Dirichlet Series Associated to Affine Manifolds

A Dissertation presented

by

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to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

August, 1976

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

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The main purposes of the thesis are to investigate the cohomology groups $H^i(M,\mathcal{S})$, of a locally affine manifold M with some special coefficient sheaves, such as sheaves of polynomial functions, sheaves of differential operators or the sheaves of exponential functions, as representation spaces of Hecke operators.

There are many examples of locally affine spaces: $\Gamma\backslash$ (the space of nxn positive definite symetric matrices), where $\Gamma \subset GL_n(\mathbb{R})$, torus bundles over some locally affine space, local Euclidean space with Bieberbach groups, some solv-manifolds, the pre-homogenous spaces of Sato-Shintani, and the many examples in Pjateckii-Sapiro.

Hecke operators in the category of local affine spaces are definable as "corespondences". When the local affine manifold M is the quotient of a domain $X \subset \mathbb{R}^n$ by a discrete group Γ of affine transformations, i.e. $M = \Gamma\backslash X$, the notion

of Hecke operators, after proper identifications, coincide with the notion of Hecke operators as is written in Shirmura's book.

And in this case of $M = \Gamma\backslash X$, the sheaf cohomology of M with the sheaves above is isomorphic to the cohomology of Γ. For example: When $p^{(m)}$ is the sheaf of local polynomial functions of degree $\leq m$ on M, then (*) $H^i(M,p^{(m)}) \cong H^i(\Gamma,P_m(\mathbb{R}^n))$ as Hecke ring modules, where $P_m(\mathbb{R}^n) = \{f(x_1, \ldots, x_n) / f$ is a polynomial of degree $\leq m\}$.

In some special cases, the action of Hecke operators and the corresponding Dirichlet series are investigated.

In the case X = (the space of 2x2 positive definite symetric matricies) and Γ contained in $GL_2(\mathbb{R})$, the above formula (*) combined with Eichler's formula and some spectral sequence arguments, allow one to prove (**) $H^i(M,p^{(m)}) \cong$ the space of cusp forms of weight $(2m+2) \oplus (2m-2) \oplus (2m-6) \oplus \ldots$, as Hecke ring modules, so that the corresponding Dirichlet series are of Hecke-Eichler type.

In the case $M = N\backslash X$, where X is the Heisenberg group $X = \{(x,y,z) / x,y,z \in \mathbb{R}\}$, with multiplication given by $(x,y,z) \times (u,v,w) = (x+u,y+v,v,z+w+y\cdot u-x\cdot v)$, and $N = \{(n,m,k)/ n,m,k \in \mathbb{Z}\}$, we consider the group Γ of automorphisms $\Gamma = \{\sigma \in Aut(X)/ \sigma(N) = N\}$ and the semi-group $\Delta = \{\sigma \in Aut(X)/ \sigma(N) \subset N\}$ of endomorphisms of M. The Hecke ring $\mathcal{R}(\Gamma,\Delta)$ is isomorphic to the Hecke ring $\mathcal{R}(SL_2(\mathbb{Z}), M_2^{\times}(\mathbb{Z}))$, and the action of $\mathfrak{R}(\Gamma,\Delta)$ on $H^i(\Gamma,H^j(M,p^{(m)}))$ are again reduced to the action of ordinary Hecke ring to the space of automorphic forms (not always cusp forms), and hence the Dirichlet series are of Hecke-Eichler type.

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Acknowledgements

I wish to acknowledge the formative influence of my thesis advisor, Professor M. Kuga, both on my mathematical interests, and on the researches represented in this thesis. Specifically, he has directed my interests in the number theory of automorphic forms, zeta functions, and classical mathematics, only part of which has directly affected the text of this thesis.

I also wish to acknowledge the technical assistance and moral support of Professor H. Sah, who has also helped to make this thesis possible. I also wish to thank the many other mathematicians and teachers who have inspired and influenced my interest in mathematics.

Chapter 1: Introduction

Number theory of automorphic forms has been investigated with the aid of the algebraic geometry of the quotient space $\Gamma\backslash X$ of the corresponding hermitian symmetric space X divided by the discrete group Γ. But when a symmetric space Y has no complex structure, perhaps because of the lack of algebraic geometry, the available arithmetic of the quotient space $\Gamma\backslash Y$ is not so deep as in the case with complex structure.

The symmetric spaces attached to quadratic forms or spaces attached to $SL_2(k)$, where k is a number field not totally real, are examples of symmetric spaces without complex structure. The arithmetic of quadratic forms is very old, and recently investigated quite thoroughly by Siegel, etc., but these results look not as deep as that of class field theory and the arithmetic of automorphic forms (complex multiplication).

In the case of positive definite quadratic forms, the situation is better; since it defines a theta-function; and via this theta-function, it relates with algebraic geometry. But still, it became so, only after the Eichler's theorem claiming "all automorphic forms are theta-functions" is established. This is established only for limited Γ. For $SL_2(k)$, the situations are similar.

Even for these cases without algebraic geometry, people defined various kinds of zeta-functions (Dirichlet series);

these are investigated analytically quite deeply, but they obviously lack an algebraic geometrical structure. A most recent endeavor of defining Dirichlet series of such kinds is Sato-Shintani's zeta-function. We will discuss them later.

As we learned from the history of the proof of Ramanujan conjecture, and the estimation of Kloosterman sums; to reach to the essential depth of arithmetic, an analytic approach alone is quite insufficient. When Y lacks complex structure, is it possible to use some other "structure" (instead of complex structure / algebraic geometry) to investigate the arithmetic of $\Gamma\backslash Y$?

We can think of two new possibilities: (1) Some real symmetric spaces Y are the manifold of real points of a hermitian symmetric doman X (with respect to some anti-holomorphic involution). If we restrict our investigation to categories of such manifolds; we can employ real algebraic geometry because the quotient space $\Gamma\backslash Y$ is a real algebraic manifold. (cf. A. Adler, H. Jaffee, S. Kudla, G. Shimura) (2) Many real symmetric spaces Y are local affine manifolds or are closely related to local affine manifolds. For example quotient spaces of $(SL_n(\mathbb{R})/SO_n(\mathbb{R})) \times \mathbb{R} = GL_n^+(\mathbb{R})/SO_n(\mathbb{R})$ is a local affine mainfold. $GL_n^+(\mathbb{R})/SO_n(\mathbb{R})$ is the space of all positive definite symmetric matrices, which is an open domain in $\mathbb{R}^{n(n+1)/2}$.

In this thesis we concentrate on the second possiblity, i.e., we like to find how well the local affine structure helps for the investigation of the arithmetic of the quotient space $\Gamma\backslash Y$. To follow analogies of algebraic geometry in the local affine structure case is not fully discussed here. It is too difficult right now. But at least we can notice that there are several fundamental sheaves attached to local affine structure. Namely sheaves of polynomial functions, sheaves of exponential or polynomial-exponential functions, sheaves of differential operators of constant or polynomial coefficients, and sheaves of "solutions" of them.

Instead of pursuing the analogy with algebraic geometry, we go to arithmetic directly. We define Hecke operators in the category of local affine manifolds, and we define the action of Hecke operators to the cohomology groups of the manifold with some of the above sheaves as coefficients. Lastly we search possibilities of constructing Dirichlet series.

I have done this in two examples. The definition in these cases are similar to the "definition" of Sato-Shintani zeta-function of prehomogeneous spaces: in the latter they used differential operators while in our case we utilize Hecke operators.

In Chapter 2, we define local affine manifolds and their sheaf cohomology groups. We show that when the topology of the local affine manifold M is determined by its fundamental group Γ, the sheaf cohomology of M may be computed by the group cohomology of its fundamental group Γ.

In Chapters 3 and 4 we compute these groups for our special cases. For Chapter 3, we consider manifolds constructed from the space of all positive definite quadratic forms, and for Chapter 4, we consider manifolds constructed from the Heisenberg group. In both of these cases we use a theorem of Eichler and spectral sequence arguments to show that the cohomology groups in question may be expressed in terms of automorphic forms.

In Chapter 5, we present our theory of Hecke operators in the category of local affine manifolds. We also consider the classical Hecke ring, and our main result is that certain elements of our Hecke ring, agree with the classical Hecke ring as operators on cohomology. Finally, we mention briefly some of the possibilities for associated Dirichlet series.

All manifolds considered here are assumed to be paracompact and C^∞. Any elementary facts about covering spaces, covering transformation groups etc., are assumed. Also elementary facts on sheaves and cohomologies are assumed. In Chapters 1-4, manifolds are always assumed to be connected, in Chapter 5, manifolds have a finite number of connected components.

An unnessesary assumption of the existence of a metric in the theorem of Chapter 2, page 10 is used. The existence is always provable. For the proof see Appendix II.

Chapter 2: Definitions; General Results on Cohomology

(I) Definitions

We begin with the basic definitions of our subject: Affine manifolds, sheaf cohomology, and group cohomology.

Let U and V be open sets in $\mathbb{R}^n$, where $\mathbb{R}^n$ is Euclian n-space. A map $f: U \to V$ is said to be an affine transformation provided there exists a linear transformation $A: R^n \to R^n$, and a vector $b \in R^n$ so that for all $x \in U$, $f(x) = A(x) + b \in V$. An affine transformation $f: U \to V$ is said to be affine isomorphism provided there exists an affine transformation $f^{-1}: V \to U$, so that $f^{-1} \circ f$ is the identity map on U, and $f \circ f^{-1}$ is the identity on V.

Definition: A system $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ of charts (i.e., atlas) on a manifold M is called an affine atlas if coordinate transformations $\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are always affine isomorphisms, whenever $U_\alpha \cap U_\beta \neq \emptyset$. Two affine atlases $\{U_\alpha, \varphi_\alpha\}$, $\{V_\beta, \Psi_\beta\}$, on a manifold M are called equivalent if $\varphi_\alpha \circ \Psi_\beta^{-1}$ and $\Psi_\beta \circ \varphi_\alpha^{-1}$ are always affine isomorphism wherever they are defined. An equivalence class of affine atlases on a manifold M is called an affine structure of M. A manifold together with an affine structure is called a local affine manifold.

For $\mathbb{R}^n$, the set of all affine isomorphism forms a topological group, $Aut(\mathbb{R}^n)$, which is known to be the semi-direct product of $GL_n(\mathbb{R})$ and $\mathbb{R}^n$, i.e., for

$A_1, A_2 \in GL_n(\mathbb{R})$, $b_1, b_2 \in \mathbb{R}^n$, $(A_1, b_1) \circ (A_2, b_2) =)A_1A_2, b_1 + A_1(b_2))$.
When $X \subseteq \mathbb{R}^n$ is an open set, define $\mathrm{Aut}(X) = \{f \in \mathrm{Aut}(\mathbb{R}^n)/f(X) = X\}$.
Then $\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Aut}(\mathbb{R}^n)$ on which we take the induced topology.

Example: Let $X \subseteq \mathbb{R}^n$ be an open set, and $\Gamma \subset \mathrm{Aut}(X)$ be a discrete subgroup which acts on X properly discontinuously, with no fixed points in X. Then the quotient $M = \Gamma\backslash X$ is a local affine manifold.

Proof: Since Γ acts properly discontinuously on X, X is a covering space of M. Let $p: X \to M$ be covering map, and pick a cover $\mathfrak{U} = (U_\alpha)$ so that each U_α is evenly covered by p. For each α, fix a connected preimage $U_{\alpha,1} \subset p^{-1}(U_\alpha)$ and let $\varphi_\alpha : U_\alpha \to U_{\alpha,1} \subset X \subseteq R^n$ be the homemorphism between U_α and $U_{\alpha,1}$. Now suppose $U_\alpha \cap U_\beta \neq \emptyset$ in M, then we may find $\delta_{\alpha,\beta} \in \Gamma$ so that $\delta_{\alpha,\beta}(U_{\alpha,1}) \cap U_{\beta,1}$ evenly covers $U_\alpha \cap U_\beta$. Then $\varphi_\beta \circ \varphi_\alpha^{-1} = \delta_{\alpha,\beta}$ and is an affine isomorphism of $\varphi_\alpha(U_\alpha \cap U_\beta) = U_{\alpha,1} \cap \delta^{-1}_{\alpha,\beta}(U_{\beta,1})$ to $\varphi_\beta(U_\alpha \cap U_\beta) = \delta_{\alpha,\beta}(U_{\alpha,1}) \cap U_{\beta,1}$.

We shall be particularly concerned with a special case of this example.

Definition: Let $X \subseteq \mathbb{R}^n$ be a contractible open set. Then we refer to X as a domain in $\mathbb{R}^n$. When Γ acts on X as in the above example, and has no finite subgroups, we refer to the local affine manifold $M = \Gamma\backslash X$ as the quotient of a domain by a discrete group acting freely and properly discontinuously on X without finite subgroups.

Remark: Let $X = \mathbb{R}^n - \{0\}$, $n \geq 3$. Since $X \cong S^{n-1} \times \mathbb{R}$, X is not contractible for $n \geq 3$. We may also consider the discrete group Γ of affine motions generated by $x \to 2 \cdot x$, $x \in \mathbb{R}^n - \{0\}$. This Γ is fixed point free and without finite subgroups, and the quotient $M = \Gamma \backslash X$ is compact. However, the topology of this manifold is not determined by its fundamental group Γ. When $n = 2$, set $M = \mathbb{R}^2 - \{0\}$. Since M is not simply connected we may consider the universal cover X, which is homeomorphic to $\mathbb{R}^2$, however the affine structure is not realized as an open set in $\mathbb{R}^2$, so this is also not of our type.

When a manifold M is local affine with charts $(U_\alpha, \varphi_\alpha)$, we have an interesting structure sheaf: $p(m)$, the sheaf of local polynomial functions of degree $\leq m$. When $U \subset M$ is an open set, define the sections over U, $\Gamma(U)$, by $\Gamma(U) = \{f: U \to \mathbb{C} /$ for all U_α with $U_\alpha \cap U \neq \emptyset$, $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U) \to \mathbb{C}$ is a polynomial of degree $\leq m$ on $\mathbb{R}^n\}$. $\Gamma(U)$ is well defined, since on $U_\alpha \cap U_\beta$, $f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})$ and the affine isomorphism $(\varphi_\alpha \circ \varphi_\beta^{-1})$ preserves the degree of polynomials on $\mathbb{R}^n$. If $U \supset V$ the restriction map $\Gamma(U) \to \Gamma(V)$ is injective, moreover if U is "sufficiently small", then the restriction map is a bijection: $\Gamma(U) \cong \Gamma(V) \cong$ the space of polynomial functions of degree $\leq m$. We define the sheaf of local polynomial functions of degree $\leq m$ on M, $p(m)$, by the local sections $\Gamma(U)$ for all U open in M.

As we shall also be concerned with other sheaves definable by virtue of the affine structure on M, we now consider a general sheaf, $\mathcal{S}$, on a manifold M with local sections $\Gamma(U,\mathcal{S})$. Let $\mathfrak{U} = (U_\alpha)_{\alpha \in A}$ be any covering by open sets of M (i.e., $M = \bigcup_{\alpha \in A} U_\alpha$, U_α open). We define a chain complex $C_* = \{C_i(M,\mathfrak{U}),\partial_i\}$ by setting $C_i(M,\mathfrak{U})$ = the free abelian group generated by $\{U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_i} / \alpha_j \in A,\ U_{\alpha_0} \cap \ldots \cap U_{\alpha_i} \neq \emptyset\}$ and $\partial_i : C_i(M,\mathfrak{U}) \to C_{i-1}(M,\mathfrak{U})$ determined by $\partial_i(U_{\alpha_0} \cap \ldots \cap U_{\alpha_i}) = \sum_{j=0}^{i} (-1)^j U_{\alpha_0} \cap \ldots \cap \hat{U}_{\alpha_j} \cap \ldots \cap U_{\alpha_i}$, where $\hat{U}_{\alpha_j}$ means omit U_{α_j}. Then we have a co-complex $\{C^i(M,\mathfrak{U},\mathcal{S}),\ \delta_i\}$, by letting $C^i(M,\mathfrak{U},\mathcal{S})$ be the free abelian group generated by the additive maps h determined by associating to every nonempty intersection $U_{\alpha_0} \cap \ldots \cap U_{\alpha_i}$ an element $h(U_{\alpha_i} \cap \ldots \cap U_{\alpha_i})$ of $\Gamma(U_{\alpha_0} \cap \ldots \cap U_{\alpha_i},\mathcal{S})$ and $\delta_i : C^i(M,\mathfrak{U},\mathcal{S}) \to C^{i+1}(M,\mathfrak{U},\mathcal{S})$ be the map defined by $(\delta_i h)(x) = h(\partial_{i+1}(x))$ for all $x \in C_{i+1}(M,\mathfrak{U})$.

<u>Definition</u>: The (Čech) <u>cohomology of M relative to the cover $\mathfrak{U}$ with coefficients in the sheaf $\mathcal{S}$</u>, $H^i_{\mathfrak{U}}(M,\mathcal{S})$ is the cohomology of the complex $\{C^i(M,\mathfrak{U},\mathcal{S}),\delta_i\}$. The (Čech) <u>cohomology of M with coefficients in $\mathcal{S}$</u>, $H^i(M,\mathcal{S})$, is given by $H^i(M,\mathcal{S}) = \varinjlim H^i_{\mathfrak{U}}(M,\mathcal{S})$, where $\varinjlim$ is the direct limit taken over all coverings $\mathfrak{U}$ of M.

For a full treatment of sheaves and Čech cohomology with coefficients in a sheaf, the reader may see Swan [8]. When our manifold M is the quotient of a domain by a discrete group of affine transformations, and the sheaf $\mathcal{S}$ is a special sheaf, such as the sheaf of local polynomial functions, p(m), the sheaf cohomology of M will be expressed in terms of the group cohomology of the fundamental group Γ of M. For certain Γ's we can compute the group cohomology, determining the sheaf cohomology $H^i(M,\mathcal{S})$.

The meaning of "special" for the coefficient sheaves p(m) is as follows. On $\mathbb{R}^n$ we have polynomial functions of degree $\leq$ m, $P_m = p_m(\mathbb{R}^n) = \{f(x_1,\ldots,x_n) \mid f$ is a polynomial of degree $\leq$ m with complex coefficients$\}$. When M is the quotient of a domain $X \subseteq \mathbb{R}^n$ by a discrete group Γ, Γ acts on $\mathbb{R}^n$ and we view $P_m(\mathbb{R}^n)$ as a Γ-module by $(\delta, f(x)) \to f(\delta^{-1}(x))$, remarking that $f(\delta^{-1}(x)) \in P_m(\mathbb{R}^n)$. We will investigate the group cohomology $H^i(\Gamma, P_m(\mathbb{R}^n))$ for certain Γ with this action.

