta) \sin^{n-2}\theta \right\}_{\theta}' +$$

(1.2)

$$\lambda[\lambda(p-1) + n - p][\lambda^2 f^2(\theta) + f'^2(\theta)]^{\frac{p-2}{2}} f(\theta) \sin^{n-2}\theta = 0 .$$

with boundary conditions

(1.3) $$f'(0) = 0 \ , \ f(\ell) = 0 \ .$$

We normalize $f(\theta)$ by the requirement that $f(0) = 1$. Those values of the parameter λ for which the problem (1.2), (1.3) has a nontrivial solution will be called its eigenvalues.

<u>Theorem 1.</u> There exist two and only two eigenvalues $\lambda_+(\ell)$ and $\lambda_-(\ell)$ of the boundary value problem (1.2), (1.3)

$$\lambda_+(\ell) > 0 \ , \quad \lambda_-(\ell) < (p-n)/(p-1) \ , \quad p \le n$$

$$\lambda_+(\ell) > (p-n)/(p-1) \ , \quad \lambda_-(\ell) < 0 \ , \quad p > n$$

for which the corresponding eigenfunctions $f_{\lambda_+}(\theta)$ and $f_{\lambda_-}(\theta)$ are positive on $[0, \ell]$ and[1]

(1.4) $$|f'_{\lambda_\pm}(\theta)| \sim \varphi(\lambda)\, \theta\, (\pi - \theta)^{\frac{2-n}{p-1}} \ , \quad \theta \in [0, \ell] .$$

These eigenfunctions are unique, in the sense that $\lambda_\pm(\ell)$ corresponds uniquely to $f_{\lambda_\pm}(\theta)$.

The proof of Theorem 1 will be given in section 3.

<u>1</u>/ Here and below, $a \sim b$ denotes that $C_1 \le a b^{-1} \le C_2$ where C_1 and C_2 are positive constants depending on n and p only.

SECTION II

The Phragmen-Lindelöf Type Theorem

In this section we shall study the solution of the Dirichlet problem (0.3) for the equation (0.1) in the neighborhood of a conical point (defined below) on the boundary. The behavior of the solution is characterized by one of the Phragmen-Lindelöf type alternatives: the solution either increases at least as rapidly as a negative power of the distance to the conical point, or decreases faster than a positive power of the distance.

Lemma 2.1. Let Ω be a bounded domain in R^n and δ a number such that for each point $0 \in \partial\Omega$ a sphere of radius δ can be drawn, which passes through 0 and whose interior belongs to $C\Omega$.

Let $u(x) \in L_p^1(\Omega)$ be a solution of the Dirichlet problem for the equation (0.1), $(u-\varphi) \in \overset{\circ}{L}{}_p^1(\Omega)$, $\varphi(x) \in L_p^1(\Omega)$, $\varphi = 0$ in a d- neighborhood of the point 0. Then the estimate

$$(2.1) \qquad \max_{x \in \Omega \cap D_d(0)} |Du| \le C\delta^{-1} \max_{x \in \Omega \cap D_d(0)} |u(x)|$$

remains true in a d-neighborhood of the point 0.

Proof. Let us denote by $d(x)$ the distance from a point $x \in \Omega$ to the boundary

$$d(x) = \operatorname{dist}(x, \partial\Omega).$$

It suffices to obtain the estimate

(2.2) $$|u(x)| \leq c\, d(x)$$

in a d-neighborhood of the point 0 .

Let d be greater than or equal to 3δ and $p \leq n$. We take the origin of our coordinate system to be at the point $P \in C\Omega$ and consider a function

$$\Gamma(y) = \delta^{\frac{p-n}{p-1}} - |y|^{\frac{p-n}{p-1}}.$$

It is clear that $\Gamma(y)$ is a solution of the equation (0.1) in $C D_\delta(P)$, such that $\Gamma(y) \geq 0$, $\Gamma(P) = 0$. If $p > n$ we take the function $-\Gamma(y)$.

Let us introduce a function

$$W(y) = \frac{\max\limits_{y \in \Omega \cap \partial D_d(0)} |u(y)|}{\min\limits_{y \in \Omega \cap \partial D_d(0)} \Gamma(y)} \Gamma(y)$$

and compare $u(y)$ and $W(y)$ in the domain $\Omega \cap D_d(0)$ whose boundary is

$$\partial[\Omega \cap D_d(0)] = [\partial\Omega \cap D_d(0)] \cup [\Omega \cap \partial D_d(0)]$$

Note, that by our choice

$$u(y) = 0 \quad , \quad W(y) \geq 0$$

for all $y \in \partial\Omega \cap D_d(0)$.

