ISOMORPHISM CLASSES OF KNOT-LIKE GROUPS

A Dissertation presented

by

Rabindra Kumar Patnaik

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

May, 1975

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

RABINDRA KUMAR PATNAIK

We, the dissertation committee for the above candidate for the Ph.D. degree, hereby recommend acceptance of the dissertation.

Bernard Maskit, Professor
Committee Chairman

Elvira R. Strasser, Professor
Thesis Advisor

Jack Morava, Assistant Professor

Alan Tucker, Professor

The dissertation is accepted by the Graduate School.

Herbert Weisinger, Dean
Graduate School

April, 1975

ABSTRACT OF THE DISSERTATION

ISOMORPHISM CLASSES OF KNOT-LIKE GROUPS

BY

RABINDRA KUMAR PATNAIK

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

1975

If $P = (x, a_1, \ldots, a_n ; r_1, \ldots, r_n)$ is a presentation of a group G with commutator quotient group, G/G', infinite cyclic then G is called a knot-like group [13]. If G' is finitely generated free then its rank is equal to the degree of the Alexander polynomial of G [10]. The following question is asked in [10]: Given an integral monic polynomial $\wp(x)$ of degree d satisfying $\wp(0) = \pm 1$, $\wp(1) = \pm 1$, how many isomorphism classes of knot-like groups G are there whose Alexander polynomial is $\wp(x)$ and G' is free of rank d ? If c is the cardinality of the family of isomorphism classes then its lower and

upper bounds are determined.These isomorphism classes are shown to be in one to one correspondence with the conjugacy classes of the admissible automorphisms (cf. definition 2.1,page 11) of the group of automorphisms of the free group of rank d. My conjecture is that c = 1 in case of one-relator knot-like groups.I was able to prove (theorem 5.1) that c = 1 in case of one-relator knot-like groups modulo the second commutator subgroup. A necessary and sufficient condition for a knot-like group to be a one-relator group, i.e. a group having a one-relator presentation, modulo the second commutator subgroup is determined in Theorem 5.2. In the sequel I have given a new simple proof of the well known fact that the Alexander polynomial $\Delta_G(x)$ of a knot-like group G satisfies the condition $\Delta_G(1) = \pm 1$ (theorem 3.1) and determined a structure theorem (theorem 3.4) for the commutator subgroup of a one-relator knot-like group. The proofs are combinatorial except for certain number theoretic results.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I wish to express my sincere gratitude and appreciation to Professor Elvira Rapaport Strasser for her generous help, continuous guidance, constant encouragement, and immense patience in achieving my goal.

INTRODUCTION

The definition of groups in terms of generators and relators became important when H.Poincaré discovered the fundamental group of a pathwise connected topological space.In most cases,these fundamental groups can be defined easily in terms of generators and relators but not otherwise in a purely group theoretical manner without reference to the underlying topological object. This way of defining a group is called presenting the group.

We shall mean by an ordered pair (S ; R), the presentation of a group with generators the elements of the set S and relators R,a collection of "words" on the elements of S and their inverses.By the group G presented by (S ; R) we mean the quotient group of the free group on S by the normal closure of the words in R.We usually call P = (S ; R) a presentation of the group G.

Obviously there is more than one presentation of a group G.A group is said to be finitely generated if it has a presentation P = (S ; R) in which S is finite, and finitely presentable if it has one in which both S and R are finite. Being generated by a finite set of elements or having a finite presentation is an algebraic property of groups. If $P = (S ; R)$ is a finite presentation then the cardinality of the set S minus the cardinality of the set R is called the deficiency of the presentation P.

The fundamental group of the complement of the homeomorphic image of a circle in the three sphere is called a knot group. If the homeomorphic image is the union of a finite number of closed straight line segments then that knot group is finitely presented. A finitely presentable knot group has two important properties — (i) it has at least one presentation whose deficiency is one, and (ii) its commutator quotient group is infinite cyclic. However, there are many finitely presented groups having these two properties that are not knot groups. This leads to the following definition [13].

<u>Definition 1.1</u> A finitely presented group satisfying two conditions (i) and (ii) above is a knot-like group.

If a knot-like group G is finitely presentable, then a polynomial is associated with G and this polynomial is called the Alexander Polynomial of G and is denoted by $\Delta_G(x)$ [2]. If G is a knot group then $\Delta_G(x)$ satisfies the conditions — (i) $\Delta_G(1) = \pm 1$, (ii) $\Delta_G(x)$ is a reciprocal polynomial [1] i.e. if $\Delta_G(x)=c_d x^d+\ldots+c_1 x+c_o$ then $c_i=c_{d-i}$, $i=0,1,\ldots,d$. This second condition forces d to be a multiple of two. As a matter of fact these two conditions characterize the Alexander Polynomial of a knot group, called the knot polynomial. But in case of a knot-like group the second condition need not be true. However, the first condition is a necessary condition for any integral polynomial to be the Alexander Polynomial of a knot-like group. We shall give a very short and simple proof of this fact in Section 3. Though in general this condition is not sufficient for a group to be knot-like we show in Theorem 3.2 that it is sufficient in case of a one-relator knot-like presentation.

There is a very close relationship between the structure of the commutator subgroup G' of a knot-like group G and its Alexander Polynomial $\Delta_G(x)$. If $c_o c_d = \pm 1$ then G'/G" is free abelian of rank d ; if $c_o c_d \neq \pm 1$ then G'/G" is not finitely generated [10]. Thus a necessary condition for G' to be finitely generated is that $c_o c_d = \pm 1$. Further, if G' is finitely generated then it is free of rank d, provided G is a knot group [7]. But in case of a knot-like group G we only know that if G' is finitely generated free then its rank is d [13]. In either case for G' of a knot-like group G to be free of rank d it is necessary that the Alexander Polynomial $\Delta_G(x)$ of G should satisfy the condition $c_o c_d = \pm 1$. We shall prove (Theorem 3.4) a necessary and sufficient condition for the commutator subgroup of a one-relator knot-like group to be free of rank d in Section 3.

Thus given an integral monic polynomial $\phi(x) = x^d + c_{d-1}x^{d-1} + \ldots + c_1 x + c_o$ satisfying $\phi(0) = \pm 1$, and $\phi(1) = \pm 1$ it is natural to expect the existence of a knot-like group G such that G' is free group of rank d and $\Delta_G(x) = \phi(x)$. Indeed such a knot-like group is given by the presentation

$$P = (x, a_o, \ldots, a_{d-1} \ ; \ \bar{x}a_o x = a_1, \ldots, \bar{x}a_{d-2}x = a_{d-1} \ ,$$
$$\bar{x}\, a_{d-1} x = \bar{a}_{d-1}^{\,c_{d-1}} \bar{a}_{d-2}^{\,c_{d-2}} \ldots \bar{a}_o^{\,c_o} \ ).$$

Then naturally the question arises, which was asked in [10]: Given an integral monic polynomial $\phi(x)$ of degree d, satisfying $\phi(0) = \pm 1$, $\phi(1) = \pm 1$, how many isomorphism classes of knot-like groups are there whose commutator subgroups are all free of rank d and whose Alexander Polynomials are all equal to $\wp(x)$?

We prove several theorems regarding this problem in Sections 4 and 5.Since this problem is tied up with many unsolved problems such as the isomorphism problem of groups, the conjugacy problem of the group of automorphisms of a free group or a free abelian group of finite rank greater than one,and the determination of class number of algebraic extensions,we conclude that a complete solution to this problem is possible at present only under well-chosen restrictions.

SECTION 2

NOTATION AND SUMMARY OF RESULTS

2.1 INTRODUCTION.

In this section we shall prepare our setting and outline most of the background material that will be used from now on. In article 2.2 we shall list all the notation that can be explained at this stage and leave the rest to be explained as the material unfolds. All other terms and notation used in the sequel without definition or reference may be found in [6]. In article 2.4 we list the summary of our results.

2.2 NOTATION.

We fix the follwing notation once for all.

$\bar{x} = x^{-1}$, inverse of an element x.

$F_d = (a_1, \ldots\ldots, a_d \ ; \quad)$, free group of rank d.

$a_{ij} = \bar{x}^j \ a_i \ x^j = a_i^{x^j}$, $i=1,2,\ldots\ldots, j=0,\pm 1,\pm 2,\ldots\ldots$

$b_i = \bar{x}^i \ b \ x^i = b^{x^i}$, $i = 0,\pm 1,\pm 2,\ldots\ldots\ldots$ Note the difference between a_i and b_i.

$\varphi(x) = x^d + c_{d-1}x^{d-1} + \ldots\ldots\ldots + c_1 x + c_o$, an integral monic polynomial in x i.e. a polynomial in x with integral coefficients and leading coefficient one, satisfying $\phi(0)=\pm 1$, $\phi(1)=\pm 1$.

$d = \deg\phi(x)$ = degree of the polynimial $\phi(x)$.

$a_i^{\phi(x)} = a_{id}^{c_{d-1}} a_{id-1} \cdots\cdots a_{io}^{c_o} = \bar{x}^d a_i x^d \bar{x}^{d-1} a_i^{c_{d-1}} x^{d-1} \cdots\cdots x a_i^{c_o}$.

$b^{\phi(x)} = b_d b_{d-1}^{c_{d-1}} \cdots\cdots b_1^{c_1} b_o^{c_o} = \bar{x}^d b x^d \; \bar{x}^{d-1} b \; x^{d-1} \cdots x b^{c_o}$. For example $b^{x^2-2x+1} = b_2 b_1^{-2} b_o = \bar{x}^2 \; b \; x^2 \; \bar{x} \; b^{-2} x \; b = \bar{x}^2 b x \bar{b}^2 x b$.

