METRIC DEFORMATIONS OF RICCI AND SECTIONAL CURVATURE ON COMPACT RIEMANNIAN MANIFOLDS

A Dissertation presented

by

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to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at

Stony Brook

May, 1974

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

The Graduate School

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Jerome E. Singer, Dean

Abstract

Motivated by a corollary to a theorem stated but proved incompletely by T. Aubin, we decided to study the following problem: is it possible to find a standard deformation $g(t)$ of a given metric g_o defined on a metric disk D agreeing with g_o in M - D with positive first derivative of sectional or Ricci curvature near Bd(D) and to use this deformation to uniformly increase the region of positive curvature? Computations led to the idea that D should be a convex metric ball, and hence to the study of "local convex deformations", i.e., local deformations with convex support.

We prove some facts about the space of Riemannian metrics for a given smooth manifold M. For instance, if a complete metric for a non-compact manifold is changed only on a compact set, the resulting metric is complete. Hence the proof of the Ricci curvature deformation theorem we give and our paper on the injectivity radius function imply that non-negatively Ricci curved complete Riemannian manifolds "positively Ricci curved at infinity" have only one end. We show the injectivity radius function and the convexity radius function on the space of Riemannian metrics of a compact manifold are locally minorized; that is given a metric g_o for M, there exists a C^2 neighborhood of g_o

on which these functions are bounded away from zero.

We study the first derivative of the Ricci curvature for all possible local convex deformations $g(t)$ of a given metric g_0 with support in a metric disk D. This then leads to the standard conformal deformation with which we prove the Ricci curvature deformation theoresm. For instance, a compact manifold admitting a metric of non-negative Ricci curvature and all Ricci curvatures positive at a point admits a metric of everywhere positive Ricci curvature.

Next we show that in general for dim $M \geq 3$ there do not exist any local convex deformations for sectional curvature that are positive at first order. Essentially we reverse the argument for the Ricci curvature. This non-existence is partly a result of the convexity of D and partly a result of the fact that dim $M \geq 3$.

Since the difficulties already arise in dimension 3, we study global metric deformations on compact 3-manifolds. A calculation shows that

(*) given (M^3,g) compact with non-negative sectional curvature and positive Ricci curvature, M admits a metric with everywhere positive sectional curvature.

Following a recent philosophy of "rigidity" in Riemannian geometry we study how this result fails if non-negative

Ricci curvature is allowed in the hypothesis of (*). This leads to the study of Ricci-productlike metrics and a formula for D*D Ric which shows in this simple case how the second covariant derivative of the curvature tensor is related to the integrability of the conullity distribution induced on M^3 by the curvature tensor. Then we define a notion of a critical metric for a class of 3-manifolds modelled on $(S^1 \times S^2, g)$ just as in Berger's result that all non-negative variations at first order of the standard metric on $S^2 \times S^2$ vanish identically at first order. If g_o is a critical metric for M^3, then (M^3, g_o) is locally isometrically a product.

Acknowledgements

I would like to thank my thesis advisor Professor Detlef Gromoll for his advice and encouragement and for many stimulating discussions of geometry and mathematics in general.

I would like to thank Professor Eugene Zaustinsky for his kind encouragement during my graduate studies, and for suggesting that I study differential geometry.

I would like to thank Jean-Pierre Bourguignon for giving generously of his time in teaching me how to make computations, in understanding how to do research through his example, for sharing his ideas on metric deformations stemming from his own work, and for helping with the calculations of Chapters 4 and 6 which were the first part of my thesis.

I would like to thank Professors James Simons, Jeff Cheeger, and Wolfgang Meyer for helping with the beginnings of my study of and work in differential geometry.

Thanks for financial support during my work on my dissertation are due to the Department of Mathematics at SUNY at Stony Brook for my National Science Foundation Traineeship and to the Centre de Mathematiques, Ecole Polytechnique, Paris, France for financial support October through December, 1973 during my stay in Paris.

Table of Contents

Introduction

In [2], Aubin stated a theorem which implied as a corollary that if a manifold M admits a Riemannian metric with non-negative Ricci curvature and all Ricci curvatures positive at some point, then M admits a metric of everywhere positive Ricci curvature. It appears the proof in [2] is incomplete and the uniformity of Aubin's estimates even in the compact case are not clear.

Motivated by what we thought was the general method suggested by Aubin's paper, we decided to study the following problem: can we find a standard deformation $g(t)$ defined on a metric disk D agreeing with the given metric g_o in M - D to spread positive Ricci and/or sectional curvature from near the center of D to all of D? Motivated by [11] and a conversation with J. Simons, we considered a weaker question: can we find a deformation $g(t)$ of the given metric g_o with $K' > 0$ or $Ric' > 0$ near $Bd(\bar{D})$?

This question led to the study of "local convex deformations" which we discuss in Chapter 2, along with the motivation for studying metric deformations of curvature and some technical lemmas used later.

In Chapter 3 we make some observations on the space of Riemannian metrics. First we need to make precise the idea that for "close" metrics, metric balls centered at the same point are similar. The precise statements here will be used in proving the Ricci curvature deformation theorems in Chapter 5. Second we observe that if a complete metric for a non-compact manifold is changed only on a compact set, then the resulting metric is also complete. Using this remark and a Ricci curvature deformation theorem from Chapter 5, we show that non-negatively Ricci curved complete Riemannian manifolds "positively Ricci curved at infinity" are connected at infinity, generalizing a result of [23], p. 80. Finally, we remark that the convexity radius function and the injectivity radius function on the space of Riemannian metrics are locally minorized.

In Chapter 4 we study Ric' for all local convex deformations. Since an aribtrary local convex deformation through C^4 metrics with support in D may be written in D as

$$g(t) = g_o + t\,\rho^3 h$$

where

$$\rho = \text{distance to } \mathrm{Bd}(\bar{D})$$

is essentially the smoothing function which forces $g(t) = g_o$ in M - D, we see that all that is needed for Ric' > 0 near $\mathrm{Bd}(\bar{D})$ is that the "tangential projection" of h should be negative definite near $\mathrm{Bd}(\bar{D})$. The presence of the smoothing function ρ means that in Ric' near $\mathrm{Bd}(\bar{D})$ the terms in ρ

will dominate the terms in ρ^2 and ρ^3. This is precisely what makes Ric' > 0 since the tangential projection of $-h$ is essentially the leading term in ρ in Ric'.

Alternatively, we might say that local convex deformations work for the Ricci curvature because Ricci curvature is the trace of the sectional curvature. The deformation we use in Chapters 4 and 5 has the property that

$$K'(x,\nabla\rho) > 0 \qquad \text{if} \quad x(\rho) = 0$$

and

$$K'(x,y) < 0 \qquad \text{if} \quad x(\rho) = y(\rho) = 0.$$

But the order in ρ of the contribution to Ric' from the "radial two-plane" $\{x,\nabla\rho\}$ is lower than the order in ρ of the sum of the n-2 tangential two-planes $\{x,y\}$ in Ric'. Also, since we are essentially taking the trace of K' in computing Ric', we always have a radial two-plane $\{x,\nabla\rho\}$ contributing a positive lowest order term to Ric'.

In Chapter 5 we use the results of Chapters 3 and 4 to prove various Ricci curvature deformation theorems. We emphasize that since $g(t) = g_0$ in M - D we only have Ric' > 0 on D, an open set, so some care is needed to be sure that we can find a t > 0 such that $\mathrm{Ric}^t(v) > 0$ for all v near $\mathrm{Bd}(\bar{D})$ simultaneously.

In Chapter 6, we show that in general there do not exist any local convex deformations for sectional curvature that are positive at first order. Essentially we just turn the argument of Chapter 4 around. The failure to find a local convex deformation $g(t)$ of g_o with $K' > 0$ near $Bd(D)$ is a consequence of the following two statements.

(1) the convexity of D implies that $\delta*(d\rho) < 0$.

(2) Since dimension $M \geq 3$, we have two families of two planes, the infinite family of radial two planes $\{x,\nabla\rho\}$ with $x(\rho) = 0$ and the tangential two planes $\{x,y\}$ with $x(\rho) = 0$ and $y(\rho) = 0$. Then $K'(x,\nabla\rho) > 0$ forces $K'(x,y) < 0$ and vice versa.

Since the difficulties arise already in dimension 3, we study global metric deformations on compact 3-manifolds. A calculation in Part 1 of Chapter 7 shows that given (M^3,g_o) compact with $K_{g_o} \geq 0$ and $Ric_{g_o} > 0$, M^3 admits a metric g with $K_g > 0$ via the deformation $g(t) = g_o+t(-Ric_{g_o})$.

There is a notion of "rigidity" that has been suggested to us by Gromoll. This concept of "rigidity" is best explained by example. The Sphere Theorem asserts that given (M^n,g_o) simply connected with

$$1/4\ \pi < K_{g_o} \leq \pi, \quad \pi > 0$$

that M^n is homeomorphic to S^n, i.e., δ-pinching with $\delta > 1/4$ implies M is homeomorphic to S^n. The principal of "rigidity" in this case asserts that if (M^n,g_o) is simply connected and 1/4-positively pinched, then M can only fail to be homeomorphic to S^n in a very special way. Explicitly, it is known that any manifold which is 1/4 positively pinched which is not homeomorphic to a sphere is isometric to a symmetric space of rank 1 with the usual metric.

Following this philosophy, in Part 2 of Chapter 7 and in Chapter 8 we ask what happens when $Ric_{g_o} \geq 0$ is allowed in the hypothesis of the theorem of Part 1 of Chapter 7. In Part 2 of Chapter 7, this leads to the study of Ricci-product like metrics and a formula for D*D Ric which shows in this simple case how the second covariant derivative of the curvature tensor is related to the integrability of the conullity "foliation" induced on M^3 by the curvature tensor. In Chapter 8, we define a notion of a critical metric for (M^3,g_o) motivated by Berger's result on the non-existence of positive variations at first order for $(S^2 \times S^2, g_{can})$. We show that critical metrics are precisely the metrics for which (M^3,g_o) is locally isometrically a product metric.

Finally, in Chapter 9 we make three miscellaneous observations on metrics and curvature.

We close with a warning (for our European readers) that following the American convention we say that a function $f : A \to R$ is <u>positive</u> if $f > 0$ on A (in lieu of "strictly positive"), <u>negative</u> if $f < 0$ on A (in lieu of "strictly negative"), <u>non-negative</u> if $f \geq 0$ on A and <u>non-positive</u> if $f \leq 0$ on A.

Chapter 2: The general theory of local convex deformations

Let M^n be a C^∞ manifold. Let TM be a tangent bundle of M with smooth sections $C^\infty(TM)$ called smooth vector fields on M. Let T*M be the co-tangent bundle of M with smooth sections $C^\infty(T^*M)$ called 1-forms on M. Let $G_2(M) \xrightarrow{\pi} M$ be the Grassman bundle with fiber at p in M all two dimensional vector subspaces of M_p which we will call two-planes. Let R(M) be the convex cone of all Riemannian metrics on M. There is a natural action $R(M) \times \mathrm{Diff}(M) \to R(M)$ given by $(g,f) \to f^*g$ where $(f^*g)(v,w) = g(f_*v,f_*w)$. Let $\mathcal{R}(M) = R(M)/\mathrm{Diff}(M)$ be the space of Riemannian structures on M. (Ebin has studied this in [17].) Given a Riemannian metric g on M, there is a unique Levi-Civita connection $D : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ satisfying for X,Y and Z in $C^\infty(TM)$

$$(1) \qquad 2g(D_XY,Z) = X\cdot g(Y,Z) + Y\cdot g(Z,X) - Z\cdot g(X,Y) + g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z]).$$

With our sign convention, the curvature tensor R of g is defined by

$$R(X,Y)Z = D_XD_YZ - D_YD_XZ - D_{[X,Y]}Z \text{ for } X,Y, \text{ and } Z$$

in $C^\infty(TM)$. But $R(X,Y)Z|_p$ depends only on $X|_p$, $Y|_p$, and $Z|_p$ so we may define for $x,y,z,w \in M_p$ the tensors $R(x,y)z$ and $R(x,y,z,w) = g(R(x,y)z,w)$ by extending x,y, and z to local fields and computing the value of R at p using these local extensions. For non-zero, non-collinear vectors $x,y \in M_p$ we find a number $K(x,y) = \frac{g(R(x,y)y,x)}{g(x,x)\cdot g(y,y) - g(x,y)^2}$.

Now $K(x,y)$ depends only on two plane spanned by x and y so that K is really a function $K : G_2(M) \to R$ called the sectional curvature function of g. The Ricci tensor for g is defined to be $Ric(x,y) = tr\{z \to R(z,x)y\}$ where $x,y,z \in M_p$. If $e_1,\ldots,e_n$ are a basis for M_p, then $Ric(x,y) = \sum_{i=1}^{n} R(e_i,x,x,e_i)$. The Ricci curvature is then defined to be $Ric(x) = \frac{Ric(x,x)}{g(x,x)}$ so if $e_1,\ldots,e_n$ are an orthonormal basis for M_p and $g(x,x) = 1$ then

$$(2) \qquad ric(x) = \sum_{i=1}^{n} K(e_i,x)\cdot(1 - g(e_i,x)^2)$$

where $K(e_i,v)$ is defined to be zero if e_i and v are collinear When there is danger of confusion, we will write R_g, K_g, Ric_g, and ric_g for the curvature tensors and functions defined by g. We will denote the scalar curvature function of (M,g) by τ or τ_g.

Let g(t) be a 1-parameter family of metrics for t in some open interval about 0 with $g(0) = g_0$. We will write D^t, R^t, Ric^t, and ric^t for the operators, tensors, and functions defined by g(t). If we fix $x,y \in M_p$, then $t \to K^t(x,y)$ is a real valued function so we can define

$$K'(x,y) = \frac{d}{dt} K^t(x,y)\Big|_{t=0} = \lim_{t\to 0} \frac{K^t(x,y) - K(x,y)}{t}$$

if the limit exists. Similarly we can define $Ric' = \frac{d}{dt} Ric^t\Big|_{t=0}$ and $ric' = \frac{d}{dt}\, ric^t\Big|_{t=0}$.

Given a Riemannian metric g on M, g defines an isomorphism of the tangent and cotangent bundles $TM \cong T^*M$ which on the fibers is given by $M_p \ni v \to g(v,-) \in M_p^*$ where $g(v,-)(w) := g(v,w)$. If $\xi \in C^\infty(T^*M)$ is a 1-form, then the vector field associated to ξ by this isomorphism which we will denote by $\xi^{\#g}$ is given by $g(\xi^{\#g},X) = \xi(X)$ for all $X \in C^\infty(TM)$. Given a vector field X, there is a 1-form $X^{\flat g}(Y) = g(X,Y)$. When there is no danger of confusion, we will write $\xi^{\#}$ for $\xi^{\#g}$ and $X^{\flat}$ for $X^{\flat g}$. Let $S^2(M)$ be the bundle of symmetric two-tensors on M. There is a differential operator $\delta^* : C^\infty(T^*M) \to C^\infty(S^2(M))$ depending on g given by

$$\delta^*\xi(x,y) = \frac{1}{2} \mathcal{L}_{\xi^{\#g}} g(x,y) = \frac{1}{2}((D_x\xi)(y) + (D_y\xi)(x))$$

where D is the Levi-Civita connection determined by g and $\mathcal{L}$ is the Lie derivative. (See [5]) Properly, we should write δ^*_g instead of δ^*. But given a variation $g(t)$ with $g(0) = g_o$, δ^* will always be the operator defined by g_o so our notation will not be ambiguous. Let $f : M \to R$ be a smooth function. We will write ∇f for the gradient vector field associated to f which is defined by $g(\nabla f, X) = X(f)$ for any $X \in C^\infty(TM)$. Then $\delta^*(df)(x,y) = g(D_x \nabla f, y)$ is the Hessian form of f. The LaPlacian Δf of f is $\Delta f = \operatorname{tr}\delta^*(df)$. We have

(3) $\delta^*(d(fg)) = f\delta^*(dg) + g\delta^*(df) + 2\, df \circ dg$ where

$df \circ dg = 1/2(df \otimes dg + dg \otimes df)$, and

(4) $\delta^*(f\xi) = f\delta^*\xi + df \circ \xi$

where $f, g : M \to R$ are C^2 functions and $\xi \in C^\infty(T^*M)$.

The importance of δ^* in the theory of metric deformations stems from the decomposition of Berger and Ebin [5], for $C^\infty(S^2(M))$. Let $\delta' = D^* : C^\infty(S^2(M)) \to C^\infty(T^*M)$ be the adjoint of the Levi-Civita connection determined by g. Then

$$C^\infty(S^2(M)) = \ker \delta' \oplus \operatorname{im} \delta^*.$$

Let us call a deformation $g(t)$ of g_0 whose 1-jet is not in the image of δ^* a "geometric deformation". Let $\xi \in C^\infty(T^*M)$ and let $g(t) = g + t\delta^*\xi = g + \frac{t}{2} \mathcal{L}_{\xi^{\#_g}} g$.

Consider the variation $g(t) = \varphi^*_{-t} g$ where $\varphi_t : M \to M$ is the 1-parameter group generated by $\xi^{\#_g}$. Then $(M, g(t)) \xrightarrow{\varphi_{-t}} (M, g)$ is an isometry and $g(t)$ determines the same coset in $\mathcal{R}(M)$ for all t. This deformation $g(t)$ can thus be thought of as a coordinate change in $\mathcal{R}(M)$. We will later be interested in (M,g) with $K_g \geq 0$. We can explicitly see at first order that deformations in $\mathrm{Im}\delta^*$ do not improve sectional curvature by

<u>Proposition 1</u>. Given (M,g) with $K = K_g \geq 0$. Let $g(t) = g + t\delta^*\xi$. Then $K(\sigma) = 0$ implies $K'(\sigma) = \frac{d}{dt}K^t(\sigma)\Big|_{t=0} = 0$.