Since

$$\min_{y \in \Omega \cap \partial D_d(0)} \Gamma(y) \geq c\delta^{\frac{p-n}{p-1}} , \quad \max_{y \in \Omega \cap \partial D_d(0)} |u(y)| < \infty ,$$

it follows that

(2.3) $$|u(y)| \leq c\delta^{\frac{n-p}{p-1}} \Gamma(y)$$

for all $y \in \partial[\Omega \cap D_d(0)]$.

As long as $W(y)$ is a solution of the equation (0.1) in $\complement D_\delta(P)$, the inequality (2.3) is valid for all $x \in \Omega \cap D_d(0)$ as a corollary of the maximum principle.

We denote by $N_i(P)$ an inward normal to $\partial\Omega$ at the point P and let $x \in N_i(P) \cap D_d(0)$. The following computation

$$|D\Gamma(x)| = c[\delta + d(x)]^{\frac{1-n}{p-1}} \leq c\delta^{\frac{1-n}{p-1}}$$

leads to the estimate

$$\Gamma(x) = \Gamma(x) - \Gamma(P) \leqslant c\delta^{\frac{1-n}{p-1}} d(x). \tag{2.4}$$

Combining inequalities (2.3) and (2.4), we obtain (2.2), which provided the result of Lemma 2.1.

Remark. Actually, we could have referred the reader to the monograph [1] by O. A. Ladyzenskaya or paper [4] by J. Serrin for the Lemma, but these results are proved under more general conditions and hence are less explicit.

Theorem 2. Let $\varepsilon > 0$ be an arbitrary number and $\ell > 0$ be a number such that the cone $K(\ell) \subset C\Omega$ has a point $0 \in \partial\Omega$ as its vertex.

Let $u(x) \in L_p^1(\Omega \setminus D_\rho)$ be a non-negative solution of the equation (0.1), $\varphi \in C(\overline{\Omega}) \cap L_{p_0}^1(\Omega)$, $\varphi = 0$ in the neighborhood of the point 0, $(u-\varphi) \in \overset{\circ}{L}{}_p^1(\Omega \setminus D_\rho)$ for all $\rho > 0$ sufficiently small. Then either

$$\limsup_{x \to 0} \left\{ |x|^{-\lambda_-(\ell)-\varepsilon} \max_{x \in \Omega \cap \partial D_\rho} u(x) \right\} = \infty \tag{2.5}$$

or

$$|u(x)| \leqslant c|x|^{\lambda_+(\ell)} \tag{2.6}$$

where $\lambda_+(\ell)$ and $\lambda_-(\ell)$ are the eigenvalues for problems (1.2), (1.3) of Theorem 1.

Proof. Here, as before, $K(\ell+\varepsilon)$ is a closed spherical cone in R^n with its vertex at the point 0 :

$$K(\ell+\varepsilon)=\{x:\ 0\le\theta\le\ell+\varepsilon,\ \cos\theta=x_n|x|^{-1}\}$$

and $\partial K(\ell+\varepsilon)$ is its boundary.

We denote by $V(x)$ an "increasing" solution of the form (1.1) of the equation (0.1) in the cone $K(\ell+\varepsilon)$, which vanishes on $\partial K(\ell+\varepsilon)$

$$V(x)=|x|^{\lambda_-+\varepsilon}f(\theta).$$

$$V(x)\ge 0,\quad V(x)=0 \quad \text{for all} \quad x\in\partial K(\ell+\varepsilon).$$

Let us consider a function

$$W(x)=\max_{x\in\Omega\cap\partial D_1}u(x)+\frac{\max\limits_{x\in\Omega\cap\partial D_\rho}u(x)}{\min\limits_{x\in\Omega\cap\partial D_\rho}V(x)}V(x).$$

Evidently $W(x)\in L_p^1(\Omega\setminus D_\rho)$. We compare $u(x)$ and $W(x)$ in the domain $\Omega\cap(D_1\setminus D_\rho)$ with the boundary

$$\partial[\Omega\cap(D_1\setminus D_\rho)]=[\partial\Omega\cap(D_1\setminus D_\rho)]\cup(\Omega\cap\partial D_1)\cup(\Omega\cap\partial D_\rho)$$