Note that $b^{x^2-2x+1} \neq b^{x^2+1-2x}$ since $b_1^{-2} b_o \neq b_o b_1^{-2}$ unless of course the b_i commute.

$L(w)$ = the length of the element $w = w(a_1, \ldots, a_d)$ of F_d, defined as the sum of the absolute values of the exponents of the generators appearing in w. $L(1) = 0$.

G' = the commutator subgroup of the group G.

$\mathbb{Z}$ = the ring of integers.

$\mathbb{Z}[x]$ = the ring of polynomials in x with integer coefficients.

$X = F_1(x)$, the free group generated by x.

$\mathbb{Z}X$ = Integral group ring of the group X, i.e. the ring of L-polynomials in x.

$\langle \phi(x) \rangle$ = the ideal generated by $\phi(x)$ in $\mathbb{Z}[x]$.

$\Delta_G(x)$ = Alexander Polynomial of the group G.

Aut G = the group of automorphisms of the group G.

c = the cardinality of the family of isomorphism classes of knot-like groups G for which $\deg\Delta_G(x) = d$ and $G' \cong F_d$.

c^* = the cardinality of the family of isomorphism classes of knot-like groups for which $\deg\Delta_G(x) = d$ and $G' \cong F_d/F_d''$.

Unless otherwise mentioned small Greek letters denote homomorphisms, isomorphisms, or automorphisms.

Unless otherwise mentioned $G, G_i, G_\alpha, G_\beta, \ldots$ denote knot-like groups. Also by a group we shall mean a knot-like group.

$B(n)$ = the partition of the integer n.

$B(n_1, \ldots, n_k)$ = the partition of the multipartite number $(n_1, \ldots, n_k)$ defined as follows: Suppose we have n_i objects of the i^{th} kind, $i = 1, \ldots, k$; $n = n_1 + \ldots + n_k$ and we have r identical cells. Let B_r be the number of ways all these n objects can be distributed among r cells such that each cell has at least one object. Then $B(n_1, \ldots, n_k) = \sum_{r=1}^{n} B_r$. There is no explicit formula for this [5].

2.3 BACKGROUND.

Let $P = (x, a_1, \ldots, a_n ; r_1, \ldots, r_n)$ (1)

be a presentation of a knot-like group G. Without any loss of generality we may assume that $a_i \in G'$, $i = 1, \ldots, n$;

that r_i has the form $a_i^{\epsilon_i} Q_i, i = 1,\ldots\ldots,n,\ \epsilon_i = \pm 1$, for some element $Q_i \in G'$ [6]; and that G/G' is generated by the coset containing x. Then $\{x^r\}, r = 0, \pm 1, \ldots\ldots$, forms a Schreier representative system for G modulo G'. Then a presentation for G' is given by

$$P' = (\ a_{ij}\ ;\ R_j,\ i = 1,\ldots.,n, j = 0,\ \pm 1,\ldots.) \qquad (2)$$

where R_j is the rewrite(cf. page 153 [6]) of the set of words $\{r_{ij}\}$ i.e. $\bar{x}^j r_i x^j$.

We assume all the words r_i are reduced and cyclically reduced ; and suppose m_i and M_i are integers such that R_o contains a_{im_i} but not a_{ik} if $k < m_i$ and a_{iM_i} but not a_{it} if $t > M_i$. Let s be the smallest second subscript occuring in the rewrite of r_1, regardless of the first subscripts; then the rewrite of $\bar{x}^s r_1 x^s$ contains a_{i0} for some i but no negative subscripts. Replacing r_1 by $\bar{x}^s r_1 x^s$ and proceeding similaly with the rest of the relators makes m_i non-negative for each i. If now $m_1 \neq 0$, then replacing a_1 by $x^{m_1} a_1\ \bar{x}^{m_1}$ every where in the relator set we get $m_1 = 0$ in R_o. This gives us a new presentation isomorphic to P. Similarly we can assume every $m_i = 0$ in R_o. Thus (2) now reduces by Tietze transformations to a presentation (a_{ij} ; R_j , $i = 1,\ldots,n$, $j = 0,\pm 1,\ldots$) (3) with $m_i = 0$ for all i.

Let $M = M_1+\ldots+M_n$. This M varies with the presentation [13]. So let us choose that presentation (1) of a knot-like group for which M is least possible.

On the other hand if G is a knot-like group such that $G' \cong F_d = (a_1,\ldots,a_d ; \ )$ and $G/G' \cong F_1(x)$ then a presentation of G may be obtained from the presentation of G' by adjoining a generator x and relations $\bar{x}a_i x = \alpha(a_i)$, $i = 1,\ldots,d$, where α isasuitable automorphism of G' induced by the element x. This is because of the fact that we may look at G as an extension of F_d by $F_1(x)$, such that $G' \cong F_d$, and since $F_1(x)$ is free, the extension is a split extension and consequently equivalent to a semidirect product of F_d by $F_1(x)$ with a suitable homomorphism of $F_1(x)$ into Aut F_d [11]. Thus G has a presentation

$$P_\alpha = (x,a_1,\ldots,a_d ; a_1^x = \alpha(a_1),\ldots,a_d^x = \alpha(a_d)) \qquad (4)$$

The automorphism α uniquely determines this presentation and therefore we shall denote this presentation by P_α with the subscript α and the knot-like group it defines by G_α. This suggests that given a free group F_d, we can manufacture presentations P_α for suitable automorphisms α of F_d, which would define knot-like groups G_α such that $G'_\alpha \cong F_d$. It is clear that not all automorphisms of F_d will give a presentation P_α which defines a knot-like group. So we make the following

<u>Definition 2.1</u> An automorphism of F_d for which the presentation (4) is a knot-like group, is called an admissible automorphism of F_d.

For example, α defined by $\alpha(a_1) = a_2, \alpha(a_2) = a_3$ $\alpha(a_3) = a_3^{-3} a_2^2 a_1$ is an admissible automorphism of F_3, because α defines a presentation P_α which is an extension of F_3 by $F_1(x)$. On the other hand the automorphism β defined by $\beta(a_1) = \bar{a}_1$, $\beta(a_2) = a_2$, $\beta(a_3) = a_2 a_3$ is not an admissible automorphism of F_3, for,

$$P_\beta = (x, a_1, a_2, a_3 \ ; \ a_1^x = \bar{a}_1, \ a_2^x = a_2, a_3^x = a_2 a_3) \quad \text{is}$$

not a knot-like group since its commutator quotient group is not infinite cyclic.

Thus for a given natural number d, and for each admissible automorphism α of F_d we have a unique presentation (4) of a knot-like group G_α with $G'_\alpha \cong F_d$ and coversely. This fact will be useful in Section 4.

If the elements of G' are allowed to commute, that is if the second commutator subgroup G" of G is factored out then in the presentation (4)

$P_\alpha = (x,a_1,\ldots\ldots,a_d \ ; \ a_1^x=\alpha(a_1),\ldots\ldots,a_d^x = \alpha(a_d) \ ),$

becomes an admissible automorphism of F_d/F'_d. In this case the relations $a_i^x = \alpha(a_i)$ take the form $a_i^x = a_1^{\alpha_{i1}} \ldots a_d^{\alpha_{id}}$, $i=1,\ldots\ldots,d$, where the α_{ij} are integers and α is uniquely determined by the matrix $\|\alpha_{ij}\|$, $i,j=1,2,\ldots\ldots,d$. We shall denote the matrix $\|\alpha_{ij}\|$ by $\|\alpha\|$, the matrix of the automorphism α. We have a new presentation

$$P_\alpha = (x,a_1,\ldots\ldots,a_d \ ; \ a_i^x = a_1^{\alpha_{i1}} \ldots\ldots a_d^{\alpha_{id}} \ ;i=1\ldots d \ ) \qquad (5)$$

We shall use the same notation P_α for both the presentations (4) and (5) for obvious reasons and it will be clear form the context whether α is an admissible automorphism of F_d or F_d/F'_d.

<u>Definition 2.2</u> The dxd square matrix $\|\alpha\|$, with integer entries and det $\|\alpha\| = \pm 1$, is called an admissible matrix if the automorphism α is an admissible automorphism of F_d/F'_d.

It is known that a square matrix is always similar to a diagonal block matrix, called the Jordan Canonical From, if the entries are from a field [3]. By similarity of two square matrices A and B over a ring we mean that there exists an invertible matrix C with entries from the ring such that $CAC^{-1} = B$. By the Jordan Canonical form we mean a matrix which has block matrices along the diagonal and each block matrix is the companion matrix of some monic integral polynomial which is a factor of the characteristic polynomial of the whole matrix. The companion matrix of the polynomial

$x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_o$ is

$$\begin{Vmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ -a_o & -a_1 & -a_2 & \ldots & -a_{n-1} \end{Vmatrix}$$

But it is an unsolved problem to decide whether an integer square matrix is similar to a matrix in the Jordan Canonical Form over $\mathbb{Z}$. Since we shall be dealing with integral matrices over $\mathbb{Z}$, we make the following

<u>Definition 2.3</u> An integer square matrix A is called a Jordan Matrix if there exists an invertible integral square matrix C such that $C\,A\,C^{-1}$ is in Jordan Canonical From.