<u>Proof</u>. In general, if $g(t) = g + t\,h^1 + \frac{t^2}{2}\,h^2 + \ldots + \frac{t^k}{k!}\,h^k$ then only the tensor h^1 effects the first derivative of sectional curvature. ([11], p. 8) Now $\tilde{g}(t) = \frac{1}{2}\,\varphi^*_{-t}g =$ $g + \frac{t}{2}\frac{d}{dt}(\varphi^*_{-t}g)\Big|_{t=0} + O(t^2) = g + t\delta^*\xi + O(t^2)$, so $\tilde{g}(t) = g(t) + O(t^2)$. For the proof of the Proposition only, let $\tilde{K}^t$ be the sectional curvature function determined by $\tilde{g}(t)$ and K^t the sectional curvature function determined by $g(t)$. Then

$$\frac{d}{dt}\,K^t\Big|_{t=0} = \frac{d}{dt}\,\tilde{K}^t\Big|_{t=0}\;.$$

Given $\sigma \in G_2(M)$, let $\sigma = \{x,y\}$. Then for small $t, \varphi_t(\sigma) = \{\varphi_{-t_*}x, \varphi_{-t_*}y\}$ is in $G_2(M)$. Suppose $K(\sigma) = 0$. Since $\tilde{K}^t(\sigma) = K(\varphi_t(\sigma)) \geq 0$, we have $\frac{d}{dt}\,\tilde{K}^t(\sigma)\Big|_{t=0} = 0$. Hence $K'(\sigma) = \frac{d}{dt}K^t(\sigma)\Big|_{t=0} = \frac{d}{dt}\,\tilde{K}^t(\sigma)\Big|_{t=0} = 0$.

Q. E. D.

<u>Remark</u>: A similar proposition holds if $K \geq A$, $K \leq B$, or $A \leq K \leq B$.

In [11] Bourguignon, Deschamps, and Sentenac studied for a fixed metric g_o for M arbitrary variations

$$g(t) = g_o + t\, h^1 + \frac{t^2}{2} h^2 + \ldots + \frac{t^k}{k!} h^k \text{ with } h^i \in C^\infty(S^2(M)).$$

They defined a differential operator $\Sigma : C^\infty(S^2(M)) \to C^\infty(S^2(\Lambda^2(T^*M)))$ depending on g_o as follows. Given $h \in C^\infty(S^2(M))$, let $g(t) = g_o + t\, h$ and define $(\Sigma h)(x,y) = \frac{d}{dt} R^t(x,y,y,x)\Big|_{t=0}$.

Then they proved that

$$(5) \quad (\Sigma\, h)(x,y) = DDh(x,y,x,y) - \tfrac{1}{2}DDh(x,x,y,y) - \tfrac{1}{2}DDh(y,y,x,x) + h(R(x,y)y,x).$$

In particular, $(\Sigma h)(x,x) = 0$. Suppose $K_{g_o} \geq 0$ on M. We call a two plane σ at which $K(\sigma) = 0$, an <u>extremal value</u> for K_{g_o}.

<u>Lemma 2</u>: (Bishop, Goldberg, [3]).

Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis for M_p. Suppose $\{e_1, e_2\}$ is an extremal two-plane. Then $R(e_1, e_2, e_1, e_j) = 0$ for all j. In particular, if x and y are an orthonormal basis for an extremal two-plane, then $R(x,y)y = R(y,x)x = 0$.

Hence, if x and y are a g_o-orthonormal basis for an extremal two-plane of K_{g_o}, then for any tensor h in $C^\infty(S^2(M))$, the last term in formula (5) vanishes. If $g(t) = g_o + t\, h^1 + \ldots$, then if x and y are g_o-orthonormal

(6) $K'(x,y) = (\Sigma h^1)(x,y) - K(x,y)(h^1(x,x)h^1(y,y) - h^1(x,y)^2)$.

Thus, if $K(x,y) = 0$, then $K'(x,y) = (\Sigma h^1)(x,y)$.

One of the fascinating aspects of Riemannian geometry is the relationship between curvature and topology. Let M^n, $n \geq 2$, be a non-compact smooth manifold. Let $\varepsilon > 0$ be given. Then work of Gromov, [24], shows that for any $K \in R$, there is a metric g for M for which

$$K - \varepsilon \leq K_g \leq K + \varepsilon.$$

This means that not all Riemannian metrics on a non-compact manifold are of interest. The interesting metrics from the point of view of global geometry are called complete metrics. An outstanding classical result in global geometry relating curvature and topology is:

(1) If M admits a complete metric g with $ric_g \geq c > 0$ then M is compact and $\pi_1(M)$ is finite.

A good example showing that completeness is needed in (1) is the cylinder $M := S^1 \times R$. Since M is diffeomorphic to

$$N := S^2 - \{\text{North pole, South pole}\}$$

M can be given a non-complete metric g of constant positive curvature 1 by pulling back the standard metric on S^2 restricted to N. Of course, $\pi_1(M) = Z$ so (1) fails to hold for (M,g).

This classical result (1) which is a topological restriction indicates one reason why the problem of trying to produce a metric of everywhere positive Ricci curvature from a metric on non-negative Ricci curvature and all Ricci curvatures positive at some point, is of interest. A related question is: given (M,g_o) complete with $K_{g_o} \geq 0$, when does M admit a complete metric g with $K_g > 0$? For instance, a possible condition that was suggested to us by James Simons is the following:

Conjecture: Let M be a smooth manifold admitting a complete metric with non-negative sectional curvature and all sectional curvatures positive at some point. Then M admits a complete metric of everywhere positive sectional curvature.

Since the result (1) which holds for positive Ricci curvature already does not indicate the full geometric significance of perturbing a metric from $K \geq 0$ to $K > 0$, we mention two other reasons for the interest of this question. The first is that Gromoll and Meyer have produced a metric on an exotic sphere with $K \geq 0$ and all sectional curvatures positive at some point. Secondly, given (M,g) complete with $K_g \geq 0$, Poor [29] has shown that M is diffeomorphic to the normal bundle of a soul S (constructed by Cheeger and Gromoll) in M.

If $K_g > 0$, it is known that a soul is a point and hence M is diffeomorphic to R^n. Thus, suppose it were true that given (M,g_o) complete with $K_{g_o} \geq 0$ and all sectional curvatures positive at some point, M admitted a complete metric g with $K_g > 0$. Then given (M,g_o) complete with $K_{g_o} \geq 0$ and all sectional curvatures positive at some point, it would follow that M was diffeomorphic to R^n.

In [2], p. 397, T. Aubin stated a theorem for which a corollary was that given any manifold with non-negative Ricci curvature and all Ricci curvatures positive at some point, M admits a metric of everywhere positive Ricci curvature. However, the proof given in [2] appears to be in doubt even in the compact case.

Let us fix some more notation for (M,g) once and for all.

Put

$$B_{g,R}(p) := \{q \text{ in } M;\ \text{dist}_g(p,q) < R\}$$

and

$$A_{g,R,\eta}(p) := \{q \text{ in } B_{g,R}(p);\ (1-\eta)R \leq \text{dist}_g(p,q) < R\}.$$

We will call $A_{g,R,\eta}(p)$ the g-<u>outer annulus of</u> g-<u>width</u> ηR for $B_{g,R}(p)$. We will let

$$r_{g,p}(q) := \text{dist}_g(p,q) = \text{the g-distance from p}$$

and we will let

$$\rho = \rho_{g,R,p}(q) = R - r_{g,p}(q) = \text{the g-distance from}$$

$Bd(\overline{B_{g,R}(p)})$ which we will define only for q in $B_{g,R}(p)$.

Convention: All deformations g(t) of a given metric g_o for M will be at least C^3 in t through at least C^4 metrics for M.

Definition: We will say that a set D contained in (M,g) is g-convex (or just convex when it is clear what metric we mean) iff for all p,q in D, there is exactly one normal minimal geodesic in D from p to q.

The theorem stated in [2] referred to above suggests that a possible method of solving the Conjecture for sectional curvature would be to find a deformation g(t) of g_o with $g(t) = g_o$ off a small disk D centered at the point p_o of everywhere positive sectional curvature which would spread the positive curvature from a slightly smaller disk D' centered at p_o to the annulus $A := D - D'$. The most obvious way to consider whether a geometric deformation g(t) is a "good" deformation is to compute K'.

More generally, let D be a "nice" connected open set in M with D compact so that if

$$\rho : D \to R \geq 0$$

is given by

$$\rho(q) := dist_g(q, Bd(\bar{D}))$$

then $grad_g \rho$ is smooth in some one-sided tubular neighborhood $U \subset D$ of $Bd(\bar{D})$.

We will say $t \to g(t)$ with $g(0) = g_o$, t in $(-c,c)$, $c > 0$ is a <u>local deformation of</u> g_o <u>with support in</u> D iff

(1) for all t in $(-c,c)$, $g(t) = g_o$ in $M - D$,

(2) for all p in D, there exists $v \neq 0$ in M_p such that

$$g(t)(v,v) \neq g_o(v,v)$$

for all $t \neq 0$.

From [11], in order to compute K' or Ric' for a deformation $g(t)$ it is enough to know the 1-jet of $g(t)$. Thus, to study K' or Ric' for an arbitrary local deformation of g_o with support in D we may assume

$$g(t) = g_o + t\, h$$

for some symmetric two tensor h. Then the conditions $g(t) = g_o$ in $M - D$ and all metrics $g(t)$ are C^4 imply that in order to study K' or Ric' near $Bd(D)$ for an arbitrary allowable local deformation with support in D, it is enough to study all deformations of the form

$$g(t) = g_o + t\,\rho^3 h.$$

Given a two-plane $P \in G_2(M)$ with $\pi(P) \in U$ we will always choose a g_o-orthonormal basis $\{x,y\}$ for P with $x(\rho) = 0$. Then it follows from Corollary 4 given below that

$$K'(P) = -\,3\rho(y(\rho))^2 h(x,x)$$
$$+ \frac{3}{2}\rho^2[2\delta^*(d\rho)(x,y)h(x,y) - \delta^*(d\rho)(x,x)h(y,y) - \delta^*(d\rho)(y,y)h(x,x)]$$
$$+ \rho^3 y(\rho)((D_x h)(x,y) - (D_y h)(x,x)) + \rho^3(\Sigma h)(x,y).$$

The presence of the Hessian $\delta^*(d\rho)$ suggests we choose D to be a g_o-convex disk so that $\delta^*(d\rho)$ will have a definite sign. Thus, we will study <u>local convex deformations</u>, that is, local deformations with support in convex metric disks $\overline{B_{g_o,R}(p)}$.

The formula above already shows the difficulty in using local convex deformations to improve sectional curvature. (This will be made more precise in Chapter 6.) Even though it is always possible to choose a tensor h on $D = B_{g_o,R}(p)$ so that $(\Sigma h)(x,y) > 0$ if $\{x,y\}$ are a g_o-orthonormal basis for a two-plane P, Σh only enters the formula for $K'(P)$ in third order in ρ and hence, near $Bd(\bar{D})$ does not control K'. We remark here that a universal choice of a tensor h with $\Sigma h > 0$ is

$$h = d(r^2) \otimes d(r^2)$$

where $r = r_{g_o,p}$. The convexity of D implies that $\delta^*(d(r^2))$ is positive definite on $D = B_{g_o,R}(p)$ and hence

$$(\Sigma h)(x,y) = (\delta^* dr^2)(x,x) \cdot \delta^*(dr^2)(y,y) - (\delta^*(dr^2)(x,y))^2 > 0$$

by the generalized Cauchy-Schwartz inequality for positive operators. Geometrically, we can picture this deformation as follows. Let $(M,g_o) := (R^2, g_{can})$ with $p = (0,0)$. Then the metric

$$g(t) = g_o + t\, d(r^2) \otimes d(r^2)$$

on R^2 can be represented by the metric on the paraboloid of revolution

$$M_t : z = z(t) = t\ (x^2 + y^2)$$

induced from the standard metric on R^3 by the inclusion $M_t \subset R^3$ since $\delta^*(d(r^2)) = 2g_{can}$.

In later chapters with this discussion as motivation, we will consider the following two problems:

Problem I: Given $D = B_{g_o,R}(p)$ convex, what are the possible local convex deformations of g_o with support in D and with Ric' > 0 in an annular neighborhood in D of $Bd(\bar{D})$?

Problem II: Given $D = B_{g_o,R}(p)$ convex, can we find a local convex deformation of g_o with support in D so that K' > 0 in an annular neighborhood in D of $Bd(\bar{D})$?

We now derive some computational lemmas that we will use later.

Convention: Given $\xi \in C^\infty(T^*(M))$, X, Y $\in C^\infty(TM)$, we will define

$$d\xi(X,Y) = X\cdot\xi(Y) - Y\cdot\xi(X) - \xi([X,Y])$$

omitting the factor of one-half.

Definition: Given $U \subset D \underset{\neq}{\subset} M$ as above, $p \in U - Bd\ D$. For $x \in M_p$ the radial component of x written x_ρ is

$x_\rho = g_o(x,\nabla\rho)\nabla\rho$ and the tangential component is $x_T = x - x_\rho$. We will say x is radial if $x_T = 0$ and x is tangential if $x_\rho = 0$.

This definition is motivated by the following geometric model. Let D be a convex disk centered at p_o and let $S_r(p_o) = \{q \in D; \operatorname{dist}_{g_o}(p_o,q) = r\}$. Then $\nabla\rho$ is the inward pointing normal vector field to $S_r(p_o) \subset D$ and $\nabla\rho$ points in the "radial" direction towards p_o. A tangential vector lies in $(\nabla\rho)^\perp$ and is tangent to the sphere $S_r(p_o)$.

Notational Convention: In the computational lemmas to follow, we will write $\langle,\rangle$ for g_o and D for the Levi-Civita connection determined by g_o. Given $x,y \in M_p$, we will always extend x and y to local vector fields X and Y in a neighborhood V of p so that $[X,Y] = 0$ in V and $DX|_p = 0$ and $DY|_p = 0$ where $DX(v) := D_vX|_p$. We will call X and Y a good extension of x and y. For a good extension,

$$(\Sigma h)(x,y) = xY(h(X,Y)) - \tfrac{1}{2}xX(h(Y,Y)) - \tfrac{1}{2}yY(h(X,X)) - h(R(x,y)y,x).$$

Lemma 3: Let $f : M \to R$ be a C^2 function and $h \in C^\infty(S^2(M))$. Then

$$(\Sigma fh)(x,y) = \delta^*(df)(x,y)h(x,y) - \frac{1}{2}\,\delta^*(df)(x,x)h(y,y) - \frac{1}{2}\,\delta^*(df)(y,y)h(x,x)$$

$$+ x(f)((D_y h)(x,y) - (D_x h)(y,y)) + y(f)((D_x h)(x,y) - (D_y h)(x,x))$$

$$+ f(\Sigma h)(x,y)$$

<u>Proof</u>: Let X and Y be a good extension of x and y. Then

$$(D_x D_y h)(x,y) = xY(fh(X,Y)) = x(Y(f)h(X,Y) + fY(h(X,Y)))$$

$$= xY(f)h(x,y) + y(f)x(h(X,Y)) + x(f)y(h(X,Y)) + fxY(h(X,Y))$$

$$= \delta^*(df)(x,y)h(x,y) + y(f)(D_x h)(x,y) + x(f)(D_y h)(x,y) + f(D_x D_y h)(x,y).$$

Similarly, $(D_x D_x h)(y,y) = \delta^*(df)(x,x)h(y,y) + 2x(f)(D_x h)(y,y) + f(D_x D_x h)(y,y)$

and the lemma now follows from formula (5).