For all $x\in\Omega\cap\partial D_1$

$$u(x) \leq \max_{x \in \Omega \cap \partial D_1} u(x) \leq W(x).$$

Since $u(x) = 0$ and $V(x) \geq 0$ on $\partial\Omega \cap (D_1 \setminus D_\rho)$, it follows that $u(x)$ remains less than or equal to $W(x)$ for $x \in \partial\Omega \cap (D_1 \setminus D_\rho)$ as well. By our choice for $W(x)$ the latter inequality is valid for $x \in \partial D_\rho \cap \Omega$. As a corollary of the maximum principle

$$u(x) \leq W(x), \qquad x \in \Omega \cap (D_1 \setminus D_\rho),$$

hence

$$u(x) \leq \max_{x \in \Omega \cap \partial D_1} u(x) + \frac{\max\limits_{x \in \Omega \cap \partial D_\rho} u(x)}{\min\limits_{x \in \Omega \cap \partial D_\rho} V(x)} V(x). \tag{2.7}$$

Statement (2.5) of the theorem follows from (2.7) if

$$\limsup_{x \to 0} \left\{ |x|^{-\lambda_-(\ell) - \varepsilon} \max_{x \in \Omega \cap \partial D_\rho} u(x) \right\} > 0.$$

Otherwise $u(x)$ is bounded and we consider as a barrier a solution of the equation (0.1)

$$W(x) = \frac{\max\limits_{x \in \Omega \cap \partial D_1} u(x)}{\min\limits_{x \in \Omega \cap \partial D_1} v_\psi(x)} \, v_\psi(x),$$

where $v_\psi(x) \in L_p^1(K(\ell))$, $(v_\psi(x) - \psi(x)) \in \overset{\circ}{L}{}_p^1(K(\ell))$

and $\psi(x) \in C_0^\infty(R^n)$ is defined by

$$\psi(x) = \begin{cases} 1 & , \ x \in \partial K(\ell) \cap (D_1 \setminus D_{1/2}) \\ 0 \le \psi(x) \le 1, & x \in \partial K(\ell) \cap (D_{1/2} \setminus D_{1/4}) \\ 0 & , \ x \in \partial K(\ell) \cap D_{1/4} \end{cases}$$

Evidently, $v_\psi(x) \geq 0$ on $\partial K(\ell)$, and by the maximum principle $v_\psi(x) \geq v(x)$, where $v(x)$ is a "decreasing" solution of the equation (0.1) of the form (1.1) in the cone $K(\ell)$, which vanishes on $\partial K(\ell)$.

It is known that $v_\psi(x) \in C^\alpha(K(\ell))$ for a certain $\alpha > 0$. Hence, there exists a number $\delta > 0$ such that

$$v_\psi(x) \geq C > 0, \quad x \in \partial D_1 \cap [K(\ell) \setminus K(\ell - \delta)]$$

On the other hand,

$$v_\psi(x) \geq v(x) \geq C > 0, \quad x \in \partial D_1 \cap K(\ell - \delta)$$

so that

(2.8) $$v_\psi(x) \geq C$$

for all $x \in \partial D_1 \cap K(\ell)$.

Let us compare $u(x)$ and $W(x)$ in the domain $\Omega \cap D_1$ with the boundary

$$\partial(\Omega \cap D_1) = (\partial\Omega \cap D_1) \cup (\Omega \cap \partial D_1).$$

It is clear that $u(x) = 0$ and $v_\psi(x) \geq 0$ on $\partial\Omega \cap D_1$. Besides

$$u(x) \leq W(x) \quad , \quad x \in \Omega \cap \partial D_1$$

by our choice of $W(x)$.

The latter inequality is valid for all $x \in \Omega \cap D_1$ as a corollary of the maximum principle, so that

$$u(x) \leq c v_\psi(x) \quad , \quad x \in \Omega \cap D_1 .$$

In order to complete the proof, we ought to obtain an estimate

(2.9) $$v_\psi(x) \leq c v(x)$$

for all $x \in K(\ell) \cap D_{1/8}$ with the boundary

$$\partial[K(\ell) \cap D_{1/8}] = [\partial K(\ell) \cap D_{1/8}] \cup [K(\ell) \cap \partial D_{1/8}].$$

Note that for all $x \in [\partial K(\ell) \cap D_{1/8}]$

$$v_\psi(x) = v(x) = 0.$$

We conclude from the relation (1.4) that

$$-\frac{\partial V(x)}{\partial n} > C \quad , \quad x \in [\partial K(\ell) \cap \partial D_{1/8}] .$$

It follows from Lemma 2.1 and the above inequality that there exists a constant C such that

$$V_\psi(x) \leq C V(x) \quad , \quad x \in [\, K(\ell) \setminus K(\ell - \delta)] \cap \partial D_{1/8} .$$

Inequality (2.9) is valid for all $x \in \partial [\, K(\ell) \cap D_{1/8}]$ since $V(x) \geq C > 0$ for $x \in [\, K(\ell - \delta) \cap \partial D_{1/8}]$ and hence, by the maximum principle, it remains true for all $x \in K(\ell) \cap D_{1/8}$. q.e.d.