Note the difference between Jordan Canonical From and Jordan matrix. Also note that our Jordan Canonical Form is different from the usual Jordan Canonical Form.

given in [3]. Namely, in the usual Jordan Canonical Form the characteristic polynomial of any block matrix on the diagonal divides the characteristic polynomial of the block matrix preceeding to or following the above block matrix. But in our Jordan Canonical From we donot require any relationship among the characteristic polynomials of the block matrices on the diagonal.

The det ($\|\alpha\|$ - I x) is the Alexander Polynomial of the knot-like group defined by (5). We shall discuss the Alexander Polynomial of a knot-like group in Section 3.

2.4 SUMMARY OF RESULTS.

The results obtained in the present work are as follows:

1. If P = (x,b ; R) is a one-relator presentation of a group then it is a knot-like group G if and only if $\triangle_G(1) = \pm 1$, (Theorem 3.2).

2. If P = (x,b ; R) is a one-relator presentation of a knot-like group G, then G' is free of finite rank d if and only if R can be rewritten in the form $\bar{b}_o w(b_1, \ldots, b_d)$ as well as in the form $b_d v(b_o, \ldots, b_{d-1})$, (Theorem 3.4).

3. Given a polynomial $\phi(x) = \phi_1^{n_1}(x) \ldots\ldots \phi_k^{n_k}(x)$ of degree

d, satisfying $\phi(0) = \pm 1$, $\phi(1) = \pm 1$, each $\phi_i(x)$ distinct and irrreducible over $\mathbb{Z}$, and the n_i are positive integers, there exist at least B ($n_1, \ldots, n_k$) isomorphism classes of knot-like groups G whose Alexander polynomial is $\phi(x)$ and $G' \cong F_d$, (Theorem 4.1).

4. $G_\alpha \cong G_\beta$ if and only if α and β are conjugates in Aut F_d, (Theorem 4.2).

5. If $\phi(x)$ is irreducible then the cardinality of the family of isomorphism classes of knot-like groups modulo the second commutator subgroup is at most equal to the class number of the field $\frac{\mathbb{Z}[x]}{\langle \phi(x) \rangle}$, (Theorem 4.4).

6. If d = 2, then there is exactly one isomorphism class modulo the second commutator subgroup, (Corollary 4.3.1).

7. Every knot-like group G for which d = 2 and $G' \cong F_2$ has a one-relator presentation, (Corollary 4.3.2).

8. If G and G^* are two one-relator knot-like groups with the same Alexander Polynomial and commutator subgroup F_d then $G = G^*$ modulo the second commutator subgroup, (Theorem 5.1).

9. A necessary and sufficient condition for a knot-like group G having a presentation (5) to have a one-relator presentation is that $\|\alpha\|$ be conjugate to the companion

matrix of the Alexander Polynomial of G over $\mathbb{Z}$, (Theorem 5.2).

10. A cojecture: Given an integral monic polynomial $\Phi(x)$ of degree d, satisfying $\Phi(1) = \pm 1$, $\Phi(0) = \pm 1$, there exists exactly one one-relator knot-like group G for which $\Delta_G(x) = \Phi(x)$, and $G' \cong F_d$.

SECTION 3

THE ALEXANDER POLYNOMIAL AND THE COMMUTATOR SUBGROUP

3.1 INTRODUCTION .

In this section we shall give a simple proof of the fact that the Alexander Polynomial $\triangle_G(x)$ of a knot-like group G satisfies the condition $\triangle_G(1) = \pm 1$. This was proved by various authors for knot groups.Either the proof uses topological results,which are available because of the knot [7],or lengthy algebraic methods [1].In article 3.3 we shall prove a structure theorem for the commutator subgroup of a one-relator knot-like group.The article 3.4 deals with some examples illustrating Theorem 3.4,and we conclude this section with a remark in the article 3.5 regarding our main problem.

3.2 THE ALEXANDER POLYNOMIAL .

The free calculus [2] is the principal mathematical tool in the construction of useful invariants of group presentation types.Consider a group presentation $P = (x_1,x_2,\ldots\ldots ; r_1,r_2,\ldots\ldots)$.Suppose this presentation defines a group G, not necessarily a knot-like group .

$F = F(x_1, x_2, \ldots\ldots)$. Let $\varphi: F \longrightarrow G$ be the canonical homomorphism and $\psi: G \longrightarrow G/G'$ be the abelianizer. These two homomorphisms possess unique extensions to homomorphisms of their respective group rings. We shall denote these extended homomorphisms by the same symbols. We thus have the composition

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x_j}} \mathbb{Z}F \xrightarrow{\varphi} \mathbb{Z}G \xrightarrow{\psi} \mathbb{Z}(G/G'), \text{where } \frac{\partial}{\partial x_j} \text{ is}$$

defined as follows :

$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ (Kronecker delta) and if $w = x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \ldots\ldots x_{j_r}^{\epsilon_r}$ is an element of F, $\frac{\partial w}{\partial x_j} = \sum_{k=1}^{r} \epsilon_k \delta_{jj_k} x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \ldots\ldots x_{j_k}^{\frac{1}{2}(\epsilon_k - 1)}$, $\epsilon_i = \pm 1$

and $\frac{\partial (w_1 + w_2)}{\partial x_j} = \frac{\partial w_1}{\partial x_j} + \frac{\partial w_2}{\partial x_j}$, $w_1, w_2 \in F$.

<u>Definition 3.1</u> The Alexander matrix of P is the matrix $\left\| \psi \varphi \left(\frac{\partial r_i}{\partial x_j} \right) \right\|$.

The Alexander Matrices of the finitely generated presentations of a finitely generated group G all belong to a single equivalence class over $\mathbb{Z}(G/G')$ [2]. Thus it makes sense if we say the Alexander Matrix of a group.

Definition 3.2 The Alexander Polynomial of a group G is the g.c.d., if it exists, of the minor determinants of all (n-1) x (n-1) submatrices of the Alexander Matrix of G , where n is the number of columns of the Alexander Matrix.This is denoted by $\triangle_G(x)$.

For knot-like groups $\mathbb{Z}$ (G/G') is particularly simple and is a g.c.d. domain.Thus the Alexander Polynomial of a knot-like group always exists.We obtain the Alexander Polynomial of a knot-like group by another simple way.

Let us consider the presentation (1) on page 8 i.e. $P = (x, a_1, \ldots, a_n ; r_1, \ldots, r_n)$ of knot-like group G.If we rewrite the relators in terms of the conjugates a_{ij} and then allow the a_{ij} to commute then each r_i will have the form

$$r_i = a_1^{p_{i1}(x)} a_2^{p_{i2}(x)} \ldots\ldots a_n^{p_{in}(x)} , \quad i = 1,\ldots,n , \qquad (6)$$

where the $p_{ij}(x)$ are integral polynomials in x.

Definition 3.3 If $p(x) = x^q(c_0 + c_1x + \ldots\ldots + c_dx^d)$, $c_0 \neq 0$, is the determinant of the n x n matrix $\| p_{ij}(x) \|$ then $\triangle_G(x) = c_0 + c_1x + \ldots\ldots + c_dx^d$ is the Alexander Polynomial of G.

For example $P = (x,b ; \bar{x}\,\bar{b}\,\bar{x}^2 b\, x\, \bar{b}\, x^2)$

presents a knot-like group G. If we rewrite the relator in terms of the conjugates $b_i = \bar{x}^i b\, x^i$ then we have $\bar{x}\bar{b}\bar{x}^2 b x \bar{b} x^2 = (\bar{x}\bar{b}x)(\bar{x}^3 b x^3)(\bar{x}^2\bar{b}x^2) = \bar{b}_1 b_3 \bar{b}_2 = b^{-x+x^3-x^2}$. So this becomes $b^{x^3-x^2-x}$ if we factored out G". Thus $p(x)=x^3-x^2-x$ $= x\,(x^2-x-1)$. Hence the Alexander Polynomial of G is $\triangle_G(x) = x^2-x-1$.

The Alexander Polynomial of a group is an invariant of the group [2].

The following theorem is a short new proof of the known fact that $\triangle_G(1) = \pm 1$ for a knot-like group G. In order to prove this we need the following

<u>Lemma 3.1</u> For a knot-like group G, G/(G",x) is the trivial group.

<u>Proof.</u> G has a presentation $P=(x,a_1,\dots,a_n\ ;\ r_1,\dots,r_n)$ in which the exponent sum of x in each r_i is zero. Therefore letting x commute with the a_i amounts to dropping the x-symbols from the r_i. Next letting also the a_i commute with one another amounts to taking $r_1,\dots,r_n$ modulo F'_{n+1}, where $F_{n+1} = (x,a_1,\dots,a_n\ ;\ )$. Now $G/G' = (x,a_1,\dots,a_n;\ r_1,\dots,r_n,F'_{n+1}) = F_1(x)$. That is modulo F'_{n+1}, $r_1,\dots,r_n$ generate $a_1,\dots,a_n$. But since

modulo G" the a_i commute with one another, it follows that G/(G",x) is trivial.This proves the lemma.

Theorem 3.1 For any knot-like group G, $\Delta_G(1) = \pm 1$.

Proof: Since $\Delta_G(x) = \Delta_{G/G"}(x)$ we may consider G/G" only.

Thus $r_i = a_1^{p_{i1}(x)} a_2^{p_{i2}(x)} \cdots\cdots a_n^{p_{in}(x)}$, i=1,2,.....,n; so if we set x = 1, then

$$r_i = a_1^{p_{i1}(1)} a_2^{p_{i2}(1)} \cdots\cdots a_n^{p_{in}(1)}, \quad i = 1,2,...,n \qquad (7)$$

are the defining relators of G/(G",x).

