ON MODULI OF MULTIPLE CONNECTED PLANE DOMAINS

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Ignacio Guerrero

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Abstract of the Dissertation

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Let $\Omega \subseteq \mathbb{C} \cup \{\infty\}$ be an open domain of finite connectivity $m \geq 2$. Suppose also that none of its boundary components reduces to a point. Represent Ω as U/Γ, where U is the upper half plane, and Γ is a Fuchsian group of the second kind without elliptic elements. We denote by $T^{\#}(\Gamma)$ the reduced Teichmüller space of Γ. It is known that $T^{\#}(\Gamma)$ carries a canonical real analytic structure and a canonical embedding as a domain in Euclidean space (Ahlfors, Bers, Earle).

The group Γ acts properly discontinuously on $T^{\#}(\Gamma) \times U$, and $V^{\#}(\Gamma) = T^{\#}(\Gamma) \times U/\Gamma$ is a real analytic manifold. $V^{\#}(\Gamma)$ is a fiber space over $T^{\#}(\Gamma)$. The fiber over $\tau \in T^{\#}(\Gamma)$ is a Riemann surface Ω_τ, a quasiconformal image of Ω. We define locally a map $F : V^{\#}(\Gamma) \to \mathbb{C}$ as follows: Given $\tau \in T^{\#}(\Gamma)$, $F(\tau, \cdot)$ is the classical circular slit map of Ω_τ (properly normalized). Real analyticity of F is proven. The map F

induces a (locally defined) real analytic map $Z_0 : T^{\#}(\Gamma) \to \mathbb{R}^{3m-6}$ ($m \geq 3$), $Z_0(\tau)$ being the parameters determining the circular slit domain corresponding to Ω_τ. Using variational formulas (Rauch, Ahlfors) we compute the differential of the map Z_0. This turns out to be non singular. Thus Z_0 provides real analytic local coordinates for $T^{\#}(\Gamma)$, with an explicit geometrical interpretation in terms of classical conformal mappings.

The variation of Green's function due to quasiconformal distortion of the domain is studied. A variational formula in terms of the Beltrami coefficient of the quasiconformal map is derived. To obtain this result, the Fuchsian group representation, Poincaré's series and well known variational formulas for quasiconformal mappings are used.

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0. Introduction

The classical theory of moduli of Riemann surfaces began with the work of Riemann. He observed that the conformal type of a compact Riemann surface of genus $g > 1$, depends on $3g-3$ complex parameters (moduli). His remarks were far from being a satisfactory answer to the problem, which in fact was vaguely formulated. One would like to construct numerical parameters associated with each surface; in order that two surfaces be conformally equivalent, these parameters should coincide.

Fundamental work of Teichmüller (1940) showed that the problem becomes simpler if conformal equivalence is replaced by a stronger equivalence relation. Teichmüller introduced a metric on this set of equivalence classes (Teichmüller space). He proved that with this metric it is homeomorphic to Euclidean space of real dimension $6g-6$. This result was already contained in the work of Fricke (1920's), who used a quite different approach.

The existence of a natural complex structure on Teichmüller space was first proven by Ahlfors (about 1960). He was strongly influenced by Teichmüller's ideas, and he used Rauch's construction (1955) of local coordinates around points which are not hyperelliptic surfaces. Other proofs were given shortly after Ahlfors by Bers, Kodaira and Spencer, and Weil.

Earle (mid 1960's) developed the theory of reduced Teichmüller space, generalizing Teichmüller space to include non-compact surfaces. The reduced Teichmüller space carries a natural real analytic structure, and it is finite dimensional for Riemann surfaces with finite double. A different generalization of Teichmüller space to open surfaces is due to Bers.

In this thesis, we consider the particular case of a plane domain of finite connectivity. Our first result is the construction of local real analytic coordinates for the reduced Teichmüller space. These coordinates are defined geometrically by means of the classical conformal representation of a multiple connected plane domain, onto a circular slit domain. Our second result concerns the variation of Green's function, due to a quasiconformal distortion of the domain. We obtained an explicit variational formula in terms of the Beltrami coefficient of the quasiconformal map.

Chapter 1 contains a description of the classical circular slit map. In particular we observe that the endpoints of the slits correspond to zeros of a holomorphic differential. Also a review of Green's function definition and elementary properties is given. In Chapter 2 we recall some basic facts about Teichmüller spaces. In the last section, we prove some non-standard properties of fiber spaces over reduced Teichmüller space. Chapter 3 is devoted to the construction of the local coordinates for the reduced Teichmüller space, from the circular slit representation of multiple connected planar surfaces. Finally, in Chapter 4 we derive the formula for the quasiconformal variation of Green's function. At the end of the chapter, we use some recent work of Gardiner to obtain the classical Schiffer's variation of Green's function as a special case.

1. Conformal Mappings of Plane Domains

1.1 Let $\Omega \subseteq \mathbb{C} \cup \{\infty\}$ be an open set of finite connectivity $m \geq 2$, we can assume without loss of generality (see Ahlfors [2] p. 244 for details) that Ω is bounded by analytic curves $C_1,\ldots,C_m$, and that none of the connected components of the complement of Ω reduces to a point.

We denote by ω_i, $i = 1,\ldots,m$, the harmonic measure of C_i; i.e., ω_i is the solution to the Dirichlet problem in Ω, with boundary values 1 on C_i and 0 on the other boundary curves. Observe that ω_i can be continued across the boundary according to the reflection principle.

Consider the function $u = \lambda_1\omega_1 + \ldots + \lambda_{m-1}\omega_{m-1}$, where the scalars $\lambda_1,\ldots,\lambda_{m-1}$ are chosen so that

$$(1.1) \qquad \int_{C_i} {}^*du = \begin{cases} 2\pi, & j = 1 \\ 0, & j = 2,\ldots,m-1 \\ -2\pi, & j = m \end{cases} .$$

In fact, let A denote the $(m-1) \times (m-1)$ matrix with entries $\alpha_{ij} = \int_{C_i} {}^*d\omega_j$, $i,j = 1,\ldots,m-1$, it is well-known that A is symmetric and non-singular. Let λ be the column vector with components λ_i, $i = 1,\ldots,m-1$, and e the $(m-1)$ column vector with components $1,0,\ldots,0$, then λ is the solution to the system of linear equations $A\lambda = 2\pi e$, the equation (1.1) for $j = m$ being a consequence of the former ones.

Define

$$(1.2) \qquad f(z) = \int_{z_0}^{z} du + i{*}du + u(z_0), \; z_0 \in \bar{\Omega},$$

f is well defined up to periods $\pm 2\pi i$ along C_1 and C_m. The function $F(z) = e^{f(z)}$ is then single-valued.

<u>Theorem</u>. The function F effects a one to one conformal mapping of Ω onto the annulus $1 < |w| < e^{\lambda_1}$ minus $m-2$ concentric arcs situated on the circles $|w| = e^{\lambda_i}$, $i = 2,\dots,m-1$. For a given choice of C_1 and C_m the map is uniquely determined up to a rotation.

This theorem is classical, for a proof see e.g. Ahlfors [2] p. 247.

1.2 To the domain Ω we associate a compact Riemann surface Ω^d of genus $g = m-1$, called the <u>double</u> of Ω. The surface Ω^d is obtained by taking a copy $\tilde{\Omega}$ of Ω and identifying corresponding points on the boundary. If $z_0 \in \Omega$ we take as local coordinate the identity map z, if $z_0 \in \tilde{\Omega}$ we let $\bar{z}$ be a local parameter. For a point $z_0 \in \partial\Omega$, let h map a neighborhood of z_0 in $\bar{\Omega}$ conformally onto $\{ w \,|\, |w| < 1, \operatorname{Im} w \geq 0\}$, $h(z_0) = 0$ and the boundary corresponding to $\operatorname{Im} w = 0$, the local coordinate at z_0 is

$$w = \begin{cases} h(z) & z \in \bar{\Omega} \\ \overline{h(z)} & z \in \tilde{\Omega} \end{cases} .$$

It is clear that Ω^d carries a canonical anticonformal involution, the image of a point $p \in \Omega^d$ under this map will be denoted by $\tilde{p}$.

Given a meromorphic q-differential $(q \geq 0)$ ω in Ω^d, the anticonformal involution induces a meromorphic q-differential $\tilde{\omega}$ in Ω^d. If $z_0 \in \Omega^d$ and z is a local coordinate in a neighborhood of z_0, $\tilde{\omega}$ is given by $\tilde{\omega} = \overline{f(\bar{z})}dz^q$, where f is such that $\omega = f(w)dw^q$ for $w = z$ if $z_0 \in \partial\Omega$ and $w = \bar{z}$ if $z_0 \notin \partial\Omega$.

A q-differential ω in Ω^d is called <u>symmetric</u> if $\omega = \tilde{\omega}$. A q-differential ω is said to be <u>real</u> on $\partial\Omega$, if for every point $z_0 \in \partial\Omega$ and every local coordinate z near z_0 such that $z = \bar{z}$ corresponds to $\partial\Omega$ $\omega = f(z)dz^q$ with f real for $z = \bar{z}$. Note that the definition is independent of the local coordinate chosen. It is not hard to see that a q-differential in Ω^d is symmetric if and only if it is real on $\partial\Omega$.

A simple computation shows that if ω is a meromorphic differential in Ω^d, and γ a path no passing through poles of ω

$$(1.3) \qquad \int_\gamma \omega = \overline{\int_{\tilde{\gamma}} \tilde{\omega}} \,.$$

<u>Proposition</u>. The symmetric holomorphic q-differentials on Ω^d are a real vector space of dimension:

$m-1$	for $q = 1$	$m \geq 2$,
1	for $q \geq 2$	$m = 2$,
$(2q-1)(m-2)$	for $q \geq 2$	$m \geq 3$.

Proof. The complexification of the space is the space of all holomorphic q-differentials.

We define cycles $a_1,\dots,a_{m-1}$, $b_1,\dots,b_{m-1}$ on Ω^d as follows: $a_i = \alpha_i - \tilde{\alpha}_i$, where α_i is a curve in $\bar{\Omega}$ joining C_m to C_i and b_i is the curve C_i properly oriented, $i = 1,\dots,m-1$. These cycles determine a canonical homology basis on Ω^d.

It is easy to see that the holomorphic differentials $\frac{1}{2}(d\omega_j + i{*}d\omega_j)$ on Ω, are purely imaginary on $\partial\Omega$ and therefore can be extended by reflection to holomorphic differentials φ_j on Ω^d, $j = 1,\dots,m$ such that $\tilde{\varphi}_j = -\varphi_j$. Further, a computation shows

$$(1.4) \qquad \int_{a_j} \varphi_k = \delta_{jk}, \quad \int_{b_j} \varphi_k = \tfrac{1}{2} i \alpha_{kj}, \quad k,j = 1,\dots,m-1.$$

Thus $\{\varphi_1,\dots,\varphi_{m-1}\}$ is the basis for the space of holomorphic differentials dual to the canonical homology basis determined by $a_1,\dots,a_{m-1}$, $b_1,\dots,b_{m-1}$.