Remarks. 1. For $p \neq 2$, equation (0.1) is not uniformly elliptic, and so does not satisfy one of the following inequalities as $|\xi| \longrightarrow 0$:

$$\nu (1 + |\xi|^{\cdot})^{p-2} |\zeta|^2 \leq \frac{\partial a_i(x,\xi)}{\partial \xi_j} \zeta_i \zeta_j \leq \mu (1 + |\xi|^{\cdot})^{p-2} |\zeta|^2$$

where ξ and ζ are arbitrary n-dimensional vectors.

Since the modulus of gradient of the solution

(2.10) $$u(\rho, \theta) = \rho^{\lambda} f_\lambda(\theta)$$

of the equation (0.1) tends to infinity as $|x| \to 0$ for $p \leq n-1$, we can easily alter equation (0.1) in this case so that the new equation becomes uniformly elliptic, while the function (2.10) remains its solution in a neighborhood of 0.

This is a property, for instance, of the equation

$$\operatorname{div}\,[\,a(u_x^2)\,u_x\,] = 0 \tag{2.11}$$

where $a(t)$ is an arbitrary smooth function equal to $|t|^{\frac{p-2}{2}}$ for large $t>0$ and satisfies, for all $t>0$

$$\nu(1+t)^{\frac{p-2}{2}} \le a(t)\,,\quad a(t)+2ta'(t) \le \mu\,(1+t)^{\frac{p-2}{2}}.$$

Clearly equation (2.11) may be written in a nondivergence form

$$\Big[\,\delta_{ij} + 2\,\frac{a'(u_x^2)}{a\,(u_x^2)}\,u_{x_i}u_{x_j}\,\Big]\,u_{x_ix_j} = 0. \tag{2.12}$$

Other examples may be obtained from (2.11) and (2.12) via the substitution $u=\varphi(v)$.

2. Similar results may be obtained for a quasilinear equation of the form

$$\operatorname{div}\,(\,|x|^{\beta}\,|u_x|^{p-2}\,u_x\,) = 0\,,\quad \beta\in\mathbb{R} \tag{2.13}$$

in exactly the same way. One would have to consider a function $u(x)\in L^1_{p,\beta}$,

$$\|u(x)\|_{L^1_{p,\beta}} = \Big(\int_{\Omega} |x|^{\beta}\,|u(x)|^p\,dx\Big)^{\frac{1}{p}}$$

as a solution of the equation (2.13), $\varphi \in L^{1}_{p,\beta}(\Omega)$, $(u-\varphi) \in \overset{\circ}{L}{}^{1}_{p,\beta}(\Omega)$.

The maximum principle and other basic facts on solutions of the equation (2.13) from $L^{1}_{p,\beta}(\Omega)$ (and more general spaces $L^{1}_{p,\mu}(\Omega)$) can be found in [19].

SECTION III

Construction of a Particular Solution

In this section we consider the nonlinear boundary value problem (1.2), (1.3)

$$(3.1)\qquad \left\{ \left[\lambda^2 f^2(\theta) + f'^2(\theta)\right]^{\frac{p-2}{2}} f'(\theta)\sin^{n-2}\theta \right\}'_\theta +$$

$$+\lambda\left[\lambda(p-1)+n-p\right]\left[\lambda^2 f^2(\theta)+f'^2(\theta)\right]^{\frac{p-2}{2}} f(\theta)\sin^{n-2}\theta = 0$$

with boundary conditions

$$(3.2)\qquad f'(0)=0 \;, \quad f(\ell)=0 \;, \;.$$

where $f(\theta)$ is normalized by the requirement that $f(0)=1$, in order to prove Theorem 1.

In our previous papers [16]-[18], we constructed a non-negative solution of (0.1) of the form

$$u(x) = |x|^{\lambda(\ell)} f\left(x_n |x|^{-1}\right)$$

which vanishes on the boundary of the cone $\{x : x_n|x|^{-1} \geq \cos\ell\}$ where ℓ is sufficiently close to 0 or to π.