It follows from Lemma 3.1 that (7) is a set of generators of the free abelian group on the a_i, i=1,......,n.Thus the determinant of the matrix $\|p_{ij}(1)\|$ has to be ±1.Hence $\Delta_G(1) = \pm 1$.This proves the theorem.

Any polynomial $\phi(x)$ satisfying $\phi(1) = \pm 1$ is the Alexander Polynomial of some knot-like group G , for instance, the group presented by (x,b; $b^{\phi(x)}$).Yet not every group with such a polynomial is knot-like.For example (x,b ; x^2, $\bar{x}bx = b^2$) defines a group G.Since G/G' = (x ; x^2), G is not a knot-like group.

The homomorphism ψ : G $\longrightarrow$ G/G' maps b on to 1 and x^2 on to 1.The Alexander Matrix of G is

$$\left\| \begin{matrix} 1+x & 0 \\ 0 & \bar{x}-2 \end{matrix} \right\| \quad \text{i.e.} \quad \left\| \begin{matrix} 1+x & 0 \\ 0 & 1-2x \end{matrix} \right\|$$

Then the Alexander Polynomial of G is the g.c.d. of the determinant of all 1 x 1 submatrices of this matrix. Thus the Alexander Polynomial $\triangle_G(x)$ of G is one.

However, in case of a one-relator, two generator presentation, which does not necessarily define a knot-like group, we prove the following

Theorem 3.2 A presentation $P = (x,b \ ; R)$ on two generators, and one relator defines a knot-like group G if and only if its Alexander Polynomial $\triangle_G(x)$ satisfies the condition $\triangle_G(1) = \pm 1$.

Proof: If G is a knot-like group then $\triangle_G(1) = \pm 1$ by Theorem 3.1.

Conversely, suppose $\triangle_G(1) = \pm 1$ and we shall show that $G/G' = F_1(x)$.

Let $\sigma_x(R)$ be the exponent sum of x in R and $\sigma_b(R)$ the exponent sum of b in R. Then $G/G' = F_1$ if and only if $(\sigma_x(R), \sigma_b(R)) = 1$.

Now the fact that the Alexander Polynomial $\triangle_G(x)$ of G involves only one variable shows that G/G'

has one generator, for, otherwise the mapping $\psi : G \longrightarrow G/G'$ would not be surjective.

This generator is a free generator because

$$\sigma_x(R) = \left(\frac{\partial R}{\partial x}\right)^{o\psi\varphi} \quad \text{and} \quad \sigma_b(R) = \left(\frac{\partial R}{\partial b}\right)^{o\psi\varphi}, \text{where}$$

$$\mathbb{Z}F_2 \xrightarrow{\frac{\partial R}{\partial t}} \mathbb{Z}F_2 \xrightarrow{\varphi} \mathbb{Z}G \xrightarrow{\psi} \mathbb{Z}(G/G') \xrightarrow{o} \mathbb{Z},$$

where $t = x$ or b .

Since $\triangle_G(x) = \left(\left(\frac{\partial R}{\partial x}\right)^{\psi\varphi}, \left(\frac{\partial R}{\partial b}\right)^{\psi\varphi} \right), \triangle_G(1) = \pm 1$

gives $1 = \left(\left(\frac{\partial R}{\partial x}\right)^{o\psi\varphi}, \left(\frac{\partial R}{\partial b}\right)^{o\psi\varphi} \right) = (\sigma_x(R), \sigma_b(R))$.
Hence $G/G' = F_1(x)$ and then the theorem follows.

The Alexander Polynomial of a knot-like group G determines the structure of G'.

3.3 THE STRUCTURE OF THE COMMUTATOR SUBGROUP OF A KNOT-LIKE GROUP.

If $\triangle_G(x) = c_o + c_1 x + \ldots\ldots + c_d x^d$ is the Alexander Polynomial of the knot-like group G and $c_o c_d \neq 1$ then G' is not finitely generated [10]. On the other hand if G' is finitely generated free then its rank is equal to the degree of the Alexander Polynomial [10].

However, if d=M, where M (cf. page 10) is the least possible for all presentations $P = (x, a_1, \ldots, a_n ; r_1, \ldots, r_n)$ of G then G' is finitely generated if and only if it is free [13].

For knot groups, however, we always have d = M. The above result is true for knot groups. Since the only proof available at present is topological [7] it does not need the assumption M = d. In case of knot groups d turns out to be twice the genus of the knot.

For one-relator presentations of knot-like groups G, M turns out to be an invariant of G and if G' is finitely generated then d = M [13]. The following theorem is proved in [13].

Theorem 3.3 If G is a one-relator knot-like group then any one of the first two statements of the following implies the rest. (i) G' is finitely generated, (ii) G' is free, (iii) the degree of the Alexander polynomial is M.

Now we shall state and prove a combinatorial version of the above theorem 3.3, which provides an effective test of whether the commutator subgroup of a one-relator knot-like group is finitely generated and gives the rank immediately if it is finitely generated.

<u>Theorem 3.4</u> If $P=(x,b ; R)$ is a one-relator presentation of a knot-like group G, then G' is free of rank d if and only if R can be expressed as $\bar{b}\, w (b^{x}, b^{x^2}, \ldots\ldots, b^{x^d})$ and $b^{-x^d} v (b , b^{x}, \ldots\ldots, b^{x^{d-1}})$, where $d = \deg \triangle_G(x)$.

<u>Proof:</u> If $P = (x,b ; R)$ is a presentation of G then by the Reidemeister-Schreier rewriting process, as is well known, a presentation for G' is given by

$P'= (\ldots\ldots, b_{-1}, b_o, b_1, \ldots\ldots ; R_i , i = 0, \pm 1, \ldots\ldots)$, where R_i is the rewrite of $x^{-i} R\, x^{i}$.

Then $\bar{b}\, w (b^{x}, b^{x^2}, \ldots\ldots, b^{x^d})$ becomes $\bar{b}_o w (b_1, b_2, \ldots\ldots, b_d)$ and $\bar{b}^{-x^d} v (b, b^{x}, \ldots\ldots, b^{x^{d-1}})$ becomes $\bar{b}_d v (b_o, b_1, \ldots\ldots, b_{d-1})$.

Now suppose R can be expressed as

$\bar{b}\, w (b^{x}, b^{x^2}, \ldots\ldots, b^{x^d})$ and $b^{-x^d} v (b, b^{x}, \ldots\ldots, b^{x^{d-1}})$ or equivalently R can be rewritten in terms of the b_i as $b_o w(b_1, b_2, \ldots\ldots, b_d)$ and $\bar{b}_d v(b_o, b_1, \ldots\ldots, b_{d-1})$, then by Tietze transformations P' can be reduced to $(b_o, b_1, \ldots\ldots ; R_i, i = 0, 1, \ldots\ldots)$, which further reduces by Tietze transformations to $(b_o, b_1, \ldots\ldots, b_{d-1} ; \ ) \cong F_d$.

Since this presentation is obtained from P'

by Tietze transformations, they define isomorphic groups and hence $G' \cong F_d$.

Conversely, if R cannot be expressed as $\bar{b}\, w(b^x, b^{x^2}, \ldots\ldots, b^{x^d})$ and/or $\bar{b}^{x^d} v(b, b^x, \ldots\ldots, b^{x^{d-1}})$ then we shall show that G' is not finitely generated.

Without any loss of generality we may assume that R cannot be expressed as $\bar{b}^{x^d}\, v\,(\, b, b^x, \ldots\ldots, b^{x^{d-1}})$ but can be expressed as $\bar{b}w(b^x, b^{x^2}, \ldots\ldots, b^{x^d})$, i.e. in terms of the b_i, R cannot be rewritten as $\bar{b}_d v(b_o, b_1, \ldots\ldots, b_{d-1})$ but can be rewritten as $\bar{b}_o w(b_1, \ldots\ldots, b_d)$.

Because of this assumption the presentation P' can be reduced by Tietze transformations to $(\, b_i \,;\, R_i \,,\, i = 0, 1, \ldots\ldots)$.

From this presentation we form presentations of certain groups of the form

$$H_n = (\, b_o, b_1, \ldots\ldots, b_n \,;\, R_o, R_1, \ldots\ldots, R_{n-d}\,), n \geqslant d.$$

Using Tietze transformations as before these reduce to free presentations of free groups on d generators $b_{n-d+1}, \ldots\ldots\ldots, b_n$.

Let $\theta_n : H_n \longrightarrow H_{n+1}$ denote homomorphisms defined by $b_i \longrightarrow b_i$ if $i=n-d+2,\ldots\ldots,n$ and

$b_{n-d+1} \longrightarrow w(b_{n-d+2},\ldots\ldots,b_{n+1})$.

Since both H_n and $\theta_n(H_n)$ are free groups of rank d, the homomorphisms θ_n are isomorphisms.

Since G' is the direct limit of the sequence $H_d \xrightarrow{\theta_d} H_{d+1} \xrightarrow{\theta_{d+1}} H_{d+2} \xrightarrow{\theta_{d+2}} \ldots\ldots\ldots$, of groups and isomorphisms, either G' is not finitely generated or almost all the isomorphisms θ_n are surjective [4,p.53].