We will need the following general definition of group cohomology. Suppose G is a group, A a $\mathbb{Z}[G]$-module, where $\mathbb{Z}[G]$ is the integral group ring. Let C_i be free $\mathbb{Z}[G]$-modules, $\mathbb{Z}$ have trivial G action, and suppose that $\cdots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$ is an exact chain complex for i > 0, so that $\epsilon \circ \partial_1 = 0$ and ϵ induces an isormorphism of Ker $\partial_1/$Im ∂_2 onto $\mathbb{Z}$. Then defining $C^i = \mathrm{Hom}(C_i,A)$ and $\delta^i : C^i \to C^{i+1}$ by $\delta_i(h)(c) = h(\partial_{i+1}(c))$, where $h \in C^i$ and $c \in C_{i+1}$, defines a chain complex for which $\mathrm{Hom}_{\mathbb{Z}[G]}(C_i,A) = \{h \in C^i \mid h(gc) = g \cdot h(c)$, where $g \in \mathbb{Z}[G]$ and $c \in C_i\}$ is a subcomplex.

Definition: The cohomology groups $H^i(G,A)$ are the cohomology groups of the above subcomplex.

Recall that a standard such resolution is given by $C_i = \mathbb{Z}[G] \times \ldots \times \mathbb{Z}[G]$ (i+1 copies) and then $\mathrm{Hom}_{\mathbb{Z}(G)}(C_i, A)$ is determined by $\{h: G \times \ldots \times G \to A \,/\, h(gg_0, \ldots, gg_i) = g \cdot h(g_0, \ldots, g_i)\}$. We will construct a different resolution for the cohomology $H^i(\Gamma, P_m(\mathbb{R}^n))$ using the covering structure $p : X \to M$. See also Maclane [5] or Hochschild-Serre [1].

(II) A Cohomology Isomorphism

Let $M = \Gamma \setminus X$ be the quotient of a domain by a discrete group of affine transformations acting freely and properly discontinuously without finite subgroups, $\mathcal{S} = p(m)$ be the sheaf of local polynomial functions, and consider $P_m(\mathbb{R}^n)$ as a Γ-module as above.

Theorem: Let $M = \Gamma \setminus X$ as above, and suppose X has a metric d so that $\min_{\substack{x \in X \\ 1 \neq \delta \in \Gamma}} d(x, \delta(x)) = \epsilon > 0$. Then $H^i(M, p^{(m)}) \cong H^i(\Gamma, P_m)$.

Proof: Let $\pi : X \to \Gamma \setminus X = M$ be the natural projection. Let $\mathfrak{U} = \{U_\alpha\}\ \alpha \in A$ be any covering of M, and fix connected components $U_{\alpha,1}$ of $\pi^{-1}(U_\alpha)$. Set $U_{\alpha,\delta} = \delta(U_{\alpha,1})$ for all $\delta \in \Gamma$, $\alpha \in A$, then $\pi^{-1}(U_\alpha) = \bigcup_{\delta \in \Gamma} U_{\alpha,\delta}$, and $\{U_{\alpha,\delta}\}_{\substack{\delta \in \Gamma \\ \alpha \in A}}$ is a covering of X.

Now fix the cover $\mathfrak{U} = (U_\alpha)$ so that for all $U_{\alpha,\delta}$, diam $(U_{\alpha,\delta}) < \frac{\epsilon}{3}$, where the diameter is measured in the metric d. Define a chain complex $C_* = \{C_i(\Gamma)\}_i$ by $C_i(\Gamma)$ = the free abelian group generated by $\{U_{\alpha_o\,\delta_o}, U_{\alpha_1,\delta_1}, \ldots, U_{\alpha_i,\delta_i}) /$

$U_{\alpha_o,\delta_o} \cap U_{\alpha_1,\delta_1} \cap \ldots \cap U_{\alpha_o,\delta_i} \neq \emptyset\}$, with boundary ∂ as in Cech theory. Since X is contractible and Γ acts freely on X (hence on $C_i(\Gamma)$, by choice of $\mathfrak{U}$), we may define $H^*(\Gamma,P_m)$ as the cohomology of the complex $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_*,P_m)$, where the coboundary γ is naturally associated to ∂, and $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_*,P_m)$ has as elements $h = (h_i)$, with $h_i : C_i \to P_m$ so that $h_i(\gamma(x)) = \gamma\, h_i(x)$ for all $x \in C_i(\Gamma)$, $\gamma \in \mathbb{Z}[\Gamma]$ (=group ring) and for all i.

For our fixed cover $\mathfrak{U} = (U_\alpha)$ of M, suppose $U_{\alpha_o} \cap \ldots \cap U_{\alpha_i} \neq \emptyset$. To define a map $\emptyset : H^i(\Gamma,P_m) \to H^i_{\mathfrak{U}}(M, p^{(m)})$, we must define $\emptyset\,(h)\,(U_{\alpha_o} \cap \ldots \cap U_{\alpha_i})$ for all $h \in H^i(\Gamma,P_m)$. By our choice of

$\mathfrak{U} = (U_\alpha)$ and the fact that π is a covering map, there exists a unique δ_1 so that $U_{\alpha_o,1} \cap U_{\alpha_1,\delta_1} \neq \emptyset$. Further, $U_{\alpha_o,\delta} \cap$

$U_{\alpha_1,\delta'} \neq \emptyset$ implies $U_{\alpha_1,\delta'} = \delta(U_{\alpha_1,\delta_1}) = U_{\alpha_1,\delta\delta_1}$. We proceed selecting unique δ_s, $s = 1, \ldots, i$ so that $(U_{\alpha_o,1} \cap U_{\alpha_1,\delta_1} \cap \ldots$

$\cap\, U_{\alpha_{s-1},\delta_{s-1}}) \cap U_{\alpha_s,\delta_s} \neq \emptyset$ and $(U_{\alpha_o,\delta} \cap U_{\alpha_1,\delta\delta_1} \cap \ldots$

$\cap\, U_{\alpha_{s-1},\delta\delta_{s-1}}) \cap U_{\alpha_s,\delta'} \neq \emptyset$ implies $\delta' = \delta\delta_s$. Since

$h(U_{\alpha_0,\gamma}, U_{\alpha_1,\gamma\delta_1}, \ldots, U_{\alpha_i,\gamma\delta_i}) = h(\gamma(U_{\alpha_0,1}, U_{\alpha_1,\delta_1}, \ldots,$
$U_{\alpha_i,\delta_i})) = \gamma\, h(U_{\alpha_0,1}, U_{\alpha_1,\delta_1}, \ldots, U_{\alpha_i,\delta_i})$, $\gamma \in \Gamma$, we may set
$f_{\alpha_0,\ldots,\alpha_i} = h(U_{\alpha_0,\delta}, U_{\alpha_1,\delta_1}, \ldots, U_{\alpha_i,\delta_0})$ and define
$[\phi(h)]\,(U_{\alpha_0}, \ldots, U_{\alpha_i}) = f_{\alpha_0,\ldots,\alpha_i} \in \Gamma\,(U_{\alpha_0} \cap \ldots \cap U_{\alpha_i}, p^{(m)})$.
We have shown, and it is easily checked, that ϕ is well defined, and defines an isomorphism $H^i(\Gamma, P_m) \cong H^i_{\mathfrak{U}}(M, p^{(m)})$.

Now suppose $\mathfrak{U}^1 = \{U^1_\alpha\}$ is a refinement of $\mathfrak{U}$ satisfying the condition diam $(U^1_{\alpha,\delta}) < \frac{\epsilon}{3}$, for all α, δ. By general theory of group cohomology, the corresponding $C^!_*(\Gamma)$ provides an equivelent definition of $H^*(\Gamma, P_m)$ (i.e., a chain homotopy of complexes exists), and our above argument shows $H^i(\Gamma, P_m) \cong H^i_{\mathfrak{U}^1}(M, p^{(m)})$. Since refinements satisfying our condition are co-final among all coverings of M, we have $H^i(\Gamma, P_m) \cong H^i(M, p^{(m)})$.

<u>Remarks</u>: When $m = 0$, the sheaf ϕ of complex numbers, and the sheaf cohomology $H^i(M, \phi)$ coincides with the (singular) cohomology $H^i(M, \phi)$ of M with complex coefficients. In this case, the Γ action $P_0 = \phi$ is trivial, and our isomorphism may be found in Maclane, P.136 [5]. A result similar to ours ("De Rahm Theorem") when M is a C-$^\infty$ manifold and $\mathcal{S}$ = sheaf of vector valued differential forms may be found in Kuga [4].

A different result may be obtained in the case when the affine manifold M is a open set in $\mathbb{R}^n$ with a contractible universal cover, as in the example $M = \mathbb{R}^2 - \{0\}$ above. Here the sheaf $p^{(m)}$ is constant on M, and we have $H^i(M,p^{(m)}) \simeq H^i(\Gamma,P_m)$, where Γ is fundamental group of M, but the Γ action on P_m is trivial. We shall not be concerned further with this case.

When X = the upper half plane, and Γ is a Fuchsian group of the first kind even admitting elliptic elements and/or parabolic elements, acting on X as conformal maps (not affine map), a results of our type is obtainable with suitable modifications of the definition of group cohomology. In this case the cohomology of Γ (non-trivial action) amy be evaluated in terms of cusp forms. For these results see Shirmura [7], chapter 8, or the papers of Eichler refered to there. These results, together with their relation to Hecke operators, will be essential in the examples considered in this thesis where $H^i(\Gamma,P_m)$ is computed, and the action of Hecke operators determined, and should perhaps be viewed as the starting point of the thesis.

For the special choices of X and Γ to be considered later in this thesis, we shall need the following results concerning the application of our theorem.

<u>Proposition 1</u>: Let $M = \Gamma\backslash X$ be the quotient of a domain X by a discrete group of affine transformations acting freely and properly discontinuously on X without finite subgroups, and suppose further that M is compact. Then there is a metric d on X satisfying the assumption of our theorem: $\min_{\substack{x\in X \\ 1\neq\delta\in\Gamma}}$

$d(\delta(x), x) = \epsilon > 0$.

Proof: Since M is paracompact, we may pick a metric d_M on M (partition of unity). Let $p: X \to M = \Gamma\backslash X$ be the projection map and set $d = p^*(d_M)$, the "pull-back" metric. By definition Γ acts by isometries (i.e., preserves distances) for this metric.

Since M is compact, we may pick a compact fundamental domain $D \subseteq X$ for Γ acting on X. Fix a point $O \in D$ and consider the closed metric disc Δ_r of radius r about O for the metric d. Since D is compact, we have $D \subset \Delta_r$ for sufficiently large r, and we fix R so that $R > 2\,\mathrm{diam}(D)$.

For each $\delta \in \Gamma$ we have a continuous function $d_\delta : \Delta_R \to \mathbb{R}$ by $d_\delta(x) = d(x, \delta(x))$ on the compact set Δ_R, so that d_δ must assume its minimum value. But since Γ acts properly discontinuously on X, for the compact set Δ_R there are only finitely many $\delta \in \Gamma$ for which $\delta(\Delta_R) \cap \Delta_R \neq \emptyset$, say $S = \{\delta_1, \ldots, \delta_n\} = \{\delta \in \Gamma \,/\, \delta(\Delta_R) \cap \Delta_R \neq \emptyset\}$. Now for all $\delta \in \Gamma - S$, $\min\limits_{X \in D} d_\delta(x) > \mathrm{diam}(D)$, and excluding the identity element $1 = \delta_1$, $\min\limits_{\substack{X \in D \\ 1 \neq \delta \in \Gamma}} d_\delta(x) = \min(\mathrm{diam}(D), m_2, \ldots, m_n) = \epsilon > 0$, where $m_i = \min\limits_{X \in D} d_\delta(x) > 0$ since δ is fixed point free, $\delta \neq 1$.

Now let $x \in X$ be arbitrary. Since D is a fundamental domain for Γ acting on X, we may find $p \in \Gamma$, $y \in D$ so that $x = p(y)$. But since Γ acts by isometries, $d(x, \delta(x)) =$

$d(p(y),\ \delta(p(y))) = d(y, p^{-1}\delta p(y))$, so that $\min\limits_{\substack{X \in X \\ 1 \neq \delta \in \Gamma}} d(x,\ \delta(x))$

$= \min\limits_{\substack{y \in D \\ 1 \neq \delta \in \Gamma}} d(y, p^{-1}\delta p(y)) = \min\limits_{\substack{y \in D \\ 1 \neq \delta^1 \in \Gamma}} d(y, \delta^1(y)) = \epsilon > 0$, (since $\Gamma \subset$ normalizer (Γ)), which was to be shown.

<u>Corollary</u>: Let $M = \Gamma \backslash X$ as in our theorem, and suppose $X = X' \times \mathbb{R}^k$ with $X' \subseteq \mathbb{R}^{n-k}$ contractible. For $x = (x', y)$, $x \in X$, $x' \in X'$ and $y \in \mathbb{R}^k$, suppose that $\delta(x) = \delta((x', y)) = (x'', y)$, with $x'' \in X'$ for all $\delta \in \Gamma$ (i.e., Γ acts trivially on $\mathbb{R}^k$). Suppose further that $\Gamma \backslash X'$ is compact, then X has a metric so that the asumption of our theorem is satisfied.

<u>Proof</u>: On X' take the metric d as in the previous proposition, on $\mathbb{R}^k$ take the Euclidean metric d_E, and on X take the product metric $d \times d_E$. Then for $x = (x', y)$ with $x' \in X'$, $y \in \mathbb{R}^k$, $\min\limits_{\substack{x \in X \\ 1 \neq \delta \in \Gamma}} d \times d_E(x, \delta(x)) = \min\limits_{\substack{x' \in X' \\ 1 \neq \delta \in \Gamma}} d(x', \delta(x')) = \epsilon > 0$ by the previous proposition, where $\delta\ (x')$ denotes the restriction of the action of to X'.

These two results will be sufficient for the examples considered in this thesis. We remark that Sato's theory of pre-homogeneous vector spaces is a source for our examples (see Sato-Shintani [6], or Kimura [2]). A pre-homogeneous vector space is a triple $(G,\ p,\ V)$, where G is a complex (algebraic) Lie group and p denotes the action of G on a complex vector V, say $V = \mathbb{C}^n$, with an open G-orbit. We consider the real points $G_{\mathbb{R}}$ of G, $V_{\mathbb{R}}$ of V and induced action $p_{\mathbb{R}}$ of $G_{\mathbb{R}}$ on $V_{\mathbb{R}}$. We pick a connected $G_{\mathbb{R}}$-orbit in $V_{\mathbb{R}}$, denote it X, and use it as indicated above. One of our examples is of this type. Other examples are provided by simply connected nilpotent or solvable Lie groups. Chapters 3 and 4 will be concerned with computations in these cases.

We also remark that our theorem holds for other "special" sheaves on a local affine manifold M arising as a quotient $M = \Gamma\backslash X$ as above. When we use other sheaves, we will merely note the appropriate Γ-module, as for our cases an identical proof applies.

While we could discuss Hecke operators at this point, we prefer to present our examples first. The reader may wish to refer to our appendix on the Hochschild-Serre spectral sequence for use in Chapters 3 and 4.

Chapter 3: The Quadratic Form Case

(I) Statement of Example, Γ Acting Linearly

Let $X = \{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} / \text{positive definite} \}$, then we have $X \subset \mathbb{R}^3$ as a contractible open set. As in the theory of quadratic forms, we have an action of $GL_2^+(\mathbb{R})$ on X by $x \to {}^t\delta \, x \delta$ whenever $x \in X$, $\delta \in GL_2^+(\mathbb{R})$. This action is affine (in the variables x,y,z), and in fact, is linear, so we have $GL_2^+(\mathbb{R}) \subset \mathrm{Aut}(X)$. We consider $\Gamma \subseteq GL_2^+(\mathbb{R})$ discrete. Actually, the matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on X, so we should take $GL_2^+(\mathbb{R}) / (\pm I)$.

Since Γ acts linearly on X, the coordinate transformations $\varphi_\beta \circ \varphi_\alpha^{-1}$ for the quotient $M = \Gamma \backslash X$ are linear (see page 6). In this case we take the sheaf $p^{(m)}$ to be given by local sections $\Gamma(U) = \{ f: U \to \mathbb{C} \ / \ f \circ \varphi_\alpha^{-1}: \varphi_\alpha \ (U \cap U_\alpha) \to \mathbb{C}$ is a polynomial of deg $= m \}$. We let $P_m(\mathbb{R}^n) = \{$polynomial of deg $= m\}$ with Γ-module structure by $f(x) \to f(\delta^{-1}(x))$. As remarked at the end of Chapter 2, our theorem remains valid in this case by an identical proof. In the following section (V), we obtain results for the original sheaf $p^{(m)}$ considered in Chapter 2.

We now consider Γ's contained in $GL_2^+(\mathbb{R})$, for which the assumptions of our theorem hold as remarked in our proposition and its corollary in Chapter 2.

(II) Structure of Discrete Subgroups $\Gamma \subseteq GL_2^+(\mathbb{R})$.

For $\Gamma \subset GL_2^+(R)$, consider $\Gamma_{(1)} = \Gamma \cap SL_2(\mathbb{R})$ and $\Gamma'' = \Gamma_{(1)} / (\pm I)$. Also let $X_{(1)} = X \cap SL_2(\mathbb{R})$ be the upper half plane. We assume that the quotient $\Gamma'' \setminus X_{(1)}$ is a compact Riemann surface, so that Γ'' must not be solvable.

Now consider the natural action of Γ on $\mathbb{R}^2$. $\text{End}_{\Gamma}(\mathbb{R}^2,\mathbb{R}^2) = \{A \in M_2(\mathbb{R}) \,/\, \delta A(v) = A(\delta(v))$ for all $v \in \mathbb{R}^2\}$ is a division algebra and coincides with the centralizer of Γ. By the classification of division algebras, $\text{End}_{\Gamma}(\mathbb{R}^2,\mathbb{R}^2)$ must be $\mathbb{R}$ or $\mathbb{C}$, since the quaternions are not a subalgebra of $M_2(\mathbb{R})$. But since $\mathbb{C}$ is a maximal commutative subalgebra of $M_2(\mathbb{R})$, its centralizer is again $\mathbb{C}$. On the other hand, by definition, Γ centralizes its centralizer, so if centralizer $(\Gamma) = \mathbb{C}$, $\Gamma \subset \mathbb{C}$. Thus $\text{End}_{\Gamma}(\mathbb{R}^2,\mathbb{R}^2) = \mathbb{R}$, since otherwise Γ would be abelian, contradicting the fact that Γ'' is not solvable.