Examples of domains were constructed in [16] and it was shown, using this solution as a barrier function, that there exist solutions of problem (0.3) for the equation (0.1) with infinitely differentiable boundary data, which do not belong to any of the Hölder classes or are discontinuous in $\overline{\Omega}$. The examples of domains for which the above solution tends to its boundary data more rapidly than any power of the distance function were constructed in [18].

Let us observe, that the problem (1.2), (1.3) is nonlinear and complicated by the nonlinear dependance on λ , so we can not make use of any known facts of spectral theory.

The procedure below is based on a priori two-sided estimates of the derivative of a solution of the Cauchy problem for the equation (1.2)

$$f'(0) = 0\,,\quad f(0) = 1\,. \tag{3.3}$$

These estimates allow us to solve the Cauchy problem (3.1), (3.3) and to continue its solution $f(\theta)$ onto the interval, where $f(\theta)$ is nonnegative. We describe the set of λ, for which the solution $f_\lambda(\theta)$ of the problem (3.1), (3.3) is a continuous, monotonic function of its argument.

These results altogether are equal to the solvability of our original boundary value problem (3.1), (3.2).

Theorem 1 has been proved in part in our earlier papers [16]-[18] with certain restructions on λ , n and p. We recall these results to clarify the following procedure.

It was established in [16] that for an arbitrary positive number λ not exceeding a certain sufficiently small constant a which depends only on n and p, the following relations are valid for $p \leq n-1$

$$|f'(\theta)| \sim \lambda\theta(\pi-\theta)^{(2-n)/(p-1)}, \quad \theta \in [0,\pi)$$

$$(\pi-\ell) \sim \begin{cases} \lambda^{(p-1)/(n-p-1)}, & p < n-1 \\ e^{-1/\lambda}, & p = n-1 \end{cases}$$

where $f(\theta) \in C^2[0,\pi)$ is a solution of the Cauchy problem (3.1), (3.3) and $[0,\ell]$ is an interval on which $f(\theta)$ is nonnegative.

Moreover, it was shown that

$$\frac{1}{\lambda(\ell)} = \left[(n-p)\, y(\pi)\right]^{\frac{1}{p-1}} \int_{\pi/2}^{\ell} (\sin\theta)^{\frac{2-n}{p-1}} d\theta + O(1),$$

where $y(\theta)$ is a positive solution of the equation

$$\left[y'^2(\theta)\right]^{p-1} = (\sin\theta)^{2(n-2)} \left[(n-p)^2 y^2(\theta) + y'^2(\theta)\right]^{p-2},$$

which satisfies the condition $y(0) = 0$.

A similar result for negative λ, sufficiently close to $\frac{p-n}{p-1}$, $p \leq n-1$ can be found in [17]. That is, for an arbitrary

$$\mu = p-n-\lambda(p-1) \in (0, a(p-1)),$$

where $\alpha = \alpha(n,p)$ is a sufficiently small positive number, the following relations are valid

$$|f'(\theta)| \sim \begin{cases} \mu\theta(\pi-\theta)^{2-n} & , \quad \theta \in [\theta, \pi-\mu^{1/(n-2)}] \\ [\mu(\pi-\theta)^{2-n}]^{1/(p-1)} & , \quad \theta \in [\pi-\mu^{1/(n-2)}, \ell] \end{cases}$$

$$(\pi-\ell) \sim \begin{cases} \mu^{1/(n-p-1)} & , \quad p < n-1 \\ e^{-\mu^{1/(2-n)}} & , \quad p = n-1 \end{cases}$$

where $f(\theta) \in C^2[0,\pi)$ is a solution of the Cauchy problem (3.1) (3.3) and $[0,\ell]$ is the interval of non-negativity of $f(\theta)$.

The asymptotic behavior of $\lambda(\ell) < \frac{p-n}{p-1}$ for $\ell \to \pi$ is described as follows:

$$\lambda(\ell) = \frac{p-n}{p-1} - \frac{\Gamma\left(\frac{n}{2}\right)(\pi-\ell)^{n-p-1}}{\sqrt{\pi}\,(p-1)\,\Gamma(n-1)} \cdot \left(\frac{n-p-1}{n-p}\right)^{p-1} \left\{1 + O\left[(\pi-\ell)^{\frac{n-p-1}{n-2}}\right]\right\}$$

for $p < n-1$ and

$$\lambda(\ell) = \frac{1}{2-n} - \frac{\Gamma\left(\frac{n}{2}\right)\left[\ln(\pi-\ell)^{-1}\right]^{2-n}}{\sqrt{\pi}\,(n-2)^{3-n}\,\Gamma\left(\frac{n-1}{2}\right)} \left\{1 + O\left[\ln^{-1}(\pi-\ell)^{-1}\right]\right\}, \quad p = n-1.$$