Q. E. D.

Now

$$\delta^*(d\rho^n)(x,y) = \langle D_x \nabla(\rho^n), y\rangle = \langle D_x(n\rho^{n-1}\nabla\rho), y\rangle$$

$$= n(n-1)\rho^{n-2}x(\rho)y(\rho) + n\rho^{n-1}\delta^*(d\rho)(x,y)$$

so

(7) $$\delta^*(d\rho^n) = n\rho^{n-1}\delta^*(d\rho) + n(n-1)\rho^{n-2}d\rho\circ d\rho \ .$$

In particular, if $x_\rho = 0$, then

(8) $$\delta^*(d\rho^n)(x,y) = n\rho^{n-1}\delta^*(d\rho)(x,y).$$

Hence, we obtain

Corollary 4: For $n \geq 3$ in U

$$\begin{aligned}(\Sigma\rho^n h)(x,y) = \frac{n(n-1)}{2}\{&2x(\rho)y(\rho)h(x,y) - \|x_\rho\|^2 h(y,y) \\ &- \|y_\rho\|^2 h(x,x)\}\rho^{n-2} \\ + \frac{n}{2}\rho^{n-1}\{&2\delta^*(d\rho)(x,y)h(x,y) - \delta^*(d\rho)(x,x)h(y,y) \\ &- \delta^*(d\rho)(y,y)h(x,x)\} \\ + n\rho^{n-1}y(\rho)((D_xh)(x,y) &- (D_yh)(x,x)) + n\rho^{n-1}x(\rho)((D_yh)(x,y) \\ &- (D_xh)(y,y)) \\ + \rho^n(\Sigma h)(x,y).\end{aligned}$$

Let $\xi,\eta \in C^\infty(T^*M)$ and consider the variation $g(t) = g_0 + t\,\xi\circ\eta$. An elementary but lengthy calculation shows:

Lemma 5: $$\begin{aligned}(\Sigma(\xi\circ\eta))(x,y) &= \frac{1}{2}\eta(x)(D_yd\xi)(x,y) + \frac{1}{2}\xi(x)(D_yd\eta)(x,y) \\ &+ \frac{1}{2}\eta(y)(D_xd\xi)(y,x) + \frac{1}{2}\xi(y)(D_xd\eta)(y,x) \\ &+ \frac{1}{2}\delta^*\xi(x,x)\delta^*\eta(y,y) + \frac{1}{2}\delta^*\eta(x,x)\delta^*\xi(y,y) - \frac{1}{2}d\xi(x,y)d\eta(x,y) \\ &- \frac{1}{2}(D_x\xi)(y)(D_x\eta)(y) - \frac{1}{2}(D_y\xi)(x)(D_y\eta)(x).\end{aligned}$$

If $d\eta = 0$, then $D_X\eta(Y) = D_Y\eta(X)$ since $[X,Y] = 0$

so $\quad D_X\xi(Y)D_X\eta(Y) + D_Y\xi(X)D_Y\eta(X) =$

$$D_X\eta(Y)(D_X\xi(Y) + D_Y\xi(X)) = 2\delta^*\eta(x,y)\delta^*\xi(x,y).$$

We have proven

<u>Corollary 6</u>: If η is exact, i.e., $d\eta = 0$, then

$$(\Sigma(\xi\circ\eta))(x,y) = \tfrac{1}{2}(\eta(y)(D_x d\xi)(y,x) + \eta(x)(D_y d\xi)(x,y) +$$

$$+ \delta^*\xi(x,x)\delta^*\eta(y,y) + \delta^*\eta(x,x)\delta^*\xi(y,y) -$$

$$- 2\delta^*\xi(x,y)\delta^*\eta(x,y)).$$

<u>Lemma 7</u>: If ξ is a Killing form, i.e., $\delta^*\xi \equiv 0$, then

$$(D_x d\xi)(y,x) = 0.$$

<u>Proof</u>: Making a good extension $2\delta^*\xi(X,Y) = 0 = D_X\xi(Y)+D_Y\xi(X)$

so

$$(D_x d\xi)(y,x) = D_x(D_Y\xi(X) - D_X\xi(Y)) = -2D_X D_X\xi(Y)$$

$$= -2R(x,x,\xi^{\#},y) = 0.$$

Q. E. D.

<u>Lemma 8</u>: $(\Sigma(\rho^3 d\rho\circ\xi))(x,y) = \frac{3}{2}\rho^2(\|x_\rho\|^2\delta^*\xi(y,y) +$

$\|y_\rho\|^2\delta^*\xi(x,x) - 2d\rho\circ d\rho(x,y)\delta^*\xi(x,y))$

$+ \frac{\rho^3}{2}(y(\rho)(D_x d\xi)(y,x) + x(\rho)(D_y d\xi)(x,y) + \delta^*\xi(x,x)\delta^*(d\rho)(y,y)$

$\delta^*\xi(y,y)\delta^*(d\rho)(x,x) - 2\delta^*\xi(x,y)\delta^*(d\rho)(x,y)).$

Proof: By Corollary 4, we have

$$(\Sigma(\rho^3 d\rho\circ\xi))(x,y) =$$

$$3\rho\{2x(\rho)y(\rho)d\rho\circ\xi(x,y) - \|x_\rho\|^2 d\rho\circ\xi(y,y) - \|y_\rho\|^2 d\rho\circ\xi(x,x)\}$$

$$+ \frac{3}{2}\rho^2\{2\delta^*(d\rho)(x,y)d\rho\circ\xi(x,y) - \delta^*(d\rho)(x,x)d\rho\circ\xi(y,y) - \delta^*(d\rho)(y,y)d\rho\circ\xi(x,x)\}$$

$$+ 3\rho^2 y(\rho)((D_x d\rho\circ\xi)(x,y) - (D_y d\rho\circ\xi)(x,x)) + 3\rho^2 x(\rho)((D_y d\rho\circ\xi)(x,y) - (D_x d\rho\circ\xi)(y,y)) - \rho^3(\Sigma(d\rho\circ\xi))(x,y).$$

Now $(D_x d\rho\circ\xi)(x,y) = \frac{1}{2}((D_x d\rho \otimes \xi)(x,y) + (D_x\xi \otimes d\rho)(x,y))$

$$= \frac{1}{2}((D_x d\rho)(x)\cdot\xi(y) + x(\rho)(D_x\xi)(y) + y(\rho)(D_x\xi)(x) + \xi(x)(D_x d\rho)(y))$$

$$= \frac{1}{2}(\xi(y)\delta^*(d\rho)(x,x) + x(\rho)(D_x\xi)(y) + y(\rho)\delta^*\xi(x,x) + \xi(x)\delta^*(d\rho)(x,y)),$$

$$(D_y d\rho\circ\xi)(x,x) = \xi(x)\delta^*(d\rho)(x,y) + x(\rho)(D_y\xi)(x),$$

$$(D_y d\rho\circ\xi)(x,y) = \frac{1}{2}(\xi(y)\delta^*(d\rho)(x,y) + x(\rho)\delta^*\xi(y,y) + y(\rho)(D_y\xi)(x) + \xi(x)\delta^*(d\rho)(y,y)),$$

and $(D_x d\rho\circ\xi)(y,y) = \xi(y)\delta^*(d\rho)(x,y) + y(\rho)(D_x\xi)(y).$

Hence,

$$3\rho^2 y(\rho)((D_x d\rho\circ\xi)(x,y) - (D_y d\rho\circ\xi)(x,x)) + 3\rho^2 x(\rho)((D_y d\rho\circ\xi)(x,y) - (D_x d\rho\circ\xi)(y,y))$$

$$= \frac{3}{2}\rho^2(\delta^*(d\rho)(x,x)d\rho\circ\xi(y,y) + \delta^*(d\rho)(y,y)d\rho\circ\xi(x,x) - 2d\rho\circ\xi(x,y)\delta^*(d\rho)(x,y)$$

$$+ \|x_\rho\|^2\delta^*\xi(y,y) + \|y_\rho\|^2\delta^*\xi(x,x) - 2d\rho\circ d\rho(x,y)\delta^*\xi(x,y)).$$

Finally,

$$2d\rho\circ d\rho(x,y)d\rho\circ\xi(x,y)-\|x_\rho\|^2 d\rho\circ\xi(y,y)-\|y_\rho\|^2 d\rho\circ\xi(x,x) =$$

$$x(\rho)y(\rho)(x(\rho)\xi(y)+y(\rho)\xi(x))-\|x_\rho\|^2 y(\rho)\xi(y)-\|y_\rho\|^2 x(\rho)\xi(x) = 0.$$

Q. E. D.

Suppose $x_\rho = 0$. Then $d\rho\circ d\rho(x,y) = 0$ and $d\rho\circ\xi(x,x) = 0$ so that

Corollary 9: If x and y are g_o-orthonormal with $x_\rho = 0$ and

$$g(t) = g_o + t\,\rho^3 d\rho\circ\xi, \quad \text{then}$$

$$K'(x,y) = \frac{3}{2}\rho^2\|y_\rho\|^2\delta^*\xi(x,x) + \frac{\rho^3}{2}(\delta^*\xi(x,x)\delta^*(d\rho)(y,y)$$

$$+ \delta^*\xi(y,y)\delta^*(d\rho)(x,x) - 2\delta^*\xi(x,y)\delta^*(d\rho)(x,y)+\langle y,\nabla\rho\rangle(D_x d\xi)(y,x))$$

$$- \rho^6\|y_\rho\|^2(\xi(x))^2 K(x,y).$$

Proof: By formula (6) for $h^1 = \rho^3 d\rho\circ\xi$, we must compute the h^1 "area squared"

$$h^1(x,x)h^1(y,y) - h^1(x,y)^2 = -(\rho^3 d\rho\circ\xi(x,y))^2$$

$$= -\rho^6\|y_\rho\|^2(\xi(x))^2.$$

Q. E. D.

Suppose $K_{g_o} \geq 0$. Notice that if ξ is a g_o-Killing form, i.e., $\delta^*_{g_o}\xi = 0$, then by (4)

$$\delta^*(\frac{\rho^4}{4}\xi) = \rho^3 d\rho\circ\xi + \frac{\rho^4}{4}\delta^*\xi = \rho^3 d\rho\circ\xi$$

so that $\rho^3 d\rho \circ \xi$ is in the image of δ^* and hence $K(x,y) = 0$ should imply $K'(x,y) = 0$ by Proposition 1. Evidently, $K'(x,y) = \langle y, \nabla\rho \rangle (D_x d\xi)(y,x)$ but this is then zero by Lemma 7.

Chapter 3: Some Remarks on the Space of Riemannian Metrics of a Smooth Manifold

The purpose of this section is threefold. First we make explicit the notion that if two metrics for a smooth manifold are close, then metric balls in the two metrics centered at the same point of the same radius are similar. This we use in Section 5 to prove the Ricci curvature deformation theorems stated in [20].

Second we observe that if two Riemannian metrics for M agree off a compact subset of M and one metric is complete, then the other metric is complete. We apply this result to determine the end structure of complete non-compact non-negatively Ricci curved manifolds with positive Ricci curvature off a compact set.

Third we apply the results of [19] and/or [14] and the comparison theory in Riemannian geometry to obtain the local minorization of the convexity radius function $g \to i_g(M)$ and the injectivity radius function $g \to c_g(M)$ on the space of Riemannian metrics R(M) for M. Precisely, if M is compact and $g_o \in R(M)$ is given, we show that there exist constants $\delta(g_o) > 0$ and $C(g_o) > 0$ such that if $g \in R(M)$ is $\delta(g_o)$ close to g in the C^2 topology on R(M), then any g-disk of g-radius $\leq C(g_o)$ is g-convex.

Given two metrics g_1 and g_2 for a manifold M, we write

$$A\, g_1 \leq g_2 \leq B\, g_1$$

for constants A, B in R iff for all v in TM

$$A\, g_1(v,v) \leq g_2(v,v) \leq B\, g_1(v,v).$$

Recall that for p and q in (M,g), we define a distance function

$$\mathrm{dist}_g : M \times M \to R \geq 0$$

from the Riemannian metric g for TM by

$$\mathrm{dist}_g(p,q) := \inf\{L_g(c);\ c \text{ is a sectionally smooth path from p to q}\}.$$

It is elementary that

<u>Lemma 1</u>: Let $0 < A \in R$. If $A\, g_1 \leq g_2$, then

$$(*) \qquad \sqrt{A}\, \mathrm{dist}_{g_1}(p,q) \leq \mathrm{dist}_{g_2}(p,q)$$

for all p and q in M. We will write

$$\sqrt{A}\, \mathrm{dist}_{g_1} \leq \mathrm{dist}_{g_2}$$

meaning that (*) holds.

<u>Corollary 2</u>: $A\, g_1 \leq g_2 \leq B\, g_1$ implies $\sqrt{A}\, \mathrm{dist}_{g_1} \leq$ $\leq \mathrm{dist}_{g_2} \leq \sqrt{B}\, \mathrm{dist}_{g_1}$.

Now let

$$\mathcal{B}_{g_o,R}(p) := \{v \text{ in } M_p;\ g_o(v,v) < R^2\} \subset M_p,$$

$$B_{g_o,R}(p) := \{q \text{ in } M;\ \mathrm{dist}_{g_o}(p,q) < R\} \subset M, \text{ and}$$

$$S_{g_o,R}(p) = \{q \text{ in } M;\ \mathrm{dist}_{g_o}(p,q) = R\} \subset M.$$

Let $0 < \delta < 1$.

<u>Lemma 3</u>: Suppose $(1-\delta)\,\mathrm{dist}_{g_o} \le \mathrm{dist}_g \le (1+\delta)\,\mathrm{dist}_{g_o}$.

Then

(i) $S_{g_o,R}(p) \subset B_{g,(1+\delta)R}(p)$, and

(ii) $B_{g,(1-\delta)R}(p) \subset B_{g_o,R}(p) \subset B_{g,(1+\delta)R}(p)$.

<u>Proof</u>: (i) Let $q \in S_{g_o,R}(p)$. Thus, $\mathrm{dist}_{g_o}(p,q) = R$.

Then

$$\mathrm{dist}_g(p,q) \le (1+\delta)\mathrm{dist}_{g_o}(p,q) \le (1+\delta)R \quad \text{so}$$

$$q \in B_{g,(1+\delta)R}(p).$$

(ii) If $q \in B_{g_o,R}(p)$, then $\mathrm{dist}_{g_o}(p,q) = R - c$

for some $c > 0$.

Then

$$\mathrm{dist}_g(p,q) \le (1+\delta)(R-c) < (1+\delta)R$$

so

$$B_{g_o,R}(p) \subset B_{g,(1+\delta)R}(p).$$

Let $m \in B_{g,(1-\delta)R}(p)$. Then $\mathrm{dist}_g(p,m) = (1-\delta)(R-d)$ for some $d > 0$. Then

$$\mathrm{dist}_{g_o}(p,m) \le \frac{1}{(1-\delta)}\,\mathrm{dist}_g(p,m) = R - d < R$$

which implies

$$B_{g,(1-\delta)R}(p) \subset B_{g_o,R}(p).$$

Q. E. D.

Lemma 3 can be thought of as a squeezing lemma which implies that if

$$(1-\delta)^2 g_o \leq g \leq (1+\delta)^2 g_o$$

for some sufficiently small δ, then to a g-observer a g_o-ball centered at p roughly looks like a g_o-ball centered at p since the g_o-sphere $S_{g_o,R}(p)$ is contained inside the g-annulus centered at p of "g-width" $\leq 2\delta R$, namely

$$A = B_{g,(1+\delta)R}(p) - B_{g,(1-\delta)R}(p).$$

As an aside, we note

<u>Lemma 4</u>: Given any Riemannian manifold (M,g), we have

$$\mathrm{dist}_g(S_{g,R_1}(p),\ S_{g,R_2}(p)) = |R_2 - R_1|.$$

Given a g_o-outer annulus $A = A_{g_o,R,\varepsilon/4}(p)$ with ε sufficiently small, it is reasonable that if g is another metric sufficiently close to g_o, then A will be contained in a g-ball $\tilde{B}$ centered at p of g-radius $\tilde{R}$ slightly larger than R such that the g-outer annulus $\tilde{A} := A_{g,\tilde{R},\varepsilon}(p)$ of B of width $\varepsilon\tilde{R}$ will contain A.

More precisely,

Lemma 5: Given M and a fixed metric g_o for M. Fix $0 < \varepsilon \le 1/2$. There exists $\delta > 0$ with the following property. Let g be any metric for M with

$$(1-\delta)^2 g_o \le g \le (1+\delta)^2 g_o .$$

Then for any p in M and R > 0, there exists $\tilde{R} > 0$ such that

$$A_{g_o,R,\varepsilon/4}(p) \subset A_{g,\tilde{R},\varepsilon}(p).$$

Proof: From the hypothesis and Corollary 2, we have

$$(1-\delta)\,\mathrm{dist}_{g_o} \le \mathrm{dist}_g \le (1+\delta)\,\mathrm{dist}_{g_o} .$$

Given ε we want to choose δ such that

$$(*) \qquad (1-\varepsilon)(1+\delta) < (1-\delta)(1-\varepsilon/4)$$

which holds if $1 - \varepsilon - \varepsilon/4 > 0$ and $\delta < 3\varepsilon/4(1 - \varepsilon - \varepsilon/4)$.
If $\varepsilon \le 1/2$, then $1 - \varepsilon - \varepsilon/4 \ge 3/8$ so (*) holds if $\delta < 2\varepsilon$.

Now let $q \in A_{g_o,R,\varepsilon/4}(p)$. Let $\tilde{R} := (1+\delta)R$.
By definition,

$$(1-\varepsilon/4)R \le \mathrm{dist}_{g_o}(p,q) < R.$$

Hence,

$$\mathrm{dist}_g(p,q) < (1+\delta)R = \tilde{R}.$$

It remains to show that

$$\mathrm{dist}_g(p,q) \ge (1-\varepsilon)\tilde{R} = (1-\varepsilon)(1+\delta)R.$$

But

$$\mathrm{dist}_g(p,q) \geq (1-\delta)\mathrm{dist}_{g_o}(p,q) \geq (1-\delta)(1-\mathcal{E}/4)R.$$

Thus

$$\mathrm{dist}_g(p,q) \geq (1-\mathcal{E})\tilde{R} \quad \text{if} \quad (1-\delta)(1-\mathcal{E}/4)R \geq (1-\mathcal{E})(1+\delta)R$$

which holds if (*) holds. Hence, for $\delta < 2\mathcal{E}$,

$$A_{g_o,R,\mathcal{E}/4}(p) \subset A_{g,(1+\delta)R,\mathcal{E}}(p) = A_{g,\tilde{R},\mathcal{E}}(p),$$

if

$$(1-\delta)^2 g_o \leq g \leq (1+\delta)^2 g_o \ .$$

Q. E. D.

We now state and prove Lemma 6, then apply it to study the end structure of complete open manifolds (M,g) with $\mathrm{Ric}_g \geq 0$ and with "positive Ricci curvature at infinity".

Lemma 6: Let (M,g_o) be complete, non-compact. Let g be another Riemannian metric for M. Suppose there exists a compact D in M such that $g = g_o$ in $TM|_{M - \mathrm{Int}(D)}$.

Then (M,g) is complete.

Proof: Let $\pi : TM \to M$ be the tangent bundle of M. Let $E := \{v \text{ in } TM;\ \pi(v) \in D \text{ and } g_o(v,v) = 1\}$. E is compact because D is compact. Then $f : E \to R > 0$ given by $f(v) := g(v,v)$ is a continuous function. Hence, we can find constants A, B in R with $0 \leq A \leq 1$, $B \geq 1$ such that

$$A \le f(v) \le B \quad \text{for all } v \text{ in } E.$$

Then

$$A\, g_o \le g \le B\, g_o$$

in TM so that

$$\sqrt{A}\ \mathrm{dist}_{g_o} \le \mathrm{dist}_g \le \sqrt{B}\ \mathrm{dist}_{g_o} .$$

Hence, using the Cauchy sequence criterion for completeness, it follows easily that (M, g_o) complete implies (M, g) complete.

Q. E. D.

Let M^n be a non-compact Riemannian manifold, $n \ge 2$. We will consider complete metrics g for M with $\mathrm{Ric}_g \ge 0$.

Definition: (M, g) is said to be positively Ricci curved infinity iff there exists a compact $C \subset M$ such that $\mathrm{Ric}_g(v, v) = 0$ implies $\pi(v) \in C$.