Since center $(\Gamma) \subset$ centralizer $(\Gamma) = \mathbb{R} \cong \{\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} / \lambda \in \mathbb{R}\}$ is discrete, we have two cases. First the quotient $\Gamma\setminus X$ may be compact, in which case center $(\Gamma) = \{\begin{pmatrix}\epsilon a^n & 0\\ 0 & \epsilon a^n\end{pmatrix} / \epsilon = \pm 1$, $1 \neq a > 0$, $n \in Z\} \cong Z$ or $\{\pm 1\} \times Z$. Second the quotient may be non-compact, in which case center $(\Gamma) = \{1\}$ or $\{\pm 1\}$. We refer to these as the compact and non-compact cases and deal with these in sections (III) and (IV), respectively.

Further structure on Γ may be obtained by considering the determinant map det: $\Gamma \to \mathbb{R}^+$. This provides an exact sequence $1 \to \Gamma_{(1)} \to \Gamma \,/\, \Gamma_{(1)} \to 1$.

<u>Proposition</u>: This sequence splits as a direct product.

<u>Proof</u>: Since $\Gamma \,/\, \Gamma_{(1)}$ is a discrete subgroup of $\mathbb{R}^+$, which is of dimension 1, $H^2(\Gamma \,/\, \Gamma_{(1)}, -) = 0$ so that the

extension splits and any $x \in \Gamma$ may be written as $s = (x_1, \det(x))$ with $x_1 = \frac{1}{|\det(x)}.x \in \Gamma_{(1)}$. Now for $x,y \in \Gamma$, $x \cdot y = (\frac{1}{|\det(x \cdot y)}.xy, \det(x \cdot y)) = (\frac{1}{|\det(x)}.x \cdot \frac{1}{|\det(y)}.y, \det(x).\det(y)) =$ $(x_1, \det(x)) \cdot (y_1, \det(y))$, where the final multiplication is direct.

In the following, we shall be concerned with the sequence $1 \to$ center $(\Gamma) \to \Gamma \to \Gamma' \to 1$, and the coefficients $P_m = \{f(x,y,z) \ / \deg f = m\}$. Note that examples of the above types of Γ arise from quaternion algebras.

(111) The Compact Case

Theorem: When Γ is a discrete group in $GL_2^+(\mathbb{R})$ as above acting on the space X of 2 x 2 symmetric matrices, and center $(\Gamma) = \mathbb{Z}$ or $\{\pm 1\} \times \mathbb{Z}$ so that the quotient $M = \Gamma \backslash X$ is compact, the cohomology $H^i(\Gamma, P_m) = 0$, for all i, unless $m = 0$.

Proof: Since center $(\Gamma) = \{\begin{pmatrix} \epsilon a^n & 0 \\ 0 & \epsilon a^n \end{pmatrix} / \epsilon = \pm 1, 1 \neq a > 0\}$ acts on X by $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} \epsilon a^n & \\ 0 & \epsilon a^n \end{pmatrix} \times \begin{pmatrix} x & y \\ y & z \end{pmatrix} \times \begin{pmatrix} \epsilon a^n & 0 \\ 0 & \epsilon a^n \end{pmatrix} = \begin{pmatrix} a^{2n}x & a^{2n}y \\ a^{2n}y & a^{2n}z \end{pmatrix}$, its action on $f(x,y,z) \in P_m$ is by $f(x,y,z) \to f(a^{2n}x, a^{2n}y, a^{2n}z)$. The only polynomials invarient under this action are the constants, so $H^0(\text{center}(\Gamma), P_m) = 0$, $m > 0$. Picking a generator σ for A, H^1 (center(Γ), P_m) may be identified with $P_m / (\sigma\text{-id})P_m$(Maclane [5], pg. 189), but for the above action, $(\sigma\text{-id})P_m \cong P_m$, $m > 0$, so $H^1(\text{center}(\Gamma), P_m) = 0$, $m > 0$. Now Z acts freely without fixed points on $\mathbb{R}$, which is contractible of dimension 1, so $H^i(\text{center}(\Gamma),-) = 0$, $i > 1$.

If we now consider the Hochschild-Serre spectral sequence associated to $1 \to \text{center}(\Gamma) \to \Gamma \to \Gamma' \to 1$, all $E_2^{p,q} = 0$, since $H^q(\text{center}(\Gamma),P_m) = 0$, $m > 0$, for all q, so $H^i(\Gamma,P_m) = 0$ for all i, $m > 0$.

For the case $m = 0$, the Γ -action is trivial. From the Proposition of section (ii), the sequence $1 \to \text{center}(\Gamma) / (\pm 1) \to \Gamma / (\pm 1) \to \Gamma' \to 1$ also splits as a direct product and since our coefficients are trivial, we may apply the Kunneth formula.

<u>Proposition</u>: For $m = 0$, $H^0(\Gamma,\emptyset) \cong H^3(\Gamma,\emptyset) \cong \emptyset$; $H^1(\Gamma,\emptyset) \cong H^2(\Gamma,\emptyset) \cong S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')} \oplus \emptyset$; $H^i(\Gamma,\emptyset) = 0$, $i > 3$, where $S_2(\Gamma'')$ is the space of cusp forms of weight 2 for the Riemann surface $\Gamma''\backslash X_{(1)}$ and $\overline{S_2(\Gamma'')}$ is its complex conjugation.

<u>Proof</u>: For $\Gamma' \subset PGL_2^+(R)$, we have $\Gamma' = \Gamma'' \subset PSL_2(R)$. Since the Riemann surface $\Gamma''\backslash X_{(1)}$ is compact and without cusps, the Eichler cohomology ("parabolic"), $H_p^i(\Gamma'', -) = H^i(\Gamma'',-)$. So by Eichler's theorem, $H^i(\Gamma', \emptyset) = H^i(\Gamma'',\emptyset) = H_p^i(\Gamma'',\emptyset) = \emptyset$, $i = 0, 2$; $= S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')}$, $i = 1$; 0, $i > 2$, where $S_2(\Gamma'')$, $\overline{S_2(\Gamma'')}$ are as above.

Now by Kunneth (our groups and coefficients have no torsion), $H^0(\Gamma,\emptyset) = H^0(\Gamma',\emptyset) \otimes H^0(\text{center}(\Gamma) / (\pm 1),\emptyset)$, $H^1(\Gamma / (\pm 1),\emptyset) = H^0(\Gamma',\emptyset) \otimes H^1(\text{center}(\Gamma) / (\pm 1), \emptyset) \oplus H^1(\Gamma',\emptyset) \otimes H^0(\text{center}(\Gamma) / (\pm 1), \emptyset)$, $H^2(\Gamma / (\pm 1),\emptyset) = H^1(\Gamma',\emptyset) \otimes H^1(\text{center}(\Gamma) / (\pm 1),\emptyset) \oplus H^2(\Gamma',\emptyset) \otimes H^0(\text{center}(\Gamma) / (\pm 1),\emptyset)$, since $\text{center}(\Gamma) / (\pm 1) = \mathbb{Z}$ and

$H^i(\mathbb{Z}, -) = 0$, $i > 1$. Since the action of $\mathbb{Z}$ is trivial on $\emptyset$, $H^i(Z,\emptyset) = \emptyset$, $i = 0, 1$, using the identifications of $H^i(\mathbb{Z}, -)$ as in the proof of the above theorem. Evaluating the above tensor products, $H^i(\Gamma / (\pm 1), \emptyset)$ has the values prescribed in our Proposition.

If center(Γ) = $\mathbb{Z}$, we are done, since $\Gamma / (\pm 1) = \Gamma$. When center $(\Gamma) = (\pm I) \times \mathbb{Z}$, we have an exact sequence $1 \to (\pm I) \to \Gamma \to \Gamma / (\pm I) \to I$. But since $(\pm I)$ is a finite group, and the coefficients $\emptyset$ are divisible without finite elements, $H^i((\pm I), \emptyset) = 0$, $i > 0$ (Maclane [5], pg. 117), and since $-I$ acts trivially on $\emptyset$, $H^0((\pm I), \emptyset) = \emptyset$. Thus $H^i(\Gamma, \emptyset) = H^i(\Gamma / (\pm I), H^0((\pm I), \emptyset)) = H^i(\Gamma / (\pm I), \emptyset)$, and we are done.

These results are somewhat trivial, but are included for completeness. Results for the non-compact case are more interesting: $H^i(\Gamma\backslash X, p^{(m)}) \neq 0$ for non-constant $p^{(m)}$ and $H^1(\Gamma, P_m)$ involves cusp forms of higher weight.

(IV) The Non-compact Case

For the non-compact case, we have center(Γ) = I or $(\pm I)$. As remarked at the beginning of (I), $-I$ acts trivially on X, so we have only to compute $H^i(\Gamma', P_m(x, y, z))$. Set s_k = the k- th symetric tensor representation, and d_j = the representation $(\det)^j$, both of $GL_2(\mathbb{R})$.

<u>Proposition</u>: $P_m(x, y, z)$, considered as a GL_2-representation may be decomposed into the irreducible components

$s_{2m} \oplus [(\det) \otimes s_{2m-4}] \oplus \ldots \oplus [(\det)^{m-e} \otimes s_{2e}]$, where e = 0, when m is even, e + 1 when m is odd. (We have refrained from our usual Young diagram notation for ease of typing).

<u>Proof</u>: By the classical theory of representations of $GL_2(\mathbb{R})$, we have that the $s_k \otimes d_j$ are irreducible, and every irreducible representation is of this form. Also, our representation is determined by it's trace, so it suffices to check that the highest weights agree. Note that the representation $P_m(x, y, z)$ of GL_2 is the composition of the representation s_2 of GL_2 on $S^2(\mathbb{R}^2) \cong \mathbb{R}^3$, with p_m, the representation of GL_3 on $S^m(\mathbb{R}^3)$.

For weights a, b of GL_2, with a > b, the representation $s_k \otimes d_j$ is given by weights $(a^{k+j} b^j, a^{k+j-1}b^{j+1}, \ldots, a^j b^{k+j})$, and for weights u, v, w of (the maximal torus of) GL_3, with u > v > w, the representation p_m is given by weights $(u^m, u^{m-1}v, u^{m-1}w, u^{m-2}vw, u^{m-2}w^2, \ldots, nv^{m-1}, uv^{m-2}w, \ldots, uw^{m-1}, v^m, v^{m-1}w, \ldots, vw^{m-1}, w^m)$. So the representation $P_m(x, y, z) = P_m \circ s_2$ of GL_2 is given by the above weights with $u = a^2$, $v = ab$, and $w = b^2$. It is easily checked that these weights agree with the weights of $s_{2m} \oplus [s_{2m-4} \otimes d_2] \oplus \ldots \oplus [s_{2e} \otimes d_{m-e}]$.

Now since $\Gamma = \Gamma_{(1)} = \Gamma \cap SL_2(R)$, and $d_j = (\det)^j$ is trivial on $SL_2(R)$, we have $H^1(\Gamma, d_j \otimes s_k) \cong H^1(\Gamma, s_k)$. Also for the case center$(\Gamma) = (\pm I)$, $\Gamma' = \Gamma / (\pm I)$, and we have $H^1(\Gamma, s_{2k}) = H^1(\Gamma', H^0((\pm I), s_{2k})) = H^1(\Gamma', s_{2k})$, since -I acts trivially for even degrees (for a basis $x^f y^g$, $f + g = 2k$, $x^f y^g \to (-x)^f(-y)^g = x^f y^g$). If we again recall that $\Gamma = \Gamma'' \subset PSL_2(R)$, and that $\Gamma''\backslash X_{11}$, is compact so that $H^i_p(\Gamma'', s_k) = H^i(\Gamma'', s_k)$ we have for $k > 0$, by Eichler's Theorem that $H^i(\Gamma', s_k) = 0$, $i = 0$; $S_{k+2}(\Gamma'') \oplus \overline{S_{k+2}(\Gamma'')}$, $i = 1$; 0, $i = 2$; and 0, $i > 2$, where, as before $S_{k+2}(\Gamma'')$ is the space of cusp forms of weight $k+2$ for the compact Riemann surface $\Gamma''\backslash X_{(1)}$, and $\overline{S_{k+2}(\Gamma'')}$ is it's complex conjugation.

<u>Theorem</u>: When $\Gamma''\backslash X_{(1)}$ is compact, while $M = \Gamma\backslash X$ is non-compact, then if m is even $H^0(\Gamma, P_m) \cong H^2(\Gamma, P_m) = \emptyset$; and $H^1(\Gamma, P_m) = S_{2m+2}(\Gamma'') \oplus \overline{S_{2m+2}(\Gamma'')} \oplus S_{2m-2}(\Gamma'') \oplus \ldots \oplus S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')}$; while for m odd, $H^1(\Gamma, P_m) = S_{2m+2}(\Gamma'') \oplus \ldots \oplus \overline{S_4(\Gamma'')}$; and $H^i(\Gamma, P_m) = 0$, $i \neq 1$.

<u>Proof</u>: Everything follows from the above, with the remark $H^i(\Gamma', s_{2m} \oplus \ldots \oplus [d_{m-e} \otimes s_{2e}]) = H^i(\Gamma', s_{2m}) \oplus \ldots \oplus H^i(\Gamma', d_{m-e} \otimes s_{2e})$, and the note that the Γ representation on P_m extends to a $GL_2(R)$ representation.

(V) Further Remarks

While polnomials of degree m provide a natural Γ-module in view of the linear action of Γ, results may also be obtained for polynomials of degree $\leq m$, as considered in Chapter 2. For the purposes of this section, let $H_m(R^3) = \{f(x,y,z) / f$ is a polynomial of deg $= m\}$ and $P_m(R^3) = \{f(x,y,z) / f$ is a polynomial of deg $\leq$ m}. Results for $H^1(\Gamma, P_m)$ follow from the previous results for $H^1(\Gamma, H_m)$ from the observation that the direct sum $P_m = \bigoplus_{k=0}^{m} H_k$ is preserved by the Γ-action.

For the non-compact case, the decomposition of H_m into irreducible components given in section (IV) provides a decomposition for P_m, which we state as a corollary.

Corollary: Let s_k, d_j and $s_k \otimes d_j$ be the irreducible $GL_2(R)$ representations as before, and id be the trivial representation of GL_2 on $\not{C}$. Suppose $\Gamma \subset SL_2(R)$ as in the non-compact case, so that $s_k \otimes d_j$ is equivalent to s_k on Γ. Let m be even, $m = 2m'$, then the Γ representation on P_m is equivalent to the representation $(m'+1)\mathrm{id} \oplus m's_2 \oplus m's_4 \oplus (m'-1)s_6 \oplus (m'-1)s_8 \oplus \ldots \oplus 2s_{4m'-4} \oplus s_{4m'-2} \oplus s_{4m'}$. For m odd, $m = 2m' + 1$, the corresponding representation is $(m'+1)\mathrm{id} \oplus (m'+1)s_2 \oplus m's_4 \oplus \ldots \oplus 2s_{4m'-2} \oplus s_{4m'} \oplus s_{4m'+2}$, where coefficients refer to the multiplicity of the representation and the subscripts to the degree of the representation.

Proof: Apply induction to m' using the proposition of section (IV). For example, if the decomposition is known for $m = 2(m'-1) + 1$, then $P_{2m'} = P_m \oplus H_{2m'}$, $H_{2m'}$ has the effect of adding $s_{4m'}$ and increasing by one the multiplicity of id, s_4, ... , and $s_{4m'-4}$, by the Proposition of section (IV).

From this result $H^1(\Gamma, P_m)$ may be easily determined for $P_m = \bigoplus_{k=0}^{m} H_k$ in the non-compact case. As before $H^1(\Gamma, P_m)$ may be expressed in terms of cusp forms, where the representation s_k corresponds to the cusp forms, $S_{k+2}(\Gamma'')$, of weight k+2, occuring with the same multiplicity as s_k (considering id = s_0).

For the compact case, again consider the sequence $1 \to \text{center}(\Gamma) \to \Gamma \to \Gamma' \to 1$. Since for every m, $p_m = \{f(x,y,z) \,/\, \deg f \le m\}$ contains the constants, $H^i(\text{center}(\Gamma), P_m) = \emptyset$, i = 0,1; 0, i > 1 as in section (III). However, the coefficients are non-trivial so we apply Hochschild-Serre. We compute the $E_2^{p,q}$ terms by the techniques of section (III) to be the following: $E_2^{p,q} = \emptyset$ for p = 0 or 2 and q = 0 or 1; and $E_2^{1,q} = S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')}$ for q = 0 or 1; and $E_2^{p,q} = 0, p > 2$ or $q > 1$. From this we may conclude $H^i(\Gamma, P_m) = 0$, $i > 3$ and $H^i(\Gamma, P_m) = S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')} \oplus \emptyset$, i = 1 or 2, if $d_2 = 0$ or $H^i(\Gamma, P_m) = S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')}$ if $d_2 \ne 0$, i = 1 or 2.

Preliminary computations indicate that $d_2 = 0$. For this we consider the sequence $1 \to \Gamma_{(1)} \to \Gamma \to \Gamma / \Gamma_{(1)} \to 1$. As a linear space, $H^q(\Gamma_{(1)}, P_m)$ may be computed as in the non-compact case, however, the d_j factor of $H^q(\Gamma_{(1)}, d_j \otimes s_k) \cong H^q(\Gamma_{(1)}, s_k)$ must not be ignored in determining the $\Gamma/\Gamma_{(1)}$ action on $H^q(\Gamma_{(1)}, s_k)$. For example, $H^0(\Gamma_{(1)}, d_j \otimes s_0) = \emptyset$ for j = 0,1, ... , but $\Gamma/\Gamma_{(1)}$ acts trivially on $H^0(\Gamma_{(1)}, \text{id})$ and non-trivially on $H^0(\Gamma_{(1)}, d_j \otimes s_0)$ for $j > 0$. Likewise, of the (m'+1) copies of $S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')}$ occuring in $H^1(\Gamma_{(1)}, P_m)$, only the one corresponding to the trivial representation $d_0 \otimes s_0$ has trivial $\Gamma/\Gamma_{(1)}$ action.