The holomorphic differential $du + i{*}du$ on Ω, defined in 1.1, can also be extended by reflection to a holomorphic differential φ on Ω^d and the circular slit map F can be written

$$(1.5) \qquad F(z) = \exp\left(\int_{z_0}^{z} \varphi + u(z_0)\right), \quad z_0 \in \bar{\Omega}.$$

By considering in (1.5) $z \in \bar{\Omega}$ we have an extension of F to $\bar{\Omega}$, this extension is one to one when restricted to Ω and two to one on $\partial\Omega$. The path of integration is taken to be contained in $\bar{\Omega}$.

Proposition. The endpoints of the m-2 arcs in the image of $\bar{\Omega}$ under F correspond to the zeros of the differential φ. Therefore φ has simple zeros located by pairs in the boundary curves $C_2,\ldots,C_{m-1}$.

Proof. The endpoints of the arcs correspond to critical points of $\int_{z_0}^{z}\varphi$, $z \in C_j$, $j = 2,\ldots,m-1$. In fact, for $z \in C_j$ $\operatorname{Arg} F(z)=\frac{1}{i}(\int_{z_0}^{z}\varphi+u(z_0)-\lambda_j)$ attains a maximum and a minimum. Let a_0 be a critical point of $\int_{z_0}^{z}\varphi$, $z \in C_j$, choose a local coordinate $w = u+iv$ in a neighborhood of a_0 such that $w(a_0) = 0$ and near a_0, C_j corresponds to $v = 0$, then

$$0 = \frac{d}{du}\int_{z_0}^{w^{-1}(u)}\varphi\Big|_{u=0} = f(0)$$

where $\varphi = f(w)dw$. Therefore a_0 is a zero of φ. To complete the proof it is enough to observe that since Ω^d has genus m-1, φ has exactly 2m-4 zeros.

1.3 Let $\xi \in \Omega$, we denote by $\alpha_{\xi\tilde{\xi}}$ the unique abelian differential of third kind with simple poles at ξ and $\tilde{\xi}$ and residues -1 and +1 respectively, and, with purely imaginary periods over the canonical system of cycles $a_1,\ldots,a_{m-1}$, $b_1,\ldots,b_{m-1}$ on Ω^d defined in 1.2. Assume further that $\xi \notin a_j$, $j = 1,\ldots,m-1$.

Next define

$$(1.6)\qquad G(z,\xi) = -\frac{1}{2}\int_{\tilde{z}}^{z}\alpha_{\xi\tilde{\xi}},\quad z \in \bar{\Omega},$$

where the path of integration is chosen to be antisymmetric and not passing through ξ or $\tilde{\xi}$, except when $z = \xi$ (see Schiffer and Spencer [19] p. 93).

We observe that $\alpha_{\xi\tilde{\xi}} + \tilde{\alpha}_{\xi\tilde{\xi}} = 0$. Indeed, the differential $\alpha_{\xi\tilde{\xi}} + \tilde{\alpha}_{\xi\tilde{\xi}}$ is holomorphic, by (1.3) and the normalization in the definition of $\alpha_{\xi\tilde{\xi}}$ we have

$$\int_{b_j} \alpha_{\xi\tilde{\xi}} + \tilde{\alpha}_{\xi\tilde{\xi}} = \int_{b_j} \alpha_{\xi\tilde{\xi}} + \overline{\int_{b_j} \alpha_{\xi\tilde{\xi}}} = 0\,, \quad j = 1,\ldots,m-1.$$

Therefore $\alpha_{\xi\tilde{\xi}} + \tilde{\alpha}_{\xi\tilde{\xi}} = 0$. Now,

$$\overline{\int_{\tilde{z}}^{z} \alpha_{\xi\tilde{\xi}}} = \int_{z}^{\tilde{z}} \tilde{\alpha}_{\xi\tilde{\xi}} = -\int_{z}^{\tilde{z}} \alpha_{\xi\tilde{\xi}} = \int_{\tilde{z}}^{z} \alpha_{\xi\tilde{\xi}}\,,$$

thus $\int_{\tilde{z}}^{z} \alpha_{\xi\tilde{\xi}}$ is real.

The definition of $G(z,\xi)$ is independent of the choice of (antisymmetric) path of integration. To prove this claim, suppose γ_1, γ_2 are antisymmetric paths from $\tilde{z}$ to z. Let c be a small circle around ξ. The closed path $\gamma_1\gamma_2^{-1}$ is homologous in $\Omega^d - \{\xi,\tilde{\xi}\}$ to $n_1a_1 + \ldots + n_{m-1}a_{m-1} + k_1b_1 + \ldots + k_{m-1}b_{m-1} + \ell_1 c + \ell_2 \tilde{c}$ where n_j, k_j, $j = 1,\ldots,m-1$, and ℓ_1, ℓ_2 are integers. Now,

$$\int_{\gamma_1} \alpha_{\xi\tilde{\xi}} - \int_{\gamma_2} \alpha_{\xi\tilde{\xi}} = \int_{\gamma_1\gamma_2^{-1}} \alpha_{\xi\tilde{\xi}} = n_1\int_{a_1} \alpha_{\xi\tilde{\xi}} + \ldots + n_{m-1}\int_{a_{m-1}} \alpha_{\xi\tilde{\xi}}$$

$$+ k_1\int_{b_1} \alpha_{\xi\tilde{\xi}} + \ldots + k_{m-1}\int_{b_{m-1}} \alpha_{\xi\tilde{\xi}} + \ell_1\int_{c} \alpha_{\xi\tilde{\xi}} + \ell_2\int_{\tilde{c}} \alpha_{\xi\tilde{\xi}}$$

The right hand side is purely imaginary, but the left hand side is real, therefore, both are zero.

The following properties of the function $G(\cdot,\xi)$ are easily verified:

1) $G(z,\xi)$ is real harmonic in $\Omega-\{\xi\}$, continuous on $\overline{\Omega}-\{\xi\}$

2) $G(z,\xi) - \log|z-\xi|$ is harmonic in Ω

3) $G(z,\xi) = 0$ for $z \in \partial\Omega$

Therefore $G(\cdot,\xi)$ is the <u>Green's function</u> of Ω with singularity at ξ.

<u>Remark</u>. In our definition, the Green's function is negative.

2. Teichmüller Spaces

2.1 Let U be the upper half plane, and Γ be a Fuchsian group, i.e., a discrete subgroup of the group $\text{Möb}_{\mathbb{R}}$ of conformal self-maps of U. The limit set $\Lambda(\Gamma)$ is the set of points of accumulations of orbits, $\Lambda(\Gamma) \subseteq \mathbb{R} \cup \{\infty\}$. We assume that Γ is non-elementary, that is $\Lambda(\Gamma)$ contains more than two points. Γ is said to be of first kind if $\Lambda(\Gamma) = \mathbb{R} \cup \{\infty\}$, otherwise $\Lambda(\Gamma)$ is a perfect nowhere dense subset of $\mathbb{R} \cup \{\infty\}$ and Γ is called of second kind.

Let $L_\infty(\Gamma)$ be the space of Beltrami differentials for Γ. $L_\infty(\Gamma)$ consists of all $\mu \in L_\infty(U,\mathbb{C})$ such that

(2.1) $$(\mu\circ\gamma)\overline{\gamma'}/\gamma' = \mu, \text{ for all } \gamma \in \Gamma.$$

We denote by $M(\Gamma)$ the open unit ball in $L_\infty(\Gamma)$, $\mu \in M(\Gamma)$ is called a Beltrami coefficient for the group Γ.

A homeomorphism w of $U \cup \mathbb{R} \cup \{\infty\}$ into $\mathbb{C}$ is said to be μ-conformal provided it has locally integrable distributional derivatives satisfying the Beltrami equation

$$w_{\bar{z}} = \mu w_z.$$

We say that w is normalized if it fixes $0,1,\infty$. Given $\mu \in M(\Gamma)$ there is a unique normalized μ-conformal homeomorphism w_μ, mapping U onto itself (see Ahlfors-Bers [4]).

From now on, we will assume that $0,1,\infty \in \Lambda(\Gamma)$, unless otherwise explicitely stated. We say that the Beltrami coefficients μ and ν are $\mathbb{R}$-equivalent if and only if $w_\mu = w_\nu$ on $\mathbb{R}$. The Teichmüller space $T(\Gamma)$ is the set of $\mathbb{R}$-equivalence classes in $M(\Gamma)$. Two Beltrami coefficients μ and ν are called equivalent if and only if $w_\mu = w_\nu$ on $\Lambda(\Gamma)$. The reduced Teichmüller space $T^{\#}(\Gamma)$ is the set of equivalence classes in $M(\Gamma)$. Note that if T is of the first kind $T(\Gamma) = T^{\#}(\Gamma)$.

It is not hard to see that if $\mu \in M(\Gamma)$ then $w_\mu \circ \gamma \circ w_\mu^{-1} \in \text{Möb}_{\mathbb{R}}$ for all $\gamma \in \Gamma$ and $\theta_\mu : \gamma \mapsto \gamma_\mu = w_\mu \circ \gamma \circ w_\mu^{-1}$ is an isomorphism of Γ onto another Fuchsian group $\Gamma_\mu = w_\mu \Gamma w_\mu^{-1}$. Furthermore, two Beltrami coefficients μ and ν are equivalent if and only if $\theta_\mu = \theta_\nu$.

We denote by $Q^{\#}(\Gamma)$ the space of symmetric integrable holomorphic quadratic differentials; i.e., $Q^{\#}(\Gamma)$ consists of holomorphic functions f in $U \cup (\mathbb{R}-\Lambda(\Gamma))$ such that

1) f is real on $\mathbb{R}-\Lambda(\Gamma)$
2) $(f \circ \gamma)(\gamma')^2 = f$, for all $\gamma \in \Gamma$,
3) $\|f\| = \frac{1}{2} \int_{U/\Gamma} |f(z)dz \wedge d\bar{z}| < \infty$.

There is a natural real pairing between $L_\infty(\Gamma)$ and $Q^{\#}(\Gamma)$ given by

$$(2.2) \qquad (f,\mu) = \frac{1}{2} \operatorname{Re} \int_{U/\Gamma} f(z)\mu(z)|dz \wedge d\bar{z}|, \qquad f \in Q^{\#}(\Gamma),\ \mu \in L_\infty(\Gamma).$$

We define $N^{\#}(\Gamma) = \{\mu \in L_\infty(\Gamma) \mid (f,\mu) = 0, \text{ for all } f \in Q^{\#}(\Gamma)\}$.