Recall that from the Hopf-Rinow Theorem a dist_g-metric bounded set in a complete manifold (M, g) is compact. Given (M, g) with $\mathrm{Ric}_g \ge 0$ positively Ricci curved at infinity (with C as in the definition) let

$$D := \bigcup_{p \in C} \overline{B_{g,1}(p)}$$

which is dist_g-bounded (since C is compact) and hence compact. By our Ricci curvature deformation theorem of Section 5 and [19] for uniformity, produce a C^4 metric $\tilde{g}$ for M with $g = \tilde{g}$ off D and $\mathrm{Ric}_{\tilde{g}} > 0$ on M.

By Lemma 6, $\tilde{g}$ is a complete metric for M. Since $Ric_{\tilde{g}} > 0$ it follows that $(M,\tilde{g})$ has no lines (see [23], pp. 78 ff.). Hence, M has only one end, for if not the standard classical construction, c.f., Preissmann, [30], produces a line. (See [23], pp. 80 for a definition of "M has only one end".) We have shown

<u>Theorem 7</u>: Let M^n, $n \geq 2$, be an open manifold. Suppose M admits a complete C^4 metric g with $Ric_g \geq 0$ that is positively Ricci curved at infinity. Then M admits a complete C^4 metric $\tilde{g}$ with $Ric_{\tilde{g}} > 0$ everywhere. Hence, M is connected at infinity, that is, M has only one end.

Now we give an example to show that $Ric_g \geq 0$ is necessary in Theorem 7. Recall that the surface of revolution

$$z = x^2 + y^2$$

represents a complete metric on R^2 with $K > 0$. Let

$$M := \{(x,y,z) \in R^3;\ z \geq 2 \text{ and } z = x^2 + y^2\}$$

Glueing two copies of M together by a tube, we get a complete metric on $S^1 \times R^2$ which is positively Ricci curved at infinity, but with some negative Ricci curvature. Clearly $S^1 \times R^2$ has two ends.

Taking the positively curved n-dimensional hypersurface $M^n \subset R^{n+1}$ given by

$$z = (x_1)^2 + (x_2)^2 + \ldots + (x_{n+1})^2$$

and making the analogous construction we get a complete n-manifold with two ends that is positively curved at infinity but again will have some negative Ricci curvature.

We make some standard definitions. Given (M,g) complete, let

$$i_g(p) := \sup\{R > 0;\ \exp_p : B_{g,R}(p) \to B_{g,R}(p) \text{ is a diffeomorphism}\}$$

called the <u>injectivity radius of</u> (M,g) <u>at</u> p, and let

$$i_g(M) := \inf\{i_g(p);\ p \in M\}$$

called the <u>injectivity radius of</u> (M,g). We say $C \subset (M,g)$ is <u>g-convex</u> iff for all $p, q \in C$ there is exactly one normal minimal geodesic in C from p to q. We define

$$c_g(p) := \sup\{R > 0;\ B_{g,R}(p) \text{ is g-convex}\}$$

which is the <u>convexity radius of</u> (M,g) <u>at</u> p, and

$$c_g(M) := \inf\{c_g(p);\ p \in M\}$$

the <u>convexity radius</u> of (M,g). It is well known that for fixed g,

$$p \to i_g(p) \quad \text{from } M \to R \geq 0$$

and

$$p \to c_g(p) \quad \text{from } M \to R \geq 0$$

are continuous functions. Thus, if M is compact,

$$i_g(M) > 0 \quad \text{and} \quad c_g(M) > 0.$$

We want to study the local behavior of $g \to i_g(M)$ and $g \to c_g(M)$ as functions $R(M) \to R \geq 0$ on the space of metrics $R(M)$ for M. Fixing a Riemannian metric g_o for M, recall that $i_{g_o}(p)$ and $c_{g_o}(p)$ are determined by the behavior of the configuration of radial geodesics from p. The basic idea is that near p the configuration is qualitatively like that in (R^n, g_{can}).

For g close to g_o in the C^2 topology on $R(M)$, we would expect the configuration of g-radial geodesics at p should be close to the g_o-configuration of radial geodesics at p. This is because writing out the partial differential equation for a g-radial geodesic at p in initial direction $v \in M_p$ in a g_o-Riemann normal coordinate ball, g being C^2 close to g_o implies the coefficients of g-p.d.e. are close to the g_o-coefficients of the g_o-p.d.e. for the g_o-radial geodesic in direction v. Hence, it is reasonable that the function

$$g \to i_g(p), \quad R(M) \to R \geq 0$$

for a fixed p in M should be locally minorized with the C^2 topology on $R(M)$. (See [19] for details.)

For M compact we will consider the behavior of the injectivity radius $i_g(M)$ and the convexity radius $c_g(M)$

as we vary g in a neighborhood of a given metric g_o in $R(M)$.

A basic estimate in Riemannian geometry asserts for M compact

(*) $$i_g(M) \geq \min\{\pi/\sqrt{n_g},\ 1/2\ L(g)\}$$

where

$$L(g) := \inf\{L_g(c);\ c \text{ is a smooth closed } g\text{-geodesic}\}$$

and where $n_g > 0$ is any constant chosen so that

$$K_g(p) \leq n_g \quad \text{for all two-planes } P \in G_2(M).$$

Of course, for M compact, there is a geodesic c with

$$L_g(c) = L(g) > 0.$$

Theorem 8: Given M compact and a Riemannian metric g_o for M, there exists constants $\delta(g_o) > 0$, $L(g_o) > 0$, and $i(g_o) > 0$ such that if g is any Riemannian metric for M with

$$|g-g_o|_{C^2} < \delta(g_o)$$

then

(i) $L(g) > L(g_o)$, and

(ii) $i_g(M) > i(g_o)$.

Proof: Since M is compact, $\operatorname{diam}(M,g_o)$, $\operatorname{Vol}(M,g_o)$, and $h_{g_o} := \sup\{|K_{g_o}(P)|;\ P \in G_2(M)\}$ are finite. Thus, we can choose d,V,h, and $\delta(g_o) > 0$ so that

$$|g-g_o|_{C^2} < \delta(g_o) \quad \text{implies}$$

(#) $\operatorname{diam}(M,g) < d$, $\operatorname{Vol}(M,g) > V$, and $K_g > h$.

Then the existence of $L(g_o)$ follows from a theorem of Cheeger [14], minorizing the length of the shortest smooth closed geodesic for any manifold and Riemannian metric for that manifold for which (#) holds. The existence of $i(g_o)$ is then trivial, from inequality (*). It is possible to give a more direct proof using [19].

Q. E. D.

Now we turn our attention to the local minorization of the convexity radius function.

<u>Definition:</u> $B_{g,R}(p)$ is said to be g-good iff for all q in $B_{g,R}(p)$, the exponential map

$$M|_q \supset \mathcal{B}_{g,2R}(q) \xrightarrow{\exp_q} B_{g,2R}(q) \subset M$$

is a diffeomorphism.

The following lemma from [22] pp. 160, shows what is needed to minorize the convexity radius function on R(M). We make the convention that for $p \in M$, $v \in S_1(M,g)|_p$ that we write

$$c_v(t) := \exp_p tv$$

for the g-radial geodesic from p in unit direction v.

Lemma I: Suppose $B_{g,R}(p)$ satisfies

A) $B_{g,R}(p)$ is g-good, and

B) for all v in $S_1(M,g)|_p$ the index form I_{c_v} is positive definite on all Jacobi fields J along c_v with initial conditions $J(0) = 0$ and $g(c_v, J) = 0$.

Then $B_{g,R}(p)$ is g-convex.

The proof of the Index Comparison Theorem ([22], p. 174) and the fundamental inequality for Jacobi fields ([22], pp. 145, for mula (5)) easily imply that the following, which seems to be absent from the standard references, holds.

Proposition 9: Fix (M_1^n, g_1), (M_2^n, g_2) complete Riemannian manifolds. Choose $p_1 \in M_1$, $p_2 \in M_2$, and a Euclidean isometry

$$\Phi : M_1|_{p_1} \to M_2|_{p_2} \quad .$$

For all v in $S_1(M_1, g_1)|_{p_1}$, let $c_v : R \to M_1$ and $c_{\Phi(v)} : R \to M_2$ be radial geodesics from p_1 and p_2 respectively, defined as above. Let

$$Par_{v,t} : M_1|_{c_v(t)} \to M_2|_{c_{\Phi(v)}(t)}$$

be the usual parallel transplant map, i.e., $Par_{v,t}(w)$ is obtained by parallel translating w along $c_v(t)$ to p_1 to get $\tilde{w} \in M_1|_{p_1}$ and then parallel translating $\Phi(\tilde{w})$ along $c_{\Phi(v)}$ to $c_{\Phi(v)}(t)$.

Suppose $R > 0$ is chosen so that $B_{g_1,R}(p_1) \subset M_1$ is g_1-good, $B_{g_2,R}(p_2) \subset M_2$ is g_2-good, and so that

$$K_{g_1}(\dot{c}_v(t),x) \geq K_{g_2}(\dot{c}_{\Phi(v)}(t), \mathrm{Par}_{v,t}(x))$$

for all $x \in (\dot{c}_v(t))^\perp \subset M_1|_{c_v(t)}$, all $t \in [0,R)$, and all $v \in S_1(M_1,g_1)|_{p_1}$. Suppose for all $v \in S_1(M_1,g_1)|_{p_1}$ that the index form I_{c_v} is positive definite on all Jacobi fields along c_v on M_1 with $J(0) = 0$ and $g_1(J,\dot{c}_v) = 0$. Then for all $\Phi(v) \in S_1(M_2,g_2)|_{p_2}$ the index form $I_{c_{\Phi(v)}}$ is positive definite on all Jacobi fields $\tilde{J}$ along $c_{\Phi(v)}$ in M_2 with $\tilde{J}(0) = 0$ and $g_2(\tilde{J},\dot{c}_{\Phi(v)}) = 0$. In particular, given $R > 0$ with $B_{g_1,R}(p_1)$ g_1-convex and $B_{g_2,R}(p_2)$ g_2-good, $B_{g_2,R}(p_2)$ is g_2-convex.

Given (M^n,g_0) compact, $K_{g_0} \leq \varkappa_{g_0} < \varkappa$, $\varkappa > 0$, Proposition 9 applied to $(M_1,g_1) := (M^n,g_0)$ and $(M_2,g_2) := (S^n_{1/\sqrt{n}},g_{can})$ shows that the behavior of (B) of Lemma I is minorized for g close to g_0, which can be stated as

<u>Proposition 10</u>: Given M compact and $\varkappa > 0$. There is

a constant $R(\kappa) > 0$ with the following property. Let g be any Riemannian metric for M with $K_g \le \kappa$. If B is any g-good ball of g-radius $\le R(\kappa)$, then B is g-convex.

Remark: Clearly, if $K_{g_0} \le \kappa < 0$, then any Riemannian metric g sufficiently close to g_0 will satisfy $K_g \le 0$ which automatically forces any g-good ball to be convex.

Putting together lemma 1, Proposition 9, Proposition 10, and Theorem 8, we obtain

Theorem 11: (Local minorization of the convexity radius function).

Given M compact and a Riemannian metric g_0 for M, there exist constants $\delta(g_0) > 0$ and $C(g_0) > 0$ such that if $g \in R(M)$ satisfies

$$|g - g_0|_{C^2} < \delta(g_0)$$

then any g-disk on M of radius $\le C(g_0)$ is g-convex.

For the Ricci curvature deformation theorem for $\mathrm{Ric} \le 0$ to be given later, we need an estimate on

$$\delta^*(d(r_{g,p}))$$

(where $r_{g,p}(q) := \mathrm{dist}_g(p,q)$, or equivalently on the index form I_{c_v} of Lemma I for all metrics g sufficiently close to a fixed metric g_0 for M.

Given M compact and a Riemannian metric g_0 for M, choose constants $\kappa^L_{g_0}$ and $\kappa^U_{g_0}$ such that for all two-planes

$P \in G_2(M)$

$$\varkappa^L_{g_o} \le K_{g_o}(P) \le \varkappa^U_{g_o} .$$

Fixing suitable constants $\varkappa_1 < \varkappa^L_{g_o}$ and $\varkappa_2 > \varkappa^U_{g_o}$ we may choose $0 < \delta_1(g_o) \le \delta(g_o)$ where $\delta(g_o)$ is the constant of Theorem 11 such that

$$|g - g_o|_{C^2} < \delta_1(g_o)$$

implies

$$\varkappa_1 \le K_g \le \varkappa_2 .$$

Then applying the idea of the proof of Proposition 10 to compare the index form of (M,g) to that of the relevant standard model spaces of constant curvature $\varkappa_1$ and $\varkappa_2$, we see that

<u>Theorem 12</u>: Given M compact and a Riemannian metric g_o for M. There exist constants $\delta(g_o) > 0$ and $F(g_o) > 0$ such that $g \in R(M)$ and

$$|g - g_o|_{C^2} < \delta(g_o)$$

implies any g-disk centered at any $p \in M$ of g-radius $\le F(g_o)$ is g-convex, and

$$(2-1/4)g \le \delta^*(d(r_{g,p}2)) \le (2+1/4)g$$

holds any such disk.

<u>Remark</u>: As a motivation for this inequality, recall that

$$\delta^*(d(r^2_{g,p}))|_p = 2\, g|_p$$

<u>Proof</u>: Let $v \in S_1(M,g)|_p$. Then

$$\delta^*(d(r_{g,p}{}^2))(v,v) = \frac{d^2}{dt^2}(r_{g,p}(c_v(t)))^2 \Big|_{t=0} .$$

But for t small,

$$(r_{g,p}(c_v(t))^2 = (r_{g,p}(\exp_p tv))^2 = (\mathrm{dist}_g(\exp_p tv, \exp_p ov))^2$$

$$= g(tv,tv) = t^2.$$

Thus,

$$\delta^*(d(r_{g,p})^2)(v,v) = \frac{d^2}{dt^2}(t \to t^2)\Big|_{t=0} = 2.$$

Chapter 4: Calculation of Ric' for local convex deformations: the solution of Problem I.

In this chapter, given (M,g_o) and p in M we consider

Problem I: Given $D = B_{g_o,R}(p)$ convex, what are the possible geometric deformations $g(t)$ of g_o C^3 in t through C^4 metrics with support in D and with $Ric' > 0$ in an annular neighborhood of $Bd(\bar{D})$ in $\bar{D}$? and show that this problem can be solved in particular by a conformal deformation.

Recall that technically $g(t)$ geometric means that if h is the 1-jet of $g(t)$, then $h \notin Im\delta^*$. Also, in Chapter 2 we saw that it was enough to study variations of the form

$$g(t) = g_o + t\,\rho^3 h$$

to compute Ric' for all possible variations satisfying the conditions of Problem I.

Fix p in M. Remember that if r is the distance from p on M, then in $M - \{p\}$ r is a smooth function up to the cut locus $C(p)$ of p and r^2 is a smooth function on M up to $C(p)$. We may take $U = D - \{p\}$ as the one-sided tubular neighborhood of D discussed earlier. Then the distance function ρ to Bd D is just $\rho = R - r$. In $D - \{p\}$, $g_o(\nabla r,\nabla r) = 1$ so that $\delta^*(dr)(x,\nabla r) = g_o(D_x\nabla r,\nabla r) = \frac{1}{2}x(g_o(\nabla r,\nabla r)) = \frac{1}{2}x(1) = 0$ and hence we obtain

(9) $\delta^*(d\rho)(x,y) = \delta^*(d\rho)(x_T,y_T)$ where X_T is the tangential component of x defined as in Chapter 2.

Given $h \in C^\infty(S^2T^*(D))$, we may write $h = h_T + d\rho\circ\eta + fd\rho\circ d\rho$ where $f : D \to R$ is a smooth function, $\eta \in C^\infty(T^*(D))$ with $\eta(\nabla\rho) = 0$ and $h_T(x,y) = h_T(x_T,y_T)$. We will call h_T the tangential component of h. For now we combine $d\rho\circ\eta + fd\rho\circ d\rho$ as $d\rho\circ\xi$ where $\xi \in C^\infty(T^*(D))$.

We want to calculate ric'(v) for $g(t) = g_o + t\rho^3(h_T + d\rho\circ\xi)$. Fix $q \in D - \{p\}$. Choose as a g_o-orthonormal basis for M_q vectors $e_1,\ldots,e_{n-1}$ spanning $(\nabla\rho|_q)^\perp$ and $\nabla\rho|_q$. Since $ric^t(v) = \frac{Ric^t(v,v)}{g(t)(v,v)}$ by the quotient rule for differentiation we have

<u>Lemma 1</u>: $ric'(v) = \frac{Ric'(v,v)}{g_o(v,v)} - \frac{Ric(v,v)}{g_o(v,v)^2}\frac{d}{dt}g(t)(v,v)\Big|_{t=0}$

so that $ric(v) = 0$, $g_o(v,v) = 1$ implies $ric'(v) = Ric'(v,v)$.

<u>Notational Convention</u>: In all computations we will take $g_o(v,v) = 1$, and write $\langle\ ,\ \rangle$ for g_o and $\langle\ ,\ \rangle_t$ for $g(t)$.