This analysis leads to the conclusion that for the sequence $1 \to \Gamma_{(1)} \to \Gamma \to \Gamma / \Gamma_{(1)} \to 1$, the Hochschild-Serre spectral sequence has $E_2^{p,q}$ terms, $E_2^{1,0} = \emptyset$ and $E_2^{0,1} = S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')}$. Since $\Gamma/\Gamma_{(1)} = \mathbb{Z}$, d_2 is automatically 0 for this sequence, so $H^1(\Gamma, P_m) = S_2(\Gamma'') \oplus \overline{S_2(\Gamma'')} \oplus \emptyset$. From this it follows that $d_2 = 0$ also holds for the sequence $1 \to \text{center}(\Gamma) \to \Gamma \to \Gamma' \to 1$. The reader may be interested to note the similarity of this computation to the computation used for the Heisenberg group in the following Chapter 4. Note that two spectral sequences sequences are compared, one of which has readily computable $E_2^{p,q}$ -terms, but does not determine d_2, while for the other $d_2 = 0$, but the action of the quotient group on the cohomology of the subgroup is complicated.

(IV) Other Sheaves

As we have previously remarked, X may be considered as a connected component of the open orbit of the prehomogeneous vector space $(GL_2(\mathbb{R}), p, \mathbb{R}^3)$ with the action p as above on $\mathbb{R}^3$ viewed as 2 x 2 symmetric matrices. For a prehomogeneous vector space (G, p, V), the polynomials on V relatively invarient under the action of G are generated by a single polynomial χ. In this case $\chi = (\det) = (xz-y^2)$, and the relative invarients are $(\det)^m$, in view of this, we consider as in Chapter 1 a sheaf $p^{(2m)} / (\det)^m$ of local rational functions on the quotient $M = \Gamma\backslash X$ whose numerator is a polynomial of degree 2m and denominator is $(xz-y^2)^m$. This sheaf is well defined since $(\det)^m$ is a relative invarient under the Γ -action (hence under coordinate transformations $\varphi_\beta \circ \varphi_\alpha^{-1}$ by the definition of the local affine structure taken on $\Gamma\backslash X$ as in Chapter 2).

The corresponding Γ -module is $P_{2m}/(\det)^m =$ $\{f(x,y,z) \ / \ (xz-y^2)^m$ such that $\deg f = 2m\}$, which is given as a $GL_2(R)$ representation by $P_{2m} \circ (s_2 \otimes d_{-m})$. Again we have $H^i(M, p^{(2m)} \ / \ (\det)^m) \cong H^i(\Gamma, P_{2m} \ / \ (\det)^m)$ by the proof of our theorem in Chapter 1.

Proposition: In the compact case, i.e., center(Γ) = $\mathbb{Z}$ or $(\pm 1) \times \mathbb{Z}$, the evaluation of $H^i(\Gamma, P_{2m}/(\det)^m)$ also involves cusp forms of higher weight.

Proof: To see this we consider the sequence $1 \to$ center(Γ) $\to \Gamma \to \Gamma' \to 1$. Note that center($\Gamma$) acts trivially on $P_{2m}/(\det)^m$, (a factor of a^{4m} cancels for the generator $\binom{a0}{0a}$)), so that, for example, $H^0(\text{center}(\Gamma), P_{2m}/(\det)^m) = P_{2m}/(\det)^m$. Now for the corresponding spectral sequence, the $E_2^{1,0}$ -term, which s survives regardless of d_2: $E_2^{0,1} \to E_2^{2,0}$, is given by $E_2^{1,0} =$ $H^1(\Gamma', P_{2m}/(\det)^m)$ where Γ' acts by the coefficient action since it acts trivially on center(Γ).

For $m > 0$, the decomposition of $P_{2m} \circ (s_2 \otimes d_{-m})$ involves $s_k \otimes d_{-j}$ with $k > 0$. Thus $H^1(\Gamma', P_{2m}/(\det)^m)$ involves terms $H^1(\Gamma', s_k \otimes d_{-j})$. Since $\Gamma' \subset PGL_2$, the d_{-j}-factor is trivial, and we have as before, $H^1(\Gamma', s_k \otimes d_{-j}) = H^1(\Gamma', s_k) = H^1(\Gamma'', s_k) =$

$S_{k+2}(\Gamma'') \oplus \overline{S_{k+2}(\Gamma'')}$, and with $k > 0$, the space of cusp forms $S_{k+2}(\Gamma'')$ have weight $k+2 > 2$.

Having extablished our point, we refrain from a precise determination of the $E_2^{p,q}$-terms and d_2. Note that for the non-compact case, we have $\Gamma/(\pm I) = \Gamma''$, and by the preceding paragraph, $H^1(\Gamma, P_{2m}/(\det)^m)$ also involves cusp forms of weight > 2.

We also remark that $H^1(\Gamma'', s_m{*}) = H^1(\Gamma'', s_m)^*$, so that results for the module $D_k = \{D = \Sigma\, a_{i_1}, \ldots, {}_{i_k} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} /$ order $(D) = k\} = P_k \circ (s_2{*})$ (or, sheaf $\mathcal{D}_k$ of local differential operators) follow directly from the computations of sections (III) and (IV). Results for polynomial coefficient differential operators are somewhat more complicated since in the decomposition of the corresponding representation, both $H^1(\Gamma'', s_m)^*$ and $H^1(\Gamma'', s_m)$ may occur, resulting in "collapsing" : $H^1(\Gamma'', s_m)^* \otimes H^1(\Gamma'', s_m) = \emptyset$. This complication is not expected to cause essential difficulties.

We note that $H^0(M, p\mathcal{D}) \neq 0$, for $p\mathcal{D}$ = sheaf of local polynomial coefficient differential operators (in contrast to the usual situation for affine manifolds), since we have for $\Delta_1 = (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z})$ and $\Delta_2 = (xz-y^2)(\frac{\partial^2}{\partial x \partial z} - 4 \frac{\partial^2}{\partial^2 y})$, $\Delta_1, \Delta_2 \in H^0(M, p\mathcal{D})$. We have computed the eigen-polynomials of Δ_1, and found that they have eigenvalues $2m+2, 2m-2, \ldots,$ $2e+2$, with $e = 0$, when m is even and $e = 1$, when m is odd (we are considering Δ_1 as an operator on $H^i(M,p^{(m)})$). This is highly suggestive as an explanation for the results in the quadratic form case, in view of the fact that in certain circumstances the eigenspaces of $D \in H^i(M,p\mathfrak{D})$ are representation spaces for the Hecke operators. Note also that we have explicit eigenvalue information here.

Chapter 4: The Heisenberg Group

(I) Preliminaries

In the following, the space X is $\mathbb{R}^3$ considered as the underlying manifold of the Heisenberg group H (for the definition see below), the discrete group is the subgroup N of integral points in H acting on X by left translation, and the relevant N-module is the space $P_m = \{f(x,y,z)/f$ is a polynomial with complex coefficients of deg $\leq m\}$, the variables x,y,z being viewed as coordinates on $X = \mathbb{R}^3$, and N acting by $n(f(x,y,z)) = f(n^{-1}(x,y,z))$. Further, the group $H^1(N,P_m)$ will be considered as a module for a group Γ of affine automorphisms of X preserving N, having the effect that our results must be compatible with automorphisms. Our main result is that a part of the cohomology group $H^j(\Gamma,H^1(N,P_m))$ may be expressed in terms of automorphic forms for $SL_2(\mathbb{Z})$. From our theory of Hecke operators in the category of affine manifolds, we obtain the Dirichlet series derived from the action of the classical Hecke ring acting on automorphic forms.

We take the following definition for the multiplication in the Heisenberg group H : $(x,y,z)\cdot(u,v,w) = (x+u,y+v,z+w+yu-xv)$. The subgroup N of integral points may be described by the following exact sequences: (A) $0 \to \mathbb{Z} \to N \to \mathbb{Z}^2 \to 0$; and (B) $0 \to \mathbb{Z}^2 \to N \to \mathbb{Z} \to 0$. For (A), the subgroup $\mathbb{Z} \cong \{(0,0,k)/ k \in \mathbb{Z}\}$ may be identified with the center of N, as may be readily checked from the above multiplication law, hence (A) is preserved by automorphisms. For (B), the subgroup $\mathbb{Z}^2 \cong \{(0,m,k) / m,k \in \mathbb{Z}\}$. We use the Hochschild-Serre spectral sequence for (A) to calculate $H^1(N,P_m)$, where d_2 is determined by a dimension estimate from the corresponding (trivial) spectral sequence for (B).

We shall need the result, as well as the resolution used in the proof, of the following standard result on the cohomology of $\mathbb{Z}^2$.

<u>Proposition 1</u>: Let t_x, t_y be generators for the group $G = \mathbb{Z}^2$, and A be a $\mathbb{Z}^2$-module, then $H^0(G,A) = A^G$; $H^1(G,A) = \{(f,g) \in A \times A / \Delta_y f = \Delta_x g\} / \{(\Delta_x f', \Delta_y f')/f' \in A\}$; $H^2(G,A) = A / \{\Delta_x g - \Delta_y f / f,g \in A\}$; and, $H^i(G,A) = 0$, for $i > 2$, where $\Delta_x = (t_x - id)$, $\Delta_y = (t_y - id)$, and A^G are the G-invarients.

<u>Proof</u>: The standard proof may be found in MacLane, [5], ch. 4. We use the following special resolution: $0 \to \mathbb{Z}[G]\, e_1 \otimes f_1 \overset{\partial}{\to} \mathbb{Z}[G]\, e_1 \otimes f_0 \oplus \mathbb{Z}[G]\, e_0 \otimes f_1 \overset{\partial}{\to} \mathbb{Z}[G]\, e_0 \otimes f_0 \overset{\epsilon}{\to} \mathbb{Z} \to 0$, where $\partial\,(e_1 \otimes f_1) = \Delta_x e_0 \otimes f_1 = \Delta\, y\, e_1 \otimes f_0$, $\partial\,(e_1 \otimes f_0) = \Delta_x\, e_0 \otimes f_0$, $\partial\,(e_0 \otimes f_1) = \Delta_y\, e_0 \otimes f_0$, and ϵ = augmentation. This is a resolution by the Kunneth formula; and the fact that e_1, e_0, with the boundary operator $\partial\,(e_1) = \Delta_x\, e_0$, provides a resolution for the subgroup generated by t_x; and f_1, f_0, with the corresponding boundary operator, provides a resolution for the subgroup generated by t_y.

$H^i(N,m) = 0$, $i > 2$, follows from our resolution, since the chain groups are 0 in these dimensions. For $i = 0$, the result is true for any group, so we omit verification. In general, we have $Z^i = \{h: C_i \to m \,/\, \delta h = 0,\ h(gx) = g \cdot h(x)$, for all $g \in \mathbb{Z}[G]$, $x \in C_i\}$, $B^i = \{C_i \to m \,/\, h = \delta h'$, where $h': C_{i-1} \to m$, and h, h' satisfy the invarience condition$\}$, where the C_i are the appropriate chain groups, and may define $H^i = Z^i/B^i$. By the invarience condition $h(gx) = g \cdot h(x)$, we may specify cocycles by giving their values on ($\mathbb{Z}[G]$ –) generators $e_0 \otimes f_0$, $e_1 \otimes f_0$, $e_0 \otimes f_1$, and $e_1 \otimes f_1$ for the chain groups C_0, C_1, and C_2.

For $i = 2$, $Z^2 = A$, since $C_3 = 0$, and $B^2 = \{h: C_2 \to A/h = \delta h'$, for h, h' invarient$\}$, so $h(e_1 \otimes f_1) = (\delta h')(e_1 \otimes f_1) = h'(\partial(e_1 \otimes f_1)) = h'(\Delta_x e_0 \otimes f_1 - \Delta_y e_1 \otimes f_0) = \Delta_x h'(e_0 \otimes f_1) - \Delta_y h'(e, \otimes f_0)$. Since $h'(e_0 \otimes f_1) = g$ and $h'(e_1 \otimes f_0) = f$ may be specified arbitrarily in A, we have our result for $i = 2$. Finally, $Z^1 = \{h: C_1 \to A \,/\, \delta h = 0\}$, so $0 = (\delta h)(e_1 \otimes f_1) = h(\partial(e_1 \otimes f_1)) = h(\Delta_x e_0 \otimes f_1 - \Delta_y e_1 \otimes f_0) = \Delta_x h(e_0 \otimes f_1) - \Delta_y h(e_1 \otimes f_0)$ (again, using invarience). Also, $B^1 = \{h: C_1 \to A \,/\, h = \delta h'\}$, so $h(e_1 \otimes f_0) = \delta h'(e_1 \otimes f_0) = h'(\partial(e_1 \otimes f_0)) = h'(\Delta_x e_0 \otimes f_0) = \Delta_x h'(e_0 \otimes f_0)$, and similarly, $h(e_0 \otimes f_1) = \Delta_y h'(e_0 \otimes f_0)$. Setting $h(e_1 \otimes f_0) = f$, $h(e_0 \otimes f_1) = g$, and $h'(e_0 \otimes f_0) = f'$, and noting that f' may be chosen arbitrarily, we obtain our result for $i = 1$, and complete our proof of proposition 1.

(II) The Sequence (A): $0 \to \mathbb{Z} \to N \to \mathbb{Z}^2 \to 0$

Next, we compute the approximations to $H^i(N,P_m)$ provided by Hochschild-Serre applied to the sequence (A). The spectral sequence associated to the exact sequence (A) will be denoted by $E_r^{p,q}(A)$, or $E_r^{p,q}$.

<u>Lemma 1</u>: As linear spaces, we obtain the following diagram for $E_2^{p,q} = E_2^{p,q}(A) = H^p(\mathbb{Z}^2, H^q(\mathbb{Z}, P_m))$:

q \ p	0	1	2
1	$\mathbb{C}^{m+1}$	$\mathbb{C}^{m+2}$	$\mathbb{C}$
0	$\mathbb{C}$	$\mathbb{C}^{m+2}$	$\mathbb{C}^{m+1}$

<u>Proof</u>: Since we are considering the variables x,y,z as coordinates, we may take the action of a generator for $\mathbb{Z}$ to be $f(x,y,z) \to f(x,y,z+1)$. We obtain $H^0(\mathbb{Z},P_m) = P_m(x,y)$; $H^1(\mathbb{Z},P_m) = P_m(x,y,z)/P_{m-1}(x,y,z)$; and $H^q(\mathbb{Z},P_m) = 0$, for $q \geq 2$, noting that for a generator t_z of $\mathbb{Z}$, $H^1(\mathbb{Z}, A) = A/\Delta_z A$, where $\Delta_z = (t_z - id)$.

Now, results for $E^{0,0}$, $E^{2,0}$, $E^{0,1}$, and $E^{2,1}$ follow from proposition 1. For example, $E^{2,0} = H^2(\mathbb{Z}^2, P_m(x,y))$, where since $\mathbb{Z} = \text{center}(N)$, the $\mathbb{Z}^2$-action is the coeficient action: we may choose a generator t_x to act by $(x,y,z) \to (x+1,y,z-y)$, and a generator t_y to act by $(x,y,z) \to (x,y+1,z+x)$. By proposition 1, $H^2(\mathbb{Z}^2, P_m(x,y)) = P_m(x,y)/\ \{\Delta_x g = \Delta_y f/f,g \in P_m(x,y)\} =$

$P_m(x,y)/P_{m-1}(x,y) \cong \not{C}^{m+1}$. The same action applies for $E^{0,1}$, and $E^{0,1} = H^0(\mathbb{Z}^2, P_m(x,y,z)/P_{m-1}(x,y,z)) \cong (P_m(x,y) + P_{m-1}(x,y,z))/P_{m-1}(x,y,z) \cong P_m(x,y)/P_m(x,y) \cap P_{m-1}(x,y,z) \cong P_m(x,y)/P_{m-1}(x,y) \cong \not{C}^{m+1}$.

Proposition 1 applies to $E^{1,0}(A)$ and $E^{1,1}(A)$, but is insufficient to determine the dimensions in question. We resort to the exact sequence $0 \to \mathbb{Z}_y \to \mathbb{Z}^2 \to \mathbb{Z}_x \to 0$, where $\mathbb{Z}_y$ is the subgroup $\mathbb{Z}_y = \{(0,m)/\ m \in \mathbb{Z}\}$, noting that the associated spectral sequence will not be preserved by automorphisms. $E^{1,0}(A) = H^1(\mathbb{Z}^2, P_m(x,y)) \cong H^0(\mathbb{Z}_x, H^1(\mathbb{Z}_y, P_m(x_1y)) \oplus H^1(\mathbb{Z}_x, H^0(\mathbb{Z}_y, P_m(x,y,))) \cong H^0(\mathbb{Z}_x, P_m(x,y)/P_{m-1}(x,y)) \oplus H^1(\mathbb{Z}_x, P_m(x))$, where, since $\mathbb{Z}^2$ is abelian, $\mathbb{Z}_x$ acts trivially on $\mathbb{Z}_y$, and so by coeficient action on $H^q(\mathbb{Z}, P_m(x,y))$. Thus, we obtain $H^0(\mathbb{Z}_x, P_m(x,y)/P_{m-1}(x,y)) = P_m(x,y)/P_{m-1}(x,y) \cong \not{C}^{m+1}$, and $H^1(\mathbb{Z}_x, P_m(x)) \cong P_m(x)/P_{m-1}(x) \cong \not{C}$. We remark that the coeficient action of $\mathbb{Z}^2$ of the preceeding paragraph applies equally well here, and we have used it in the above. We have established: $\dim(E^{1,0}) = (m+1) + 1 = m+2$. The computation for $E^{1,1}$ is similar, and we consider our lemma as established.

For the above spectral sequence (associated to the exact sequence (A)), $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ is 0 unless $p = 0$, $q = 1$, since $E_2^{p+2,q-1} = 0$ for all other p,q. Our next two lemmas show that $d_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism.

Lemma 2: d_2 is either 0, or an isomorphism (for $p = 0$, $q = 1$).

Proof: The complete proof will be defered until we have discussed the automorphisms of N. In the proof of Lemma 1, we have shown that $E^{0,1}$ (or $E^{2,0}$) may be identified with $P_m(x,y)/P_{m-1}(x,y)$. We will show that $SL_2(\mathbb{Z})$ is contained in the automorphism group of N. Since the sequence (1) is preserved by automorphisms, $SL_2(\mathbb{Z})$ acts on the spectral sequence in question. We will show that $SL_2(\mathbb{Z})$ acts irreducibly on $E^{0,1}$ (or $E^{2,0}$). Our lemma is then an application of Shure's lemma.

(III) A Dimension Estimate for $H^1(N,P_m)$

Lemma 3: $\dim H^1(N,P_m < 2m + 3$.

Corollary: $\dim H^1(N,P_m) = m + 2$.

Proof (of Cor.): By lemmas 1 and 2, either $\dim H^1(N) = 2m + 3$ or $\dim(H^1(N)) = m + 2$.