<u>Theorem</u>. (Ahlfors, Bers, Earle). $T^{\#}(\Gamma)$ has a natural real analytic structure so that the canonical map $\Phi^{\#} : M(\Gamma) \to T^{\#}(\Gamma)$ is real analytic. The tangent space to $T^{\#}(\Gamma)$ at $\Phi^{\#}(0)$ is canonically isomorphic to $L_\infty(\Gamma)/N^{\#}(\Gamma)$. The pairing (2.2) gives a canonical isomorphism between $L_\infty(\Gamma)/N^{\#}(\Gamma)$ and the (real) conjugate space of $Q^{\#}(\Gamma)$.

For a proof see Ahlfors [3], Bers [6], Earle [10] [11].

<u>Remarks</u>.

1) $M(\Gamma)$ is an open set in the Banach space $L_\infty(\Gamma)$ therefore has a natural analytic structure.

2) If Γ is of the first kind the theorem also holds when we replace real analytic by complex analytic, and disregard Re in the pairing (2.2).

2.2 Let $\Omega(\Gamma)$ be the component of the complement of $\Lambda(\Gamma)$ which contains the lower half plane L. $\Omega(\Gamma)$ is L when Γ is of the first kind and the complement of $\Lambda(\Gamma)$ when Γ is of the second kind. $\Omega(\Gamma)$ is invariant under Γ and we can introduce the <u>Poincaré metric</u> λ with curvature -4, which satisfies

$$(2.3) \qquad (\lambda \circ \gamma)|\gamma'| = \lambda, \text{ for all } \gamma \in \Gamma$$

(see Kra [16], Chapter II).

The Banach space $B^{\#}(\Gamma)$ of <u>symmetric bounded holomorphic quadratic differentials</u> on L for Γ is the set of functions φ

holomorphic in $L \cup (\mathbb{R}-\Lambda(\Gamma))$ such that

1) φ is real on $\mathbb{R}-\Lambda(\Gamma)$,

2) $(\varphi\circ\gamma)(\gamma')^2 = \varphi$, for all $\gamma \in \Gamma$,

3) $\|\varphi\|_\infty = \sup\{|\varphi(z)|\lambda(z)^{-2};\ z \in L\} < \infty$.

<u>Theorem</u>. (Ahlfors, Bers, Earle). There is an analytic bijection j, mapping $T^{\#}(\Gamma)$ onto an open bounded domain in $B^{\#}(\Gamma)$. Every φ in $B^{\#}(\Gamma)$ with $\|\varphi\| < 2$ is of the form $j(\phi^{\#}(\mu))$ where $\mu(z) = -\frac{1}{2}\varphi(\bar{z})\lambda(\bar{z})^{-2}$.

For a proof see Ahlfors [3], Bers [6], Earle [11].

<u>Remark</u>. In the theorem the bijection is complex or real analytic according Γ is of the first or second kind.

There is a real pairing between $B^{\#}(\Gamma)$ and $Q^{\#}(\Gamma)$, given by the <u>Petersson scalar product</u>

$$(2.4) \qquad (f,\varphi) = \frac{1}{2}\mathrm{Re} \int_{U/\Gamma} f(z)\varphi(\bar{z})\lambda(\bar{z})^{-2}|dz\wedge d\bar{z}|,$$

$$f \in Q^{\#}(\Gamma),\ \varphi \in B^{\#}(\Gamma).$$

Through this pairing we have an isomorphism between $B^{\#}(\Gamma)$ and the (real) conjugate space to $Q^{\#}(\Gamma)$. For Γ of the first kind, if we disregard Re in (2.4), the pairing becomes complex.

Given $\mu \in M(\Gamma)$ we define a map (known as <u>right translation</u>) $r_\mu : M(\Gamma_\mu) \to M(\Gamma)$ by $r_\mu(\lambda) = \nu$ if and only if $w_\nu = w_\lambda \circ w_\mu$. A computation shows that the map is biholomorphic. Suppose

$\lambda_1, \lambda_2 \in M(\Gamma_\mu)$ are equivalent, then $w_{\lambda_1} = w_{\lambda_2}$ on $\Lambda(\Gamma_\mu)$ but $\Lambda(\Gamma_\mu) = w_\mu(\Lambda(\Gamma))$, therefore $w_{\lambda_1} \circ w_\mu = w_{\lambda_2} \circ w_\mu$ on $\Lambda(\Gamma)$; i.e., $r_\mu(\lambda_1)$ is equivalent to $r_\mu(\lambda_2)$. Since the projection from Beltrami coefficients to reduced Teichmüller space is real analytic it follows that the induced map $r_\mu : T^{\#}(\Gamma_\mu) \to T^{\#}(\Gamma)$ is a real analytic bijection, taking the class $\phi^{\#}(0)$ to the class $\phi^{\#}(\mu)$.

2.3 We suppose, in this section, that Γ is of the second kind. Let $\rho : U \to \Omega(\Gamma)$ be a universal covering map such that $\rho \circ J = \bar{\rho}$ where $Jz = -\bar{z}$. Define a group G

$$G = \{g \in \text{Möb}_{\mathbb{R}} \mid \text{There is } \gamma \in \Gamma \text{ such that } \rho \circ g = \gamma \circ \rho\},$$

and let H be the covering group of ρ; i.e.,

$$H = \{h \in \text{Möb}_{\mathbb{R}} \mid \rho \circ h = \rho\}.$$

Both G and H are Fuchsian groups of the first kind. We have an exact sequence

$$\{1\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\rho^*} \Gamma \longrightarrow \{1\},$$

where i the inclusion map and $\rho^*(g) = \gamma$ where $\gamma \in \Gamma$ is the unique element of Γ such that $\rho \circ g = \gamma \circ \rho$. We have $\Omega(\Gamma) \cong U/H$ and $\Omega(\Gamma)/\Gamma \cong U/G$.

We define $M'(G)$ to be the unit ball in the real Banach space $L'_\infty(G) = \{\mu \in L_\infty(G) \mid \mu \circ J = \bar{\mu}\}$ and denote by $T'(G)$ the image of $M'(G)$ under the canonical projection $\phi : M(G) \to T(G)$.

We call $T'(G)$ the symmetric part of $T(G)$.

Theorem. (Earle). Define a map $M(\Gamma) \to M'(G)$ by $\mu \mapsto \hat{\mu} = \mu \circ \rho\, \bar{\rho}'/\rho'$, where we extend μ to $\Omega(\Gamma)$ by setting $\mu(\bar{z}) = \overline{\mu(z)}$. This map is a bijection which induces a real analytic embedding $T^{\#}(\Gamma) \to T'(G) \subseteq T(G)$ with $\phi^{\#}(\mu) \mapsto \phi(\hat{\mu})$.

For a proof see Earle [10].

The tangent space to $T(G)$ at $\phi(0)$ is canonically isomorphic to $L_\infty(G)/N(G)$ where

$$N(G) = \{\mu \in L_\infty(G) \mid (f,\mu) = 0, \text{ for all } f \in Q^{\#}(G)\}.$$

Note that since G is of the first kind, $Q^{\#}(G)$ is a complex linear space. Here the pairing is the complex version of (2.2). (See remark in section 2.1.)

The tangent space to $T'(G)$ at $\phi(0)$ is canonically isomorphic to $L'_\infty(G)/N'(G)$ where

$$N'(G) = \{\mu \in L'_\infty(G) \mid (f,\mu) = 0, \text{ for all } f \in Q'(G)\}$$

and

$$Q'(G) = \{f \in Q^{\#}(G) \mid f \circ J = \bar{f}\}.$$

The maps $M(\Gamma) \to M'(G) \to M(G)$ induce maps

$$L_\infty(\Gamma)/N^{\#}(\Gamma) \to L'_\infty(G)/N'(G) \to L_\infty(G)/N(G)\,;$$

the first given by $\mu \mapsto \mu \circ \rho\, \bar{\rho}'/\rho'$ and the second given by the inclusion $L'_\infty(G) \to L_\infty(G)$. Recalling the pairing (2.4) we have the equivalent maps

$$B^{\#}(\Gamma) \to B'(G) \to B^{\#}(G)$$

where

$$B'(G) = \{\varphi \in B^{\#}(G) | \varphi\circ J = \bar{\varphi}\}.$$

The first map is $\varphi \mapsto (\varphi\circ\rho)(\rho')^2$ and the second the inclusion map. $B'(G)$ is a real linear subspace of $B^{\#}(G)$, actually $B^{\#}(G)$ is the complexification of $B'(G)$. In fact every $\varphi \in B^{\#}(G)$ can be written as $\varphi_1 + i\varphi_2$ with $\varphi_1, \varphi_2 \in B'(G)$, explicitly

$$\varphi = \frac{1}{2}(\varphi + \overline{\varphi\circ J}) + i\,\frac{1}{2i}(\varphi - \overline{\varphi\circ J}).$$

2.4 Let S_0 be a Riemann surface. A marked Riemann surface with respect to S_0 is a surface S together with a quasi-conformal homeomorphism $f : S_0 \to S$. Two marked Riemann surfaces (S_0, f_1, S_1) and (S_0, f_2, S_2) are called equivalent if there is a conformal map $h : S_1 \to S_2$ such that $f_2^{-1}\circ h\circ f_1$ is homotopic to the identity map. The reduced Teichmüller space $T^{\#}(S_0)$ is the set of equivalence classes. For the Teichmüller space $T(S_0)$ we require the homotopy to be relative to the boundary curves.

Let Γ be a Fuchsian group;then the orbit space U/Γ is in a natural way a Riemann surface.

Theorem. There is a canonical isomorphism between $T^{\#}(\Gamma)$ and $T^{\#}(U_\Gamma/\Gamma)$, where U_Γ is the upper half plane with all fixed points of elliptic elements of Γ removed.

If Γ contains no elliptic transformations see proof in Earle [11], for the general case see Bers and Greenberg [9].

A Riemann surface S is said to be finite if there is a holomorphic embedding $f : S \to \bar{S}$, where $\bar{S}$ is a compact Riemann surface and $\bar{S}-f(S)$ is a finite set of points.

<u>Theorem</u>. Let Γ be a Fuchsian group, $\Omega_0(\Gamma)$ the set of points in $\Omega(\Gamma)$ which are not fixed points of nontrivial elements of Γ. The following are equivalent

1) $T^{\#}(\Gamma)$ has finite dimension.
2) $\Omega_0(\Gamma)/\Gamma$ is a finite Riemann surface.
3) Γ is a finitely generated group.

Proof in Earle [11].

2.5 Let $\mu \in M(\Gamma)$, we recall that w_μ maps $U \cup \mathbb{R} \cup \{\infty\}$ onto itself and fixes $0, 1, \infty$. The map w_μ can be extended to $\mathbb{C} \cup \{\infty\}$ by setting $w_\mu(z) = \overline{w_\mu(\bar{z})}$ for $z \in L$, this extension is μ-conformal in $\mathbb{C}$ where $\mu(z) = \overline{\mu(\bar{z})}$ for $z \in L$.