But in the latter case all the isomorphisms have to be surjective, for, all the isomorphisms θ_n are defined in the same way. Then $G' \cong H_d$, and we claim that this is not possible. Indeed, if $G' \cong H_d$ then the automorphism of G' defined by $b_{i+1} \longrightarrow b_i$, $i = 0,\pm 1,\pm 2,\ldots\ldots$, induces the automorphism φ of H_d for which $b_d \longrightarrow b_{d-1}$, $b_{d-1} \longrightarrow b_{d-2}$, $\ldots\ldots\ldots$, $b_2 \longrightarrow b_1$, $b_1 \longrightarrow w(b_1,\ldots\ldots,b_d)$. Since H_d is freely generated by $b_1,b_2,\ldots\ldots\ldots,b_d$; $\varphi(H_d)$ contains b_d. That is $b_{d-1},\ldots\ldots,b_1,w(b_1,\ldots\ldots,b_d)$ generate freely b_d. This implies w contains b_d only once. This contradicts our assumption that R_o cannot be rewritten as $\bar{b}_d v(b_o,b_1,\ldots\ldots,b_{d-1})$. Hence $b_{d-1},\ldots\ldots\ldots,b_1,w$ do not generate H_d, i.e. φ is not an automorphism and consequently G' is not isomorphic to H_d. Hence G' is not finitely generated. Q.E.D.

This theorem will now be extended to the case when G is the free product of a finite number of one-relator knot-like groups with amalgamation of certain free cyclic groups.

Theorem 3.5 Let $P = (x,a_1,\ldots\ldots,a_n ; r_1,\ldots\ldots\ldots,r_n)$ define a knot-like group G which is the free product of one-relator knot-like groups G_i having a presentation $P_i = (x_i,a_i ; r_i), i = 1,2,\ldots\ldots,n$ with amalgamation of the free cyclic group on the x_i. Then G' is a free group if and only if each G_i' has rank d_i (and then its rank is $d = \sum_{i=1}^{n} d_i$).

Proof: Since $P = (x_1,a_1 ; r_1) \underset{(x=x_1=x_2)}{*} \ldots\ldots \underset{(x_{n-1}=x_n)}{*} (x_n,a_n ; r_n)$

and the automorphism which the element x in P induces in G' is the product of the automorphisms extended to G' which the elements x_i in P_i induce in G_i', $i = 1,\ldots\ldots,n$ we conclude that G' is the free product of the G_i'. Then theorem 3.5 follows from theorem 3.4.

3.4 EXAMPLES.

We shall give below two examples to illustrate theorem 3.4.

1. $P = (x,b ; \bar{x}^2 bx\bar{b}xb)$ is a presentation of the trefoil knot group G.

In this presentation the relator $R = \bar{x}^2 bx\bar{b}xb$ rewritten in terms of the conjugates is $R_o = (\bar{x}^2 bx^2)(\bar{x}\bar{b}x)b$. So $R_o = b_2\bar{b}_1 b_o$.

Then $R_o = 1$ can be expressed as $b_2 = \bar{b}_o b_1$ and as $b_o = b_1\bar{b}_2$. Hence by theorem 3.4 G' is a finitely generated free group of rank 2 and one pair of generators is b_o, b_1.

Also we can compute G' directly, for, by the Reidemeister-Schreier rewriting process a presentation of G' is given by $(b_i ; b_{i+2}\bar{b}_{i+1}b_i, i = 0, \pm 1, \ldots\ldots)$ which reduces by Tietze transformations to $(b_i ; b_{i+2}\bar{b}_{i+1}b_i, i = 0, 1, \ldots\ldots)$ which further reduces by Tietze transformation to $(b_o, b_1 ;)$. Thus $G' \cong F_2$.

2. $P = (x,b ; \bar{x}^2 bx^2\bar{b}^2)$ is a presentation of a knot-like group G, which is not a knot group.

Since its Alexander polynomial is x^2-2, its G' is not finitely generated [10].

On the other hand, the relator $R = \bar{x}^2 bx^2\bar{b}^2$ of this presentation rewritten in terms of the conjugates is $R_o = b_2 b_o^{-2}$,which again can be stated as $b_2 = b_o^2$ but not as $b_o = w(b_1, b_2)$.Hence by theorem 3.4 the commutator subgroup G' is not finitely generated.

3.5 A PROBLEM .

We know that whenever the commutator subgroup of a knot-like group G is free of finite rank then this rank is equal to the degree of $\triangle_G(x)$.It is now natural to ask the question: given an integral monic polynomial $\phi(x)$,of degree d ,satisfying $\phi(1)=\pm 1$, $\phi(0) = \pm 1$,how many isomorphism classes of knot-like groups G are there for which $\triangle_G(x) = \phi(x)$ and $G' \cong F_d$? This question is discussed in the next two sections.

SECTION 4

CLASSIFICATION PROBLEM : GENERAL CASE

4.1 INTRODUCTION .

The question just posed is a very difficult one as pointed out in article 1.1.So we shall go as far as our present state of knowledge will permit us.Throught the rest of our discussion we shall assume that the monic integral polynomial $\phi(x) = x^d+c_{d-1}x^{d-1}+\dots+c_1x+c_o$, satisfying $\phi(1) = \pm 1$, $\phi(0) = \pm 1$,is given.Let c be the cardinality of the family of isomorphism classes of knot-like groups G with $G' \cong F_d$ and $\Delta_G(x) = \phi(x)$. In this section we shall determine the lower and upper bounds of c,we shall show that c is the cardinality of the family of conjugacy classes of admissible automorphisms of the group of automorphisms of F_d and give certain other results modulo G".

4.2 A LOWER BOUND .

Definition 4.1 Consider a matrix A over an arbitrary commutative ring and an arbitrary non-negative integer t.The ideal generated by the minor determinants of A of order n-t where n is the number of columns, is called the t^{th} elementary ideal $\epsilon_t(A)$ of A.

It is to be understood that $\epsilon_t(A) = (1)$ for $t \geqslant n$, and that $\epsilon_t(A) = (0)$ if A has fewer than n-t rows.

Clearly $\epsilon_t(A) \subseteq \epsilon_{t+1}(A)$, thus to each matrix A there is associated its chain of elementary ideals $\epsilon_o(A) \subseteq \epsilon_1(A) \subseteq$ .... .The equivalent matrices have the same chain of elementary ideals [2].

<u>Definition 4.2</u> The smallest integer t for which $\epsilon_t(A)=(1)$ is called the length of the chain of elementary ideals.

<u>Definition 4.3</u> The t^{th} elementary ideal of a presentation P = (S ; R) is the t^{th} elementary ideal of an Alexander matrix of P.

Since the isomorphic presentations have equivalent Alexander matrices, the chain of elementary ideals is an invariant of the group.

The following theorem gives a lower bound for c.

<u>Theorem 4.1</u> If $\phi(x) = \phi_1^{n_1}(x)\ \phi_2^{n_2}(x) \ldots\ldots \phi_k^{n_k}(x)$, such that the $\phi_i(x)$ are all distinct, irreducible over $\mathbb{Z}$, and the n_i are positive integers, then c is at least $B(n_1, \ldots\ldots, n_k)$.

<u>Proof:</u> First we shall prove this theorem for the case when $n_1=n_2=\dots\dots=n_k=1$. In this case we have $\phi(x)=\phi_1(x)\dots\dots\dots\phi_k(x)$. Let us consider the presentation

$$P = (\ x, a_1, \dots\dots, a_m\ ;\ a_1^{Q_1(x)}\ ,\ a_2^{Q_2(x)}\ , \dots\dots, a_m^{Q_m(x)}\ ) \qquad (8)$$

where $Q_1(x)$ is the product of some of the $\phi_i(x)$, $Q_2(x)$ is the product of some of the remaining $\phi_i(x)$, etc., and $Q_1(x)\dots\dots Q_m(x) = \phi_1(x)\dots\dots\phi_k(x)$, $m \leqslant k$.

For each m there are $\frac{1}{m!}\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}(m-i)^k$ such presentations and in this case there are

$$B(1,1,\dots,1) = \sum_{m=1}^{k}\frac{1}{m!}\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}(m-i)^k \text{ presentation}$$

[5].

We claim that these presentations define non-isomorphic knot-like groups G, for which $\Delta_G(x) = \phi(x)$, and $G' \cong F_d$.

That these presentations define knot-like groups of Alexander polynomial $\phi(x)$ is clear.

That these presentations define knot-like groups whose commutator subgroups are free of rank d follows from theorem 3.5 and the fact that the presentation (8) can be written as

$$P=(x_1,a_1 ; a_1^{Q_1(x)}) \underset{(x=x_1=x_2)}{*} \cdots\cdots \underset{(x_{m-1}=x_m)}{*} (x_m,a_m ; a_m^{Q_m(x)})$$

in which each factor P_i has a commutator subgroup free of rank equal to the degree of $Q_i(x)$ and $d=\sum_{i=1}^{m} \deg Q_i(x)$.

Now we shall show that all these presentations are non-isomorphic.

Clearly, presentations corresponding to different values of m define non-isomorphic groups, for, the lengths of the chains of elementary ideals of the Alexander matrices of these presentations are different [2]. The length of the chain of elementary ideals of the presentation (8) is m+1.

Next, let us consider any two presentations corresponding to the same value of m. Suppose in addition to the presentation (8) we have another presentation

$$P^* = (x,a_1,\ldots\ldots,a_m ; a_1^{Q_1'(x)},\ldots\ldots\ldots,a_m^{Q_m'(x)}) \tag{9}$$

where the $Q_i'(x)$ satisfy the same conditions as the $Q_i(x)$ and at least one (hence two) of the $Q_i'(x)$ is different from the $Q_j(x)$.