Finally, it was proven in [18], that for an arbitrary λ $|\lambda| \in (a^{-1}, \infty)$, where $\alpha = \alpha(n,p)$ is a sufficiently small positive number, the following relations are valid

$$|f'(\theta)| \sim \lambda^2 \theta \quad , \quad \theta \in [0, l]$$

$$l \sim |\lambda|^{-1} \quad , \quad p > 1 .$$

Here $f(\theta) \in C^2[0, \pi)$ is a solution of the Cauchy problem (3.1), (3.3) and $[0, l]$ is the interval of non-negativity of $f(\theta)$.

The following equality holds for $l \longrightarrow 0$

$$\lambda(l) = \pm L l^{-1} + O(1),$$

where L is the first zero of the solution of the Cauchy problem for the equation

$$\left\{ [g^2(\tau) + g'^2(\tau)]^{\frac{p-2}{2}} g'(\tau) \tau^{n-2} \right\}'_{\tau} +$$

$$+(p-1)[g^2(\tau) + g'^2(\tau)]^{\frac{p-2}{2}} g(\tau) \tau^{n-2} = 0$$

with the initial data

$$g'(0) = 0 , \quad g(0) = 1 .$$

As a matter of convenience, the constant $a = a(n, p)$ remains the same throughout all the results.

In order to include the rest of the cases in our scheme, we use the following argument to reduce our consideration to the case $|\lambda| = O(1)$.

Let us observe the number line. Evidently, $|\lambda| = O(1)$ for $p \leq n-1$ in the cases under consideration, since in our previous results we dealt with λ, sufficiently close to the irregular points $\pm\infty$, $\frac{p-n}{p-1}$ and 0.

Let us consider the case $p > n-1$. The required estimates (1.4) for the large $|\lambda|$ has been already obtained in [18] for all $p-1$, so we ought to deal with λ, sufficiently close to either 0 or $(p-n)/(p-1)$.

We take, for example, an arbitrary positive λ. By verbatim repetition of the reasoning in the proof of Lemma 1.1 from [16] we obtain the relation

$$1 - f_\lambda(\theta) \sim \lambda + \lambda^{(p-n+1)/(p-1)}, \quad p > n-1,$$

integrating (1.4) over an interval of non-negativity for $f_\lambda(\theta)$. This shows that λ must be bounded away from zero in order for $f_\lambda(\theta)$ to vanish at a certain point of the interval $(0,\pi)$. In what follows, this point shall be denoted by ℓ.

The lim inferior of all λ, for which the solution of the Cauchy problem (3.1), (3.3) vanishes at a point ℓ of the interval $(0,\pi)$ we shall again denote by α. Evidently, α depends only on n and p and $\alpha > 0$.

We treat the other cases analogously, so that we need only consider the case $|\lambda| = O(1)$.

<u>Lemma 3.1</u>. Let $|\lambda| = O(1)$, $p>1$ and $f(\theta) \in C^2[0,\pi)$ be a solution of the Cauchy problem (3.1), (3.3). Then for all $\theta \in [0,\ell]$

$$|f'(\theta)| \sim \theta(\pi-\theta)^{(2-n)/(p-1)}, \tag{3.4}$$

where $[0,\ell]$ is the interval of non-negativity for $f(\theta)$.

<u>Proof</u>. We write the equation (3.1) in the form

$$-\sin^{n-2}\theta\,[\lambda^2 f^2(\theta)+f'^2(\theta)]^{\frac{p-2}{2}} f'(\theta) = \tag{3.5}$$
$$=\lambda[\lambda(p-1)+n-p]\int_0^\theta [\lambda^2 f^2(\tau)+f'^2(\tau)]^{\frac{p-2}{2}} f(\tau)\sin^{n-2}\tau\,d\tau.$$

First take the case $\theta \in [0, \min(\pi/2, \ell)]$. From (3.5) it follows that

$$[f^2(\theta)+f'^2(\theta)]^{\frac{p-2}{2}}|f'(\theta)| \le C\int_0^\theta [f^2(\tau)+f'^2(\tau)]^{\frac{p-2}{2}} f(\tau)\,d\tau \tag{3.6}$$

Let M denote the maximum of $|f'(\theta)|$ on the interval $[0, \min(\pi/2, \ell)]$. Since

$$f'(\theta) \le 0 \quad , \quad f(\theta) \le 1$$

we have for $p \geq 2$

$$M^{p-1} \le C(1+M^2)^{\frac{p-2}{2}}$$

i.e.