<u>Theorem</u>. Let $K \subseteq \mathbb{C}$ be a compact set. There exists a constant M depending on K and $\|\mu\|$ such that

$$(2.5) \qquad |w_\mu(z_1)-w_\mu(z_2)| \leq M|z_1-z_2|^\alpha, \quad |z_1-z_2| \leq M|w_\mu(z_1)-w_\mu(z_2)|^\alpha,$$

where $\alpha = (1 - \|\mu\|)/(1 + \|\mu\|)$.

See Ahlfors [3] p. 51 for a proof.

Let $[z_1, z_2]$ denote the spherical distance. The following result is proven in Ahlfors and Bers [4].

Theorem. There exists a constant c such that

$$[w_\mu(z), w_\nu(z)] \le c\|\mu-\nu\|.$$

If $\mu \in M(\Gamma)$ depends analytically on real parameters, so does $w_\mu(z)$ for every z, (Ahlfors and Bers [4]); in particular, let $\nu \in L_\infty(\Gamma)$ and denote by $\dot{w}_\mu[\nu](z)$ the directional derivative $\frac{\partial}{\partial s} w_{\mu+s\nu}(z)|_{s=0}$. We have

(2.6) $$\dot{w}_\mu[\nu](z) = -\frac{1}{\pi}\int_{\mathbb{C}} \nu(t)R(w_\mu(t), w_\mu(z))(w_\mu)_t(t))^2 dudv,$$
$$t = u+iv,$$

where we extended ν to $\mathbb{C}$ by reflexion and

(2.7) $$R(t,z) = \frac{z(z-1)}{t(t-1)(t-z)}.$$

These formulas can be found in Ahlfors [3].

Given $\mu \in M(\Gamma)$ there is a unique normalized homeomorphism w^μ of $\mathbb{C} \cup \{\infty\}$ onto itself which is μ-conformal in U and conformal in L. If μ dpends analytically on complex parameters, so does $w^\mu(z)$ for every z (see Ahlfors and Bers [4]).

2.6 Let G be a Fuchsian group of the first kind. Let $\sigma \in M(G)$ then the domain $w^\sigma(U)$ depends only on the Teichmüller class $\Phi(\sigma)$. The Bers fiber space is

$$F(G) = \{(\phi(\sigma), z) \mid \phi(\sigma) \in T(G),\ z \in w^{\sigma}(U)\}.$$

This space is an open subset of $T(G) \times \mathbb{C}$ (Bers [7]), therefore a complex manifold.

We have a map

$$\ell : T(G) \times U \to F(G)$$

given by $\ell(\phi(\sigma), z) = (\phi(\sigma), h_\sigma(z))$ where $h_\sigma = w^{\sigma} \circ w_{\sigma}^{-1}$, ℓ is a real analytic bijection (Bers [7], Earle [12]).

Now let Γ and G be as in section 2.3. Further we assume that Γ is finitely generated without elliptic elements. Therefore so is G. For $\mu \in M(\Gamma)$, $w_\mu : \Omega(\Gamma) \to \Omega(\Gamma_\mu)$; here the map is supposed extended to L by reflexion. Let ρ_μ be defined by the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ w_{\hat{\mu}}\ } & U \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_\mu} \\ \Omega(\Gamma) & \xrightarrow{\ w_\mu\ } & \Omega(\Gamma_\mu) \end{array}$$

where, as before, $\hat{\mu} = (\mu \circ \rho)\overline{\rho'}/\rho'$.

<u>Proposition</u>. The map ρ_μ is holomorphic and depends only on the class $\phi^{\#}(\mu)$.

<u>Proof</u>. $\rho_\mu = w_\mu \circ \rho \circ w_{\hat{\mu}}^{-1}$, let $s = \rho(t)$, $t = w_{\hat{\mu}}^{-1}(z)$

$$(\rho_\mu)_{\bar z}(z) = (w_\mu)_s(s)\rho'(t)(w_{\hat\mu}^{-1})_{\bar z}(z) + (w_\mu)_{\bar s}(s)\overline{\rho'(t)}\,\overline{(w_{\hat\mu}^{-1})_{\bar z}(z)}$$

$$= -(w_\mu)_s(s)\rho'(t)\frac{(w_{\hat\mu})_{\bar t}(t)}{J(t)} + (w_\mu)_{\bar s}(s)\overline{\rho'(t)}\;\frac{(w_{\hat\mu})_t(t)}{J(t)}$$

where $J(t) = |(w_\mu)_t(t)|^2 - |(w_\mu)_{\bar t}(t)|^2$,

$$(\rho_\mu)_{\bar z}(z) = \frac{(w_\mu)_s(s)\rho'(t)(w_{\hat\mu})_t(t)}{J(t)}\left[-\frac{(w_{\hat\mu})_{\bar t}(t)}{(w_{\hat\mu})_t(t)} + \frac{(w_\mu)_{\bar s}(s)}{(w_\mu)_s(s)}\,\frac{\overline{\rho'(t)}}{\rho'(t)}\right],$$

and

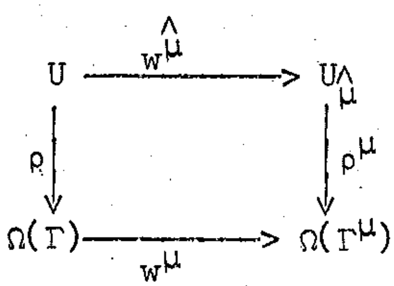
$$-\frac{(w_{\hat\mu})_{\bar t}(t)}{(w_{\hat\mu})_t(t)} - \frac{(w_\mu)_{\bar s}(s)}{(w_\mu)_s(s)}\,\frac{\overline{\rho'(t)}}{\rho'(t)} = -\hat\mu(t) + \mu(s)\,\frac{\overline{\rho'(t)}}{\rho'(t)} = 0\,.$$

Therefore, $(\rho_\mu)_{\bar z} = 0$.

Now, $\phi^\#(\mu) = \phi^\#(\nu)$ if and only if $w_\sigma = w_\mu^{-1}\circ w_\nu$ commutes with Γ. In this case $w_{\hat\sigma}$ commutes with G and $w_\sigma\circ\rho = \rho\circ w_{\hat\sigma}$ (Earle [10], Lemma 6). A computation shows that $w_{\hat\sigma} = w_{\hat\mu}^{-1}\circ w_{\hat\nu}$. Therefore we have $w_\mu^{-1}\circ w_\nu\circ\rho = \rho\circ w_{\hat\mu}^{-1}\circ w_{\hat\nu}$ which implies $\rho_\nu = w_\nu\circ\rho\circ w_{\hat\nu}^{-1} = w_\mu\circ\rho\circ w_{\hat\mu}^{-1} = \rho_\mu$.

<u>Proposition.</u> The map $h : (\phi(\hat\mu), z) \to \rho_\mu(z)$ from $T'(G) \times U$ into $\mathbb{C} \cup \{\infty\}$ is real analytic.

<u>Proof.</u> Let $M(\Gamma,\Omega(\Gamma))$ be the space of Beltrami coefficients for Γ supported in $\Omega(\Gamma)$; i.e., the elements of the unit ball in $L_\infty(\Omega(\Gamma),\mathbb{C})$ satisfying (2.1). The map $\mu \mapsto \hat\mu$ extends naturally to a bijection $M(\Gamma,\Omega(\Gamma)) \to M(G)$. Define ρ^μ by the diagram



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here, w^{μ} is the unique normalized μ-conformal map of $\mathbb{C} \cup \{\infty\}$ onto itself, Γ^{μ} is the Kleinian group $w^{\mu}\Gamma(w^{\mu})^{-1}$ and, as before, $w^{\hat{\mu}}$ is conformal in L. The map ρ^{μ} depends only on the class $\phi(\hat{\mu})$ (see e.g. Kra [17]) and $k : (\phi(\hat{\mu}),z) \to \rho^{\mu}(z)$ is a holomorphic map $F(G) \to \mathbb{C} \cup \{\infty\}$ (see Earle and Kra [13]). We have a commutative diagram

$$\begin{array}{ccccc} T'(G) \times U & \xrightarrow{i} & T(G) \times U & \xrightarrow{\ell} & F(G) \\ & \searrow^{h} & & & \downarrow k \\ & & & & \mathbb{C} \cup \{\infty\} \end{array}$$

Indeed, $k \circ \ell(\phi(\hat{\mu}),z) = k(\phi(\hat{\mu}), w^{\hat{\mu}} \circ w_{\hat{\mu}}^{-1}(z)) = w^{\mu} \circ \rho \circ (w^{\hat{\mu}})^{-1} \circ w^{\hat{\mu}} \circ w_{\hat{\mu}}^{-1}(z)$ $= w^{\mu} \circ \rho \circ w_{\hat{\mu}}^{-1}(z) = \rho_{\mu}(z)$, since $\phi(\hat{\mu}) \in T'(G)$ implies $w^{\mu} = w_{\mu}$. It follows that h is real analytic.

The domain $\Omega(\Gamma_{\mu}) = w_{\mu}(\Omega(\Gamma))$ depends only on the class $\phi^{\#}(\mu)$, we define then

$$F^{\#}(\Gamma) = \{(\phi^{\#}(\mu),z) \mid \phi^{\#}(\mu) \in T^{\#}(\Gamma),\ z \in \Omega(\Gamma_{\mu})\}.$$

Note that $T^{\#}(\Gamma) \times U \subseteq F^{\#}(\Gamma)$.

Define $\beta : T'(G) \times U \to F^{\#}(\Gamma)$ by $\beta(\phi(\hat{\mu}),z) = (\phi^{\#}(\mu), \rho_{\mu}z)$.

<u>Proposition</u>. The map β is a real analytic surjective local homeomorphism.

<u>Proof</u>. It is clear from the proposition above and theorem in section 2.3 that β is real analytic, it is obviously surjective.

A computation (see Earle and Kra [13]) shows that the differential of β at $(\phi(0),z)$ is given by

$$(\hat{\nu},\xi) \mapsto (\nu,\dot{w}[\nu](\rho(z)) - \rho'(z)\dot{w}[\hat{\nu}](z) + \rho'(z)\xi).$$

Since $\nu \in N^{\#}(\Gamma)$ if and only if $\hat{\nu} \in N'(G)$, it follows that the differential is injective. By use of the right translation maps (section 2.2), we conclude that the differential of β is everywhere nonsingular.

The group G acts properly discontinuously on F(G) by $g(\phi(\sigma),z) = (\phi(\sigma),g^{\sigma}(z))$, $g^{\sigma} = w^{\sigma}\circ g\circ(w^{\sigma})^{-1}$ (Bers [7]). Also G acts on $T(G) \times U$ by $g(\phi(\sigma),z) = (\phi(\sigma),g_{\sigma}(z))$, $g_{\sigma} = w_{\sigma}\circ g\circ w_{\sigma}^{-1}$. It is easy to see that these actions commute with ℓ, hence G acts properly discontinuously on $T(G) \times U$ as well. Obviously the action on $T(G) \times U$ induces a properly discontinuous action on $T'(G) \times U$. Similarly we have an action of Γ in $F^{\#}(\Gamma)$.