Proof (of lemma): We consider the exact sequence (B). The spectral sequence associated to the exact sequence (B) is denoted by $E_r^{p,q}(B)$ or $E_r^{p,q}$. Since $E_2^{p,q}(B) = 0$, for $p \geq 2$ (we may find a resolution for $\mathbb{Z}$, as refered to in the proof of Prop. 1 using e_0, e_1, for which the chain groups C_i vanish for $i \geq 2$), $d_2 = d_3 = \dots = d_r = 0$, and so, $H^1(N) = E_2^{1,0} \oplus E_2^{0,1}$. Now $H^0(\mathbb{Z}^2, P_m(x,y,z)) = P_m(x)$ (cannonially), and $E^{1,0} = H^1(\mathbb{Z}_x, P_m(x)) = P_m(x)/P_{m-1}(x) = \emptyset$, so our lemma may be reformulated as $\dim E^{0,1}(B) < 2m+2$.

By Prop. 1, $H^1(\mathbb{Z}^2, P_m(x,y,z)) = \{(f,g)/ \Delta_z f = \Delta_y g\}/$

$\{\Delta_y f', \Delta_z f')\}$. Also, using a spectral sequence associated to $0 \to \mathbb{Z}_z \to \mathbb{Z}^2 \to \mathbb{Z}_y \to 0$, where the subscripts refer to the generator t_z acting by $(x,y,z) \to (x,y,z+1)$ and the generator t_y acting by $(x,y,z) \to (x,y+1,z-x)$. We have determined $H^q(\mathbb{Z}_z, P_m)$ at the begining of the proof of Lemma 1, and again note that t_y acts trivially on $\mathbb{Z}_z$, so that we may compute $H^1(\mathbb{Z}_y, P_m(x,y)) \cong P_m(x,y)/P_{m-1}(x,y) \cong \mathbb{C}^{m+1}$ and $H^0(\mathbb{Z}_y, P_m(x,y,z)/P_{m-1}(x,y,z)) = (P_m(x,y) + P_{m-1}(x,y,z))/P_{m-1}(x,y,z) \cong P_m(x,y)/P_{m-1}(x,y) \cong \mathbb{C}^{m+1}$, and conclude $\dim H^1(\mathbb{Z}^2, P_m(x,y,z)) = 2m + 2$.

To prove our lemma, we must show $\dim H^0(\mathbb{Z}, H^1(\mathbb{Z}^2, P_m)) < 2m +2$, and by the preceeding paragraph, it suffices to find one cocycle $(f,g) \in H^1(\mathbb{Z}^2, P_m)$ not invarient under the action of t_x. However, our task is complicated by the fact that $\mathbb{Z}_x$ acts non-trivially on $\mathbb{Z}^2$. By our proof of Proposition 1, we may determine the action of $\mathbb{Z}_x$ on a cocycle (f,g) from an action on our resolution of $\mathbb{Z}^2$. For such an action to determine an action on cohomology, it must commute with the ∂ -operator. We shall show that picking $g = y^{m-1}z$ provides the required cocycle.

For the generator $e_0 \otimes f_0$, and a generator t_x of $\mathbb{Z}_x$, set $t_x(e_0 \otimes f_0) = a\, e_0 \otimes f_0$, $a \in \mathbb{Z}[\mathbb{Z}^2]$, then $\partial \cdot t_x(e_0 \otimes f_0) = \epsilon(a)$, where ϵ is the augmentation map, likewise, $t_x \cdot \partial\, (e_0 \otimes f_0) = t_x(1) = 1$, so we need $\epsilon\,(a) = 1$, and may take $a = 1$. For the generator $e_0 \otimes f_1$, set $t_x(e_0 \otimes f_1) = c\, e_1 \otimes f_0 + d\, e_0 \otimes f_1$, then $\partial \cdot t_x(e_0 \otimes f_1) = (c \cdot \Delta_y + d \cdot \Delta_z)e_0 \otimes f_0$ and $t_x \cdot \partial (e_0 \otimes f_1) = t_x(\Delta_z e_0 \otimes f_0) = t_x(\Delta_z) \cdot t_x(e_0 \otimes f_0) = \Delta_z e_0 \otimes f_0$, by the preceeding sentance and the fact that $t_x(t_z) = t_x o t_z o t_x{-1} = t_z$. Therefore, we may take $c = 0$, $d = 1$, so that

$$(*): \quad t_x(e_0 \otimes f_1) = e_0 \otimes f_1.$$

In a similar fashion we can solve for the action on $e_1 \otimes f_0$ and $e_1 \otimes f_1$ in a compatible manner. The resulting formula are more complicated, but these will not be relevant in the following.

By general principles, the action of t_x on cohomology is t_x: $h \to t_x(h)$, where $t_x(h)c = t_x . h(t_x^{-1}(c))$, for $c \in C_1(Z^2, P_m)$; where the outside action is the coefficient action and the inside action is the action on the resolution of $\mathbb{Z}^2$ as above. Applying t_x^{-1} to both sides of (*), $t_x^{-1}(e_0 \otimes f_1) = e_0 \otimes f_1$. Now, recall that the cocycle h has been identified with it's values on generators, $f = h(e_1 \otimes f_0)$ and $g = h(e_0 \otimes f_1)$. Under the action of t_x on cocyles, $(f,g) \to (f',g')$, where $f' = t_x \cdot h(t_x{}^{-1}(e_1 \otimes f_0))$, and $g' = t_x \cdot h(t_x{}^{-1}(e_0 \otimes f_1))$.

Assuming that $(-, g)$ defines a cocycle, the t_x-action is $(-, g) \to (-, t_x \cdot g)$, since $t_x{}^{-1}(e_0 \otimes f_1) = e_0 \otimes f_1$. Now set $g = y^{m-1}z$ (we will verify that g occurs for a pair (f,g) defining a cocycle later, page 40). Since the t_x-action is the coefficient action, $t_x(g) = y^{m-1}(z-y) = y^{m-1}z - y^m$. To have $(-, g) = (-, t_x g) \pmod{B^1}$, there must exist, by Prop. 1, $f' \in P_m(x,y,z)$ so that $g - t_x(g) = y^m = \Delta_z f'$. But

Δ_z lowers degrees by 1 (we have used this fact repeatedly in the above), so that $\deg \Delta_z f' < \deg f' \leq m$, and this is impossible, completing the proof of lemma 3.

As remarked in the corollary of lemma 3, we have $\dim H'(N,P_m) = m + 2$. In fact, returning to the exact sequence (A), $H^1(N,P_m) = E_3^{1,0} \oplus E_3^{0,1} \cong E_3^{1,0} = H^1(\mathbb{Z}^2, H^0(\mathbb{Z}, P_m)) \cong H^1(\mathbb{Z}^2, P_m(x,y))$, and similiarly, $H^2(N,P_m) \cong H^1(\mathbb{Z}^2, P_m/P_{m-1})$. We now restrict our attention to $H^1(N)$.

(IV) Further Remarks on $H^1(N)$

We now have the following special case of Proposition 1, which is helpful in determining the module structure on $H^1(N)$ under the action of automorphisms.

Proposition 2: Let $A = P_m(x,y)$, and suppose that the generators t_x, t_y of $\mathbb{Z}^2$ act by $t_x \cdot f(x,y) = f(x+1,y)$ and $t_y \cdot f(x,y) = f(x,y+1)$. Then we have an explicit isomorphism $H^1(\mathbb{Z}^2, P_m(x,y)) \cong P_{m+1}(x,y)/P_m(x,y)$.

Remark 1: Our contention that a certain cohomology group may be expressed in terms of automorphic forms will follow upon showing that $H^1(N,P_m)$ is an irreducible $SL_2(\mathbb{Z})$-module. We will show that the above isomorphism holds as $SL_2(\mathbb{Z})$-modules, where $SL_2(\mathbb{Z})$ acts on the right hand side by the dual of the $(m+1)^{st}$ symmetric power of the natural representation on $\mathbb{R}^2$.

Proof (of Prop.): By Proposition 1, we have $H^1 = Z^1/B^1$, where $Z^1 = \{(f,g)/ \Delta_y f = \Delta_x g\}$, $B^1 = \{(\Delta_x f', \Delta_y f')\}$, with

$f, g, f' \in P_m(x,y)$. We intend to show that when (f,g) satisfies our condition for cocycles, there exists $G \in P_{m+1}(x,y)$ so that $\Delta_x G = f$ and $\Delta_y G = g$. Since Δ_x and Δ_y both lower degrees by 1, the induced map from $P_{m+1}(x,y)$ to H^1 has kernel $P_m(x,y)$, yeilding our result.

The desired polynomial G is given by $G(x,y) = S_0^y(g(x,y)) + S_0^x(f(x,0)) + xf(o,0)$, where $S_0^y(g(x,y))$ and $S_0^x(f(x,0))$ are "Bernoulli integration," of which the definition is given below. We refrain from further verifications, and consider the proof complete.

For a 1-varible polynomial f(x) of degree m, the polynomial solution F(x) of the difference equation $F(x+1) - F(x) = f(x)$ with initial condition $F(0) = 0$, is denoted by $F(x) = S_0^x f(x)$ and called the Bernoulli integral of f(x). F(x) is uniquely determined and $\deg F(x) = m + 1$. Moreover for $f(x) = a_0 + a_1 x + \ldots + a_m x^m$, the coefficients c_i of $F(x) = c_0 + c_1 x + \ldots + c_{m+1} x^{m+1}$ are linear functions of the a_i's: $c_i = \sum_{j=0}^{m} \lambda_i^j a_j$, $i = 1, \ldots, m+1$, with rational coefficients λ_i^j, which are expressible in term of Bernoulli numbers. Or we can express $F(x) = \sum_{i=0}^{m} a_i(\phi_{i+1}(x)/i+1)$, where $\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = x^2 - x$, $\phi_3(x) + x^3 - (3/2)x^2 + (1/2)x$, etc., are Bernoulli polynomials; i.e., ϕ_i are defined by $t(e^{xt} - 1)/(e^t - 1) = \sum_{n=0}^{\infty} (\phi_n(x)/n!)x^n$.

For a two variable polynomial $g(x,y) = \sum_{i=1}^{n} a_i(x)y^i =$

$\sum_{k=0}^{m} b_k(y)x^k$, we define $S_0^y g(x,y) = \sum_{i=1}^{n} a_i(x)(\emptyset_{i+1}(y)/(i+1))$ and $S_0^x g(x,y) = \sum_{k=0}^{m} b_k(y)(\emptyset_{k+1}(x)/(k+1))$. For polynomials of more variables, we may define Bernoulli integrals similarly.

Remark 2: In the proof of lemma 3, we have left the assertion that $g = y^{m-1}z$. occurs for a pair (f,g) satisfying the cocycle condition, i.e., there exists f so that $\Delta_y f = \Delta_z g = y^{m-1}$. The required f is $f = S_0^y(y^{m-1}) = (1/m)y^m +$ (lower degree terms).

Remark 3: If the generators t_x, t_y acts differently on $P_m(x,y)$ from the assumptions of Proposition 2, the method of method of Proposition 2 does not apply. For example if $t_x(f(x,y)) = f(x+1,x+y)$ and t_y acts as before, Δ_x no longer lowers degrees. It is roughly this difficulty that prevents applying the method of Proposition 2 to $H^2(N)$.

(V) Automorphisms

We now have all the machinery and technical results to begin the computation of $H^j(\Gamma,H^1(N))$. We first determine the structure of Aut(X) and conclude the structure of $\text{Aut}(X,N) = \{s \in \text{Aut}(X)/\ s(N) = N\}$. Note that by Aut(X) we mean the (continuous) Lie group automorphisms of X = H. Our method is similar to that of A. Weil [10], using the sequence $1 \to \text{center}(X) \to X \to X/\text{center} \to 1$. Here, checking our multiplication law, $\text{center}(X) \cong \mathbb{R}$ and $X/\text{center} \cong \mathbb{R}^2$. Letting $w \in \mathbb{R}^2$, $z \in \mathbb{R}$, and writing elements of X as (w,z), an automorphism s may be written $s(w,z) = (\sigma w, f(w) + S(0,z))$, where $\sigma \in \mathrm{Aut}(x/\text{center}) = GL_2(\mathbb{R})$ and $f: \mathbb{R}^2 \to \mathbb{R}$ is a continuous homorphism (we have not assumed s to be trivial on center(x)). For $\sigma = \binom{ab}{cd} \in GL_2(\mathbb{R})$, we set $\Phi(\sigma)(x,y,z) = (ax+by, cx+dy, (ad-bc)z)$. The multiplication on X, with X = H, the Heisenberg group, has been chosen so that $\Phi(\sigma)$ is an automorphism. We obtain an exact sequence $0 \to \mathbb{R}^{2*} \to \mathrm{Aut}(X) \to GL_2(\mathbb{R}) \to 1$, where $\mathbb{R}^{2*}$ is the dual of $\mathbb{R}^2$, from the sequence $0 \to \mathrm{Cont}(\mathbb{R}^2,\mathbb{R}) \to \mathrm{Aut}(X) \to$ Aut(X)/center) and from the splitting $\Phi : GL_2(\mathbb{R}) \to \mathrm{Aut}(X)$. We note that for $(u,v) \in \mathbb{R}^{2*}$, the corresponding automorphism is the inner automorphism given by conjugation by (-v, u, 0). Note that $s \in \mathrm{Aut}(X)$ acts by affine isomorphisms on X so that Aut(X) is cotained in the group of all affine isomorphisms, the latter group having been denoted by Aut(X) in Chapter 2.

From the above exact sequence for Aut(X), we obtain an exact sequence $0 \to \mathbb{Z}^{2*} \to \mathrm{Aut}(X,N) \to GL_2(\mathbb{Z}) \to 1$ for Aut(X,N). We denote by Γ any subgroup of Aut(X,N), and may consider $H^j(\Gamma, H^i(N, P_m))$. For the remainder of this Chapter, we take $\Gamma = \mathbb{Z}^{2*} \times SL_2(\mathbb{Z})$, which is a subgroup because of the splitting Φ, and is the largest subgroup of Aut(X,N) trivial on center(X). We denote by (C) the sequence $0 \to \mathbb{Z}^{2*} \to \Gamma \to SL_2(\mathbb{Z}) \to 1$.

It remains to determine $H^j(\Gamma, -)$, for which we use a spectral sequence on Γ. We are particularly interested in the case j = 1, i = 1. One term that occurs in the spectral sequence is $E_2^{1,0} = H^1(SL_2(\mathbb{Z}),\ H^0(\mathbb{Z}^2, H^1(N)))$. Since $\mathbb{Z}^2$ consists of inner automorphisms of N, it acts trivially on $H^i(N,P_m)$, so $E^{1,0} = H^1(SL_2(\mathbb{Z}),\ H^1(N))$.

Proposition 3: The isomorphism $H^1(N,P_m) \cong P_{m+1}(x,y)/P_m(x,y)$ of Proposition 2 holds as $SL_2(\mathbb{Z})$-modules, where $SL_2(\mathbb{Z})$ acts on the right by the dual of the $(m+1)^{\underline{st}}$ symmetric tensor representation.

Proof: We first note that for a cocycle in $H^1(N)$, which we have identified with a cocycle $h \in H^1(\mathbb{Z}^2, P_m(x,Y))$ (compatibly with the SL_2-action!) by the proof of Lemma 1 and the result of Lemma 3, $\sigma \in SL_2(\mathbb{Z})$ acts by $\sigma(h)c = \sigma \cdot h(\sigma^{-1}(c))$, where $c \in C_1(\mathbb{Z}^2)$. Since $h(\sigma^{-1}(c)) \in P_m(x,y,z)$ is a polynomial, where the variables x,y,z are to be viewed as coordinates on the group X, and the coordinates are elements of the dual space of X ($\cong R^3$), the outside action of σ is the dual action to the action of σ on X, i.e., $\sigma(f)(x,y,z) = f({}^t\sigma^{-1}(x,y,z))$.

To show $H^1(N) \cong P_{m+1}(x,y)/P_m(x,y)$ as an $SL_2(\mathbb{Z})$-module, we consider the action of the elements $\binom{11}{01}$ and $\binom{10}{11}$ (in fact these generate $SL_2(Z)$). By Proposition 1, and our initial remarks, we may identify the relevant cocycle h with the pair of polynomials (f,g), where $f = h(e_1 \otimes f_0)$ and $g = h(e_0 \otimes f_1)$. We now consider $\sigma = \binom{11}{01}$.

By the method used to determine the equation (*) in Lemma 3, we find that $\sigma^{-1}(e_1 \otimes f_0) = e_1 \otimes f_0$ and $\sigma^{-1}(e_0 \otimes f_1) =$

$e_0 \otimes f_1 - yx^{-1}e_1 \otimes f_0$. Thus, $(f,g) \to (\sigma \cdot h(e_1 \otimes f_0),$ $\sigma \cdot h(e_0 \otimes f_1) - \sigma(yx^{-1}) \cdot \sigma h(e_1 \otimes f_0)) = (\sigma f, \sigma g - y \ \sigma f)$, where σ acts by the outside described above, and y, as the generator of the group $\mathbb{Z}_y$ acts by $f(x,y) \to f(x,y+1)$.

Now let $G(x,y) \in P_{m+1}(x,y)$ be a polynomial of degree $\leq m+1$, and (f,g) be the corresponding cocycle with $f = \Delta_x G$, $g = \Delta_y G$. Our claim for the action of σ is that $(\sigma \Delta_x G, \ \sigma \Delta_y G - y \cdot \sigma \Delta_x G) = (\Delta_x \sigma G, \Delta_y \sigma G)$, where the action of σ on G is the dual of the natural action and the right hand side of the equation is the cocycle corresponding to σ G. For the dual action, we have ${}^t\sigma^{-1} = \left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right)$, so that for a polynomial $f(x,y)$, $(\sigma f)(x,y) = f(x-y,y)$. $\sigma \Delta_x G = \Delta_x \sigma G$ follows since these operators commute by the calculation $\sigma \Delta_x G(x,y) = \sigma [G(x+1,y) - G(x,y)] = G(x-y+1,y) - G(x-y,y) = \Delta_x[G(x-y,y))] = \Delta_x \sigma G$. The operators σ and Δ_y fail to commute by precisely the term $-y \sigma \Delta_x$ arising from the action of σ on our resolution for the cohomology of $\mathbb{Z}^2$. The calculation is as follows: $\sigma \Delta_y G(x,y) - y \cdot \sigma \Delta_x G = \sigma [G(x,y+1) - G(x,y)] - y [G(x-y+1,y) - G(x-y,y)] = G(x-y,y+1) - G(x,-y,y) - G(x-(y+1)+1,y+1) + G(x-(y+1),y+1) = G(x-(y+1),y+1) - G(x-y,y) = \Delta_y [G(x-y,y)] = \Delta_y \sigma G(x,y)$.