$$M \le C. \tag{3.7}$$

So for $\theta \in [0,\varepsilon]$ where $\varepsilon = \varepsilon(n,p)$ is a sufficiently small positive constant

$$1-f(\theta) \leq c\varepsilon < \frac{1}{2}. \tag{3.8}$$

In the case $p<2$ we get from (3.6)

$$M \leq c\,(1+M^2)^{\frac{2-p}{2}}$$

and we obtain once more the estimates (3.7) and (3.8). Moreover, it follows from (3.6) and (3.8) that for $p>1$

$$|f'(\theta)| \leq c\theta. \tag{3.9}$$

Now we bound $|f'(\theta)|$ from below on the interval $[0, \min(\pi/2, \ell)]$. By virtue of (3.5)

$$|f'(\theta)|\left[f^2(\theta)+f'^2(\theta)\right]^{\frac{p-2}{2}} \geq c\int_{c\theta}^{\theta}\left[f^2(\tau)+f'^2(\tau)\right]^{\frac{p-2}{2}} f(\tau)\,d\tau.$$

For all $p>1$ it follows from the last inequality, together with (3.7) and (3.8), that

$$|f'(\theta)| \geq c\theta. \tag{3.10}$$

Hence, the relation (3.4) is established for all $\theta \in [0, \min(\pi/2, \ell)]$.

Now let us estimate $|f'(\theta)|$ on the interval $[\pi/2, \ell]$, if $\ell > \pi/2$. Let $p \geq 2$. Then (3.5) yields

$$\begin{aligned}|f'(\theta)|^{p-1}\sin^{n-2}\theta &\leq c\int_0^{\theta}\left[f^2(\tau)+f'^2(\tau)\right]^{\frac{p-2}{2}} f(\tau)\sin^{n-2}\tau\,d\tau \leq \\ &\leq c\left[1+\int_0^{\theta}|f'(\tau)|^{p-2}\sin^{n-2}\tau\,d\tau\right].\end{aligned} \tag{3.11}$$

Integrating the last inequality from 0 to θ and using Hölder's inequality we find

$$\int_0^\theta |f'(\tau)|^{p-1} \sin^{n-2}\tau\, d\tau \le C\left\{1 + \left[\int_0^\theta |f'(\tau)|^{p-1} \sin^{n-2}\tau\, d\tau\right]^{\frac{p-2}{p-1}}\right\}$$

Hence

$$\int_0^\theta |f'(\tau)|^{p-1} \sin^{n-2}\tau\, d\tau \le C,$$

which, together with (3.11), gives the estimate

$$|f'(\theta)| \le C\,(\sin\theta)^{(2-n)/(p-1)}. \tag{3.12}$$

If $p<2$, we use (3.5) to get the inequality

$$|f'(\theta)| \sin^{n-2}\theta \le C\left[f^2(\theta)+f'^2(\theta)\right]^{\frac{2-p}{2}} \cdot \int_0^\theta \frac{f(\tau)\sin^{n-2}\tau\, d\tau}{\left[f^2(\tau)+f'^2(\tau)\right]^{\frac{2-p}{2}}} \le$$
$$\le C\left[1+f'^2(\theta)\right]^{\frac{2-p}{2}},$$

which once again yields (3.12).

Now let us estimate $|f'(\theta)|$ from below on the interval $[\pi/2, \ell]$. Let $p \ge 2$. We get from (3.5)

$$\left[f^2(\theta)+f'^2(\theta)\right]^{\frac{p-2}{2}} |f'(\theta)| \sin^{n-2}\theta \ge C\int_0^\theta \left[f^2(\tau)+f'^2(\tau)\right]^{\frac{p-2}{2}} f(\tau)\sin^{n-2}\tau\, d\tau \ge$$
$$\ge C\int_0^\theta f^{p-1}(\tau)\sin^{n-2}\tau\, d\tau \ge C,$$

from which we deduce

(3.13) $$|f'(\theta)| \geq c\,(\sin\theta)^{(2-n)/(p-1)}$$

For $p<2$

$$|f'(\theta)|\sin^{n-2}\theta \geq c\left[f^2(\theta)+f'^2(\theta)\right]^{\frac{2-p}{2}}\int_0^{\theta}\frac{f(\tau)\sin^{n-2}\tau\,d\tau}{\left[f^2(\tau)+f'^2(\tau)\right]^{\frac{2-p}{2}}} \geq c\,|f'(\theta)|^{2-p},$$

and this gives the estimate (3.13) once again. This proves the lemma.