Let $g(t) = g_o + t\,h$. Then

$$Ric^t(v,v) = \sum_{i=1}^{n-1}\langle R^t(e_i,v)v,e_i\rangle + \langle R^t(\nabla\rho,v)v,\nabla\rho\rangle$$

$$= \sum_{i=1}^{n-1}\langle R^t(e_i,v)v,e_i\rangle_t + \langle R^t(\nabla\rho,v)v,\nabla\rho\rangle_t$$

$$- t\sum_{i=1}^{n-1}h(R^t(e_i,v)v,e_i) - t\,h(R^t(\nabla\rho,v)v,\nabla\rho).$$

Thus,

$$\mathrm{Ric}'(v,v) = \sum_{i=1}^{n-1} (\Sigma h)(v,e_i) + (\Sigma h)(v,\nabla\rho)$$

$$- \sum_{i=1}^{n-1} h(R(v,e_i)e_i,v) - h(R(v,\nabla\rho)\nabla\rho,v).$$

Thus, for $g(t) = g_o + t\,\rho^3(h_T + d\rho\circ\xi)$, we have

$$\mathrm{Ric}'(v,v) = \mathrm{tr}((\Sigma(\rho^3(h_T+d\rho\circ\xi)))(-,v)) + \rho^3 \text{ (terms in curvature).}$$

Let $A(h)(x,y) := 3y(\rho)((D_x h)(x,y) - (D_y h)(x,x))$

$$+ 3x(\rho)((D_y h)(y,x) - (D_x h)(y,y)).$$

Now $d\rho\circ d\rho(v,\nabla\rho) = v(\rho)$, $h_T(v,\nabla\rho) = h_T(\nabla\rho,\nabla\rho) = 0$, $\delta^*(d\rho)(v,\nabla\rho) = 0$, and $\delta^*(d\rho)(\nabla\rho,\nabla\rho) = 0$. Using Lemma 8 and Corollary 4 of Section 2 we have for $h = \rho^3(h_T+d\rho\circ\xi)$ that

$$(\Sigma h)(\nabla\rho,v) = -3\rho\, h_T(v,v) + \rho^2 A(h_T)(\nabla\rho,v) + \rho^3(\Sigma h_T)(\nabla\rho,v)$$

$$+ \frac{3}{2}\rho^2(\|v_\rho\|^2\delta^*\xi(\nabla\rho,\nabla\rho) + \delta^*\xi(v,v) - 2v(\rho)\delta^*\xi(\nabla\rho,v))$$

$$+ \frac{\rho^3}{2}(\langle v,\nabla\rho\rangle(D_{\nabla\rho}d\xi)(v,\nabla\rho) + (D_v d\xi)(\nabla\rho,v) + \delta^*\xi(\nabla\rho,\nabla\rho)\delta^*(d\rho)(v,v)).$$

Using Corollary 4 and Corollary 9 of Section 2, since $\langle e_i,\nabla\rho\rangle = 0$,

$$(\Sigma h)(e_i,\nabla\rho) = -3\rho\|v_\rho\|^2 h_T(e_i,e_i) + \frac{3}{2}\rho^2[2\delta^*(d\rho)(e_i,v)h_T(e_i,v)$$

$$- \delta^*(d\rho)(e_i,e_i)h_T(v,v) - \delta^*(d\rho)(v,v)h_T(e_i,e_i)] + \rho^2 A(h_T)(e_i,v)$$

$$+ \rho^3(\Sigma h_T)(e_i,v) + \frac{3}{2}\rho^2\|v_\rho\|^2\delta^*\xi(e_i,e_i) + \frac{\rho^3}{2}[\langle v,\nabla\rho\rangle(D_{e_i}d\xi)(v,e_i)$$

$$+ \delta^*\xi(e_i,e_i)\delta^*(d\rho)(v,v) + \delta^*\xi(v,v)\delta^*(d\rho)(e_i,e_i) - 2\delta^*\xi(e_i,v)\delta^*(d\rho)(e_i,v)$$

To compute $\operatorname{tr}\{x \to \delta^*(d\rho)(x,v)h_T(x,v)\}$, we may choose $e_1,\ldots,e_{n-1}$ with $h_T(e_i,v) = 0$ and $v,e_1,\ldots,e_{n-1}$ a basis for M_q. Then

$$\operatorname{tr} \delta^*(d\rho)(-,v)h_T(-,v) = \delta^*(d\rho)(v,v)h_T(v,v)$$
$$+ \sum_{i=1}^{n-1} \delta^*(d\rho)(e_i,v)h_T(e_i,v)$$
$$= \delta^*(d\rho)(v,v)h_T(v,v)$$

If we define $\delta\xi = \operatorname{tr}D\xi = \operatorname{tr}\delta^*\xi$, then

Lemma 2: For $g_o(v,v) = 1$ and $g(t) = g_o + t\rho^3(h_T + d\rho\circ\xi)$,

$$\operatorname{Ric}'(v,v) = -3\rho[h_T(v,v) + \|v\rho\|^2 \operatorname{tr}h_T] + \rho^2 \operatorname{tr}A(h_T)(-,v)$$
$$+ \frac{3}{2}\rho^2[2\delta^*(d\rho)(v,v)h_T(v,v) - \Delta\rho\cdot h_T(v,v) - \delta^*(d\rho)(v,v)\operatorname{tr}h_T]$$
$$+ \frac{3}{2}\rho^2[\delta^*\xi(v,v) + \|v_\rho\|^2\delta\xi - 2\langle v,\nabla\rho\rangle_o \delta^*\xi(v,\nabla\rho)]$$
$$+ \rho^3[\langle v,\nabla\rho\rangle_o \operatorname{tr}\{x \to (D_x d\xi)(v,x)\} + (D_v d\xi)(\nabla\rho,v) +$$
$$+ \delta^*(d\rho)(v,v)\delta\xi + \delta^*\xi(v,v)\Delta\rho - 2\operatorname{tr}\delta^*\xi(-,v)\delta^*(d\rho)(-,v)].$$
$$- \rho^3\{z \to h_T(R(v,z)z,v)\}.$$

From Lemma 2, we see that if h_T is negative definite near $Bd(\bar{D})$, say $h_T(v,v) \leq -\lambda g_o(v,v)$, with $\lambda > 0$ a constant, then Ric' will be positive in an annular neighborhood of $Bd(\bar{D})$ in $\bar{D}$. Thus, the answer to Problem I is YES. The most obvious choice of h_T satisfying this condition is

$$h_T := - g_o|(\nabla\rho)^{\perp}.$$

This leads us to consider the deformations $g_1(t) = g_o + t\rho^3 g_o$ and $g_2(t) = g_o - t\rho^3 g_o$.

From Lemma 2 we observe that for the variation $g_1(t) = g_o + t\rho^3 g_o = (1+t\rho^3)g_o$, $Ric'(v,v) = -3\rho - 3(n-2)\rho\|v_\rho\|^2 + O(\rho^2)$ and for $g_2(t) = g_o - t\rho^3 g_o = (1-t\rho^3)g_o$, $Ric'(v,v) = 3\rho + 3(n-2)\rho\|v_\rho\|^2 + O(\rho^2)$. Thus, in an open annulus with the outer boundary circle Bd $\bar{D}$ for $g_1(t)$ we have $Ric'(v,v) < 0$ and for $g_2(t)$ we have $Ric'(v,v) > 0$. Hence, up to second order in t near Bd D the variation $g_1(t)$ is decreasing all Ricci curvatures and $g_2(t)$ is increasing all Ricci curvatures. By taking t sufficiently small in the Taylors expansion

$$Ric^t = Ric + t\ Ric' + \frac{t^2}{2} Ric'' + \ldots$$

we should hopefully be able to make Ric' dominate the higher order time derivatives of Ric^t. We shall shortly see that this can be done.

In [11] Bourguignon, Deschamps, and Sentenac consider C^k variations

$$g(t) = g_o + t\ h^1 + \frac{t^2}{2} h^2 + \ldots + \frac{t^k}{k!} h^k, h^i \in C^\infty(S^2(T^*M)).$$

The difference tensor of the Levi-Civita connections D^t and D^o can be written $D^t - D^o = t\ C^1 + \frac{t^2}{2} C^2 + \ldots + \frac{t^k}{k!} C^k$ for two-tensors C^i. Then Bourguignon, Deschamps, and

Sentenac give formulas for calculating $\frac{d^i}{dt} R^t \Big|_{t=0}$ from the tensors C^j.

For $g_1(t) = (1+t\rho^3)g_o$ we calculate these tensors directly since the formulas of [11] are complicated for $i > 2$. We write $\langle , \rangle$ for g_o, D for D^o, and $\langle , \rangle_t$ for $g(t)$. Given vectors x,y, and z at M_q make good extensions to local fields X,Y, and Z. Then

$$2\langle D^t_X Y, z\rangle = x\langle Y,Z\rangle_t + y\cdot\langle Z,X\rangle_t - z\langle X,Y\rangle_t$$

$$= x((1+t\rho^3)\langle Y,Z\rangle) + y((1+t\rho^3)\langle Z,X\rangle) - z((1+t\rho^3)\langle X,Y\rangle)$$

$$= 3t\rho^2(x(\rho)\langle y,z\rangle + y(\rho)\langle x,z\rangle - \langle x,y\rangle\langle \nabla\rho\big|_q, z\rangle)$$

$$+ 2\langle D_X Y, z\rangle + 2t\rho^3\langle D_X Y, z\rangle .$$

Let $C_\rho(x,y) = x(\rho)\cdot y + y(\rho)\cdot x - \langle x,y\rangle\nabla\rho$. Then we have

$\langle D^t_X, Y, z\rangle_t = (1+t\rho^3)\langle D_X Y, z\rangle + \frac{3}{2} t\rho^2\langle C_\rho(x,y), z\rangle$. But

$\langle D^t_X Y, z\rangle_t = (1+t\rho^3)\langle D^t_X Y, z\rangle$ so that

$$D^t_X Y - D_X Y = \frac{3}{2} \; \frac{t\rho^2}{1+t\rho^3} C_\rho(x,y) = \frac{3}{2} t\rho^2(1-t\rho^3+(t\rho^3)^2-\dots)C_\rho(x,y)$$

$$= \frac{3}{2} t\rho^2 C_\rho(x,y) - \frac{3}{2} t^2\rho^5 C_\rho(x,y) + \frac{3}{2} t^3\rho^8 C_\rho(x,y) - \dots$$

so that $C^k(x,y) = (-1)^{k+1} \frac{3}{2} t^k \rho^{3k-1} C_\rho(x,y)$.

Consider the conformal variation $\tilde{g}_1(t) = e^{t\rho^3} g_o$. Using the power series expansion for the exponential,

$$\tilde{g}_1(t) = g_o + t\rho^3 g_o + \frac{t^2}{2}\rho^6 g_o + \dots \quad .$$

Since $\tilde{g}_1(t) = g_1(t) + O(t^2)$, both variations have the same first derivative of Ricci and sectional curvature. But for $\tilde{g}_1(t)$, a calculation similar to that for $g_1(t)$ shows that

$$\tilde{D}^t_X Y - D_X Y = \frac{3}{2} t\rho^2(x(\rho)y + y(\rho)x - \langle x,y\rangle \nabla\rho)$$

so $\tilde{C}^1(x,y) = \frac{3}{2}\rho^2 C_\rho(x,y)$ and $\tilde{C}^i(x,y) = 0$ for $i \geq 2$. Although $g_1(t)$ looks like a simpler variation than $\tilde{g}_1(t)$ the Levi-Civita connection defined by $g_1(t)$ is more complicated than the Levi-Civita connection defined by $\tilde{g}_1(t)$. Hence we will use the conformal variation $\tilde{g}_1(t)$ to perturb the Ricci tensor.

We now consider the conformal variation $g(t) = e^{-2t\rho^5} g_o$ on convex disks.

The reason for using ρ^5 instead of ρ^3 is only to make $g(t)$ a C^4 metric on M and hence ric^t a C^2 function on the sphere bundle $S_1(M,g_o)$ of g_o-unit vectors in TM. If $g(t) = e^{2tf} g_o$, then it is well-known that

$$ric^t(v) = e^{-2tf}\{ric(v) - t(n-2)\delta^*(df)(v,v) - t\Delta f + t^2(n-2)((v(f))^2 - \|\nabla f\|^2)\}.$$

(See for instance [22], p. 90.) Hence for $f = -2\rho^5$, $\|v\| = 1$,

$$ric^t(v) = e^{2t\rho^5}\{ric(v) + t(n-2)\delta^*(d\rho^5)(v,v) + t\Delta(\rho^5) + t^2(n-2)((v(\rho^5))^2 - \|\nabla(\rho^5)\|^2)\}.$$

But $\delta^*(d\rho^5)(v,v) = 20\rho^3\|v_\rho\|^2 + 5\rho^4\delta^*(d\rho)(v,v)$,

$$\Delta(\rho^5) = 20\rho^3 + 5\rho^4\,\Delta\rho, \text{ and}$$

$$(v(\rho^5))^2 - \|\nabla(\rho^5)\|^2 = -25\rho^8\|v_T\|^2.$$

Hence

$$\begin{aligned}\mathrm{ric}^t(v) &= e^{2t\rho^5}\{\mathrm{ric}(v) + 20t\rho^3(1+(n-2)\|v_\rho\|^2)\\ &+ 5t\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))-25t^2(n-2)\rho^8\|v_T\|^2\}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}\mathrm{ric}^t(v) &= 2\rho^5 e^{2t\rho^5}\{\mathrm{ric}(v)+20t\rho^3(1+(n-2)\|v_\rho\|^2)\\ &+ 5t\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v)) - 25t^2(n-2)\rho^8\|v_T\|^2\}\\ &+ e^{2t\rho^5}\{20\rho^3(1+(n-2)\|v_\rho\|^2)+5\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))\\ &- 50t(n-2)\rho^8\|v_T\|^2\}, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{ric}^t(v) &= 4\rho^{10}e^{2t\rho^5}\{\mathrm{ric}(v) + 20t\rho^3(1+(n-2)\|v_\rho\|^2)\\ &+ 5t\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v)) - 25t^2(n-2)\rho^8\|v_T\|^2\}\\ &+ 4\rho^5 e^{2t\rho^5}\{20\rho^3(1+(n-2)\|v_\rho\|^2) + 5\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))\\ &- 50t(n-2)\rho^8\|v_T\|^2\}\\ &- 50(n-2)\rho^8\|v_T\|^2 e^{2t\rho^5}.\end{aligned}$$

Hence, $\frac{d^2}{dt^2}\mathrm{ric}^t$ is of order ρ^8 and

$$\mathrm{ric}'(v) = \frac{d}{dt}\,\mathrm{ric}^t(v)\Big|_{t=0} = 20\rho^3(1+(n-2)\|v_\rho\|^2)+5\rho^4(\Delta\rho$$

$$+ (n-2)\delta^*(d\rho)(v,v)) + 2\rho^5\mathrm{ric}(v)$$

$$\geq 5\rho^3(4+\rho(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))).$$

Let $p \in (M,g_o)$ be a point where all Ricci curvatures are positive (assuming $\mathrm{Ric}_{g_o} \geq 0$). Thus, in some small closed disk about p all Ricci curvatures are positive. We want to change the metric so that all Ricci curvatures will be positive on a larger convex disk $D = B_{g_o,R}(p)$ of radius R and so that the new metric agrees with g_o off D.

If we let r be the g_o-distance from p on M, then $\rho = R-r$ so $\delta^*(d\rho) = -\delta^*(dr)$ and $\Delta\rho = -\Delta r$. By choice of R according to our convention, $\delta^*(d\rho)$ will be negative definite on $D - \{p\}$. At p, the function r^2 is smooth and $\delta^*(d(r^2))(v,v) = 2g_o(v,v)$ for all v in M_p, as we showed in Chapter 3. We may then by continuity and/or the Index Comparison Theorem technique of Chapter 3 choose R small enough so that on D

$$(9) \qquad (2-1/4)g_o \leq \delta^*(d(r^2)) \leq (2+1/4)g_o.$$

Hence on $D - \{p\}$ we have

$$(9') \qquad \frac{(1-1/8)}{r}\, g_o(x_T,x_T) \leq \delta^*(dr)(x,x) \leq \frac{1+1/8}{r} g_o(x_T,x_T).$$

Consider the Taylor series with remainder

$$\mathrm{ric}^t = \mathrm{ric} + t\,\mathrm{ric}' + \frac{t^2}{2}\,\frac{d^2}{dt^2}\,\mathrm{ric}^t\Big|_{t=s} \quad \text{for some } s$$

with $0 < s < t$ where $g(t) = e^{-2t\rho^5} g_o$ in D centered at p.

From above for v with $g_o(v,v) = 1$,

$$\mathrm{ric}^t(v) \geq (1+2\rho^5)\mathrm{ric}(v) + 5t\rho^3(4+\rho(\Delta\rho+(n-2)\delta^*(d\rho)(v,v)))$$

$$+ \frac{t^2}{2}[4\rho^{10}e^{2s\rho^5}(5s\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))-25\rho^8(n-2)s^2\|v_T\|^2)$$

$$+ 4\rho^5e^{2s\rho^5}(5\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))-50\rho^8 s(n-2)\|v_T\|^2)$$

$$- 50(n-2)\rho^8\|v_T\|^2e^{2t\rho^5}].$$

Choose t small enough so that on D, $1 \leq e^{2t\rho^5} \leq 2$. Then

$$\mathrm{ric}^t(v) \geq \mathrm{ric}(v)(1+t\rho^5)+5t\rho^3[4+\rho(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))$$

$$+ t\{4\rho^7[s\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))-5\rho^8(n-2)s^2\|v_T\|^2]$$

$$+ 4\rho^5[\rho^4(\Delta\rho+(n-2)\delta^*(d\rho)(v,v))-10\rho^8 s(n-2)\|v_T\|^2]$$

$$- 10\rho^5(n-2)\|v_T\|^2\}].$$

Remember that $R = r+\rho$. We will assume $R \leq 1$ so that $\rho \leq 1$.

From (9), $|\delta^*(d\rho)(v,v)| \leq \dfrac{9\|v_T\|^2}{8(R-\rho)} \leq \dfrac{9}{8(R-\rho)}$ so $|\Delta\rho| \leq \dfrac{9(n-1)}{8(R-\rho)}$.

(Remember that $\delta^*d\rho(\nabla\rho,\nabla\rho) = 0$.) Thus

$$|\Delta\rho+(n-2)\delta^*(d\rho)(v,v)| \leq \frac{9(2n-3)}{8(R-\rho)}$$

so $\mathrm{ric}^t(v) \geq \mathrm{ric}(v)(1+t\rho^5)+5t\rho^3[4-\rho\{ \frac{9(2n-3)}{8(R-\rho)} + 4ts\,\frac{9(2n-3)}{8(R-\rho)}$

$+ 5ts^2(n-2)+4t\cdot \frac{9(2n-3)}{8(R-\rho)} -40ts(n-2)-10t(n-2)\}]$.