<u>Proposition</u>. Γ acts properly discontinuously on $F^{\#}(\Gamma)$.

<u>Proof</u>. An easy computation shows that $\beta\circ g = \rho^*(g)\circ\beta$. Given an arbitrary point in $F^{\#}(\Gamma)$, choose an open set $V \subseteq T'(G) \times U$ over it such that

1) $\beta|_V$ is injective.
2) $g(V) \cap V = \phi$, for all $g \in G$.

Suppose there is $\gamma \in \Gamma$ such that $\gamma\beta(V) \cap \beta(V) \neq \phi$. Let $g \in G$ such that $\rho^*(g) = \gamma$. Then $\beta(g(V)) \cap \beta(V) \neq \phi$, so there are $(\phi(\hat{\mu}),z)$, $(\phi(\hat{\nu}),w) \in V$ so that $\beta g(\phi(\hat{\mu}),z) =$

$= \beta(\phi(\hat{\nu}),w)$; i.e., $\beta(\phi(\hat{\mu}),g_{\hat{\mu}}(z)) = \beta(\phi(\hat{\nu}),w)$, this implies $(\phi^{\#}(\mu),\rho_{\mu}g_{\hat{\mu}}(z)) = (\phi^{\#}(\nu),\rho_{\nu}w)$. Therefore $\rho_{\mu}g_{\hat{\mu}}(z) = \rho_{\mu}(w)$, so there exists $h \in H$ such that $g_{\hat{\mu}}(z) = h_{\hat{\mu}}(w)$, which implies that $g = h$. Then $\gamma = \rho^*(g) = \rho^*(h) = 1$.

<u>Theorem</u>. $V(\Gamma) = F^{\#}(\Gamma)/\Gamma$ and $V^{\#}(\Gamma) = T^{\#}(\Gamma) \times U/\Gamma$ are real analytic manifolds with real analytic projections $V(\Gamma) \to T^{\#}(\Gamma)$, $V^{\#}(\Gamma) \to T^{\#}(\Gamma)$. The fibers over $\phi^{\#}(\mu)$ are $\Omega(\Gamma_{\mu})/\Gamma_{\mu} = (U/\Gamma_{\mu})^d$ and U/Γ_{μ} respectively.

<u>Proof</u>. It follows from the propositions above.

3. Local Coordinates for Reduced Teichmüller Space

3.1 Let $\Omega \subseteq \mathbb{C} \cup \{\infty\}$ be as in 1.1, it is well-known that Ω is conformally equivalent to U/Γ, where Γ is a finitely generated normalized Fuchsian group of the second kind without elliptic elements. We are assuming that $m \geq 3$.

In $(U/\Gamma)^d$ we have a canonical homology basis $a_1,\ldots,a_{m-1}$, $b_1,\ldots,b_{m-1}$ so that $b_1,\ldots,b_{m-1}$ correspond to boundary curves in U/Γ (see 1.2). For every $\phi^{\#}(\mu) \in T^{\#}(\Gamma)$, we have the Riemann surface $(U/\Gamma_\mu)^d$ with a canonical homology basis $a_1(\mu),\ldots,a_{m-1}(\mu)$, $b_1(\mu),\ldots,b_{m-1}(\mu)$ determined by the isomorphism $\theta_\mu : \Gamma \to \Gamma_\mu$, therefore depending only on $\phi^{\#}(\mu)$.

On $(U/\Gamma_\mu)^d$ we can construct an abelian differential of the first kind $\varphi(\mu)$ as it was done in sections 1.1 and 1.2. Thus, we have a function $f : F^{\#}(\Gamma) \to \mathbb{C}$ such that $f(\gamma(\phi^{\#}(\mu),z))\gamma_\mu'(z) = f(\phi^{\#}(\mu),z)$.

<u>Proposition</u>. The differential $\varphi(\mu)$ depends real analytically on $\phi^{\#}(\mu) \in T^{\#}(\Gamma)$; i.e., f is real analytic.

<u>Proof</u>. $\varphi(\mu) = 2\sum_{k=1}^{m-1} \lambda_k(\mu)\varphi_k(\mu)$, where $\varphi_1(\mu),\ldots,\varphi_{m-1}(\mu)$ is the canonical basis for the space of abelian differentials of the first kind dual to $a_1(\mu),\ldots,a_{m-1}(\mu)$, $b_1(\mu),\ldots,b_{m-1}(\mu)$. The coefficients $\lambda_1(\mu),\ldots,\lambda_{m-1}(\mu)$ are solutions of $P\lambda = \pi i e$, where $P(\mu)$ is the period matrix

of the normalized basis $\varphi_1(\mu),\dots,\varphi_{m-1}(\mu)$, (see 1.1 and 1.2). Thus, it suffices to study the dependence of $\varphi_k(\mu)$ $k = 1,\dots,m-1$, on $\phi^{\#}(\mu)$. Again, we use the notation of 2.3. $\varphi_k(\mu)$ can be lifted to $f_k : F^{\#}(\Gamma) \to \mathbb{C}$ such that $f_k(\gamma(\phi^{\#}(\mu),z))\gamma_\mu'(z) = f_k(\phi^{\#}(\mu),z)$. Define $\hat{f}_k : T'(G) \times U \to \mathbb{C}$ by $\hat{f}_k(\phi(\hat{\mu}),z) = f_k(\phi^{\#}(\mu),\rho_\mu(z))\rho_\mu'(z)$. It is easily seen that $\hat{f}_k(g(\phi(\hat{\mu}),z))g_{\hat{\mu}}'(z) = \hat{f}_k(\phi(\hat{\mu}),z)$. Now, $\hat{f}_k$ can be extended to $T(G) \times U$ in the obvious way, denote the extension by the same symbol.

The function $\hat{f}_k$ is the pull back of $p_k : F(G) \to \mathbb{C}$ via $\ell : T(G) \times U \to F(G)$; i.e., $\hat{f}_k(\phi(\mu),z) = p_k(\phi(\mu),h_\mu(z))h_\mu'(z)$ (see 2.6). Bers [8] proved that p_k is holomorphic, it follows that $\hat{f}_k$ is real analytic and hence so is f_k.

We recall, that the differential $i\varphi(\mu)$ is symmetric and it has 2m-4 simple zeros located by pairs in the boundary curves $b_2(\mu),\dots,b_{m-1}(\mu)$. The harmonic differential Re $\varphi(\mu)$ is exact in U/Γ_μ and Re $\varphi(\mu) = du(\mu)$, where $u(\mu)$ is the unique harmonic function in U/Γ_μ with boundary values $\lambda_1(\mu),\dots,\lambda_{m-1}(\mu)$ on $b_1(\mu),\dots,b_{m-1}(\mu)$ and 0 on the m-th boundary component.

Let $z_0(\mu)$ denote a real analytic section $T^{\#}(\Gamma) \to V^{\#}(\Gamma)$. We define

$$F : V^{\#}(\Gamma) \to \mathbb{C}$$

by

$$(3.1) \qquad F(\phi^{\#}(\mu),z) = \exp\left(\int_{z_o(\mu)}^{z} \varphi(\mu) + u(\mu)(z_o(\mu))\right).$$

For fixed $\phi^{\#}(\mu) \in T^{\#}(\Gamma)$, $F(\phi^{\#}(\mu),\cdot)$ is the circular slit map of U/Γ_μ. It is clear that the map F is real analytic.

3.2 It follows from the implicit function theorem, that in a neighborhood of $\phi^{\#}(\mu) = 0$ there are 2m-4 real analytic local sections $z_j : T^{\#}(\Gamma) \to V(\Gamma)$, $j = 0,\ldots,2m-5$ so that $z_o(\mu),\ldots,z_{2m-5}(\mu)$ are the zeros of $\varphi(\mu)$. Assume that we have ordered them so that

$$z_{2j}(\mu),\ z_{2j+1}(\mu) \in b_{j+2}(\mu), \quad j = 0,\ldots,m-3.$$

Now, define in a neighborhood of $\phi^{\#}(\mu) = 0$ a map

$$Z_o : T^{\#}(\Gamma) \to \mathbb{R}^{3m-6}$$

as follows

$$(3.2) \qquad Z_o(\phi^{\#}(\mu)) = (\rho_1(\mu),\ldots,\rho_{m-1}(\mu),\alpha_1(\mu),\ldots,\alpha_{2m-5}(\mu)),$$

where

$$(3.3) \qquad \rho_j(\mu) = \exp \lambda_j(\mu), \quad j = 1,\ldots,m-1,$$

$$(3.4) \qquad \alpha_k(\mu) = \operatorname{Im} \int_{z_o(\mu)}^{z_k(\mu)} \varphi(\mu),\ k = 0,\ldots,2m-5.$$

In (3.4) the integral is taken over the path in $(U/\Gamma_\mu)^d$ that we now define. Choose an arbitrary but fixed path from $z_o(0)$ to $z_k(0)$ in $(U/\Gamma)^d$, to this path we add straight line segments (in some local coordinates) from $z_o(\mu)$ to to $z_o(0)$ and from $z_k(0)$ to $z_k(\mu)$. This construction is possible, provided $\phi^{\#}(\mu)$ is sufficiently close to 0.

Proposition. The map Z_o is real analytic.

Proof. It follows from the remarks in section 3.1, and the fact that z_j $(j = 0,\ldots,2m-5)$ is real analytic.

Note that $\rho_j(\mu) = |F(\phi^{\#}(\mu),z)|$, $z \in b_j(\mu)$, $j = 1,\ldots,m-1$, and $\alpha_k(\mu) = \operatorname{Arg} F(\phi^{\#}(\mu),z_k(\mu))$, $k = 0,\ldots,2m-5$, where Arg is the branch of the argument determined by the above choice of integration paths. It is clear then that ρ_1 is the radius of the outer circle in the corresponding circular slit domain, $\rho_2,\ldots,\rho_{m-1}$ are the radii of the circles containing the m-2 slits, and $\alpha_o,\ldots,\alpha_{2m-5}$ are the arguments of the slits endpoints (by the choice of z_o, $\alpha_o = 0$).

3.3 We collect now, some variational formulae due to Rauch and Ahlfors. For proofs and details see Rauch [18] and Ahlfors [1].

Let $p_{ij}(\mu)$ denote the (i,j) entry $(i,j = 1,\ldots,m-1)$ of the period matrix $P(\mu)$ of the normalized basis $\varphi_1(\mu),\ldots,\varphi_{m-1}(\mu)$. The differential of p_{ij} at $\phi^{\#}(\mu) = 0$ is given by

$$(3.5)\quad d_o p_{ij}[\mu] = -i \int_{(U/\Gamma)} d\varphi_i(z)\varphi_j(z)\mu(z)|dz\wedge d\bar{z}|,$$

$$\mu \in L_\infty(\Omega(\Gamma),\mathbb{C}),$$

or

(3.6) $$d_0 p_{ij}[\mu] = -2i \ \mathrm{Re} \int_{U/\Gamma} \varphi_i(z)\varphi_j(z)\mu(z)\,|dz\wedge d\bar{z}|,$$

$$\mu \in L_\infty(\Gamma).$$

Formula (3.6) follows from (3.5) since $\varphi_i\varphi_j$ is real on $\partial(U/\Gamma)$ and $\mu(z) = \overline{\mu(\bar{z})}$.