Let us denote by A and A' the Alexander

matrices of the presentations (8) and (9) respectively. Then these two matrices are not equivalent because the m^{th} elementary ideals of these two matrices are different [2]. For, the m^{th} elementary ideal $\epsilon_m(A)$ of A is $(Q_1(x), \ldots\ldots, Q_m(x))$ and the m^{th} elementary ideal $\epsilon_m(A')$ of A' is $(Q_1'(x), \ldots\ldots\ldots, Q_m'(x))$. These are different ideals in the group ring $\mathbb{Z}X$, because of the assumptions on the $Q_i(x)$ and on the $Q_i'(x)$.

Hence the presentations (8) and (9) define non-isomorphic knot-like groups [2].

Now, the proof in all other cases follows from the above, by properly defining the polynomials $Q_i(x)$ because of the fact that $B(k) \leqslant B(n_1, \ldots, n_k) \leqslant B(1, \ldots, 1)$. This completes the proof of the theorem.

Though in general there is no explicit formula for $B(n_1, \ldots\ldots, n_k)$, in specific cases it can be computed [5]. When $\phi(x)$ is irreducible then clearly there is at least one such group, namely, the one defined by

$$P = (x, b \; ; \; b^{\phi(x)}) \; .$$

When $\phi(x) = \phi_1(x)\,\phi_2(x)$, both $\phi_i(x)$ irreducible, then there are at least two such groups defined by the presentations $P_1 = (x,a \; ; \; a^{\phi(x)})$ and

$P_2 = (x,a,b \; ; \; a^{\phi_1(x)}, b^{\phi_2(x)})$.

4.3 CONJUGACY THEOREM .

We have remarked before that for a given natural number d and for each admissible automorphism α of F_d there is a presentation

$P_\alpha = (x,a_1,\ldots\ldots,a_d \; ; \; a_1^x = \alpha(a_1),\ldots\ldots\ldots,a_d^x = \alpha(a_d))$ of a knot-like group G_α with $G'_\alpha \cong F_d$. Given a polynomial $\phi(x)$ of degree d there may be another admissible automorphism β of F_d with $P_\beta = (x,a_1,\ldots\ldots,a_d \; ; \; a_1^x = \beta(a_1),\ldots\ldots,a_d^x = \beta(a_d))$ such that $G'_\alpha \cong F_d \cong G'_\beta$ and $\Delta_{G_\alpha}(x) = \phi(x) = \Delta_{G_\beta}(x)$. Then the natural question is: Are G_α and G_β always isomorphic ? The answer is in the negative; for example:

Take $\phi(x) = (x^2-3x+1)(x^2-x+1) = x^4-4x^3+5x^2-4x+1$.

$P_\alpha = (x,a_0,a_1,a_2,a_3; a_0^x = a_1, a_1^x = a_2, a_2^x = a_3, a_3^x = \bar{a}_0 a_1^4 a_2^{-5} a_3^4)$ and

$P_\beta = (x,a_0,a_1,a_2,a_3; a_0^x = a_1, a_1^x = a_2, a_2^x = a_3, a_3^x = \bar{a}_2 a_3)$. Then

$P_\alpha \simeq (x,a_0 \; ; \; a_0^{\phi(x)})$ and the chain of elementary ideals is $\epsilon_0(G_\alpha) = (0)$, $\epsilon_1(G_\alpha) = (x^4-4x^3+5x^2-4x+1)$, $\epsilon_2(G_\alpha) = (1)$,

while $P_\beta \simeq (x,a_0,a_1 \; ; \; a_0^{x^2-3x+1}, a_2^{x^2-x+1})$, and the chain of elementary ideals is $\epsilon_0(G_\beta)=(0)$, $\epsilon_1(G_\beta)=(x^4-4x^3+5x^2-4x+1)$, $\epsilon_2(G_\beta)=(x^2-3x+1, x^2-x+1)$, $\epsilon_3(G_\beta)=(1)$. Since the length of the chain of elementary ideals is an invariant of the group and for P_α this length is 2, for P_β this length is 3, we see that $G_\alpha \not\cong G_\beta$.

Next the question is: When are G_α and G_β isomorphic ? The following theorem answers this question and transfers the burden on to another unsolved problem.

<u>Theorem 4.2</u> $G_\alpha \cong G_\beta$ if and only if $\tau\alpha\tau^{-1}=\beta$ for some automorphism τ of F_d.

<u>Proof:</u> We may look at G_α and G_β as extensions of F_d by F_1 such that $G'_\alpha \cong G'_\beta = F_d$.

If $G_\alpha \cong G_\beta$ then let θ be this isomorphism and we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_d & \xrightarrow{\alpha} & G & \longrightarrow & F_1 & \longrightarrow & 0 \\ & & \downarrow \tau & & \downarrow \theta & & \downarrow & & \\ 1 & \longrightarrow & F_d & \xrightarrow{\beta} & G & \longrightarrow & F_1 & \longrightarrow & 0 \end{array}$$

in which both rows are exact and $\tau = \theta|_{G'_\alpha}$, the restriction of θ to $G'_\alpha \cong F_d$.

Now for any element $z \in F_d$ we have $\theta\alpha(z)=\beta\tau(z)$. Since $\alpha(z) \in G'_\alpha$ so $\theta(\alpha(z))=\tau(\alpha(z))$.

If $\tau(z) = y$ then $z = \tau^{-1}(y)$, so that

$$\theta\alpha(z) = \beta\tau(z) \Rightarrow \theta(\alpha(z)) = \beta(\tau(z)) \Rightarrow \tau(\alpha(z)) = \beta(\tau(z))$$
$$\Rightarrow \tau\alpha\tau^{-1}(y) = \beta(y).$$

Thus $\tau\alpha\tau^{-1} = \beta$. Hence α and β conjugates.

Conversely, let us suppose there exists an automorphism τ of F_d such that $\tau\alpha\tau^{-1} = \beta$, and we shall show that $G_\alpha \cong G_\beta$.

$$\tau\alpha\tau^{-1} = \beta \Rightarrow \tau\alpha^s\tau^{-1} = \beta^s \Rightarrow \tau\alpha^s = \beta^s\tau, \text{ for}$$

every integer s.

Since $G_\alpha/G'_\alpha \cong F_1(x)$ is free, G'_α is a split extension and hence is equivalent to a semidirect product of $G'_\alpha \cong F_d$ by $G_\alpha/G'_\alpha \cong F_1(x)$ with a homomorphism $\varphi: F_1(x) \longrightarrow \text{Aut } F_d$ defined by $\varphi(x) = \alpha$ i.e. $\alpha(a_i) = \varphi(x)(a_i) = a_i^x$, $i = 1, \ldots\ldots, d$ [11, p.250].

Then the elements of G will have the form $x^r a$, where $x^r \in F_1(x)$ for some integer r, $a \in F_d$ and the composition law is given by $(x^r a)(x^s a') = (x^{r+s})(\alpha^s(a).a')$ for all integers r, s and some $a, a' \in F_d$.

Similarly, we may look at G_β as the semidirect product of $G'_\beta \cong F_d$ by $G_\beta/G'_\beta \cong F_1(x)$ with a homomorphism $\varphi': F_1(x) \longrightarrow \text{Aut } F_d$ defined by $\varphi'(x) = \beta$, i.e. $\beta(a_i) = \varphi'(x)(a_i) = a_i^x$, $i=1,\ldots,d$.

Then the elements of G_β will have the form $x^r a$ with $x^r \in F_1(x)$, $a \in F_d$ and the law of composition is given by $(x^r a)(x^s a') = (x^{r+s})(\beta^s(a).a')$.

Now we define a mapping $\theta: G_\alpha \longrightarrow G_\beta$ by $\theta(x^r a) = x^r \tau(a)$.

Since τ is an automorphism of F_d, this mapping θ is a bijection. Therefore we have to check only that θ is actually a homomorphism.

Indeed, $\theta\left[(x^r a)(x^s a')\right] = \theta\left[x^{r+s}(\alpha^s(a).a')\right]$
$=(x^{r+s})(\tau(\alpha^s(a).a')) = (x^{r+s})(\tau\alpha^s(a).\tau(a'))$
$=(x^{r+s})(\beta^s\tau(a).\tau(a')) = (x^r\tau(a))(x^s\tau(a'))=\theta(x^r a).\theta(x^s a')$.

Hence $G_\alpha \cong G_\beta$. This proves the theorem.

Note that in the above proof the automorphism τ need not be an admissible automorphism of F_d.

This theorem suggests that the cardinality c of the family of isomorphism classes of knot-like groups G with $G' \cong F_d$ and $\Delta_G(x)=\phi(x)$ is equal to the cardinality of the family of conjugacy classes of admissible automorphisms of the group of automorphisms of F_d. Since the conjugate of an admissible automorphism is again admissible, the set of all presentations of the form under consideration is in one-one correspondence with these automorphisms and their isomorphism classes correspond to the conjugacy classes of the latter. Therefore c is equal to the cardinality of the family of conjugacy classes of all admissible automorphisms of the group F_d.

Furthermore, this theorem also suggests that the solution of our problem is directly hinged to the solution of the conjugacy problem in the group of automorphisms of the free group F_d. Since the solution to the latter problem is unknown, if indeed the problem is not unsolvable, we are forced to weaken our conditions.