By choosing t and hence s small enough, in $D - \{p\}$ we obtain

$$(10) \qquad \mathrm{ric}^t(v) \geq \mathrm{ric}(v)(1+t\rho^5)+5t\rho^3[2 - \frac{9(2n-3)}{8(R-\rho)}\,\rho].$$

Now if $2 - \frac{9(2n-3)}{8(R-\rho)}\,\rho > 0$ and $0 < \rho < R$, then $\mathrm{ric}^t(v) > 0$ for all v. But this is true if $0 < \rho < \frac{1}{1+\frac{9}{16}(2n-3)}\,R$.

Hence we have shown

Theorem 3: There exists a constant $\varepsilon = \varepsilon(n) > 0$ with the following property. Given (M^n, g_o) with $\mathrm{Ric}_{g_o} \geq 0$, let $D = B_{g_o,R}(p)$ be any disk for which (9) holds and $R \leq 1$. Let

$$g(t) = \begin{cases} e^{-2t\rho^5} g_o & \text{in } D \\ g_o & \text{in } M - D. \end{cases}$$

Then there exists $t_o > 0$ such that for all $t \in (0, t_o]$, $\mathrm{ric}^t > 0$ on the g_o-outer annulus $A_{g_o,R,2\varepsilon(n)}(p)$.

Remark: We need to consider the second derivative of ric^t since $\mathrm{ric}'(v) > 0$ only on some open annulus about $\mathrm{Bd}(\bar{D})$ since $g(t) = g_o$ on $\mathrm{Bd}(\bar{D})$. From $\mathrm{ric}'(v) > 0$ on an open set we cannot conclude that there exists a small t such that $\mathrm{ric}^t(v) > 0$ everywhere on the open set even though for each point q in the open set we can find a $t = t(q)$ so that $\mathrm{ric}^t(v) > 0$ for all v in M_q. Of course, if $\mathrm{ric}' > 0$ on a compact set, we can find a $t > 0$ so that $\mathrm{ric}^t > 0$ everywhere on the set.

A trivial modification of the argument of [15], p. 121, shows

Lemma 4: (Cheeger, Gromoll) Given (M,g_o) complete with

$$\mathrm{Ric}_g \geq A\, g$$

for some constant $A \geq 0$. Then for any p in M, up to the cut locus of p,

$$\Delta r_{g,p} \leq \frac{n-1}{r_{g,p}} - A.$$

Hence, if $\mathrm{Ric}_g \geq 0$, then $\Delta r_{g,p} \leq \frac{n-1}{r_{g,p}}$.

Remark: This lemma should be compared with the computation in Berger, Gauduchon, and Mazet, [6], p. 134 ff. (noting that their Laplacian has the opposite sign from ours) which shows

$$\Delta r = \frac{n-1}{r} + \frac{\theta'}{\theta} .$$

In particular, in (R^n, g_{can}), $\Delta r = (n-1)/r$. Also, Lemma 4 says that $\mathrm{Ric}_g \geq 0$ implies $\frac{\theta'}{\theta} \leq 0$.

Using the estimate of Lemma 4 in place of (9), we obtain by an analogous calculation

<u>Theorem 5</u>: There exists a constant $\varepsilon = \varepsilon(n) > 0$, with the following property. Let M^n be any n-dimensional smooth manifold. Let g be any C^4 Riemannian metric for M with $Ric_g \geq 0$ and let $D := B_{g,R}(p)$ be any g-convex disk with $R \leq 1$. Let

$$g(t) = \begin{cases} e^{-2t\rho^5} g_o & \text{in } D \\ g_o & \text{in } M - D. \end{cases}$$

Then there exists a $t_o > 0$ such that $Ric_{g(t)} > 0$ in the g-outer annulus $A_{g,R,\varepsilon}(p)$ for all t with $0 < t \leq t_o$.

We remark that the analogue to Theorem 4 holds for $Ric_{g_o} \leq 0$ using the deformation

$$g(t) = \begin{cases} e^{2t\rho^3} g_o & \text{in } D \\ g_o & \text{in } M - D, \end{cases}$$

<u>Remark</u>: In Theorems 3 and 5 since $\rho = R - r$ is not smooth at p, the deformation given is not smooth at p. But we are interested only in the positivity of the Ricci curvature in the outer annulus. Clearly given $D = B_{g,R}(p)$, g(t) can be smoothed off near p to produce a metric satisfying the conclusions of Theorems 3 and 5. We will call this the "standard deformation of Theorem 4.3" in Chapter 5.

Chapter 5: Proof of the Ricci Curvature Deformation Theorems.

In this section, we use local convex deformations to prove several theorems on deformation of Ricci curvature. First we will give a detailed proof of the corollary to the theorem of [2] using the results of Chapters 3 and 4. Then the same method of proof together with some additional local calculations like that of Chapter 4 will imply the remaining theorems.

We prove

Theorem 1: Let M^n be compact and let g_0 be a C^4 metric for M with $ric_{g_0} \geq 0$ and all Ricci curvatures positive at some point. Then M admits a C^4 metric of everywhere positive Ricci curvature. Hence $\pi_1(M)$ is finite and $b_1(M) = 0$.

Proof: The basic idea to prove Theorem 1 is to use the standard deformation of Theorem 4.3 to spread the positive Ricci curvature from the point of positive curvature to all of M.

Explicitly, Theorem 4.3 tells us that given positive Ricci curvature on a small enough disk, we can spread the positive Ricci curvature to a slightly larger disk centered at the same point, provided the larger disk is convex.

But if we just start changing the metric naively, even though we can always find a convex disk in which to apply Theorem 4.3, it is not obvious that the radii of the disks in the original metric can be chosen to remain bounded away from zero so that we can cover all of M with even an infinite number of deformations. In particular, the convexity radius is changing with each deformation.

However, we can use the compactness of M to overcome this difficulty as follows. Cover M by N balls $\{B_i = B_{g_o,R_i}(p_i)\}_{i=1}^{N}$ with $R_i \le F(g_o)/2$, $F(g_o)$ as in Theorem 3.12, satisfying the following properties. Let $\varepsilon = \varepsilon(n)$ be as in Theorem 4.3.

(1) B_1 is chosen so that p_1 is a point with all g_o-Ricci curvatures positive in $B_1 - A_{g_o,R_1,\varepsilon/4}(p_1)$ and thus the standard deformation of Theorem 4.3 applied to B_1 will produce a metric g_1 for M with $ric_{g_1} \ge 0$ and $ric_{g_1} > 0$ in B_1.

(2) Inductively for $n \ge 2$, if ric_{g_o} <u>were</u> positive on

$$B_1 \cup B_2 \cup \ldots \cup B_{n-1} \cup (B_n - A_{g_o,R_n,\varepsilon/4}(p_n))$$

then the standard deformation of Theorem 4.3 applied to B_n would produce a metric g for M with $ric_g \ge 0$ and $ric_g > 0$ on $B_1 \cup \ldots \cup B_n$.

(3) the g_o-outer annuli $\{A_{g_o,R_i,\varepsilon/4}(p_i)\}_{i=1}^{N}$ cover all points q in M for which there is a $v \ne 0$ in M_q with

$ric_{g_o}(v) = 0$.

For instance, assume $ric_{g_o} > 0$ in $B_{g_o,R}(p)$ to begin with. We can choose a finite number of balls as required to cover all possible points with zero Ricci curvatures in $B_{g_o,(1+\varepsilon/8)R}(p)$. In fact, it is clear that for all n, with a finite number of balls we can extend the cover of all possible "zero points" from $B_{g_o,(1+n\varepsilon/8)R}(p)$ to $B_{g_o,(1+(n+1)\varepsilon/8)R}(p)$. Since M is compact, we can thus clearly produce the required sequence $\{B_i\}_{i=1}^{N}$ stipulated above.

Let $\delta(g_o)$ satisfy Theorem 3.12 and Lemma 3.5. Use the standard deformation of Theorem 4.3 to produce from g_o a metric g_1 for M with

$$|g_o - g_1|_{C^2} < \delta(g_o)/N$$

with $ric_{g_1} \geq 0$ on M, and $ric_{g_1} > 0$ on B_1 (which is possible since $ric^t > 0$ on $A_{g_o,R_1,2\varepsilon}(p_1)$ by Theorem 4.3.

By Lemma 3.5 applied to g_1, we can find a disk $\tilde{B}_2 = B_{g_1,\tilde{R}_2}(p_2)$ which is g_1-convex and with

$$A = A_{g_o,R_2,\varepsilon/4}(p_2) \subset A_{g_1,\tilde{R}_2,\varepsilon}(p_2) = \tilde{A}$$

$$\text{and} \quad \tilde{R} \leq F(g_o).$$

Hence we can apply the standard deformation of Theorem 4.3 to g_1 and $\tilde{B}_2$ to produce a metric g_2 for M with

$$|g_1 - g_2|_{C^2} < \delta(g_o)/N,$$

$ric_{g_2} \geq 0$ and $ric_{g_2} > 0$ on $B_1 \cup \tilde{B}_2$ and hence on $B_1 \cup B_2$, spreading the positive Ricci curvature to $A_{g_o, R_2, \varepsilon/4}(p_2)$. Note that by construction

$$|g_o - g_2|_{C^2} < \delta(g_o).$$

Carrying on this way we construct metrics $g_3, \ldots, g_N$ applying the standard deformation to balls $\tilde{B}_3 \supset B_3, \ldots, \tilde{B}_N \supset B_N$ to spread the positive Ricci curvature from g_o-outer annulus to g_o-outer annulus making each new metric g_n $\delta(g_o)/N$ close to the preceeding metric g_{n-1} and hence $\delta(g_o)$ close to g_o so Lemma 3.5 and Theorem 3.12 apply to enable us to carry out the next step.

Q. E. D.

Remarks:

(1) Using $g(t) = e^{-2t\rho^m} g_o$ as the standard deformation, we can produce a C^m metric for M for any $m \geq 4$.

(2) If g_o was a C^∞ metric for M, we can produce a C^4 metric for M with $ric_g > 0$ by Theorem 1. Since M is compact, there is a $c > 0$ with $ric_g \geq c > 0$. Hence we can approximate g by a C^∞ metric $\tilde{g}$ with $ric_{\tilde{g}} > 0$ everywhere.

We state

<u>Theorem 2</u>: Let M^n be compact and let g_o be a C^4 metric for M with $ric_{g_o} \leq 0$ and all Ricci curvatures negative at some point. Then M admits a C^4 metric of everywhere negative Ricci curvature. ($ric<0$)

Theorem 2 has a completely analogous proof to Theorem 1, the only modification being that the local convex deformation $g(t) = e^{2t\rho^5} g_o$ is used.

Suppose (M,g_o) is a compact Riemannian manifold with isometry group $I_{g_o}(M)$. Suppose $ric_{g_o} \geq 0$ and at some point $ric_{g_o} > 0$. Then we could apply Theorem 1 to produce a metric g on M with $ric_g > 0$. But if $I_{g_o}(M)$ is not discrete, it is not obvious that the construction outlined in the proof of Theorem 1 will result in $I_{g_o}(M) \subset I_g(M)$. However, using the idea of Alan Weinstein in [33] to integrate the metric deformation over the isometry group, we can use local deformations to produce a new metric g with $ric_g > 0$ and $I_{g_o}(M) \subset I_g(M)$. Let $g(t) = e^{-2t\rho^5} g_o$ as before. Let $d\nu$ be Haar measure on $I_{g_o}(M)$ normalized so that

$$\int_{\varphi \in I_{g_o}(M)} d\nu(\varphi) = 1.$$

Let

$$\tilde{g}(t) := \int_{\varphi \in I_{g_o}(M)} \varphi^*(g(t))\,d\nu(\varphi).$$

By the invariance of $d\nu$, $\tilde{g}(t)$ is invariant under $I_{g_0}(M)$. But

$$\widetilde{Ric}^t = \int_{\varphi \in I_{g_0}(M)} \varphi^*(Ric^t)\, d\nu(\varphi)$$

and

$$\widetilde{Ric}' = \int_{\varphi \in I_{g_0}(M)} \varphi^*(Ric')\, d\nu(\varphi)$$

as Weinstein observed for the sectional curvature in [33]. Hence, the proof of Theorem 1 carries through using the deformations $\tilde{g}_i(t)$ obtained by integrating the deformation $g_i(t) = e^{-2t\rho^5}\tilde{g}_{i-1}$ over the isometry group $I_{\tilde{g}_{i-1}}(M)$ of the metric $\tilde{g}_{i-1}$ constructed at the previous step. Theorem 2 has an equivariant version also obtained by integrating the deformations $g(t) = e^{2t\rho^5}g_0$ over the isometry group.

We want to consider whether local convex deformations can be used to improve Ricci pinching.

<u>Definition</u>: (M^n, g_0) is <u>positively Ricci pinched</u> with pinching constant $0 < A \leq 1$ iff for all $v \in TM$,
$A\, k\, g_0(v,v) \leq Ric_{g_0}(v,v) \leq k\, g_0(v,v)$ for some $k > 0$.
If (M, g_0) is Ricci pinched, then multiplying the metric by a constant we may assume that $k = 1$.

If $A = 1$, then (M, g_0) is an Einstein manifold. If

(M_1,g_1) and (M_2,g_2) are Einstein manifolds with positive Ricci curvature, then $(M_1 \times M_2,\ g_1 \times g_2)$ will be Ricci pinched manifold with the upper and lower bounds for the pinching attained at each point. Recently H. Hernandez [25] has given an example of manifolds which are not product manifolds with non-Einstein pinching that is attained at each point. Evidently local convex deformations cannot be used to improve the pinching in such situations.

Suppose however that (M^n,g_o) is a compact manifold with $A\, g_o \le \mathrm{Ric}_{g_o} \le g_o$ and suppose there exists $p \in M$ such that the pinching is not attained for any $v \in M_p$. By compactness arguments, we can find a closed convex disk D' centered at p of radius $R \le \frac{1}{4}\, c_{g_o}(M)$ and $\lambda > 0$ so that for all $v \in S_1(M,g_o)_{|D}$ either

(11) $$A + \lambda \le \mathrm{Ric}(v,v) \le 1, \quad \text{or}$$

(12) $$A \le \mathrm{Ric}(v,v) \le 1 - \lambda.$$

Let D be a disk centered at p with $D' \subset D$. Consider the local convex deformation $g(t) = e^{-2t\rho^5} g_o$ on D. For $g(t) = e^{2tf} g_o$ if $e_1,\ldots,e_{n-1}$ are a g_o-orthonormal basis for $v^\perp$, then $e_i(t) = e^{-tf} e_i$ will be a $g(t)$-orthonormal basis for $v^\perp$ and

$$\mathrm{Ric}^t(v,v) = \sum_{i=1}^{n-1} \langle R^t(e_i(t),v)v, e_i(t)\rangle_t = \sum_{i=1}^{n-1} \langle R^t(e_i,v)v,e_i\rangle$$

$$= \mathrm{Ric}(v,v) - t(n-2)\delta^*(df)(v,v) - t\Delta f + t^2(n-2)((v(f))^2 - \|\nabla f\|^2).$$

Hence, for $g(t) = e^{-2t\rho^5} g_o$, $f = -\rho^5$ and we have

$$\mathrm{Ric}^t(v,v) = \mathrm{Ric}(v,v) + 20t\rho^3(1+(n-2)\|v_\rho\|^2)$$

$$+ 5t\rho^4(\Delta\rho + (n-2)\delta^*(d\rho)(v,v)) - 25t^2(n-2)\rho^8\|v_T\|^2$$

$$= \mathrm{Ric}(v,v) + 20t\rho^3(1+(n-2)\|v_\rho\|^2) + tO(\rho^4) + t^2O(\rho^8).$$

Then, fixing $q \in M - \{p\}$, $\max_{\substack{g_o(v,v)=1 \\ v\in M_q}} \{\mathrm{Ric}^t(v,v)\} \le 1 + 20t\rho^3(n-1) + tO(\rho^4) + t^2O(\rho^8)$

and $\min_{\substack{g_o(v,v)=1 \\ v\in M_q}} \{\mathrm{Ric}^t(v,v)\} \ge A + 20t\rho^3 + tO(\rho^4) + t^2O(\rho^8).$

As in the proof of Theorems 1 and 2, by making D and the annulus D - D' small, the term in ρ of order 3 will dominate the two higher order terms. The condition for improving the Ricci pinching is that the ratio $\dfrac{\min \mathrm{Ric}^t|_q}{\max \mathrm{Ric}^t|_q}$ be greater than A. But we have

$$\frac{\min \mathrm{Ric}^t|_q}{\max \mathrm{Ric}^t|_q} \ge \frac{A + 20t\rho^3}{1+20t\rho^3(n-1)} + tO(\rho^4) =$$

$$(A+20t\rho^3)(1-20t\rho^3(n-1)) + t^2O(\rho^6)) + tO(\rho^4)$$

$$= A + 20t\rho^3(1-A(n-1)) + tO(\rho^4)$$

By choosing t and ρ small we can make this ratio greater than A if $1 - A(n-1) > 0$, that is, if $A < 1/(n-1)$. Hence if the annulus $D - D'$ is small and $A < 1/(n-1)$, we can improve the pinching in the annulus. It is clear that we can obtain a uniform estimate for the size of the annulus depending only on the dimension n of M as we did in proving Theorem 1. Also, since $\delta^*(d\rho^5)$ is bounded in D' we can keep the pinching greater than A in D'. Explicitly, choose an integer $m > 2$ so that $A > 1/(m-1)$. Choose t small enough so that $|Ric^t - Ric| < \lambda/m$ on $S_1(M,g_o)_{|D'}$. Suppose v satisfies (11), i.e., $A + \lambda \le Ric(v,v) \le 1$. Then $A + \frac{m-1}{m}\lambda \le Ric^t(v,v) \le 1 + \frac{\lambda}{m}$. We must check that

$$\frac{A + \frac{m-1}{m}\lambda}{1 + \lambda/m} > A.$$

But $A < 1 < m-1$ implies that this inequality holds. Suppose v satisfies (12) and hence $A \le Ric(v,v) \le 1 - \lambda$. Then $A - \frac{\lambda}{m} \le Ric^t(v,v) \le 1 - \frac{m-1}{m}\lambda$. But this holds iff $A > 1/(m-1)$. Hence, for a small t, the metric $g_1 = e^{-2t\rho^5}g_o$ spreads the area on M where the Ricci pinching is greater than A from D' to D. By our remark above, we can perform this construction in a "uniform annulus" and hence spread the greater pinching from p to all of M in a finite number of deformations for compact M. Doing this, we get a C^4 metric $\tilde{g}$ for M so that

$$\text{pinch}(\tilde{g}) = \inf_{p \in M}\left\{ \frac{\min \text{Ric}_{\tilde{g}}|_p}{\max \text{Ric}_{\tilde{g}}|_p} \text{, both extrema over } v \in M_p \text{ with } \tilde{g}(v,v) = 1 \right\} > A.$$

Then by compactness of M, there exists $d > 0$ so that pinch $(\tilde{g}) \geq A - d$. Now we can choose a C^∞ metric g approximating $\tilde{g}$ so that $|\text{pinch}(g) - \text{pinch}(g)| \leq d/4$ and hence $\text{pinch}(g) > A$. We have proven

Theorem 3: Let (M^n, g_o) be a compact Riemannian manifold. Suppose there exists $k > 0$ and A with $0 < A < 1/(n-1)$ so that on TM the inequality $A\, k\, g_o \leq \text{Ric}_{g_o} \leq k\, g_o$ holds. If at some point for all non-zero vectors the pinching is not achieved, then it is possible to improve the Ricci pinching.