Now let

(3.7) $$\tau_j(\mu) = \int_{z_0(\mu)}^{z_j(\mu)} \varphi(\mu),$$

then

(3.8) $$d_0\tau_j[\mu] = \frac{1}{2\pi} \int_{(U/\Gamma)^d} \omega_j(z)\varphi(z)\mu(z)\,|dz\wedge d\bar{z}|,$$

$$\mu \in L_\infty(\Omega(\Gamma), \mathbb{C}),$$

where ω_j is the differential of the third kind which has 0 b-periods and simple poles of residue 1 at z_j and -1 at z_0, the periods being computed on a system of representative cycles of the canonical homology basis on $(U/\Gamma)^d$, chosen so that they do not pass through the points $z_0, \ldots, z_{2m-5}$.

Let w_j be the differential of third kind having the same singularities as ω_j, but purely imaginary periods. The differential w_j is symmetric; in fact, $w_j - \tilde{w}_j$ is holomorphic and

$$\int_{a_k} w_j - \tilde{w}_j = \int_{a_k} w_j - \int_{a_k} \tilde{w}_j = \int_{a_k} w_j - \overline{\int_{\tilde{a}_k} w_j} = \int_{a_k} w_j + \overline{\int_{a_k} w_j} = 0.$$

We can write

$$(3.9) \qquad w_j = \omega_j + \sum_{k=1}^{m-1} \beta_{kj}\varphi_k ,$$

where

$$(3.10) \qquad \beta_{kj} = - \operatorname{Re} \int_{a_k} \omega_j .$$

Now, (3.8) and (3.9) yield

$$(3.11) \qquad d_o\tau_j[\mu] = \frac{1}{2\pi} \int_{(U/\Gamma)} dw_j(z)\varphi(z)\mu(z)\,|dz\wedge d\bar{z}|$$

$$- \frac{1}{2\pi} \sum_{k=1}^{m-1} \beta_{kj} \int_{(U/\Gamma)} d\varphi_k(z)\varphi(z)\mu(z)\,|dz\wedge d\bar{z}| .$$

The last integrals in (3.11) are real, thus

$$d_o \operatorname{Im} \tau_j[\mu] = \frac{1}{2\pi} \operatorname{Im} \int_{(U/\Gamma)} dw_j(z)\varphi(z)\mu(z)\,|dz\wedge d\bar{z}|,$$

or

$$(3.11) \qquad d_o\alpha_j[\mu] = - \frac{1}{\pi} \operatorname{Re} \int_{U/\Gamma} iw_j(z)\varphi(z)\mu(z)\,|dz\wedge d\bar{z}|, \quad \mu \in L_\infty(\Gamma).$$

3.4 We want to prove now that the map Z_o has a nonsingular differential at $\phi^{\#}(\mu) = 0$. It is clear that it suffices to look at the map

$$Z : \phi^{\#}(\mu) \mapsto (\lambda_1(\mu),\ldots,\lambda_{m-1}(\mu),\ \alpha_1(\mu),\ldots,\alpha_{2m-5}(\mu)) .$$

Since $P\lambda = \pi i e$, we have

$$d_oP[\mu]\lambda + Pd_o\lambda[\mu] = 0.$$

Hence

$$d_o\lambda[\mu] = - P^{-1} d_o P[\mu]\lambda,$$

(3.12) $$d_o\lambda[\mu] = - P^{-1}(-2i \ \mathrm{Re} \int_{U/\Gamma} Q(z)\lambda\mu(z)|dz\wedge d\bar{z}|)$$

To obtain the latter expression, we used (3.6) and denoted by $Q(z)$ the matrix with entries $\varphi_i(z)\varphi_j(z)$.

The j-th component of the vector $Q(z)\lambda$ is

$$[Q(z)\lambda]_j = \sum_{k=1}^{m-1} \varphi_j(z)\varphi_k(z)\lambda_k = \varphi_j(z) \sum_{k=1}^{m-1} \lambda_k\varphi_k(z) = \frac{1}{2}\varphi_j(z)\varphi(z).$$

Then from (3.12) we get

(3.13) $$d_o\lambda[\mu] = \mathrm{Re} \int_{U/\Gamma} iP^{-1}\Phi(z)\varphi(z)\mu(z)|dz\wedge d\bar{z}|,$$

where $\Phi(z)$ is the vector with components $\varphi_1(z),\ldots,\varphi_{m-1}(z)$, note that the matrix iP^{-1} has real entries.

From (3.11) and (3.13) and the definition of Z we get

(3.14) $$d_oZ[\mu] = \mathrm{Re} \int_{U/\Gamma} [iP^{-1}\Phi(z)\varphi(z), \frac{1}{\pi i} w_1(z)\varphi(z),\ldots, \frac{1}{\pi i} w_{2m-5}(z)\varphi(z)] \ \mu(z) \ |dz\wedge d\bar{z}| \ .$$

It is clear that the components of the vector in brackets in (3.14) are a basis for $Q^{\#}(\Gamma)$. From the theorem in section 2.1 characterizing the tangent space to $T^{\#}(\Gamma)$ at $\phi^{\#}(0)$, we conclude that d_oZ is nonsingular.

<u>Theorem</u>. The map Z_o provides real analytic coordinates

for $T^{\#}(\Gamma)$ in a neighborhood of the origin.

Proof. Inverse function theorem.

4. Variation of the Green's Function.

4.1 Let $\Omega \subseteq \mathbb{C} \cup \{\infty\}$ be as before. Represent Ω as U/Γ where Γ is a finitely generated Fuchsian group of the second kind. Assume further that ∞ is an ordinary point for Γ, and that -1, 1, 0 are limit points.

In this chapter, we redefine w_μ by requiring that it fixes -1, 1, 0, with this definition formula (2.6) is still true provided we redefine the kernel R by

$$R(t,z) = \frac{(z+1)z(z-1)}{(t+1)t(t-1)(t-z)} \tag{4.1}$$

The proof of this claim is a computation.

Consider now the series

$$G(z,\xi) = \sum_{\gamma\in\Gamma} \log \left| \frac{z-\gamma(\xi)}{z-\gamma(\bar\xi)} \right|, \quad z,\ \xi \in U. \tag{4.2}$$

Let $\Delta \subseteq U$ be an open disc such that none of the points $\gamma(\xi)$, $\gamma \in \Gamma$, is in Δ. Then on Δ

$$\sum_{\gamma\in\Gamma} \log \left| \frac{z-\gamma(\xi)}{z-\gamma(\bar\xi)} \right| = \operatorname{Re} \sum_{\gamma\in\Gamma} \operatorname{Log} \frac{z-\gamma(\xi)}{z-\gamma(\bar\xi)}, \tag{4.3}$$

where Log is the principal branch of the logarithm function. Now, if $z \in \Delta$, $|z-\gamma(\bar\xi)| > \delta > 0$, for all $\gamma \in \Gamma$, and since $|\gamma(\xi) - \gamma(\bar\xi)|$ tends to zero, except for a finite number of elements $\gamma \in \Gamma$,

$$\left|\operatorname{Log}\left(\frac{z-\gamma(\xi)}{z-\gamma(\bar\xi)}\right)\right| = \left|\operatorname{Log}\left(1 - \frac{\gamma(\xi)-\gamma(\bar\xi)}{z-\gamma(\bar\xi)}\right)\right| < \frac{2}{\delta}\,|\gamma(\xi) - \gamma(\bar\xi)|\,.$$

If $\gamma = \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix}$ then $|\gamma(\xi) - \gamma(\bar{\xi})| < k \frac{1}{c^2}$ for some constant k. It follows from the theory of Poincaré's series (see e.g. Ford [14]) that $\sum_{\gamma\in\Gamma} |\gamma(\xi) - \gamma(\bar{\xi})|$ converges, and hence the series $\sum_{\gamma\in\Gamma} \operatorname{Log} \frac{z-\gamma(\xi)}{z-\gamma(\bar{\xi})}$ converges uniformly on Δ, thus its limit is holomorphic. We conclude from (4.3) that the series (4.2) converges to a harmonic function on $U - \{\gamma(\xi) | \gamma \in \Gamma\}$, and has logarithmic singularities at the points $\gamma(\xi)$.

Let $\alpha \in \Gamma$, $\alpha = \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix}$, then

$$\left|\frac{\alpha(z)-\gamma(\xi)}{\alpha(z)-\gamma(\bar{\xi})}\right| = \left|\frac{(z-\alpha^{-1}\circ\gamma(\xi))/(cz+d)(c\alpha^{-1}\circ\gamma(\xi)+d)}{(z-\alpha^{-1}\circ\gamma(\bar{\xi}))/(cz+d)(c\alpha^{-1}\circ\gamma(\bar{\xi})+d)}\right|$$

$$= \left|\frac{z-\alpha^{-1}\circ\gamma(\xi)}{z-\alpha^{-1}\circ\gamma(\bar{\xi})}\right| ,$$

this relation shows that

$$(4.4) \qquad G(\alpha(z),\xi) = G(z,\xi), \quad \text{for all } \alpha \in \Gamma .$$

It is easy to see that $G(z,\xi)$ tends to zero as z tends to $(\mathbb{R} \cup \{\infty\}) - \Lambda(\Gamma)$, so we have constructed the Green's function for U/Γ, with singularity at the orbit $\Gamma\xi$.

Given a section $s : T^{\#}(\Gamma) \to V^{\#}(\Gamma)$, we define

$$(4.5) \qquad G_s : V^{\#}(\Gamma) \to \mathbb{R} \cup \{\infty\}$$

by $G_s(\phi^{\#}(\mu),z) = G_\mu(z,\xi_\mu)$ where

$$(4.6) \qquad G_\mu(z,\xi_\mu) = \sum_{\gamma\in\Gamma} \log \left|\frac{z-\gamma_\mu(\xi_\mu)}{z-\gamma_\mu(\bar{\xi}_\mu)}\right|$$

and ξ_μ is a point in the orbit $s(\phi^{\#}(\mu)) \in U/\Gamma_\mu$. In other words $G_s(\phi^{\#}(\mu),\cdot)$ is the Green's function for U/Γ_μ with singularity at $s(\phi^{\#}(\mu))$.