For $\tau = \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$, we have $\tau^{-1}(e_0 \otimes f_1) = e_0 \otimes f_1$ and $\tau^{-1}(e_1 \otimes f_0) = e_1 \otimes f_0 - xy^{-1}e_0 \otimes f_1$. Therefore the action of τ on a cocycle (f,g) is $\tau(f,g) = (\tau f - x \tau g, \tau g)$. A computation similar to the one above establishes that τ and

Δ_y commute, while $\Delta_x \tau G = \tau \Delta_x G - x \tau \Delta_y G$, so that the action of τ on $Z^1(\mathbb{Z}^2, P_m(x,Y))$ agrees with the prescribed action on $P_{m+1}(x,y)$.

Now since σ, τ generate $SL_2(\mathbb{Z})$, the $SL_2(\mathbb{Z})$ action on $Z^1(\mathbb{Z}^2, P_m(x,y))$ agrees with the dual action on $P_{m+1}(x,y)$. But $SL_2(\mathbb{Z})$ preserves degrees in $P_{m+1}(x,y)$, so we have that the dual action of $SL_2(\mathbb{Z})$ on $P_{m+1}(x,y)/P_m(x,y)$ agrees with the action of $SL_2(\mathbb{Z})$ on $H^1(\mathbb{Z}^2, P_m(x,Y))$ arising from the action of $SL_2(\mathbb{Z})$ as automorphisms on the manifold $N\backslash X$.

<u>Theorem</u>: The $E_2^{1,0}(C)$ term for the spectral sequence associated to the exact sequence (C) is given by $E^{1,0}(C) = H^1(SL_2(\mathbb{Z}), s^*_{m+1})$, where s^*_{m+1} is the dual of the $(m+1)^{\underline{st}}$ symmetric power of the natural representation. Thus $E^{1,0}(C)$ may be expressed in terms of automorphic forms.

<u>Proof</u>: For the subgroup $H^1_P(SL_2(\mathbb{Z}), s^*_{m+1})$ of "parabolic" classes in $H^1(SL_2(\mathbb{Z}), -)$, we have by Eichler's theorem $H^1_p(SL_2(\mathbb{Z}), s^*_{m+1}) = H^1_p(SL_2(\mathbb{Z}), s_{m+1})^* = (S_{m+3}\ (SL_2(\mathbb{Z})) \oplus \overline{S_{m+3}\ (SL_2(\mathbb{Z}))})^* = \overline{SL_{m+3}\ (SL_2(\mathbb{Z}))} \oplus SL_{m+3}\ (SL_2(\mathbb{Z}))$, where $S_{m+3}\ (\ )$ is the space of cusp forms of weight $(m+3)$ on the (compactified) Riemann surface $SL_2(\mathbb{Z}) \backslash$ (upper half plane) and $\overline{S_{2m+4}(\ )}$ is its complex conjugation. (Shirmura [7]). By results of I. Kra [3], the remainder of $H^1(SL_2(\mathbb{Z}), -)$ may be expressed in terms of Eisenstein series, completing our proof.

Having established that $SL_2(\mathbb{Z})$ acts as automorphisms on

N, we may easily prove Lemma 2. By our previous remarks we need only establish that $E_2^{0,1}$ is an irreducible $SL_2(Z)$-module.

Proof (of Lemma 2): For the spectral sequence in question, $E_2^{0,1} = H^0(\mathbb{Z}^2, H^1(\mathbb{Z}, P_m(x,y,z)) = H^0(\mathbb{Z}^2, P_m(x,y,x)/P_{m-1}(x,y,z))$ where $H^1(\mathbb{Z}, P_m(x,y,z))$ is as previously computed in section (II), and since $\mathbb{Z}$ is the center of N, $\mathbb{Z}^2$ acts by the natural coefficient action on $P_m(x,y,z)/P_{m-1}(x,y,z)$. Since $H^0(\mathbb{Z}^2, -)$ may be cannonically identified with the $\mathbb{Z}^2$-invarients, $H^0(\mathbb{Z}^2, P_m(x,y,z)/P_{m-1}(x,y,z)) = P_m(x,y)/P_{m-1}(x,y)$ (see section (II)). Note that the induced action of $SL_2(\mathbb{Z})$ on the factor group $\mathbb{Z}^2 = N/\text{center}(N)$ is natural, while the coefficient action on $P_m(x,y,z)/P_{m-1}(x,y,z)$ induces the dual of the m^{th} symmetric power of the natural representation $P_m(x,y)/P_{m-1}(x,y)$. Now since the identification of $H^0(\mathbb{Z}^2, -)$ is canonical, $SL_2(\mathbb{Z})$ acts on $E^{0,1} = P_m(x,y)/P_{m-1}(x,y)$ by the dual of the m^{th} symmetric power of the natural representation, hence irreducibly.

We observe that the comparative difficulty of the proof of Proposition 3 results from the fact that our identification of $H^1(\mathbb{Z}^2, -)$ is not canonical: We have picked generators $e_1 \otimes f_0$ and $e_0 \otimes f_1$ for the chain group $C_1(\mathbb{Z}^2)$ as in Proposition 1 of section (I). A similar difficulty arises for the bar resolution: We must pick generators for $\mathbb{Z}^2$ to identify $H^1(\mathbb{Z}^2)$. If we pick generators t_x and t_y as in Proposition 2, a similar result follows upon identifying a (inhomogeneous) cocycle with its values $f = h(t_x)$, $g = h(t_y)$.

However, these generators are not preserved by non-trivial elements of $SL_2(\mathbb{Z})$.

We also note that the $E_2^{0,1}$ (C)-term for $H^j(\Gamma, -)$ described in our theorem actually occurs in the resulting cohomology, since this sequence degenerates at the E_2 -term. This remark follows from the observation that $SL_2(\mathbb{Z})$ has a free subgroup of finite index (the commutator subgroup) and the fact our coefficients are without torsion. We have not yet computed the other $E_2^{p,q}$ terms for $H^j(\Gamma, H^1(N))$. Also we have $H^0(N) = H^3(N) = \emptyset$, so $H^j(\Gamma, H^i(N))$ provides trivial results for these i. The case i = 2 also remains to be determined. Note that these cohomology groups may be considered as $E_2^{p,q}$ terms for a group G described by a sequence $1 \to N \to G \to \Gamma \to 1$. This G may be considered as the fundamental group of a certain 7-dimensional manifold, whose cohomology is of interest for our proposed analogue of the Sato-Shintani zeta function.

Chapter 5: Hecke Operators

(I) The Local Affine Hecke Ring $\mathcal{H}(M)$

Fix a local affine manifold M. Let M' be any other local affine manifold for which we may select two local affine covering maps $f,g: M' \to M$ of finite degree. We consider a category $\Rightarrow(M)$ whose objects are such triples (M', f, g), where a morphism between triples (M_1', f_1, g_1) and $M_2', f_2, g_2)$ is a local affine map $\varphi : M_1' \to M_2'$ so that $f_1 = f_2 \circ \varphi$ and $g_1 = g_2 \circ \varphi$.

On the objects of $\Rightarrow(M)$, we define an equivalence relation by $(M_1', f_1, g_1) \sim (M_2', f_2, g_2)$ if and only if there exists a morphism $\varphi : M_1' \to M_2'$ so that φ is a local isomorphism. We consider the quotient of the objects of $\Rightarrow(M)$ by this equivalence relation, and obtain a set which we denote $(\Rightarrow(M)/\sim)$. From the set $(\Rightarrow(M)/\sim)$ we form an abelian group $\mathbb{Z}(\Rightarrow(M)/\sim)$ consisting of finite formal sums $\sum_{i=1}^{k} m_i A_i$ with $m_i \in \mathbb{Z}$, $A_i \in (\Rightarrow(M)/\sim)$ with addition performed formally under the provision that $(m_1 A) + (m_2 A) = (m_1 + m_2)A$ for a fixed $A \in (\Rightarrow(M)/\sim)$ and $m_1, m_2 \in \mathbb{Z}$.

Let $A, B \in (\Rightarrow(M)/\sim)$ be represented by triples (M_1', f_1, g_1) and (M_2', f_2, g_2), respectively. We define another addition $A \dotplus B$ by letting $A \dotplus B$ be the equivalence class of the triple $(M_1' \amalg M_2', f_1 \amalg f_2, g_1 \amalg g_2)$, where $M_1' \amalg M_2'$ is the disjoint union of M_1' and M_2' with $f_1 \amalg f_2(m_i') = f_i(m_i')$ and $g_1 \amalg g_2(m_i') = g_i(m_i')$ whenever $m_i' \in M_i'$. In the abelian group $\mathbb{Z}(\Rightarrow(M)/\sim)$ we form the subgroup S generated by elements of the form $1(A \dot{+} B) - 1(A) - 1(B)$ where A and B range over all elements of $(\Rightarrow (M)/\sim)$. Finally, we consider the quotient group $\mathcal{H}(M) = \mathbb{Z}(\Rightarrow (M)/\sim)/S$. Note that $\mathcal{H}(M)$ is generated by the equivalence classes of triples (M', f, g) with M' connected, since taking the quotient by the subgroup S has the effect of identifying a triple having M' disconnected with the " $\dot{+}$ " sum of its connected components with the appropriate maps.

We now define the structure of a ring on $\mathcal{H}(M)$. We also denote this ring $\mathcal{H}(M)$ and call it the local affine Hecke ring of M. To define a product on $\mathcal{H}(M)$, it suffices to define the product of generators $A \cdot B$, since for formal sums $s = \sum_i m_i A_i$, $t = \sum_i n_j B_j$, we set $s \cdot t = \sum_{i,j} m_i n_j A_i . B_j$. For generators A,B given as the cosets of equivalence classes with representatives M_1', f_1, g_1) and M_2', f_2, g_2) respectively, we let $A \cdot B$ be the coset of the equivalence class of $(M_1' \underset{g_1=f_2}{\times} M_2', f_1 \circ P_{r_1}, g_2 \circ P_{r_2})$, where $M_1' \underset{g_1=f_2}{\times} M_2'$ is the fibered product of M_1' and M_2' over M along the maps g_1, f_2, and $P_{r_1} : M_1' \underset{g_1=f_2}{\times} M_2' \to M_1'$ and $P_{r_2} : M_1' \underset{g_1=f_2}{\times} M_2' \to M_2'$ are the canonical projections. We note that this definition is independent of choices of representatives since we may identify the cosets A and B with an equivalence class of triples with M_1' and M_2'

connected and then equivalences φ_1 and φ_2 induce an equivalence $\varphi_1 \underset{g_1=f_2}{x} \varphi_2$. Note also that even when M_1 and M_2 are connected, their fiber product may not be.

<u>Proposition</u>: For the product defined above, ℋ (M) is an associative ring with identity.

<u>Proof</u>: The identity is the coset of the equivalence class of the triple (M, id, id), i.e., M' = M and f = g = the identity map. For representatives, we note that for a triple (M', f,g), $M \underset{id=f}{x} M' = M'$ and that since $P_{r_1} = f$, $P_{r_2} = id_{M'}$, $id_M \circ P_{r_1} = id_M \circ f = f$, while $g \circ P_{r_2} = g \circ id_{M'} = g$, where id_M and $id_{M'}$ are the identity maps of M and M' respectively.

For associativity, it suffices to check that for triples (M_1', f_1, g_1), (M_2', f_2, g_2) and $M_3', f_3, g_3)$, the fiber products in the appropriate orders with their maps agree. For this we note that $M_1' \,x_{g_1=f_2} M_2') \underset{g_2 \circ Pr_2=f_3}{x} M_3' = \{((x,y),z) \,/\, (x,y) \in M_1' \underset{g_1=f_2}{x} M_2',\ z \in M_3' \text{ and } g_2(y) = g_2 \circ Pr_2(x,y) = f_3(z)\} =$

$\{(x,y,z) \,/\, x \in M_1',\ y \in M_2',\ z \in M_3' \text{ and } g_1(x) = f_2(y),\ g_2(y) = f_3(z)\} = \{(x, (y,z)) \,/\, x \in M_1',\ (y,z) \in M_2' \underset{g_2=f_3}{x} M_3', \text{ and }$

$g_1(x) = f_2(y) = f_2 \circ Pr_2(y,z)\} = M_1' \underset{g_1=f_2 \circ Pr_2}{x} (M_2' \underset{g_2=f_3}{x} M_3'),$

where the projections Pr_i are the canonical maps for the fiber products being considered, so that we need only check the maps. But, observing that these are the appropriate maps to be checked, we have $f_1 \circ Pr_1 \circ Pr_{1,2} = f_1 \circ Pr_1$ and $g_3 \circ Pr_3 = g_3 \circ Pr_3 \circ Pr_{2,3}$, where for the left hand side of these equation $Pr_1, Pr_{1,2}$ and Pr_3 are the projections associated with $(M_1' \underset{g_1=f_2}{x} M_2') \underset{g_2 \circ Pr_2 = f_3}{x} M_3'$, and on the right hand sides, Pr_1, Pr_3 and $Pr_{2,3}$ are the projections associated with $M_1' \underset{g_1=f_2 \circ Pr_2}{x} (M_2' \underset{g_2=f_3}{x} M_3')$.

(II) The Action of ℍ (M) on the Sheaf Cohomology $H^p(M, p^{(m)})$

As a step towards an explanation of the choice of the term "local affine Hecke ring" for ℍ (M), we define an action of ℍ (M) on $H^p(M, p^{(m)})$. Note that our definitions work as well for sheaves $\mathcal{S}$ and $\mathcal{S}'$ on M and M' compatible with the maps f and g. In section (III), we restrict to the case where $M = \Gamma\backslash X$ is the quotient of a domain X and consider special triples (M', f, g) corresponding to the double cosets $\Gamma \alpha \Gamma$ belonging to the classical Hecke ring of Γ.

We first define an action of (M', f, g) ∈ ℍ (M). Note that for any local affine covering map $g: M' \to M$, we may define a map $g^*: H^p(M,p^{(m)}) \to H^p(M',p^{(m)})$ by $[g^*(w)] (U_0' \cap \ldots \cap U_p') = w(g(U_0') \cap \ldots \cap g(U_p'))\circ g$, for $U_0', \ldots, U_p'$ open sets in M' with $U_0' \cap \ldots \cap U_p' \neq \emptyset$, where $w \in H^p(M,p^{(m)})$. To define a "push-forward" $f_!: H^p(M', p^{(m)}) \to H^p(M, p^{(m)})$, for $f: M' \to M$ a local affine covering map of finite degree, we first select a special cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M compatible with the covering $f: M' \to M$.

A set S in M is called f-copiable if there exist a connected set $B \subset M$, with properties: (1) $B \supset A$, (2) $f^{-1}(B)$ is a disjoint union of connected sets $B_1, B_2, \ldots, B_d$ in M' such that $f / _{B_i}$ is an isomorphism of B_i onto B for each i. In this case, $f^{-1}(A) = \bigcup_i A_i$ where $A_i \subset B_i$. We call A_i the "components" of $f^{-1}(A)$. A covering $\{U_\alpha\}_{\alpha \in A}$ of M is called a f-copiable covering if every finite union $U_{\alpha_0} \cup \ldots \cup U_{\alpha_j}$ is f-copiable when $U_{\alpha_0} \cap \ldots \cap U_{\alpha_j} \neq \emptyset$.

Lemma: Let $f: M' \to M$ be a local affine covering map of finite degree. Then M has a f-copiable covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$.

Proof: This is a standard result from topology. See, for example, A. Weil [9].

Having fixed a cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M as above and noting that such covers are co-final among all coverings of M, we consider a non-empty intersection $U_0 \cap \ldots \cap U_p$ with $U_0, \ldots, U_p \in \mathfrak{U}$. By the choice of the cover $\mathfrak{U}$, we may index the inverse images $f^{-1}(U_j) = \bigcup_{i=1}^{d} U_{j,i}$ for $j = 0, \ldots, p$, so that $U_{0,i} \cap U_{1,i} \cap \ldots \cap U_{p,i} \neq \emptyset$ for all i and $U_{0,i} \cap \ldots \cap U_{j,k} \cap \ldots \cap U_{p,i} = \emptyset$ whenever $k \neq i$. We may now define

$f_!: H^p(M', p^{(m)})\ H^p(M, p^{(m)})$ by setting

$$[f_!(w')]\ (U_0 \ldots U_p) =$$

$$\sum_{i=1}^{d} w'(U_{0,i} \cap \ldots \cap U_{p,i}) \circ (f|_{U_{0,i} \ldots U_{p,i}})^{-1},$$ where $w' \in H^p(M', p^{(m)})$, since $f|_{U_{0,i} \cap \ldots \cap U_{p,i}}$ is an isomorphism.

Note that $(f|_{U_{0,i} \cap \ldots \cap U_{p,i}})^{-1}$ maps the section $w'(U_{0,i} \cap \ldots \cap U_{p,i}) \in \Gamma'(U_{0,i} \cap \ldots \cap U_{p,i}, p^{(m)})$ to $\Gamma(U_0 \cap \ldots \cap U_p, p^{(m)})$, where $\Gamma'(\ )$ denotes sections of the sheaf $p^{(m)}$ on M' and $\Gamma(\ )$, sections of the sheaf $p^{(m)}$ on M.

We remark that the formulas given for g^* and $f_!$ hold for cocycles, and may easily be seen to give well defined maps on cohomology classes. Finally we may define the action of (M', f, g) on $H^p(M, p^{(m)})$ by $[(M', f, g)]\ w = f_!(g^*(w))$.

To show that our action is well defined for the equivalence class of (M', f, g) in $(\Rightarrow (M)/\sim)$, suppose $(M', f, g) \sim (M_1', f_1, g_1)$, i.e., that there exists a local affine isomorphism $\varphi : M' \to M_1'$ so that $f = f_1 \circ \varphi$ and $g = g_1 \circ \varphi$. The caculation $f_! \circ g^* = (f_1 \circ \varphi)_! \circ (g_1 \circ \varphi)^* = f_{1!}\ \varphi_! \circ \varphi^*\ g_1^* = f_{1!} g_1^*$ may be easily verified. We note that by our definition, the fact that $\varphi_! \circ \varphi^* =$ identity depends upon φ being an isomorphism, so that the degree of φ is one. Similarly, the formula $(f_1 \amalg f_2)_!(g_1 \amalg g_2)^* = f_1! g_2^*$ shows that the generating elements $1(A \dotplus B) - 1(A) - 1(B)$ of $S \subset \mathbb{Z}(\Rightarrow (M)/\sim)$ act trivially on $H^p(M, p^{(m)})$, so we obtain a well defined action of generators for $\aleph(M)$.