We have established relation (1.4) for all λ, $p>1$, for which a solution of the Cauchy problem vanishes at a certain point of the interval $(0,\pi)$. In the following two lemmas, we prove, on the basis of reltion (1.4), the solvability of the Cauchy problem (3.1), (3.3), and the fact that the first zero of the solution of the Cauchy problem is a continuous monotonic function of parameter λ.

Lemma 3.2. For any λ

$$\lambda \in \left[\min\left(0,\ \frac{p-n}{p-1}\right),\ \max\left(0,\ \frac{p-n}{p-1}\right)\right]$$

there exists in a neighborhood of $\theta=0$ one and only one solution $f(\theta)\in C^2[0,\pi)$ of the Cauchy problem (3.1), (3.3). This solution may be continued in a unique way throughout the entire interval of its non-negativity.

Proof. For the proof, it suffices to rewrite equation (1.2) in the equivalent form

$$f(\theta) = 1 - \lambda[\lambda(p-1)+n-p] \times$$

(3.14)

$$\times \int_0^\theta \frac{d\tau}{[\lambda^2 f^2(\tau)+f'^2(\tau)]^{\frac{p-2}{2}} \sin^{n-2}\tau} \int_0^\tau [\lambda^2 f^2(t)+f'^2(t)]^{\frac{p-2}{2}} f(t) \sin^{n-2} t\, dt$$

where $\theta \in [0,\varepsilon]$ and to observe that the operator on the right side of (3.14) maps the set

$$\mathcal{M} = \left\{ f : f \in C^2[0,\varepsilon],\ f(0)=1,\ f'(0)=0,\ -\tfrac{1}{2} \le f'(\theta) \le 0 \right\}$$

into itself for small ε , and is a contraction operator on $\mathcal{M}$.

The continuability of $f(\theta)$ to any interval $[0,\ell]$, $\ell<\pi$ where $f(\theta)>0$, follows from the relation (1.4). q.e.d.

The continuity of the first zero $\ell(\lambda)$ of the solution of the Cauchy problem (3.1), (3.3) we establish by verbatim repetition of the reasoning in the corresponding lemmas from [16]-[18], making use of relation (1.4), proved in Lemma 3.1.

It was proved in [16], that the first zero $\ell(\lambda)$ of the Cauchy problem (3.1), (3.3) is a decreasing function of for small positive λ . This result requires the relation (1.4) only and can be extended to all positive by verbatim repetition, since relation (1.4) is now established for all $\lambda \in \left[\min\left(0, \frac{p-n}{p-1}\right), \max\left(0, \frac{p-n}{p-1}\right)\right]$.

To conclude the proof of Theorem 2, we need only the following:

Lemma 3.3. The first zero $\ell(\lambda)$ of the solution of the Cauchy problem (3.1), (3.3) is an increasing function of for all negative λ, $\lambda \in \left[\min\left(0, \frac{p-n}{p-1}\right), \max\left(0, \frac{p-n}{p-1}\right)\right]$.

Proof. Let $\lambda_1 > \lambda_2$. Let ℓ_i, $i = 1,2$ denote the first zero of the solution $f_i(\theta)$ of the Cauchy problem (3.1), (3.3). Put $u_i = \rho^{\lambda_i} f_i(\theta)$. Obviously, u_i is a solution of (3.1) in the cone

$$K_i = \{ x : 0 \le \theta \le \ell_i , \cos\theta = x_n |x|^{-1} \}$$

and it is equal to zero on ∂K_i. Suppose that $\ell_1 \le \ell_2$. By (1.4)

$$C = \max_{\theta \in [0, \ell_1]} \frac{f_1(\theta)}{f_2(\theta)} < \infty$$

and, hence, for $x \in K_1 \cap \partial D_1$

(3.15) $$u_1(x) \le C u_2(x)$$

Besides,

$$0 = u_1(x) \le C u_2(x)$$

for $x \in \partial K_1 \cap CD_1$. By virtue of the maximum principle, the inequality (3.15) remains true for all $x \in K_1 \cap CD_1$. Since $\lambda_1 > \lambda_2$ and $f_i(0) = 1$, this implies that for sufficiently large ρ

$$u_1(\rho, 0) = \rho^{\lambda_1} > c\,\rho^{\lambda_2} = c\,u_2(\rho, 0).$$

This contradiction proves Lemma 3.3 and concludes the proof of Theorem 2.

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