4.4 REDUCTION MODULO THE SECOND COMMUTATOR SUBGROUP .

The above problem together with the fact that G and G/G" have the same Alexander polynomial lead us to factor out G". In the rest of this section a knot-like group G will mean G/G" and therefore by G' we shall understand $G'/G'' \cong F_d/F_d'$. Then the elements of G' commute and G_α will have a presentation

$$P_\alpha = (x, a_1, \ldots\ldots, a_d \ ; \ a_i^x = \alpha(a_i) = a_1^{\alpha_{i1}} a_2^{\alpha_{i2}} \ldots\ldots a_d^{\alpha_{id}}, i=1,\ldots,d)$$

similarly for G_β a presentation

$P_\beta = (x, a_1, \ldots, a_d \; ; \; a_i^x = \beta(a_i) = a_1^{\beta i1} a_2^{\beta i2} \ldots a_d^{\beta id}, i=1,\ldots,d)$

with α and β admissible automorphisms of F_d/F'_d. So now we can talk of the matrix $\|\alpha\|$ of the automorphism α as defined in the article 2.3 on page 12.

In this case theorem 4.2 becomes the following

<u>Theorem 4.3</u> $G_\alpha \cong G_\beta$ modulo G" if and only if the matrices $\|\alpha\|$ and $\|\beta\|$ are conjugate over the integers.

Thus the cardinality c^* of the family of isomorphism classes of knot-like groups G′ modulo G" for which $G' \cong F_d$ and $\Delta_G(x) = \phi(x)$ of fixed degree d, is equal to the cardinality of the family of conjugacy classes of d x d admissible matrices in the group of all unimodular d x d integer matrices. The problem of similarity of d x d integer matrices over the integers runs into the theory of numbers and is again an unsolved problem. Therefore we cannot go beyond this theorem at the present state of our knowledge.

Now, if $\phi(x)$ is irreducible over the integers then we have the following theorem regarding the cardinality c^*.

Theorem 4.4 If $\phi(x)$ is irreducible then c^* is at most equal to the class number [15] of the field $\frac{\mathbb{Z}[x]}{(\phi(x))}$.

Proof: By theorem 4.3 above there is a one to one correspondence between the isomorphism classes of knot-like groups G, satisfying $G' \cong F_d$ and $\Delta_G(x) = \phi(x)$, and the conjugacy classes of admissible matrices with characteristic polynomial $\phi(x)$.

On the other hand O.Taussky proved in [14] that there is a one to one correspondence between the conjugacy classes of matrix solutions of $\phi(x)=0$ and the ideal classes of the integer ring of $\frac{\mathbb{Z}[x]}{(\phi(x))}$ i.e. the class number of the field $\frac{\mathbb{Z}[x]}{(\phi(x))}$. Hence the theorem.

Since in general there is no explicit formula or method to compute the class number of a field, in practice the above theorem will not help us to find c^*. However, the above theorem gives an upper bound for c^* in case of an irreducible polynomial.

On the other hand the quadratic fields of class number one have been completely determined [12], so the result in [10] regarding d=2 is immediately obtained as

<u>Corollary 4.4.1</u> If $d = 2$ then $c^* = 1$.

<u>Proof:</u> If $d = 2$ then $\phi(x) = x^2 + c_1 x + c_o$, $\phi(1) = \pm 1$ $\phi(0) = \pm 1$, so that $c_o = \pm 1$, $c_1 = 3, 1, \text{or} -1$ and in each case the class number of $\frac{\mathbb{Z}[x]}{(\phi(x))}$ is one [12]. Then the corollary follows from theorem 4.4.

<u>Corollary 4.4.2</u> The set of all knot-like groups G, for which $\Delta_G(x) = x^2 + c_1 x \pm 1$ and $G' \cong F_2$ are isomorphic modulo G" and have a one-relator presentation modulo G".

<u>Proof:</u> Let G be such a group with $G" = 1$. Then there exists an admissible automorphism α of F_2/F_2' such that

$P_\alpha = (\ x, a, b\ ;\ a^x = b, b^x = a^{\pm 1} b^{-c_1}\ )$ defines G.

Replacing b by a^x and dropping $a^x = b$ and b, P_α reduces to $(\ x, a\ ;\ x^{-2} a x a^{c_1} x a^{\pm 1}\ )$.

Thus G has a one-relator presentation. Then from corollary 4.4.1 it follows that all the knot-like groups satisfying the hypothesis are isomorphic, so they are all isomorphic to G, hence have one-relator presentation.

In the case of one-relator knot-like groups we can find the exact value of c^*. As a matter of fact one-relator knot-like groups turn out to be much more interesting than the general case. The classification problem of these groups is the subject matter of the next section.

SECTION 5

CLASSIFICATION PROBLEM : ONE-RELATOR CASE

5.1 INTRODUCTION .

From the definition it follows that a one-relator knot-like group has a presentation $P = (x,b\ ;\ R)$ with two generators and one relator, where R may always assumed to be a reduced and cyclically reduced word in x and b such that the exponent sum of x in R is zero and that of b is ± 1. In this section we shall show that the cardinality c^* of the family of one-relator knot-like groups G for which $G' \cong F_d$ and $\triangle_G(x) = \phi(x)$ is 1, modulo the second commutator subgroup. Also we shall determine a necessary and sufficient condition for a knot-like group to be a one-relator group. We conclude this section with a conjecture.

5.2 THE ISOMORPHISM PROBLEM .

Throughout this section we assume that $P = (x,b\ ;\ R)$ and $P^* = (x,b\ ;\ R^*)$ are two presentations which define knot-like groups G and G^* respectively, such $G' \cong G^{*\prime} \cong F_d$ and $\triangle_G(x) = \triangle_{G^*}(x) = \phi(x) = c_0 + \ldots + c_{d-1}x^{d-1} + x^d$.

Let R_o and R_o^* be the rewrites of R and R^* respectively in terms of the conjugates b_i. Without any loss of generality we may assume $R_o = b_d b_{d-1}^{c_{d-1}} \ldots\ldots b_o^{c_o}$, because modulo the second commutator subgroup $R_o \equiv b_d b_{d-1}^{c_{d-1}} \ldots\ldots b_o^{c_o}$ (cf. lemma 5.1). We shall prove that for a given $\phi(x)$ there is exactly one isomorphism class of one-relator knot-like groups, modulo the second commutator subgroup, whose commutator subgroups are free of rank= deg $\phi(x)$ and Alexander polynomials are $\phi(x)$. For this we need the following

<u>Lemma 5.1</u> R_o^* is obtained from R_o by permuting the b_i, $i = 0,1,\ldots,d$ and inserting pairs $b_i, \bar{b}_i, i \neq 0,d$.

<u>Proof:</u> R_o and R_o^* are words in the b_i. Let $L(R_o)$ and $L(R_o^*)$ be their lengths as b_i-words respectively. The proof is by induction on $L(R_o^*)$.

Since P^* defines a one-relator group whose commutator subgroup is free of rank d, theorem 3.4 applies. Hence b_o and b_d must occur in R_o^* and each of them must occur only once (with exponent ± 1). Furthermore, 0, d are the minimum and maximum among the subscripts of the b_i in R_o^* (if not it can be made so the way we have done in article 2.3). Thus R_o^* can be rewritten as $b_o^{\epsilon} w(b_1,\ldots,b_d)$ and $v(b_o,\ldots\ldots,b_d)b_d$, $\epsilon = \pm 1$.

$L(R_o) = 2 + \sum_{i=1}^{d-1} |c_i|$, c_i the coefficients of the Alexander polynomial of G or G^*.

Clearly, $L(R_o^*)$ cannot be less than $L(R_o)$, otherwise the Alexander polynomial of G^* will be different from $\phi(x)$. Thus $L(R_o^*) \geqslant L(R_o)$.

Now, let us assume that $L(R_o^*) = L(R_o)$. R_o^* must be obtained from R_o just by permuting the b_i, $i = 0 \ldots \ldots, d$ and by inserting no pair of elements, that is $R_o^* \equiv R_o \bmod G''$ and our lemma is true in this case.

Next let us suppose our lemma is true for $L(R_o^*) = L(R_o) + n$, for even $n \geqslant 2$. Here n is even because of the fact that $\Delta_G(x) = \Delta_{G^*}(x)$. We shall show that the lemma is also true for $L(R_o^*) = L(R_o) + 2 + n$.

Since $n \geqslant 2$, R_o^* contains at least one pair $b_i, \bar{b}_i$, $i \neq 0,d$ more than R_o. Let us suppose that the pair is $b_1, \bar{b}_1$. R_o^* cannot contain a pair like b_1, b_1 or b_1, b_2, because in that case $\Delta_G(x) \neq \Delta_{G^*}(x)$. When we drop this pair from R_o^* we obtain $R_o^{*\prime}$ whose length is $L(R_o) + n$, and hence by our assumption $R_o^{*\prime}$ is obtained from R_o by permuting b_i, $i = 0,1,\ldots,d$ and inserting pairs $b_i,\bar{b}_i$, $i \neq 0,d$. Now R_o^* is a word which is obtained from $R_o^{*\prime}$ by inserting the pair $b_1,\bar{b}_1$. Thus R_o^* is a word which can be obtained from R_o by permuting b_i, $i = 0,1,\ldots,d$ and inserting various pairs $b_i,\bar{b}_i$, $i \neq 0,d$. Hence our lemma is proved.

This lemma says that $R_o^* \equiv R_o$ modulo G''. We have the following theorem from this lemma.