We now define a map $\rho_2: \mathcal{H}(M) \to \mathrm{End}(H^p(M,p^{(m)}))$ by setting $\rho_2(\Sigma\, n_i A_i) = \Sigma\, n_i\, \rho_2(A_i)$, where by the above remarks $\rho_2(A_i)$ is well defined by the action of a representative $(M_i^!, f_i, g_i)$ for the equivalence class defining the coset A_i.

<u>Proposition</u>: The map $\rho_2: \mathcal{H}(M) \to \mathrm{End}(H^p(M,p^{(m)}))$ defining the action of $\mathcal{H}(M)$ on $H^p(M,p^{(m)})$ is a ring homomorphism.

<u>Proof</u>: ρ_2 is additive by definition. To show that ρ_2 is multiplicative, i.e., $\rho_2(A\cdot B) = \rho_2(A)\cdot \rho_2(B)$ whenever $A, B \in \mathcal{H}(M)$, we may take representatives $(M_1^!, f_1, g_1)$ and $(M_2^!, f_2, g_2)$ for A and B respectively. For representatives so chosen, $A\cdot B$ may be represented by $(M_1^! \underset{g_1=f_2}{\times} M_2^!, f_1 \circ Pr_1, g_2 \circ Pr_2)$ (cf, the definition of $A\cdot B$).

By the definition of ρ_2, we must show $(f_1 \circ pr_1)_! \circ (g_2 \circ Pr_2)^* = (f_{1!} \circ g_1^*) \circ (f_{2!} \circ g_2^*)$. We check that this is the same as $f_{1!} \circ Pr_{1!} \circ Pr_2^* \circ g_2^* = f_{1!} \circ g_1^* \circ f_{2!} \circ g_2^*$. If $Pr_{1!} \circ Pr_2^* = g_1^* \circ f_{2!}$ we are done.

<u>Lemma</u>: Let $M_1^! \underset{g_1=f_2}{\times} M_2^!$ be the fiber product of $M_1^!$ and $M_2^!$ over M along the maps $g_1: M_1^! \to M$ and $f_2: M_2^! \to M$ (of finite degree), with the canonical projections $Pr_1: M_1^! \underset{g_1=f_2}{\times} M_2^! \to M_1^!$ and $Pr_2: M_1^! \underset{g_1=f_2}{\times} M_2^! \to M_2^!$, then $Pr_{1!} \circ Pr_2^* = g_1^* \circ f_{2!}$.

<u>Proof</u>: We first select coverings of these manifolds compatibly with their maps. In particular, we require that

a non-empty intersection of open sets from the covering of M (and of M_1') lift uniquely to a collection of intersections of open sets for the covering of M_2' under f_2 (and of the covering of $M_1' \times_{g_1=f_2} M_2'$ under Pr_1) as in the definition of $f_{2!}(Pr_{1!})$; and that, for all maps, the image of an intersection of p open sets for these covers in an intersection of p open sets for the cover of the range manifold.

Having fixed coverings satisfying our requirements we consider $w \in H^p(M_2', p^{(m)})$ and $S = U_0 \cap \ldots \cap U_p \neq \emptyset$ an intersection of open sets selected from the covering of M_1'. Now, $[(Pr_{1!} \circ Pr_2^*)w]\ (S) = \sum_i [(Pr_2^*(w)]\ (S_i) \circ (Pr_1' \mid_{S_i})^{-1} = \sum_i w(Pr_2(S_i)) \circ (Pr_1 \mid_{S_i})^{-1}$, and $[(g_1^* \circ f_{2!})w]\ (S) = [f_{2!}(w)]\ g_1(S) = \sum_j w(T_j) \circ (f_2 \mid T_j)^{-1}$, where $Pr_1^{-1}(S) = \bigcup_i S_i$ and $f_2^{-1}(g_1(S)) = \bigcup_j T_jA$ are decompositions into disjoint "components", where "components" are in the sense of the definitions preceding the definition of $f_!$. If we can show that $i = j$ and that $\{T_1, \ldots, T_j\} = \{Pr_2(S_1), \ldots, Pr_2(S_i)\}$, we will be done, since for $T_s = Pr_2(S_r)$ we have $w(Pr_2(S_r)) \circ (Pr_1 \mid_{S_r})^{-1} = w(T_j) \circ (f_2 \mid_{T_s})^{-1}$ by the commutivity of the diagram defining the fiber product ($g_1 \circ Pr_1 = f_2 \circ Pr_2$ implies that $(f_2 \mid_{T_s})^{-1}\ [g(S)] = Pr_2\ [(Pr_1 \mid_{S_r})^{-1}(S)]$).

To establish our remaining claims: that $i = j$ and $\{T_1, \ldots, T_j\} = \{Pr_2(S_1), \ldots, Pr_2(S_i)\}$; we consider the special nature of the projections Pr_1 and Pr_2 of the fiber product. Fix $x_0 \in M_1'$ and note that $\deg(Pr_1) =$
$\# \{z \in M_1' \times_{g_1=f_2} M_2' \ / \ Pr_1(z) = x_0\} = \# \{(x,y) \in M_1' \times M_2' \ / \ x = x_0$

and $g_1(x_0) = f_2(y)\} = \# \{y \in M_2' \ / \ f_2(y) = g(x_0)\} =$
$\# \{y \in M_2' \ / \ y \in f_2^{-1}(g(x_0))\} = \deg(f_2)$, by the definitions of degree and the fiber product, where for a set A, $\#A$ denotes the cardinality of the set A. By our choice of covers with open sets sufficiently small, we have $i = \deg Pr_1 = \deg f_2 = j$. We denote by d_2 the degree $d_2 = \deg Pr_1 = \deg f_2$; and, note that a similar proof establishes $\deg (g_1) = \deg(Pr_2)$.

Finally, we note that Pr_2 must not collapse the fiber of Pr_1. For this we let $x \in M_1'$, and $z_1, z_2 \in Pr_1^{-1}(x)$, $z_1 \neq z_2$, then $Pr_2(z_1) \neq Pr_2(z_2)$ since we have for $z_1 = (x, y_1)$ and $z_2 = (x,y_2)$, $Pr_2(z_i) = y_i$ and $z_1 \neq z_2$ implies $y_1 \neq y_2$. We again remark that the same holds for our sufficiently small open sets. Now for $x \in S = U_0 \cap \ldots \cap U_p$ let $Pr_1^{-1}(x) = \{z_1, \ldots, z_{d_2}\}$, then for $Pr_2(z_i) = y_i$ we have $y_i \neq y_k$ for $i \neq k$. Since f_2 is of degree d_2 and $f_2(y_1) = f_2(y_2) = \ldots = f_2(y_{d_2}) = g_1(x)$, we have $f^{-1}(g(x)) = \{y_1, \ldots, y_{d_2}\}$. Applying this argument to our sufficiently small open sets, we have for $f^{-1}(g(S)) = \bigcup_{i=1}^{d2} T_i$ and $Pr_1^{-1}(S) = \bigcup_{i=1}^{d2} S_i$ that

$\{T_1, \ldots, T_{d_2}\} = \{Pr_2(S_1), \ldots, Pr_2(S_{d_2})\}$. Since this was the sole remaining claim necessary to establish our lemma, we are done with its proof; and, as remarked above, our lemma suffices to prove our proposition.

Our proposition establishes that $\mathcal{H}(M)$ acts as a ring of operators on $H^p(M,p^{(m)})$. In section (III), we consider an action ρ_1 of the ring of double cosets $\Gamma\alpha\Gamma$ on $H^p(M,p^{(m)})$, in the case $M = \Gamma\backslash X$, arising from their action on $H^p(\Gamma,P_m)$ and our isomorphism $H^p(M,p^{(m)}) = H^p(\Gamma,P_m)$ in this case.

(III) The Relation of $\mathcal{H}(M)$ to the Classical Hecke Ring

For this section, we assume that $M = \Gamma\backslash X$ is the quotient of a domain $X \subset R^n$ by a discrete group Γ without finite subgroups acting freely and properly discontinuously as in the theorem of Chapter 1. By our theorem, we have $H^p(\Gamma\backslash X,p^{(m)}) \cong H^p(\Gamma, P_m)$. Since we have $\Gamma \subset Aut(X)$, we may take the commensurator $\tilde{\Gamma}$ of Γ in $Aut(X)$, and for a semi-group Δ, so that $\Gamma \subset \Delta \subset \tilde{\Gamma}$, may form the Hecke ring $\mathcal{R}(\Gamma,\Delta)$ as in Shirmura [7].

We let the ring $\mathcal{R}(\Gamma,\Delta)$ act on the group cohomology $H^p(\Gamma,-)$ in a similar manner to that used in Kuga [4], page 14, Vol. II, and obtain a representation $\rho_1\colon \mathcal{R}(\Gamma,\Delta) \to End(H^p(\Gamma\backslash X,p^{(m)}))$. We propose to define a map $\varphi\colon \mathcal{R}(\Gamma,\Delta) \to \mathcal{H}(M)$, for $M = \Gamma\backslash X$, so that for $\Gamma\alpha\Gamma \in \mathcal{R}(\Gamma,\Delta)$,

$\rho_2(\varphi(\Gamma\ \alpha\ \Gamma)) = \rho_1(\Gamma\ \alpha\ \Gamma)$, where ρ_2 is the action defined in section (II).

For the definition of $\rho_1(\Gamma\ \alpha\ \Gamma)$, we first define an action of $\Gamma\ \alpha\ \Gamma \in \mathcal{R}\ (\Gamma,\Delta)$ on a resolution for $H^p(\Gamma,-)$ of the form used to prove $H^p(\Gamma\backslash X,\ -) = H^p(\Gamma,\ -)$.

For a double coset $\Gamma\ \alpha\ \Gamma \in \mathcal{R}\ (\Gamma,\Delta)$, we take a left coset decomposition $\Gamma\ \alpha\ \Gamma = \bigcup_{i=1}^{d} \Gamma\ \alpha_i$, where $\alpha_i = \alpha\ g_i$ for distinct coset representatives g_i of $\Gamma/\Gamma \cap \alpha^{-1}\Gamma\ \alpha$. When $\gamma \in \Gamma$ we use our decomposition to define $\tau_i(\gamma) \in \Gamma$ as in Kuga [4], so that $\alpha_i\gamma = \tau_i(\gamma)\ \alpha\ _{i^\gamma}$, where $\{1^\gamma,\ 2^\gamma,\ \dots\ ,d^\gamma\}$ is a permutation of $\{1,\ 2,\ \dots\ ,d\}$.

We pick a Γ-invarient metric d on X so that $\min_{\substack{1\neq\gamma\in\Gamma \\ x\in X}} d(x,\gamma(x)) = \epsilon > 0$; the existence of such metric is proved in Appendix II. And we define an "index set" A, by A = {open sets U in X/ diam $U < \epsilon$, and diam $(\alpha_i U) < \epsilon$, $i = 1,\ \dots\ ,d$}. The condition: "diam $U < \epsilon$; diam $(\alpha_i U) < \epsilon$ $(i = 1,\ \dots\ ,d)$", is equivalent with the condition "diam $U < \epsilon$, and diam $(\xi\ U) < \epsilon$ for all $\xi \in \Gamma\ \alpha\ \Gamma$", as we can see easily. So our condition is Γ-invarient, and Γ acts on the set A. The set A itself is a covering of X, so we also denote A by $\mathfrak{U}$; and an index $\nu \in A$ considered as an open set in X is also denoted by U_ν, i.e., $\mathfrak{U} = A = \{\nu\} = \{U_\nu\}_{\nu\ \in\ A}$. Also we take another covering $\mathfrak{U}'$ of X, defined by $\mathfrak{U}' = \{$open sets U' in X $\mid$ dim $U' < \epsilon\}$. We define cocheim groups $C^p(\Gamma,\mathfrak{u},P_m)$, $C^p(\Gamma,\mathfrak{u}',P_m)$, both gives the same $H^p(\Gamma,P_m)$.

When $h \in C^p(\Gamma,\mathfrak{u},P_m)$ is a cocyle, we may define

$$[\Gamma\alpha\Gamma(h)]\, U_0 \cap \dots \cap U_p = \sum_{i=1}^{d} \alpha_i^{-1} h(\alpha_i(U_0) \cap \dots \cap \alpha_i(U_p))$$ by

the above. The proof that this action takes cocycles to cocycles, and commutes with the coboundary operator δ, so that it gives a well defined action on cohomology classes, may be directly copied from the proof on page 14 of Kuga [4] for the action used there.

We use cover $\mathfrak{v} = \{V_\alpha\}_{\alpha \in \Gamma\backslash A}$ of $M = \Gamma\backslash X$, where A is the index set used for the cover $\mathfrak{u}$ of X and $V_\alpha = Pr_M(U_\alpha)$. Then the isomorphism ϕ: $H^p(M,P^m) \to H^p(\Gamma,P_m)$ is given by $[\phi(w)](U_0 \cap \dots \cap U_p) = w(V_0 \cap \dots \cap V_p) \circ \pi|_{U_0 \cap \dots \cap U_p}$ where $V_i = pr_M(U_i)$. Now we define the action ρ_1 of $\Gamma\alpha\Gamma \in \mathcal{R}(\Gamma,\Delta)$ on $H^p(M,P^{(m)})$ by the commutativity of the diagram:

$$\begin{array}{ccc} H^p(M,P^{(m)}) & \xrightarrow{\ \phi\ } & H^p(\Gamma,P_m) \\ \downarrow \rho_1(\Gamma\alpha\Gamma) & & \downarrow \Gamma\alpha\Gamma \\ H^p(M,P^{(m)}) & \xrightarrow{\ \phi\ } & H^p(\Gamma,P_m) \end{array},$$

i.e. for a cochain w of $H^p(M,P^{(m)})$, we define, a cochain $\rho_1(\Gamma\alpha\Gamma)w$, by $(\rho_1(\Gamma\alpha\Gamma)w)(V_0 \cap \dots \cap V_p) = (\sum_i \alpha_i h(\alpha_i V_0, \dots, \alpha_i U_p)) \circ (\pi|_{U_0 \cap \dots \cap U_p})^{-1}$ where $U_0, \dots, U_p \in \mathfrak{u}$, with $Pr_M(U_i) = V_i$, $U_0 \cap \dots \cap U_p \neq \phi$, $h = \phi(w)$. We can show easily that this is well defined, i.e., the right side is independent of the choice of copies U_i of V_i with $U_0 \cap \dots \cap U_p \neq \phi$.

For $\Gamma \alpha \Gamma \in \mathcal{R}(\Gamma,\Delta)$, let $M' = \Gamma'\backslash X$ with $\Gamma' = \alpha \Gamma \alpha^{-1} \cap \Gamma$. Corresponding to the map $\alpha^{-1}: X \to X$, we have a map $\overline{\alpha^{-1}}$: $\Gamma'\backslash X \to \alpha^{-1}(\Gamma')\backslash X$, where $\alpha^{-1}(\Gamma') = \Gamma \cap \alpha^{-1}(\Gamma') = \Gamma \cap \alpha^{-1} \Gamma \alpha$. We define $\bar{\varphi}: \mathcal{R}(\Gamma,\Delta) \to \Rightarrow (M)$ on generators $\Gamma \alpha \Gamma$ by setting $\bar{\varphi}(\Gamma \alpha \Gamma) = (M', f, g)$, with $M' = \Gamma'\backslash X$ as above, $g = Pr$, where $Pr: M' \to M$ is the projection induced by the inclusion $\Gamma' = \alpha \Gamma \alpha^{-1} \cap \Gamma \subset \Gamma$ and $f = Pr_\alpha \circ \overline{\alpha^{-1}}$, where $\overline{\alpha^{-1}}$ is as defined above and $Pr_\alpha : \alpha^{-1}(M) = \alpha^{-1}(\Gamma')\backslash X \to M$ is the projection induced by the inclusion $\alpha^{-1}(\Gamma') = \Gamma \cap \alpha^{-1} \Gamma \alpha \subset \Gamma$. Then the map $\varphi : \mathcal{R}(\Gamma,\Delta) \to \mathcal{H}(M)$ is defined by letting $\varphi(\Gamma \alpha \Gamma)$ be the coset determined by the quivalence class of $\bar{\varphi}(\Gamma \alpha \Gamma) = (\Gamma'\backslash X, Pr_\alpha \circ \overline{\alpha^{-1}}, Pr)$, and setting $\varphi(\Sigma n_i \Gamma \alpha_i \Gamma) = \Sigma n_i \varphi(\Gamma \alpha_i \Gamma)$ for an arbitrary finite formal sum $\Sigma n_i \Gamma \alpha_i \Gamma \in \mathcal{R}(\Gamma,\Delta)$.

Theorem: Let $M = \Gamma\backslash X$, $\varphi: \mathcal{R}(\Gamma,\Delta) \to \mathcal{H}(M)$, and ρ_1, ρ_2 be as above. Then $\rho_1(\Gamma \alpha \Gamma) = \rho_2(\varphi(\Gamma \alpha \Gamma))$ as operators on $H^p(M,p^{(m)})$.

Proof: Notations are as above. Then we need only to show that: $((f!\circ g^*)(w)\, V_0 \cap \dots \cap V_p) \circ Pr_M|_{U_0 \cap \dots \cap U_p} = \sum_i \alpha_i^{-1} h(\alpha_i U_0, \dots, \alpha_i U_p)$. The definition of $h = \phi(w)$, implies $w(V_0 \cap \dots \cap V_p) \circ Pr_M|_{U_0 \cap \dots \cap U_p} = h(U_0, \dots, U_p)$.