The analogous theorem holds for negative Ricci pinching using the local convex deformation $g(t) = e^{2t\rho^5} g_o$ for which

$$\text{Ric}^t(v,v) = \text{Ric}(v,v) - 20t\rho^3(1+(n-1)\|v_\rho\|^2) + tO(\rho^4).$$

Hence, if $-g_o(v,v) \leq \text{Ric}_{g_o}(v,v) \leq -A\, g_o(v,v)$ with $0 < A < 1$, then

$$\frac{\min \text{Ric}^t|_q}{\max \text{Ric}^t|_q} = \frac{-A - (n-1)20t\rho^3}{-1-(n-1)20t\rho^3} + tO(\rho^4) = \frac{A + 20t\rho^3}{1+(n-1)20t\rho^3} + tO(\rho^4)$$

so again this ratio can be made greater than A for small

t and ρ if $A < 1/(n-1)$.

We remark that if $K_{g_o} \geq 0$ and we perform a sequence of local convex deformations to improve the Ricci pinching as above getting a metric $\tilde{g}$ with pinch($\tilde{g}$) > A, we cannot be sure that $K_{\tilde{g}} \geq 0$. The reason is the following.

From Corollary 4, Section 2, if $g(t) = e^{-t\rho^5} g_o = g_o - t\rho^5 g_o + \ldots$ on some convex disk D and if x and y are g_o-orthonormal basis for σ with $x_\rho = 0$, then

$$K(\sigma) = 0 \Rightarrow K'(x,y) = 10\rho^3 \|y_\rho\|^2 + \frac{5}{2}\rho^4(\delta^*(d\rho)(x,x) + \delta^*(d\rho)(y,y))$$

Hence, if $y_\rho = 0$, then $K'(x,y) = \frac{5}{2}\rho^4(\delta^*(d\rho)(x,x)+\delta^*(d\rho)(y,y))<0$.

Now consider for $A\, g_o \leq Ric_{g_o} \leq g_o$, $0 < A < 1$, what effect the variation $g(t) = e^{2t\rho^5} g_o$ has on the Ricci pinching. In this case,

$$Ric^t(v,v) = Ric(v,v) - 20t\rho^3(1+(n-2)\|v_\rho\|^2) + tO(\rho^4).$$

Thus, $\dfrac{\min Ric^t|_q}{\max Ric^t|_q} \geq \dfrac{A - 20t(n-1)\rho^3}{1 - 20t\rho^3} + tO(\rho^4)$, so we need

$\dfrac{A - 20t(n-1)\rho^3}{1 - 20t\rho^3} > A$. But this inequality holds iff $A > n-1$.

Hence, the variation $g(t) = e^{2t\rho^5} g_o$ worsens positive Ricci pinching.

N. Hitchin [26] has shown that the signature of a 4-manifold M^4 is an obstruction to M^4 admitting an Einstein metric. S. T. Yau [34] generalized the result of Hitchin to show that the signature represents an obstruction to improving negative Ricci pinching to be Einstein. Yau gave an example of a compact negatively Ricci curved manifold which does not admit a $\sqrt{5/6}$ Ricci-pinched metric.

For $n = 4$, the bound of Theorem 3 is 1/3 so there is a gap between the "geometric" obstruction of Theorem 3 and Yau's topological obstruction of $\sqrt{5/6}$. It is important to see whether the "geometric" obstruction of $1/(n-1)$ in improving Ricci pinching using conformal deformations in a convex disk can be improved by using some other type of local convex deformation.

We shall show, however, by studying $\frac{\max(1 + t\mathrm{Ric}')}{\min(A - t\mathrm{Ric}')}$ that assuming only that the pinching does not hold at some point we cannot improve the bound of $1/(n-1)$ from conformal variation. Thus, conformal deformation is the "best" variation for improving Ricci pinching by the method of local convex deformations.

Let $p \in M$, let D be a closed convex disk about p, and let $g(t) = g_o + t\rho^5 h$ in a one-sided tubular neighborhood of Bd D. Recall that we remarked that h may be written as

$$h = h_T + f d\rho \circ d\rho + d\rho \circ \xi \quad \text{where} \quad \xi(\nabla\rho) = 0.$$

Then $\mathrm{Ric}'(v,v) = -20\rho^3(h_T(v,v) + \|v_\rho\|^2 \mathrm{tr}\, h_T) + O(\rho^4)$.

If $t > 0$ and $\min \mathrm{Ric}'(v,v) > 0$, we have the inequality

$$\frac{A + t \min \mathrm{Ric}'}{1 + t \max \mathrm{Ric}'} > A$$

which must be satisfied to improve the Ricci pinching. This is equivalent to the inequality

$A < \dfrac{\min \mathrm{Ric}' |M_q}{\max \mathrm{Ric}' |M_q}$. But $\min\{\mathrm{Ric}'(v,v);\ v \in M_q \text{ and } g_o(v,v) = 1\}$ is the smallest eigenvalue of $\mathrm{Ric}'|_{M_q}$ and $\max\{\mathrm{Ric}'(v,v);\ v \in M_q \text{ and } g_o(v,v) = 1\}$ is the largest eigenvalue of $\mathrm{Ric}'|_{M_q}$. Thus, the inequality for improving the pinching is that

$$(13) \qquad A < \frac{\text{smallest eigenvalue of } \mathrm{Ric}'|M_q}{\text{largest eigenvalue of } \mathrm{Ric}'|M_q} .$$

We can estimate (13) by the leading terms in ρ in the expansion for Ric' in terms of ρ.

Suppose h_T is negative definite with eigenvalues $-\lambda_1, \ldots, -\lambda_{n-1}$ where $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_{n-1}$. Then up to $O(\rho^3)$ the inequality (13) becomes $A < \lambda_1 / \sum_{i=1}^{n-1} \lambda_i$. Let $B(h_T) = \lambda_1 / \sum_{i=1}^{n-1} \lambda_i$. Then

$B(h_T) \le 1/n-1$ and $B(h_T) < 1/(n-1)$ unless $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$. In this case, $h_T = -\lambda_1\, g_o|_{(\nabla\rho)^\perp} = -\lambda_1 (g_o)_T$ so that $g(t) = g_o + t\rho^5(h_T + f d\rho \circ d\rho + d\rho \circ \xi)$ and

and $\tilde{g}(t) = e^{-2t\rho^5} g_o$ have up to a constant the same first derivative Ric' in the leading term in ρ. If h_T is taken to be positive definite with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_{n-1}$, then to improve the pinching, A must satisfy the inequality $A > \sum_{i=1}^{n-1} \lambda_i/\lambda_1 \ge n-1$. Thus, this variation does not improve positive Ricci pinching (as we saw for $g(t) = e^{2t\rho^5} g_o$ above). Suppose h_T has positive and negative eigenvalues

$-\mu_j \le -\mu_{j-1} \le \ldots \le -\mu_1 < 0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_s$. Then

$$\frac{A + t \min \mathrm{Ric}'}{1 + t \max \mathrm{Ric}'} \le \frac{A - t\rho^3 \sum_i \lambda_j}{1 + t\rho^3 \sum_i \mu_i} \le A - t\rho^3 \Sigma\lambda_j < A$$

so the pinching is worsened. If for some v, the term

$$h_T(v,v) + \|v_\rho\|^2 \delta^*(d\rho)(v,v) = 0$$

then the terms in order ρ^4 are important. For such a vector v,

$$\mathrm{Ric}'(v,v) = -\frac{5}{2}\rho^4 \delta^*(d\rho)(v,v) \operatorname{tr} h_T + \rho^4 \operatorname{tr} A(h_T)(-,v)$$

$$+ \rho^4 \text{ (terms from } d\rho \circ \xi \text{ and } f d\rho \circ d\rho) + O(\rho^5)$$

so that we cannot be sure that all eigenvalues of Ric' will have a definite sign. But we just saw that if there are eigenvalues of opposite sign, then the pinching may be worsened. Hence, to achieve the bound of $1/(n-1)$ with h_T not identically zero we must have h_T collinear with $g_o|(\nabla\rho)^{\perp}$.

Suppose $h_T \equiv 0$ so that $g(t) = g_o + t\rho^5(d\rho\circ\xi + f d\rho\circ d\rho)$.

Then $\mathrm{Ric}'(v,v) = \frac{5}{2}\rho^4[f\delta^*(d\rho)(v,v) + f\ \mathrm{tr}\ \delta^*(d\rho)\|v_\rho\|^2 + \delta^*\xi(v,v)$

$+ \|v_\rho\|^2\ \mathrm{tr}\ \delta^*\xi - \langle v,\nabla\rho\rangle\delta^*\xi(v,\nabla\rho)] + O(\rho^5)$.

Since $\xi(\nabla\rho) = 0$, $\delta^*\xi(\nabla\rho,\nabla\rho) = 0$. Let $S_r(p) = \{q\in M; \mathrm{dist}_{g_o}(p,q) = r\}$. Then, writing g for g_o, we have $g = g_T + d\rho\circ d\rho$ (geodesic polar coordinates!). Let

$$\delta_T^*\xi := \frac{1}{2}\delta_{\xi^{\#}g_T}\, g_T$$

be the differential operator on $S_r(p)$ from restricting the metric g to $S_r(p)$. Then since $\xi(\nabla\rho) = 0$, $\delta_T^*\xi(x,y) = \delta^*\xi(x,y)$ for x and y tangential vectors. If $\delta^*\xi(v,v) \geq 0$ for all non-zero vectors v tangent to $S_r(p)$ and for some v, $\delta^*\xi(v,v) > 0$, then $\delta_T\xi = \mathrm{tr}\ \delta_T^*\xi = \mathrm{tr}\ \delta^*\xi \geq 0$ and at $\pi(v)$, $\delta_T\xi > 0$ so $\int_{S_r(p)} \delta_T\xi > 0$. But $\delta_T\xi$ is a divergence on $S_r(p)$ so must integrate to zero. Hence either for all tangential v we must have $\delta^*\xi(v,v) = 0$, or $\delta^*\xi(v,v)$ does not have a definite sign on tangential vectors. Since $\delta^*\xi(\nabla\rho,\nabla\rho) = 0$, this implies that either ξ must be a Killing 1-form on a tubular neighborhood of D or $\delta^*\xi$ does not have a definite sign on tangential vectors. Since the existence of Killing 1-forms implies that D is "symmetric" near Bd D, we must suppose that the second alternative holds and hence $\delta^*\xi$ has positive and negative eigenvalues. Thus, by the argument we gave above for h_T with positive and negative eigenvalues, we conclude that the deformation $g(t) = g_o + t\rho^5 d\rho \circ \xi$ cannot improve the Ricci pinching near Bd D. If we consider $g(t) = g_o - t\rho^5 f\, d\rho \circ d\rho$, then if $\delta^*(d\rho)$ has eigenvalues $-\lambda_1, \ldots, -\lambda_{n-1}$ with $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1}$ (remember $\delta^*(d\rho)(\nabla\rho, \nabla\rho) = 0$), then to improve the Ricci pinching near Bd D we need

$$A < \frac{\lambda_1}{\sum_i \lambda_i} \leq \frac{1}{n-1}$$

so that at best this is no better than a conformal variation. In the variation $g(t) = g_o - t\rho^5(f d\rho \otimes d\rho + d\rho \circ \xi)$ the addition of the tensor $d\rho\ \xi$ to the variation $g(t) = g_o - t\rho^5 f d\rho \otimes d\rho$ only worsens the lower bound $\dfrac{\lambda_1}{\sum_i \lambda_i}$ from the tensor $f\ d\rho \otimes d\rho$.

If we consider a variation $g(t)$ so that only terms in ρ^5 appear in Ric', then since $\delta^*(d\rho)$ is negative definite we must have $g(t) = g_o + t\rho^5 d\rho \circ \xi$ where $\delta^*\xi \equiv 0$ near Bd D. Even if such a Killing 1-form exists near Bd D, one can check that Ric' will not have a definite sign. Hence, Theorem 3 using local conformal convex deformations is the best theorem we can obtain.

Chapter 6: The solution to Problem II: The non-existence of positive at first order local convex deformations for sectional curvature

Since we saw that Ricci curvature can be deformed from being non-negative and positive at a point to being everywhere positive using local convex deformations, it is reasonable to consider whether local convex deformations can be used to solve the following conjecture for $n \geq 3$.

Conjecture: Let (M^n, g_0) be a compact Riemannian manifold with all sectional curvatures non-negative and suppose there is a point p on M with all sectional curvatures positive. Then M admits a metric of everywhere positive sectional curvature.

For $n = 2$, the conjecture is true by Aubin's Theorem.

Recall that if $f : R^{n+1} \to R$ is a smooth function and 0 is not a critical value for f, then $M^n = f^{-1}(0)$ is a smooth n-manifold imbedded in R^{n+1}. By the Gauss curvature equations, [22], p. 10, for x and y orthonormal vectors in M_p

$$K(x,y) = \frac{1}{\|\nabla f|_p\|^2} (\delta^*(df)(x,x)\delta^*(df)(y,y) - \delta^*(df(x,y)^2).$$

Hence, if f is strictly convex (iff all eigenvalues of f are positive), then $K > 0$ by the generalized Cauchy-Schwarz

Inequality.

Suppose $M^n = f^{-1}(0)$ is a compact hypersurface where f is weakly convex, i.e., the eigenvalues of $\delta^*(df)$ are all non-negative. Let $g : R^{n+1} \to R$ be given by $g(x_1,\dots,x_{n+1}) = \frac{1}{2}(\sum_{i=1}^{n+1} x_i^2 - 1)$, so that $\delta^*(dg) = \mathrm{Id}$. Then for small $\epsilon > 0$, if $f_\epsilon = f + \epsilon g$, then $\delta^*(df_\epsilon)$ will have positive eigenvalues so $M_\epsilon = f_\epsilon^{-1}(0)$ will be a compact hypersurface with everywhere positive sectional curvature. If $\varphi_\epsilon : M \to M_\epsilon$ is a diffeomorphism and g is the standard Euclidean metric on R^{n+1} restricted to M, then $(M, \varphi_\epsilon{}^* g)$ will be a metric on M with everywhere positive sectional curvature. Hence the conjecture is probably true for compact manifolds imbedded isometrically in Euclidean space in codimension 1.

We can also use a single global convex deformation to handle the following simple case of the conjecture. (This was noticed by Bourguignon and myself independently.) Let

$$D_s(p) = \{q \in M;\ \mathrm{dist}_{g_0}(p,q) \le s\} \text{ and } Z_{g_0} := K_{g_0}{}^{-1}(0).$$

Given (M,g_0) suppose there exists $p \in M$ so that $\pi(Z_{g_0}) \subset \mathrm{Int}(D_R(p)) \subset D_{R+d}(p)$ where $d > 0$ and $D_{R+d}(p)$ is convex. Let r be the g_0 distance from p in M. Let $f : R^{\ge 0} \to R^{\ge 0}$ be a smooth function with

$f(t) = 0$ for $t \geq R+d$, $f(t) = d$ for $t \leq R+d/2$.

Consider the deformation $g(t) = g_o + tf(r)d(r^2) \otimes d(r^2)$. Then for $0 \leq r \leq R$, where $f(r) = d$

$$(\Sigma(f(r)dr^2 \otimes dr^2))(x,y) = d^2[\delta^*(d(r^2))(x,x)\delta^*(d(r^2))(y,y) - \delta^*(d(r^2))(x,y)^2] > 0$$

by the generalized Cauchy-Schwarz Inequality for positive operators. Hence $g(t)$ is a positive variation of Z_{g_o}. As in Theorem 1, Chapter 4, we can find $t_o > 0$ so that $0 < t < t_o$ implies $K^t(x,y) = K(x,y) + tK'(x,y) + \frac{t^2}{2} \frac{d^2}{dt^2} K^t(x,y)\Big|_{t=s} > 0$ in $D_{R+d}(p)$. Off $D_{R+d}(p)$, of course $K^t = K_{g_o} > 0$ by our assumption that $\pi(Z_{g_o}) \subset \operatorname{Int}(D_R(p))$. In general, if g_o is a metric for M with $K_{g_o} \geq 0$, then this construction will deform isolated points of $\pi(Z_{g_o})$ to points of everywhere positive sectional curvature. Notice that this method is different from the methods of Section 4 in that here we took $p \in \pi(K_{g_o}^{-1}(0))$ instead of taking a point of positive sectional curvature and spreading the curvature out from p while decreasing the magnitude of the positive curvature at p.