<u>Theorem 5.1</u> For a given $\phi(x)$ there is exactly one isomorphism class of one-relator knot-like groups, modulo the second commutator subgroup, whose commutator subgroups are free of rank $= \deg \phi(x)$ and Alexander polynomials are $\phi(x)$.

<u>Proof:</u> This is immediate from the lemma 5.1, since we have exactly one presentation, modulo the second commutator subgroup, satisfying the hypothesis of the theorem.

Thus we are now faced with a new problem i.e. which of the groups defined by the presentations

$$P_\alpha = (\ x, a_1, \ldots, a_d\ ;\ a_i^x = \alpha(a_i) = a_1^{\alpha_{i1}} a_2^{\alpha_{i2}} \cdots a_d^{\alpha_{id}}\ ,\quad i = 1,\ldots,d\ .)$$

are one-relator groups ? We give a necessary and sufficient condition for this in terms of the matrix $\|\alpha\|$.

5.3 A NECESSARY AND SUFFICIENT CONDITION.

Theorem 5.2 A necessary and sufficient condition for a knot-like group G given by the above presentation P_α to have a one-relator presentation modulo G" is that $\|\alpha\|$ be conjugate to the companion matrix of the Alexander polynomial of G over the integers.

Proof: This follows immediately from theorem 4.3 and the fact that the automorphism corresponding to the companion matrix of $\triangle_G(x)$ defines a one-relator knot-like group satisfying the hypothesis of the theorem. For, the companion matrix of $\triangle_G(x)$ is

$$\left\| \begin{matrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \cdot & \cdot & \cdot \cdots & \cdot \\ \cdot & \cdot & \cdots\cdots & \cdot \\ -c_0 & -c_1 & -c_2 \cdots & -c_{d-1} \end{matrix} \right\|$$

and the automorphism of F_d/F'_d corresponding to this matrix defines a presentation

$$(x, a_1, \ldots, a_d \; ; \; \bar{x} a_1 x = a_2, \ldots, \bar{x} a_{d-1} x = a_d,$$

$$\bar{x} a_d x = a_d^{-c_{d-1}} a_{d-1}^{-c_{d-2}} \cdots a_1^{-c_0})$$

$$\simeq (x, a_1 \; ; \; a_1^{c_0 + c_1 x + \cdots + c_{d-1} x^{d-1} + x^d}) .$$

The conjugacy problem for the group of automorphisms of F_d/F_d' is unsolved, so this is the best one can say at the present state of our knowledge.

Corollary 5.2 If $\|\alpha\|$ is a Jordan matrix then G/G'' is the free product of one-relator knot-like groups with amalgamated free cyclic groups.

Proof: If $\|\alpha\|$ is a Jordan matrix (cf. definition 2.3, page 13) then it is similar over $\mathbb{Z}$ to a matrix $\|\beta\|$, which is in the Jordan canonical form. Suppose $\|\beta\|$ is the direct sum of square matrices B_i of dimension $d_i \times d_i$, where B_i is the companion matrix of the polynomial $\phi_i(x)$ of degree d_i, $i = 1, \ldots, k$.

Now corresponding to each polynomial $\phi_i(x)$, free group F_{d_i} and matrix B_i we have an admissible automorphism β_i on F_{d_i}/F_{d_i}', such that $\left| \|\beta_i\| - Ix \right| = \phi_i(x)$, which defines a one-relator knot-like group G_i given by the presentation $P_i = (x_i , a_i \; ; \; a_i^{\phi_i(x)})$. Clearly $G_i' \cong F_{d_i}$, $\Delta_{G_i}(x) = \phi_i(x)$, $i = 1,\ldots\ldots,k$. Therefore

$$P_\beta = \prod_{(x)}^{*} P_i \; .$$

By theorem 4.3 P_α and P_β define isomorphic groups, so $G \cong \prod_{(x)}^{*} G_i$. This proves the corollary.

Note that in the above proof $\beta = \beta_1 \ldots\ldots \beta_k$ and this product of automorphisms on the free groups does not depend on the order in which the β_i appear i.e. $\beta_i \beta_j = \beta_j \beta_i$ because $F_{d_i} \cap F_{d_j} = 1$ for $i \neq j$.

5.4 A CONJECTURE .

Theorem 5.1 says that given an integral monic polynomial $\phi(x)$ of degree d, satisfying $\phi(0) = \pm 1$, $\phi(1) = \pm 1$, there exists exactly one one-relator knot-like group G, modulo the second commutator subgroup, for which $\Delta_G(x) = \phi(x)$ and $G' \cong F_d$. I believe this theorem can be improved by dropping "modulo the second commutator subgroup". Therefore my conjecture is :Given an integral monic polynomial $\phi(x)$ of degree d, satisfying $\phi(1) = \pm 1$, $\phi(0) = \pm 1$, there exists exactly one one-relator knot-like group G, for which $\Delta_G(x) = \phi(x)$ and $G' \cong F_d$.

For instance, let us consider the example of page 36 together with a new presentation P_γ as given below,

$$P_\alpha = (x, a_o, a_1, a_2, a_3 \ ; \ a_o^x = a_1, a_1^x = a_2, a_2^x = a_3, a_3^x = \bar{a}_o a_1^4 \bar{a}_2^5 a_3^4),$$

$$P_\beta = (x, a_o, a_1, a_2, a_3 \ ; \ a_o^x = a_1, a_1^x = \bar{a}_o a_1^3, a_2^x = a_3, a_3^x = \bar{a}_2 a_3),$$

$$P_\gamma = (x, a_o, a_1, a_2, a_3 \ ; \ a_o^x = a_1, a_1^x = a_2, a_2^x = a_3, a_3^x = a_1^6 a_2^{-3} a_3^2 \bar{a}_o a_1^{-2} a_3^2 a_2^{-2}).$$

We know that P_α is not isomorphic to P_β. Similarly we can show that P_γ is not isomorphic to P_β.

$P_\alpha, P_\beta, P_\gamma$ define knot-like groups $G_\alpha, G_\beta, G_\gamma$ respsctively, for which $G'_\alpha \cong G'_\beta \cong G'_\gamma \cong F_4$ and

$$\Delta_G(x) = \Delta_G(x) = \Delta_G(x) = x^4 - 4x^3 + 5x^2 - 4x + 1.$$

Also we can see that

$$P_\alpha \simeq (x, a_o \ ; \ a_o^{x^4 - 4x^3 + 5x^2 - 4x + 1}),$$

$$P_\beta \simeq (x, a_o, a_2 \ ; \ a_o^{x^2 - 3x + 1}, a_2^{x^2 - x + 1}),$$

$$P_\gamma \simeq (x, a_o \ ; \ a_o^{x^4 + 2x^2 - 2x^3 + 2x + 1 - 2x^3 + 3x^2 - 6x}).$$

Thus P_α and P_γ define one-relator knot-like groups where as P_β defines a knot-like group which has no one-relator presentation. For, suppose this is isomorphic to a one-relator presentation then modulo the second commutator subgroup, i.e. G_β / G_β'' has a one-relator presentation. If τ is the automorphism of F_4/F_4' which defines this presentation then $\|\tau\|$ should be conjugate to $\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 4 & -5 & 4 \end{Vmatrix}$ and $\|\beta\| = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{Vmatrix}$.

But these two matrices are not conjugates. Hence P_β is not isomorphic to a one-relator presentation.

P_α and P_γ are the same if we allow the a_i to commute i.e. if we factor out the second commutator subgroup. But if we donot factor out the second commutator subgroup P_α and P_γ are two different presentations. My conjecture says $P_\alpha \simeq P_\gamma$.

If G has a nontrivial centre then my conjecture can probably be proved by similar techniques to those used in [8]. A proof of my conjecture, however, at this time, seems out of reach.

BIBLIOGRAPHY

[1] Crowell,R.H. and Fox,R.H : Introduction to Knot theory,Blaisell Publishing Co.,N.Y.1963

[2] Fox,R.H : Free differencial calculus II,Ann.Math.59 (1954) 196-210.

[3] Jacobson,N : Lectures in Abstract Algebra II , Van Nostrand , N.Y. , 1951 .

[4] Kurosh, A.G : The theory of groups I , Chelsea Pub. Co., N.Y., 1960.

[5] Mac Mahan,P.A: Combinatory Analysis I , Chelsea Pub. Co., N.Y. , 1960.

[6] Magnus,W;Karrass,A;Solitar,D: Combinatorial group, Interscience Pub.Co.,N.Y.,1966.

[7] Neuwirth,L.P : Knot groups,Ann Math. Studies #56, Princ.Univ.Press, 1965.

[8] Pietrowski,A: The isomorphism problem for one-relator groups with nontrivial centre,Math.Z, 136 (1974) 95-106 .

[9] Rapaport,E.S: Note on Nielsen Transformations,A M S, Proc.10 (1959) 228-235.

[10] Rapaport,E.S: On the commutator subgroup of a knot group,Ann Math.71 (1960) 157-162.

[11] Scott,W,R: Group theory,Prentice-Hall,N.J.,1964.

[12] Stark,H.M: A complete determination of the complex quadratic fields of class number one, Mich.J.Math. 14 (1967) 1-27 .

[13] Strasser,E.R: Knot-like groups, Ann Math.,R.H.Fox Memorial vol. (1975) 1-16.

[14] Taussky,Olga: On a theorem of Latimer and MacDuff, Cand.J.Math. 1 (1949) 300-302.

[15] Weiss,E: Algebraic Number Theory, Mac Graw-Hill, N.Y., 1963.