By the definition of α_i $(i=1, \dots, d)$, we have $f^{-1}(Pr_M(x)) = \{Pr_{M'}\alpha_1^{-1}(x), \dots, Pr_{M'}\alpha_\alpha^{-1}(x)\}$, for $x \in X$. So $f^{-1}(V_0 \cap \dots \cap V_p) = \bigcup_{i=1}^{d} Pr_{M'}(\alpha_i(U_0 \cap \dots \cap U_p))$, and; denoting

$\alpha_i(U_j)$ by $U_{i,j}$, $(f|_{U_{i,0}\cap \dots \cap U_{i,p}})^{-1} \circ Pr_M|_{U_0\cap \dots \cap U_p}$

$= Pr_{M'} \circ \alpha_i|_{U_0\cap \dots \cap U_p}$. So we have $[(f_! \circ g^*)w]\ V_0\cap \dots \cap V_p$

$$= \sum_{i=1}^{d} [g^*(w)](V_{i0}\cap \dots \cap V_{i,p}) \circ (f|V_{i,0}\cap \dots \cap V_{i,p})^{-1}$$

$$= \sum_i w(g(V_{0,0}) \cap \dots \cap g(V_{p,i})) \circ g \circ (f|V_{i,0}\cap \dots \cap V_{i,p})^{-1}$$

where $V_{i,j} = Pr_{M'}(U_{i,j}) = Pr_{M'}(\alpha_i U_j)$. Now put

$$F_i = w(g(V_{i,0}) \cap \dots \cap g(V_{i,p})) \circ g \circ Pr_{M'}|_{\alpha_i(U_0\cap \dots \cap U_p)}$$

$$= w(g(V_{i,0}) \cap \dots \cap g(V_{i,p})) \circ Pr_M|\ \alpha_i(U_0\cap \dots \cap U_p)$$

$$= h(\alpha_i U_0, \dots , \alpha_i U_p).$$

So $\sum_i \alpha_i^{-1} h(\alpha_i U_0, \dots , \alpha_i U_p)$

$$= \sum \alpha_i^{-1} F_i = \sum F_i \circ \alpha_i$$

$$= \sum w(g(V_{i,0}) \cap \dots \cap g(V_{i,p})) \circ g \circ Pr_{M'}|_{\alpha_i(U_0\cap \dots \cap U_p)} \circ \alpha_i$$

$$= \sum w(g(V_{i,0}) \cap \dots \cap g(V_{i,p})) \circ (f|_{V_{i,0}\cap \dots \cap V_{i,p}})^{-1} \circ$$

$Pr_M|_{U_0\cap \dots \cap U_p} = [(f_! \circ g^*)w](V_0\cap \dots \cap V_p) \circ Pr_M|_{U_0\cap \dots \cap U_p}$,

as was to be shown.

<u>Corollary</u>: $\mathcal{R}(\Gamma,\Delta)$ and $\mathcal{H}(M)$ are compatible as rings of operators on $H^p(M,p^{(m)})$.

<u>Proof</u>: Let $\Gamma\alpha\Gamma$, $\Gamma\beta\Gamma \in \mathcal{R}(\Gamma,\Delta)$, then we mean that $\rho_2(\varphi(\Gamma\alpha\Gamma\cdot\Gamma\beta\Gamma)) = \rho_2(\varphi(\Gamma\alpha\Gamma))x\varphi(\Gamma\beta\Gamma)))$, where $\Gamma\alpha\Gamma\cdot\Gamma\beta\Gamma$ is the product in $\mathcal{R}(\Gamma,\Delta)$ and $\varphi(\Gamma\alpha\Gamma)x\varphi(\Gamma\beta\Gamma)$ is the product in $\mathcal{H}(M)$. Note that we may modify the proof of the preceding theorem by taking the

cover $\mathfrak{U} = \{U \subset X/U$ is an open set with diam $U < \epsilon$, diam $\alpha_i(U) < \epsilon$ and diam $\beta_j(U) < \epsilon\}$, so that the same resolution works for both $\Gamma \alpha \Gamma$ and $\Gamma \beta \Gamma$. Since ρ_1 is a ring homomorphism we have $\rho_1(\Gamma \alpha \Gamma \cdot \Gamma \beta \Gamma) = \rho_1(\Gamma \alpha \Gamma)\ \rho_1(\Gamma \beta \Gamma)$. Substituting in the result of our theorem to both sides of this equation, we have $\rho_2\ \varphi\ (\Gamma \alpha \Gamma \cdot \Gamma \beta \Gamma) = \rho_2\ \varphi\ (\Gamma \alpha \Gamma) \circ \rho_2\ \varphi\ (\Gamma \beta \Gamma) = \rho_2\ (\varphi\ (\Gamma \alpha \Gamma) \times \varphi\ (\Gamma \beta \Gamma))$, where the last equality holds by the proposition of section (II).

Remark: We state this corollary as a substitute for asserting that $\varphi : \mathcal{R}\ (\Gamma, \Delta) \to \mathcal{H}\ (M)$ is a ring homorphism. In fact, as presently defined the image of φ in $\mathcal{H}\ (M)$ is not closed under multiplication: the components of $\varphi\ (\Gamma \alpha \Gamma) \times \varphi\ (\Gamma \beta \Gamma)$ consist of triples $(\Gamma''\backslash X, f, g)$, where $\Gamma'' \subset \delta\Gamma\ \delta^{-1} \cap \Gamma$ for some $\delta \in \Delta$, but $\Gamma'' \neq \delta\ \Gamma\ \delta^{-1} \cap \Gamma$ for any $\delta \in \Delta$. We conjecture that upon further identifications in $\mathcal{H}\ (M)$ (triples $(\Gamma''\backslash X, f, g)$ with Γ'' "redundant" as above are to be identified with $m\cdot(\tilde{\Gamma}\backslash X, \tilde{f}, \tilde{g})$, where $\tilde{\Gamma}$ is a group of the form $\delta_0\ \Gamma\ \delta_0^{-1} \cap \Gamma$ with $\delta_0 \in \Delta$ so that the index $[\Gamma'' : \delta_0\ \Gamma\ \delta_0^{-1} \cap \Gamma] = m$ is minimal) φ is a homomorphism.

In the next section we mention briefly the result towards which our efforts have been directed.

(IV) Sketch of Associated Dirichlet Series

Hecke, Eichler, and Shimura considered Dirichlet series associated to Hecke operators; let Γ be a Fuchsian group defined by a quaternion algebra; and denote by S_m the space of cusp forms of Γ. Their Dirichlet series are the matrix valued Dirichlet series: $\sum_{n=1}^{\infty} (T_n \mid S_m(\Gamma))\, n^{-s}$ where T_n are "Hecke operators" of Γ.

In general, denote by $\mathcal{R}\ (\Gamma,\Delta)$ the Hecke ring, properly defined for the Fuchsian group Γ, and take a representation module A of $\mathcal{R}\ (\Gamma,\Delta)$, we have a End(A)-valued $\sum_{n} (T_n \mid A)\, n^{-s}$. Now in our "general" case of local affine manifolds, we have the Hecke ring $\mathcal{H}$ (M), and representation spaces $H^i(M,p^{(m)})$, so we like to find a "generalization" of associated Dirichlet series.

For example, if we have some natural sequence $\{T_n\}$ of elements in $\mathcal{H}$ (M), indexed by positive integers n, we have a Dirichlet series: $\sum_{n=1}^{\infty} (T_n \mid H^i(M,p^{(m)}))\, n^{-s}$. Or we may use other formulations. We have no final formulation, but we like to discuss here two trials; one is defined for the examples in Chapter 3, and the other is good for the examples in Chapter 4.

In the quadratic form case $M = \Gamma\backslash X$; where $\Gamma \subset SL_2(\mathbb{R})$ is a Fuchsian group given by a quaternion algebra, then for Shimura's Hecke ring $\mathcal{R}\ (\Gamma,\Delta)$, we have $\mathcal{R}\ (\Gamma,\Delta)$ contained in $\mathcal{H}$ (M) (as in section (III)). In $\mathcal{R}\ (\Gamma,\Delta)$ we have T_n (by Eichler-Shimura) for n relatively prime to the discriminant and level of Γ. So we could define $\sum_n (T_n \mid H^i(M,p^{(m)}))\, n^{-s}$.

The compatibility of the actions ρ_1, and ρ_2 on $H^i(M,p^{(m)})$, and the isomorphism

$$(*): H^1(M,p^{(m)}) = (S_{2m+2} \oplus \overline{S_{2m+2}} \oplus S_{2m-2} \oplus \cdots \oplus \overline{S_{2e+2}}),$$

with e = 1, when m is odd, e = 0, when m is even, and an analysis of the action of T_n to both sides; we can prove the actions of the Hecke operators $\Gamma\alpha\Gamma$ on both sides of (*) are compatible with a twist of $(\det \alpha)^{2\nu}$ multiplied on $S_{2m-2\nu+2} \oplus \overline{S_{2m-2\nu+2}}$ $\nu = 0, 1, \ldots$); so we have $\sum_n (T_n \mid H^i(M,p^{(m)}))\, n^{-s} =$ $\sum_n (T_n \mid S_{2m+2})\, n^{-s} \oplus \sum_n (T_n \mid \overline{S_{2m+2}})\, n^{-s} \oplus \sum_n (T_n \mid S_{2m-2})$ $n^{-(s-2)} \oplus \cdots \oplus \sum_n (T_n \mid S_{2e+2})\, n^{-s-m+e)} \oplus \sum_n (T_n \mid \overline{S_{2e+2}}) n^{-s-m+e)}$.

The right side is a sum of Hecke-Eichler-Shimura Dirichlet series $\xi\,(s-2\nu, S_{2m-4\nu+2})$ and its "complex conjugations" $\overline{\xi\,(\bar{s}-2\nu, S_{2m-4\nu+2})}$.

For the Heisenberg case: $M = N\backslash X$ (for notation see Chapter 4); we consider a group Γ of automorphisms of M, instead of the group N. Define $\Gamma_1 = \mathrm{Aut}(M) = \mathrm{Aut}(X,N)$; and $\Delta_1 = \{\alpha \in \mathrm{Aut}(X)/\ \alpha\,(N) \subset N\}$; then $\Gamma_1 \subset \Delta_1$, and Δ_1 is a semi-group. For a proper choice of Γ,Δ such that (1) $\Gamma \subset \Gamma_1$, $\Delta \subset \Delta_1$, (2) for all $\alpha \in \Delta$, $\alpha^{-1}\,\Gamma\,\alpha$ is commensurable with Γ in Aut(X), we can define the Hecke ring $\mathcal{R}\,(\Gamma,\Delta)$ as in Shirmura; which operates naturally on $H^i(\Gamma, H^j(M,p^{(m)}))$. Analizing the actions of $\mathcal{R}\,(\Gamma,\Delta)$, and the isomorphism:

$(\#)$: $E^{1,0}(\Gamma, H^1(M, p^{(m)})) = H^1(SL_2(\mathbb{Z}), s^*_{m+1}) = S_{m+3}(SL_2(\mathbb{Z}))$ $\oplus$ {Eisenstein series} $\oplus$ $\overline{S_{m+3}(SL_2(\mathbb{Z}))}$ (for notation see Chapter 4), we have that the action of $T_n \in \mathcal{R}(SL_2(\mathbb{Z}), M_2^x(\mathbb{Z})) = \mathcal{R}(\Gamma, \Delta)$ agrees on both sides. So we have again $\sum_n (T_n \mid E^{1,0})\, n^{-2} = \oplus$ sum of Hecke-Eichler type Dirichlet series $\xi(S_{m+3}, s)$, ξ(Eisenstein, s) and its "complex conjugations" $\xi(\overline{S_{m+3}}, s) = \xi\overline{(S_{m+3}, \bar{s})}$.

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Appendix I: The Hochschild-Serre Spectral Sequence

As the material presented here is not part of the author's original work, and is certainly known to experts in the subject, we provide only enough detail to justify the computations of Chapters 3 and 4. As noted below the spectral sequence we use is a specified version of the "usual" spectral sequence, while one of our applications requires a more general formulation. We have found the paper of Hochschild-Serre [1] to be the most useful reference.

In general, we take a spectral sequence to be a collection $\{E_r^{p,q}, d_r\}$, with indicies $p,q,r \in \mathbb{Z}$, where the $E_r^{p,q}$ are groups and the d_r are maps $d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ so that $d_r^2 = 0$ and $E_{r+1}^{p,q} = \mathrm{Ker}(d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}) / \mathrm{Im}(d_r: E_r^{p-r,q+r-1} \to E_r^{p,q})$. The usual formulation of the Hoschschild-Serre spectral sequence as given in Maclane [5], or Hochschild-Serre [1] arises from the fact that for a G-module A and an exact sequence $0 \to K \to G \to G/K \to 0$ of groups with K normal in G, G/K has a well defined action on $H^q(K,A)$ determined by an action of G/K on the bar resolution of G.

Theorem (Lyndon, Hochschild-Serre): For a G-module A and an exact sequence $0 \to K \to G \to G/K \to 0$ as above, the action of G/K on the bar resolution of G provides a spectral sequence $E_r^{p,q}$, d_r so that

(1) $E_2^{p,q} = H^p(G/K, H^q(K,A))$; and

(2) $H^{p+q}(G,A) \underset{p}{\Leftarrow} E_r^{p,q}$,.

where (2) means that the "approximations" $E_r^{p,q}$ converge to $H^{p+q}(G,A)$ in the sense provided by a filtration on the group of cochain for the bar resolution of G.

<u>Proof</u>: See Maclane [5], or Hochschild-Serre [1].

As a corollary we obtain the "specialized" version of the conclusion of the Hochschild-Serre spectral sequence, that we have used.

<u>Corollary</u>: When the G-module A is a finite dimensional vector space over a field of characteristic 0, we may replace the conclusion (2) by

$$(2') \quad H^i(G,A) = \bigoplus_{p+q=i} E_\infty^{p,q},$$

where for sufficiently large r, r ≥ R, we have $E_\infty^{p,q} \cong E_R^{p,q} \cong E_{R+1}^{p,q} \cong \ldots$

<u>Proof</u>: Since vector spaces over a field of characteristic 0 split, the filtration of (2) may be replaced by $\bigoplus_{p+q=i}$.

For the required "generalization", we note that as in Chapter 1 of Hochschild-Serre, an action of G/K on any resolution of G determines a spectral sequence, which when A is a vector space of characteristic 0 has properties (1) and (2'). For the spectral sequence used in Chapter 4, we note that the actions used there, in particular the action used to determine equation (*) on page 37 arise from an action of G/K on a resolution of the full group G = N, although we have not specified the resolution used for N.

Appendix II:

In the proof of the theorem in Chapter 2 page 10, we have utilized a metric d on X, satisfying conditions (1), (2):

(1) d is Γ-invariant

(2) $\min\limits_{\substack{1 \neq \delta \in \Gamma \\ x \in X}} d(x, \delta(x)) = \epsilon > 0.$

Here we shall prove the existence of such a metric d on X.

Lemma (A. Weil): Let M be a (C^∞- paracompact) manifold; and let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be a locally finite covering of M by relatively compact open sets U_α. Then there exist open sets W_α and W'_α for each index $\alpha \in A$, such that

(1) $\bar{W}_\alpha \subset W'_\alpha$, $\bar{W}'_\alpha \subset U_\alpha$.

(2) $\{W_\alpha\}_{\alpha \in A}$ is also a covering of M and there exists a Riemann metric ds on M such that

(3) distance of $\bar{W}_\alpha$ to $M - W'_\alpha$ is bigger than 1 for all $\alpha \in A$.

Proof: See A. Weil [9].

Let M be a mfd, or $N \xrightarrow{f} M$ be a covering of M. A connected set A in M is called f-copiable if:

(1) $f^{-1}(A) = \bigcup\limits_i A_i$ A_i are connected,

(2) $f|_{A_i}$ is a homeomorphism of A_i with A.

A_i are called copies of A. Also a subset S in M is called f-copiable if there exist a connected set A such that

(1) $S \subset A \subset M$

(2) A is copiable.

Then, $f^{-1}(S)$ is a disjoint sum of subsets S_i such that:

(1) $f^{-1}(S) = \bigcup_i S_i$, $f^{-1}(A) = \cup A_i$ disjoint sum

(2) $S_i \subset A_i$.

(3) $f|_{S_i}\ S_i \to S$ is a bijection. S_i are called components or copies of S.

Let $X \overset{\pi}{\to} M$ be the universal covering of M, and Γ be the covering transformation gr of $X \overset{\pi}{\to} M$. By the definition of covering space, every point P of M has a neighborhood U which is π-copiable. Therefore M has a locally finite covering $\mathfrak{u} = \{U_\alpha\}_{\alpha \in A}$ by f-copiable open sets U_α. Take a system $\{W_\alpha\}_{\alpha\in A}$, $\{W'_\alpha\}_{\alpha\in A}$ of open sets in M, and a Riemann metric ds^2 on M, satisfying conclusions of Weil's lemma. And take the pull back $d\tilde{s}^2$ of the Riemann metric ds^2 from M to X. Then the metric d on X measured by $d\tilde{s}^2$ satisfies desired condition: i.e.

(1) $d(x,y) = d(\delta x, \delta y)$ $\delta \in \Gamma$

(2) $d(x, \delta x) \geq 1$ if $\delta \neq 1$.

<u>Proof</u>: (1) is obvious since $d\tilde{s}^2$ is the pull back from M. To see (2), put $\pi(x) = P$, and take $\alpha \in A$, with $W_\alpha \ni P$. Take the copies $\tilde{W}_\alpha$, $\tilde{W}'_\alpha$, $\tilde{U}_\alpha$ of W_α, W'_α, U_α (respectively), containing the pt x. Since $\pi(\delta x) = \pi(x) = P$, and since U_α is copiable, δx must lie outside of $\tilde{U}_\alpha$. So $d(x, \delta x) \geq d(\bar{W}_\alpha, X - W'_\alpha) \geq 1$, as was to be shown.

The metric ds^2 on M constructed by Weil is also convienent to show the following statement: A covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M by open sets U_α are called f-copiable if (1) all U_α are copiable and (2) any finite union $U_{\alpha_0} \cup U_{\alpha_1} \cup \dots \cup U_{\alpha_r}$ of U_α's in $\mathfrak{U}$ is copiable as long as they meet: $U_{\alpha_0} \cap \dots \cap U_{\alpha_r} \neq \emptyset$.

<u>Proposition</u>: If $N \xrightarrow{f} M$ is any covering of a (C^∞-paracompact) manifold M, M has a locally finite f-copiable covering $\mathfrak{U} = \{U_\alpha\}$.

<u>Proof</u>: Let $\mathfrak{U} = \{U_\alpha\}$ be a locally finite covering of M by relatively compact f-copiable open set U_α. Take $\{W_\alpha\}$, $\{W'_\alpha\}$, ds^2 on M as in Weil's lemma. Then as we can see easily any subset $S \subset M$, with diam $(S) < 1$ is f-copiable. Consider the covering $\mathfrak{W} = \{W_\xi\}$ of M by all open sets $\mathfrak{W}_\xi$ with diameter $(W_\xi) < \frac{1}{2}$. Then $\mathfrak{W}$ is a f- copiable covering of M, since diam $(W_{\alpha_0} \cup \dots \cup W_{\alpha_\nu}) < 1$ if $W_{\alpha_0} \cap \dots \cap W_{\alpha_\nu} \neq \emptyset$.