4.2 Now, we look at the function $\mu \mapsto G_\mu(w_\mu(z),w_\mu(\xi))$, for fixed $\xi \in U$, $z \in U - \Gamma\xi$.

$$(4.7) \qquad G_\mu(w_\mu(z),w_\mu(\xi)) = \sum_{\gamma\in\Gamma} \operatorname{Re} h_\mu(z,\xi,\gamma)$$

where

$$(4.8) \qquad h_\mu(z,\xi,\gamma) = \operatorname{Log}\left(\frac{w_\mu(z)-w_\mu\circ\gamma(\xi)}{w_\mu(z)-w_\mu\circ\gamma(\bar{\xi})}\right),$$

here Log is the principal branch of the logarithm function. The representation (4.7) is valid for $\|\mu\|$ small, so that $w_\mu(z) \in \Delta$, with Δ being a neighborhood of z such that $w_\mu\circ\gamma(\xi) \notin \Delta$, for all $\gamma \in \Gamma$. That μ can be chosen sufficiently small to satisfy these conditions is a consequence of the properties of quasiconformal mappings stated in section 2.5.

Let $\dot{h}_\nu[\mu](z,\xi,\gamma)$ be the derivative of $h_\mu(z,\xi,\gamma)$ at $\nu \in M(\Gamma)$ in the direction μ, from (4.8) using the chain rule we get

$$(4.9) \qquad \dot{h}_\nu[\mu](z,\xi,\gamma) = \frac{\dot{w}_\nu[\mu](z)-\dot{w}_\nu[\mu](\gamma(\xi))}{w_\nu(z)-w_\nu\circ\gamma(\xi)} - \frac{\dot{w}_\nu[\mu](z)-\dot{w}_\nu[\mu](\gamma(\bar{\xi}))}{w_\nu(z)-w_\nu\circ\gamma(\bar{\xi})}$$

or, by (2.6)

$$(4.10) \quad \dot{h}_\nu[\mu](z,\xi,\gamma) = -\frac{1}{\pi}\int_{\mathbb{C}}\left[\frac{R(w_\nu(t),w_\nu(z))-R(w_\nu(t),w_\nu\circ\gamma(\xi))}{w_\nu(z)-w_\nu\circ\gamma(\xi)}\right.$$

$$\left.-\frac{R(w_\nu(t),w_\nu(z))-R(w_\nu(t),w_\nu\circ\gamma(\bar{\xi}))}{w_\nu(z)-w_\nu\circ\gamma(\bar{\xi})}\right](w_\nu(t)_t)^2\mu(t)\,du\,dv,$$

$t = u+iv$.

From (4.10) we can compute $\operatorname{Re}\dot{h}_\nu[\mu](z,\xi,\gamma)$. A straightforward, but rather long calculation yields

$$(4.11) \quad \operatorname{Re}\dot{h}_\nu[\mu](z,\xi,\gamma) = -\frac{1}{2\pi}\int_{\mathbb{C}}\left[\frac{1}{w_\nu(t)-w_\nu\circ\gamma(\xi)} - \frac{1}{w_\nu(t)-w_\nu\circ\gamma(\bar{\xi})}\right]$$

$$\times\left[\frac{1}{w_\nu(t)-w_\nu(z)} - \frac{1}{w_\nu(t)-w_\nu(\bar{z})}\right](w_\nu(t)_t)^2\mu(t)\,du\,dv\,.$$

4.3 We want to show now, that the series

$$(4.12) \qquad \sum_{\gamma\in\Gamma}\operatorname{Re}\dot{h}_\nu[\mu](z,\xi,\gamma)$$

converges uniformly in a neighborhood of $\nu = 0$. Denote by $J_{w_\nu}(t)$ the Jacobian of the map w_ν. We have

$$|w_\nu(t)_t|^2\left(1 - \frac{|w_\nu(t)_{\bar{t}}|^2}{|w_\nu(t)_t|^2}\right) = J_{w_\nu}(t),$$

from where

$$(4.13) \qquad |w_\nu(t)_t|^2 \le \frac{J_{w_\nu}(t)}{1-\|\nu\|^2}\,.$$

From (4.11) and (4.13) we get

$$|\mathrm{Re}\ \dot{h}_\nu[\mu](z,\xi,\gamma)| \le \frac{1}{2\pi}\ \frac{\|\mu\|}{1-\|\nu\|} \int_{\mathbb{C}} \left|\frac{1}{w_\nu(t)-w_\nu\circ\gamma(\xi)} - \frac{1}{w_\nu(t)-w_\nu\circ\gamma(\bar{\xi})}\right|$$

$$\times \left|\frac{1}{w_\nu(t)-w_\nu(z)} - \frac{1}{w_\nu(t)-w_\nu(\bar{z})}\right| J_{w_\nu}(t)\,du\,dv,$$

by change of integration variable

$$(4.14)\quad |\mathrm{Re}\ \dot{h}_\nu[\mu](z,\xi,\gamma)| \le \frac{1}{2\pi}\ \frac{\|\mu\|}{1-\|\nu\|} \int_{\mathbb{C}} \left|\frac{1}{t-w_\nu\circ\gamma(\xi)} - \frac{1}{t-w_\nu\circ\gamma(\bar{\xi})}\right| \left|\frac{1}{t-w_\nu(z)} - \frac{1}{t-w_\nu(\bar{z})}\right| du\,dv\ .$$

Using the triangle inequality, to estimate the integral in (4.14), we have to look at

$$(4.15)\quad I(z,\xi,\gamma) = \int_{\mathbb{C}} \left|\frac{1}{t-w_\nu\circ\gamma(\xi)} - \frac{1}{t-w_\nu\circ\gamma(\bar{\xi})}\right| \left|\frac{1}{t-w_\nu(z)}\right| du\,dv$$

and $I(\bar{z},\xi,\gamma)$. It will be sufficient to study $I(z,\xi,\gamma)$.

First, note that

$$(4.16)\qquad I(z,\xi,\gamma) =$$

$$|w_\nu\circ\gamma(\xi)-w_\nu\circ\gamma(\bar{\xi})| \int_{\mathbb{C}} \frac{du\,dv}{|t-w_\nu\circ\gamma(\xi)|\ |t-w_\nu\circ\gamma(\bar{\xi})|\ |t-w_\nu(z)|}\ .$$

We will need the following lemmas:

<u>Lemma</u>. Given $\epsilon > 0$ then for all $\|\nu\|$ sufficiently small

$|w_\nu \circ \gamma(\xi) - w_\nu \circ \gamma(\bar{\xi})| < \epsilon$, except for a finite number $N(\epsilon)$ of elements $\gamma \in \Gamma$.

Proof. Let $r \in \mathbb{R}$ such that $\Lambda(\Gamma) \subseteq [-r,r]$. If $\eta < \epsilon/2$ then for all $\|\nu\|$ sufficiently small $w_\nu([-r,r]\times[-\eta,\eta]) \subseteq [-R,R] \times [-\epsilon/2, \epsilon/2]$ for some $R \geq r$. This follows from the second theorem in section 2.5 noting that on a compact set $|z_1-z_2| \leq A[z_1,z_2]$. Now, except for a finite number $N(\epsilon)$ of elements $\gamma \in \Gamma$, $\gamma(\xi)$, $\gamma(\bar{\xi}) \in [-r,r] \times [-\eta,\eta]$ and hence $w_\nu\circ\gamma(\xi)$, $w_\nu\circ\gamma(\bar{\xi}) \in [-R,R] \times [-\epsilon/2, \epsilon/2]$. The proof is completedby observing that $w_\nu\circ\gamma(\bar{\xi}) = \overline{w_\nu\circ\gamma(\xi)}$.

Lemma. Let $K \subseteq \mathbb{C}$ compact then for $a,b \in K$, $0 < |a-b|$ small, there exists constants α, $\beta \in R$ such that

$$\int_K \frac{dudv}{|t-a|\;|t-b|} \leq \alpha + \beta \log \frac{1}{|a-b|} \;.$$

Proof. Let R be the radius of circle centered at 0 and containing K. Set $t - a = (a-b)s$, $s = x+iy$.

$$\int_K \frac{dudv}{|t-a|\;|t-b|} \leq \int_{|t|\leq R} \frac{dudv}{|t-a|\;|t-b|} \leq \int_{|s|\leq \frac{2R}{|a-b|}} \frac{dxdy}{|s|\;|s+1|} \leq$$

$$\int_{|s|\leq 3/2} \frac{dxdy}{|s|\;|s+1|} + (\text{const.}) \int_{\frac{3}{2}\leq|s|\leq\frac{2R}{|a-b|}} \frac{dxdy}{|s|^2} = \alpha+\beta\,\text{Log}\,\frac{1}{|a-b|} \;.$$

Lemma. Let $K \subseteq \mathbb{C}$ compact then

$$g(c) = \int_K \frac{dudv}{|t-c|}, \quad c \in \mathbb{C} \text{ is continuous.}$$

Proof. Let $c, c' \in K$ then

$$|g(c)-g(c')| \leq \int_K \frac{||t-c'|-|t-c||}{|t-c|\,|t-c'|} dudv \leq |c-c'| \int_K \frac{dudv}{|t-c|\,|t-c'|}$$

$$\leq |c-c'|(\alpha + \beta \log \frac{1}{|c-c'|}) \to 0, \text{ as } c \to c'.$$

If $c \notin K$, or $c' \notin K$ $\int_K \frac{dudv}{|t-c|\,|t-c'|}$ is bounded.

Lemma. Let $K_1, K_2 \subseteq \mathbb{C}$ be disjoint compact sets then there exist constants $\alpha, \beta \in \mathbb{R}$ such that

$$\int_{\mathbb{C}} \frac{dudv}{|t-a|\,|t-b|\,|t-c|} \leq \alpha + \beta \log \frac{1}{|a-b|}$$

for all $c \in K_1$, and $a,b \in K_2$, $0 < |a-b|$ small.

Proof. Take $\tilde{K}_1, \tilde{K}_2 \subseteq \mathbb{C}$ compact sets so that $K_i \subseteq \tilde{K}_i$, distance $(\partial\tilde{K}_i, K_i) > 0$, $i = 1,2$, and $\tilde{K}_1 \cap \tilde{K}_2 = \emptyset$.

$$\int_{\mathbb{C}} \frac{dudv}{|t-a|\,|t-b|\,|t-c|} = \int_{\mathbb{C}-(\tilde{K}_1 \cup \tilde{K}_2)} \frac{dudv}{|t-a|\,|t-b|\,|t-c|} + \int_{\tilde{K}_1} \frac{dudv}{|t-a|\,|t-b|\,|t-c|}$$

$$+ \int_{\tilde{K}_2} \frac{dudv}{|t-a|\,|t-b|\,|t-c|} \leq A + B \int_{\tilde{K}_1} \frac{dudv}{|t-c|} + C \int_{\tilde{K}_2} \frac{dudv}{|t-a|\,|t-b|}$$

$$\leq \alpha + \beta \log \frac{1}{|a-b|} .$$

From (4.16) and the above lemmas we obtain

$$(4.17) \quad I(z,\xi,\gamma) \leq |w_\nu \circ \gamma(\xi) - w_\nu \circ \gamma(\bar{\xi})| \left(A + B \log \frac{1}{|w_\nu \circ \gamma(\xi) - w_\nu \circ \gamma(\bar{\xi})|}\right),$$

for all $\|\nu\|$ small and except for a finite number of elements $\gamma \in \Gamma$.

Now, on a compact set and for all $\|\nu\|$ small

$$(4.18) \quad |w_\nu(z_1) - w_\nu(z_2)| \leq M|z_1 - z_2|^\alpha, \quad |z_1 - z_2| \leq M|w_\nu(z_1) - w_\nu(z_2)|^\alpha,$$

where $\alpha = \dfrac{1-\|\nu\|}{1+\|\nu\|}$ and M is independent of ν (see section 2.5).

Then we have

$$\log \frac{1}{|z_1 - z_2|} \geq \log \frac{1}{M|w_\nu(z_1) - w_\nu(z_2)|^\alpha}$$

or

$$(4.19) \quad \log \frac{1}{|w_\nu(z_1) - w_\nu(z_2)|} \leq \frac{1}{\alpha} \log \frac{1}{|z_1 - z_2|} - \frac{1}{\alpha} \log \frac{1}{M} .$$

From (4.17), (4.18), (4.19) we get

$$I(z,\xi,\gamma) \leq |\gamma(\xi) - \gamma(\bar{\xi})|^\alpha \left(C + D \log \frac{1}{|\gamma(\xi) - \gamma(\bar{\xi})|}\right)$$

$$= |\gamma(\xi) - \gamma(\bar{\xi})|^{\alpha-\delta} |\gamma(\xi) - \gamma(\bar{\xi})|^\delta \left(C + D \log \frac{1}{|\gamma(\xi) - \gamma(\bar{\xi})|}\right) .$$

(4.20) $$I(z,\xi,\gamma) \le K'_\delta \, |\gamma(\xi)-\gamma(\bar{\xi})|^{\alpha-\delta} \le K_\delta \frac{1}{c^{2(\alpha-\delta)}} ,$$

where $\gamma = \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix}$.

For a given finitely generated Fuchsian group Γ of the second kind with ∞ as an ordinary point, there exists $t_o < 2$ so that

$$\sum_{\gamma\in\Gamma} \frac{1}{c_\gamma^t} < \infty, \quad \text{for } t > t_o, \ \gamma = \begin{pmatrix} \cdot & \cdot \\ c_\gamma & \cdot \end{pmatrix} .$$

This is a hard theorem due to Beardon [5]. Therefore, we can choose in (4.20) $\|\nu\|$ and δ sufficiently small so that $\sum_{\gamma\in\Gamma} \frac{1}{c^{2(\alpha-\delta)}} < \infty$. We have proven that the series (4.12) is uniformly convergent in a neighborhood of $\nu = 0$.

4.4. From well known results of differential calculus in a Banach space, and from our discussion in sections 4.2 and 4.3 we conclude that the map $\mu \to G_\mu(w_\mu(z),w_\mu(\xi))$ is differentiable at the origin and its differential can be computed by differentiating term by term the series (4.7). From (4.11) we get

(4.21) $$\dot{G}[\mu](z,\xi) = -\frac{1}{2\pi} \sum_{\gamma\in\Gamma} \int_{\mathbb{C}} \left[\frac{1}{t-\gamma(\xi)} - \frac{1}{t-\gamma(\bar{\xi})}\right] \times \left[\frac{1}{t-z} - \frac{1}{t-\bar{z}}\right]\mu(t)\,du\,dv.$$

Now, let ω be a fundamental region for Γ in $\Omega(\Gamma)$ then

$$\dot{G}[\mu](z,\xi) = -\frac{1}{2\pi}\sum_{\beta\in\Gamma}\sum_{\gamma\in\Gamma}\int_{\beta(\omega)}\left[\frac{1}{t-\gamma(\xi)}-\frac{1}{t-\gamma(\bar{\xi})}\right]$$

$$\times\left[\frac{1}{t-z}-\frac{1}{t-\bar{z}}\right]\mu(t)\,du\,dv\,.$$

By a change of variable in the integral we get

$$\dot{G}[\mu](z,\xi) = -\frac{1}{2\pi}\sum_{\beta,\gamma\in\Gamma}\int_{\omega}\left[\frac{1}{\beta(t)-\gamma(\xi)}-\frac{1}{\beta(t)-\gamma(\bar{\xi})}\right]$$

$$\times\left[\frac{1}{\beta(t)-z}-\frac{1}{\beta(t)-\bar{z}}\right]\mu\circ\beta(t)|\beta'(t)|^2\,du\,dv$$

$$= -\frac{1}{2\pi}\sum_{\beta,\gamma\in\Gamma}\int_{\omega}\left[\frac{1}{\gamma^{-1}\circ\beta(t)-\xi}-\frac{1}{\gamma^{-1}\circ\beta(t)-\bar{\xi}}\right](\gamma^{-1}\circ\beta)'(t)$$

$$\times\left[\frac{1}{\beta(t)-z}-\frac{1}{\beta(t)-\bar{z}}\right]\beta'(t)\mu(t)\,du\,dv$$

$$= -\frac{1}{2\pi}\int_{\omega}\left[\sum_{\gamma\in\Gamma}\left(\frac{1}{\gamma(t)-\xi}-\frac{1}{\gamma(t)-\bar{\xi}}\right)\gamma'(t)\right]$$

$$\times\left[\sum_{\beta\in\Gamma}\left(\frac{1}{\beta(t)-z}-\frac{1}{\beta(t)-\bar{z}}\right)\beta'(t)\right]\mu(t)\,du\,dv\,.$$

(4.22) $$\dot{G}[\mu](z,\xi) = -\frac{1}{2\pi}\int_{(U/\Gamma)^d}\alpha_{\xi\tilde{\xi}}(t)\alpha_{z\tilde{z}}(t)\mu(t)\,du\,dv\,,$$

where $\alpha_{w\tilde{w}}$ $(w = \xi, z)$ denotes the unique abelian differential of third kind in $(U/\Gamma)^d$ with simple poles at w and $\tilde{w}$ and residues -1 and $+1$ respectively, and, having purely imaginary periods. By symmetry (4.22) can be written as

$$(4.23)\qquad \dot{G}[\mu](z,\xi) = -\frac{1}{\pi}\,\mathrm{Re}\int_\Omega \alpha_{\xi\tilde{\xi}}\,\alpha_{z\tilde{z}}\,\mu\ ,$$

or from the definition of the Green's function

$$(4.24)\qquad \dot{G}[\mu](z,\xi) = -\frac{4}{\pi}\,\mathrm{Re}\int_\Omega \frac{\partial G}{\partial t}(t,\xi)\,\frac{\partial G}{\partial t}(t,z)\mu(t)\,du\,dv\ .$$

If we look at the function $\mu \to G_\mu(z,\xi)$ for z,ξ fixed and $\|\mu\|$ small, using (4.24) and the chain rule we get the differential at the origin

$$(4.25)\qquad d_0G[\mu](z,\xi) = -\frac{4}{\pi}\,\mathrm{Re}\int_\Omega \frac{\partial G}{\partial t}(t,\xi)\,\frac{\partial G}{\partial t}(t,z)\mu(t)\,du\,dv$$

$$-\ 2\mathrm{Re}\Big\{\frac{\partial G}{\partial z}(z,\xi)\dot{w}[\mu](z) + \frac{\partial G}{\partial \xi}(z,\xi)\dot{w}[\mu](\xi)\Big\}\ .$$

4.5 We now use our general variational formula for the Green's function, to obtain as a special case the Schiffer variational formula (see Schiffer and Spencer [19]).

The Schiffer interior variation can be obtained as a quasiconformal variation as follows. Let $z_0 \in \Omega$ and γ a simple, closed, analytic curve bounding a cell which contains z_0. By the Riemann mapping theorem we can assume without loss of generality that γ is the unit circle and $z_0 = 0$. Let $r(z)$ be a function holomorphic in the complement of the open unit disc, then $r(z)$ has a representation:

$$(4.26)\qquad r(z) = \sum_{n=-\infty}^{\infty} a_n z^n\ .$$

Define on $|z| \leq 1$

$$(4.27) \qquad F(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n}\bar{z}^n ,$$

$F(z)$ is continuous $F(z) = r(z)$ on $|z| = 1$, define

$$(4.28) \qquad w(z) = \begin{cases} z + \epsilon\, r(z) & |z| \geq 1 \\ z + \epsilon\, F(z) & |z| \leq 1 \end{cases} .$$

The Beltrami coefficient of w is

$$(4.29) \qquad \mu_\epsilon(z) = \begin{cases} 0 & |z| \geq 1 \\ \dfrac{\epsilon \sum_{n=1}^{\infty} a_{-n} n \bar{z}^{n-1}}{1+\epsilon \sum_{n=1}^{\infty} a_n n z^{n-1}} & |z| < 1 \end{cases} .$$

For sufficiently small ϵ, $z \mapsto z + \epsilon\, r(z)$ is schlicht, and $\|\mu\| < 1$. Thus w is quasiconformal in the unit disc and agrees with a homeomorphism on the boundary, we conclude that w is a homeomorphism on $|z| \leq 1$. The image of Ω under w is conformally equivalent to Schiffer's variation of Ω. This description of Schiffer's variation is due to Gardiner [15].

We note that

$$(4.30) \qquad \mu_\epsilon(z) = \epsilon \sum_{n=1}^{\infty} a_{-n} n \bar{z}^{n-1} + o(\epsilon) , \quad |z| < 1.$$

Stokes' theorem gives

$$(4.31)\quad \int_{|t|\le 1} n\bar{t}^{n-1}h(t)\,du\,dv = \frac{1}{2i}\int_{|t|=1}\bar{t}^{n}h(t)\,dt = \frac{1}{2i}\int_{|t|=1}\frac{h(t)}{t^{n}}\,dt$$

for h holomorphic on $|t| \le 1$.

From (4.24), (4.30) and (4.31) we get, for $z, \xi \in \Omega$ $z, \xi \notin \{t| \ |t| \le 1\}$,

$$\dot{G}[\mu_\varepsilon](z,\xi) = -\frac{4}{\pi}\,\mathrm{Re}\,\frac{\varepsilon}{2i}\int_{|t|=1}\left(\sum_{1}^{\infty} a_{-n}t^{-n}\right)G_t(t,z)G_t(t,\xi)\,dt + o(\varepsilon)$$

or

$$(4.32)\quad \dot{G}[\mu_\varepsilon](z,\xi) = -\,\mathrm{Re}\,\frac{2\varepsilon}{\pi i}\int_{|t|=1} r(t)G_t(t,z)G_t(t,\xi)\,dt + o(\varepsilon)\ .$$

This is the Schiffer's variational formula for the Greens' function.

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