Inspired by the proof of Theorem 5.1, a reasonable way to try to prove the conjecture would be to find a deformation $g(t)$ with support on a convex disk D centered at p so that on some annular neighborhood of $\operatorname{Bd}(\bar{D})$, $K(\sigma) = 0$ would imply $K'(\sigma) > 0$. However, we will see that the convexity of D and the condition that $g(t) = g_o$ off D means no such deformation can be found.

We will consider C^3 variations with support in a closed convex disk D with center p and $D \neq M$. (If $D = M$, then the variation $g(t) = g_o + t\, d(r^2) \otimes d(r^2)$ with r the distance from p will prove the conjecture.) We saw above since we are only interested in computing K' that we need only consider variations $g(t) = g_o + th$. Further, $g(t) = g$ in M-Int(D) and $g(t)$ a C^3 variation with support in D implied $h = \rho^3 \tilde{h}$ in an annular neighborhood of Bd D.

Definition: Given $g \in R(M)$, let $Z_g = K_g^{-1}(0)$ where $K_g : G_2(M) \to R$ is the sectional curvature function defined by g.

Definition: A C^3 local convex variation $g(t)$ of g_o with support in D will be called positive at first order iff $K' > 0$ on $Z_{g_o} \cap \pi^{-1}(\mathrm{Int}(D))$. Let (M^3, g_o) be the 3-sphere flattened near the North pole so that if g_o is the metric induced by the inclusion $S^3 \subset (R^3, g_{can})$ then $K_{g_o} = 0$ on the closed 3-ball N of radius 1/4 about the North pole and $K_{g_o} > 0$ in S^3-N. Let p in S^3-N be a point near N in the Northern hemisphere. Thus $K_{g_o} > 0$ on $\pi^{-1}(p)$. Let D be any convex disk centered at p so that $\mathrm{Bd}(D) \cap \mathrm{Int}(N)$ is non-empty. In $D \cap N$ all curvatures are zero.

Theorem 1: There is no positive at first order local convex deformation with support in D. Thus the answer to Problem II is NO.

Hence, the argument for the Ricci curvature given in Theorem 1 of Chapter 5 does not generalize to the sectional curvature. In fact, if D is any closed convex set with $Bd(D) \cap Int(N) \neq \emptyset$, then our proof of Theorem 1 shows that there is no positive variation with support in D. Hence, non-negative sectional curvature and positi curvature at a point is rigid under local convex deformations. Our proof is presented for n=3 only for convenier and Theorem 1 is true with the analogous construction of N and D on the flattened n-sphere for all $n \geq 3$.

Fix an arbitrary convex variation $g(t)$ on D. Then there exists a one-sided tubular neighborhood U of $Bd(D) \cap N$ in D so that $g(t) = g_o + t\rho^3 h$. Decompose $h = h_T + d\rho \circ \xi$. By Corollary 4 of Section 2, if x and y are a g_o-orthonormal basis for a two-plane σ with $\pi(\sigma) \in N \cap U$, and $x_\rho = 0$, writing $\langle , \rangle$ for g_o

$$(14) \qquad K'(x,y) = (\Sigma(\rho^3 h))(x,y) = -3\rho\|y_\rho\|^2 h_T(x,x)$$

$$+ \frac{3}{2}\rho^2(2\delta^*(d\rho)(x,y)h_T(x,y) - \delta^*(d\rho)(x,x)h_T(y,y) - \delta^*(d\rho)(y,y)h_{\text{T}}$$

$$+ 3\rho^2\langle y,\nabla\rho\rangle((D_x h_T)(x,y) - (D_y h_T)(x,x)) + \rho^3(\Sigma h_T)(x,y) + \frac{3}{2}\rho^2\|y_\rho\|^2$$

$$+ \frac{\rho^3}{2}\{\delta^*(d\rho)(x,x)\delta^*\xi(y,y) + \delta^*(d\rho)(y,y)\delta^*\xi(x,x) - 2\delta^*\xi(x,y)\delta^*($$

$$+ \langle y,\nabla\rho\rangle(D_x d\xi)(y,x)\}.$$

It is easy to find tensors h with $(\Sigma h)(x,y) > 0$ like $h = d\rho^2 \circ d\rho^2$. However, (14) shows that the derivatives of the smoothing function ρ are of order 1 and 2 in ρ but Σh is a term in order 3 in ρ. We are thus led to ask what conditions must be imposed on h in order that $\Sigma(\rho^3 h) = \rho^3 \Sigma h$ near Bd D. It follows from (14) that this happens iff $h_T \equiv 0$ and $\delta^*\xi(x,y) = 0$ for all tangential vectors x and y near Bd D. We will say ξ is a <u>horizontal Killing 1-form</u> iff $\delta^*\xi(x,y) = 0$ for all tangential vectors x and y. Hence $\Sigma(\rho^3 h) = \rho^3 \Sigma h$ near Bd D iff $h = d\rho \circ \xi$ where ξ is a horizontal Killing 1-form near Bd D.

<u>Lemma 2</u>: For $K' > 0$ in $\mathrm{Int}(D) \cap N$, we must have $h_T \equiv 0$ in some neighborhood $U_1 \subset U$ of $\mathrm{Bd}(D) \cap N$.

<u>Proof</u>: If not, then there exists a sequence $\{p_i\}_{i=1}^{\infty}$ so that $\lim_{i\to\infty} \mathrm{dist}_{g_0}(p_i, \mathrm{Bd}(D) \cap N) = 0$ with $h_T \neq 0$ at p_i. Since h_T is a symmetric two-tensor, we may choose g_0-orthonormal tangential vectors x_i and y_i at p_i so that $h_T(x_i,y_i) = 0$. For <u>all</u> tangential vectors v, we have the radial zero two-plane $\sigma_v = \{v, \nabla\rho\}$. From (14) we have

$$(15) \qquad K'(v,\nabla\rho) = -3\rho h_T(v,v) + O(\rho^2)$$

and also

$$(16) \quad K'(x_i,y_i) = -\frac{3}{2}\rho^2\big(\delta^*(d\rho)(y_i,y_i)h_T(x_i,x_i) + \delta^*(d\rho)(x_i,x_i)h_T(y_i,y_i)\big) + O(\rho^3)$$

From (15) for $v = x_i$ and y_i, for i large, we must have $h_T(x_i,x_i) < 0$ or $h_T(y_i,y_i) < 0$ and $h_T(v,v) \leq 0$ for $v = x_i, y_i$ since $h_T(x_i,x_i) = h_T(y_i,y_i) = 0$ implies $h_T \equiv 0$ at p_i contrary to choice of p_i. Then near $Bd(D) \cap N$ from (16) we have $K'(x_i,y_i) < 0$.

Q. E. D.

Now in U_1, $g(t) = g_o + t\rho^3 d\rho \circ \xi$. Suppose we have $\{p_i\} \subset Int(U_1) \cap N$ with $\lim_{i\to\infty} dist_{g_o}(p_i, Bd(D) \cap N) = 0$ and $\delta^*\xi(x,x) \neq 0$ at p_i for some tangential $x \in M_{p_i}$ for all i. For all tangential vectors, v, we have

$$(17) \qquad K'(v,\nabla\rho) = \frac{3}{2}\rho^2\delta^*\xi(v,v) + O(\rho^3).$$

Let x_i and y_i in M_{p_i} be g_o-orthonormal vectors diagonalizing $\delta^*\xi|(\nabla\rho)^\perp$. Then

$$K'(x_i,y_i) = \frac{\rho^3}{2}(\delta^*(d\rho)(x_i,x_i)\delta^*\xi(y_i,y_i)+\delta^*(d\rho)(y_i,y_i)\delta^*\xi(x_i,x_i))$$

From (17) with $v = x_i, y_i$ for p_i near Bd D, we must have $\delta^*\xi(v,v) \geq 0$ for $v = x_i, y_i$ and either $\delta^*\xi(x_i,x_i) > 0$ or $\delta^*\xi(y_i,y_i) > 0$ for $K' > 0$. But then near $Bd(D) \cap N$, $K'(x_i,y_i) < 0$. We have shown

<u>Lemma 3</u>: For $g(t) = g_o + t\rho^3 d\rho \circ \xi$ to have $K' > 0$ on some neighborhood of $Bd(D) \cap N$, ξ must be a tangential Killing 1-form in some neighborhood $U_2 \subset U_1$ of $Bd(D) \cap N$.

Using Lemmas 2 and 3 we can now prove Theorem 1. To be a positive variation on D, on some neighborhood U_2 of $Bd(D) \cap N$, $g(t) = g_o + t\rho^3 d\rho \circ \xi$, where ξ is a tangential Killing 1-form in U_2. Then for any g_o-orthonormal tangential vectors x and y in U_2

$$K'(x,y)= \frac{\rho^3}{2}[\delta^*(d\rho)(x,x)\delta^*\xi(y,y)+\delta^*(d\rho(y,y)\delta^*\xi(x,x)-2\delta^*(d\rho)(x,y)\delta^*\xi(x,y$$

$$= 0$$

so Theorem 1 is proven. It is not at all clear that tangential Killing forms exist near $Bd(D) \cap N$. If there are no tangential Killing forms, then in fact Lemma 3 shows that K' does not have a definite sign in $Bd(D) \cap N$ for any local convex variation in D.

Let $f : R \to R>0$ be a smooth function with $f(t) = 1$ for $t \le 0$ and $f''(t) > 0$ for $t > 0$. Form the Bishop-O'Neill warped product 3-manifold $M^3 = R \times_f T^2$ where $(T^2, \langle , \rangle_{Torus})$ is the 2-torus with the standard flat metric $\langle , \rangle_{Torus}$. (See [9] for the warped product). Then M^3 is complete in the metric g given by: for $w_1 = \alpha_1 \frac{d}{dt} + v_1$ and $w_2 = \alpha_2 \frac{d}{dt} + v_2$ in $M^3|_{(t,p)}$ define

$$g(w_1,w_2)_{M^3} = \alpha_1\alpha_2 + (f(t))^2\langle v_1,v_2\rangle_{Torus} .$$

Given a two-plane σ at (t,p) we can choose a basis of the form $\sigma = \{\alpha \frac{d}{dt} + v, w\}$ where $\alpha^2 + (f(t))^2\|v\|^2_{Torus} = 1$

and $\|w\|^2_{Torus} = 1/(f(t))^2$. Then from Bishop and O'Neill [9] p. 27, the sectional curvature is given by

$$K(\sigma) = -\frac{f''(t)}{f(t)}\alpha^2 - \frac{(f'(t))^2}{(f(t))^2}\|v\|^2_{Torus} .$$

Hence, $K_{g_{M^3}} \equiv 0$ for (t,p) with $t \le 0$, and $K_{g_{M^3}} < 0$ for (t,p) with $t > 0$. Let $N = (-\infty,0]\times_f T^2$ so $K_{g_{M^3}} \equiv 0$ on N. At the point $p_o = (1/100,p)$ we have $K_{g_{M^3}} < 0$ for all two-planes. Let $\tilde{D}$ be a closed convex disk of finite radius centered at p_o with $Bd(\tilde{D}) \cap Int(N) \neq \emptyset$.

<u>Definition</u>: A C^3 convex deformation of g_{M^3} with support in $\tilde{D}$ will be called <u>negative at first order</u> iff $K' < 0$ on $Z_{g_{M^3}} \cap \pi^{-1}(Int(\tilde{D}))$. The proof of Theorem 1 just changing the signs shows

<u>Theorem 4</u>: There does not exist any negative deformation at first order on $\tilde{D}$. Hence, the method of using local convex deformations to prove Theorem 2 of Chapter 6 for the Ricci curvature does not generalize to non-positive sectional curvature.

Let us again consider convex variations on D on the flattened n-sphere M^n, $n \ge 3$. From the proof of Theorem 1, a necessary condition for $K' \ge 0$ in $D \cap N$ is that in some neighborhood $U_2 \subset D$ of Bd D, the local convex variation must be of the form

$g(t) = g_o + t\rho^3 d\rho \circ \xi$ where ξ is a horizontal Killing form on U_2. If ξ is the restriction of a Killing form to U_2, then $K' \equiv 0$ on U_2. We derive some necessary conditions for $\xi \notin \mathrm{Im}\delta^*$ to satisfy in order that $K' \geq 0$ in $\mathrm{Bd}(D) \cap N$.

Notational Convention: We will write $g = \langle , \rangle$ for g_o in the following computation.

For $g(t) = g + t\rho^3 d\rho \circ \xi$ with ξ a horizontal Killing form, $K(x,y) = 0$, $x_\rho = 0$ implies

$$K'(x,y) = \frac{\rho^3}{2}\{\delta^*(d\rho)(x,x)\delta^*\xi(y,y) - 2\delta^*\xi(x,y)\delta^*(d\rho)(x,y) + y(\rho)(D_x d\xi)(y,x)\}$$

For x and y g-orthonormal vectors, let $z_{a,b} = a\cdot y + b\nabla\rho$, $a^2+b^2 = 1$. Then we can define a quadratic form $Q(a,b)$ in a and b in $TM|U_2 \cap N$ by $Q(a,b) := \frac{2}{\rho^3} K'(x, z_{a,b})$. If $Q(a,b) = h(a,a)a^2 + 2h(a,b)ab + h(b,b)b^2$ and we write

$$\mathrm{mat}(Q(a,b)) = \begin{pmatrix} h(a,a) & h(a,b) \\ h(a,b) & h(b,b) \end{pmatrix}$$

then a necessary condition for $Q(a,b)$ to be positive semi-definite is than $\mathrm{tr}(\mathrm{mat}(Q(a,b))) \geq 0$ and $\det(\mathrm{mat}(Q(a,b))) \geq 0$. Using

$$\delta^*(d\rho)(x,\nabla\rho) = \delta^*\xi(x,y) = \delta^*\xi(y,y) = 0,$$

$$Q(a,b) = b^2[\delta^*(d\rho)(x,x)\delta^*\xi(\nabla\rho,\nabla\rho) + (D_x d\xi)(\nabla\rho,x)]$$
$$+ 2ab[\delta^*(d\rho)(x,x)\delta^*\xi(y,\nabla\rho) - \delta^*(d\rho)(x,y)\delta^*\xi(x,\nabla\rho) - \tfrac{1}{2}(D_x d\xi)(y,x)]$$

so $h(a,a) = 0$, $h(b,b) = \delta^*(d\rho)(x,x)\delta^*\xi(\nabla\rho,\nabla\rho)+(D_x d\xi)(\nabla\rho,x)$,
and

$$h(a,b) = \delta^*(d\rho)(x,x)\delta^*\xi(y,\nabla\rho)-\delta^*(d\rho)(x,y)\delta^*\xi(x,\nabla\rho) - \tfrac{1}{2}(D_x d\xi)(y,x).$$

Now $\det(\mathrm{mat}(Q(a,b))) = -h(a,b)^2 \geq 0$ iff $h(a,b) = 0$,
and $\mathrm{tr}(\mathrm{mat}(Q(a,b))) = h(b,b)$ so we obtain for
$g(t) = g_o + t\rho^3 d\rho \circ \xi$ with $\xi \notin \mathrm{Im}\delta^*$.

Proposition 5: A necessary condition for $K' \geq 0$ in U_2 is that the following three conditions are satisfied:

(i) ξ is a horizontal Killing form on U_2, but not a Killing form.

(ii) for all g_o-orthonormal tangential vectors x and y

$$2\delta^*(d\rho)(x,x)\delta^*\xi(y,\nabla\rho)-\delta^*(d\rho)(x,y)\delta^*\xi(x,\nabla\rho)+(D_x d\xi)(y,x) = 0.$$

(iii) for all tangential vectors x,

$$\delta^*(d\rho)(x,x)\delta^*\xi(\nabla\rho,\nabla\rho) + (D_x d\xi)(\nabla\rho,x) \geq 0.$$

We make a final comment on (i). We may write $\xi = \eta + f d\rho$ where $\eta = \xi^T$ satisfies $\eta(\nabla\rho) = 0$. Since $g_o = g_T + d\rho \circ d\rho$, if δ^*_T is the differential operator defined as in Chapter 5, by restricting δ^* to $S_r(p)$, then condition (i) becomes

(i') $\delta^*_T\eta(x,y) + f\,\delta^*(d\rho)(x,y) = 0$ for all tangential x and y.

From the geometry, it is clear that there does not exist a tangential 1-form η with $\delta^*_T\eta = 0$ near $\mathrm{Bd}(N) \cap U_2$, since any sphere $S_r(p)$ with $S_r(p) \cap \mathrm{Int}(N) \neq \emptyset$ is more curved in the tangential metric in $D - N$ than in $D \cap N$. Hence, if $f \equiv 0$, then condition (i') cannot be satisfied. If $\eta \equiv 0$, then for condition (i') to hold since $\delta^*(d\rho)$ is negative definite, we need $f \equiv 0$. So for condition (i') to hold, $\delta^*_T \eta$ must be collinear with $\delta^*(d\rho)$ near $U_2 \cap \mathrm{Bd}(N)$ on the spheres $S_r(p)$ with $S_r(p) \cap \mathrm{Int}(N) \neq \emptyset$.

Theorems 1 and 4 suggest the following open questions:

Question A: In the context of Theorem 1, do there exist local convex deformations non-negative at first order?

More interesting are:

Question B: Does there exist a manifold M admitting a complete metric g_o with everywhere non-negative sectional curvature and all sectional curvatures positive at a point such that no complete metric $g \in R(M)$ in a C^2 neighborhood of g (suitably defined in the non-compact case) has everywhere positive sectional curvature?

Question C: Does there exist a manifold M admitting a complete metric g_o with everywhere non-positive sectional curvature and all sectional curvatures negative at a point such that no complete metric $g \in R(M)$ in a C^2 neighborhood of g has everywhere negative sectional curvature?

Perhaps these questions make more sense for M compact in view of the technical difficulties involved in defining a topology on R(M) for M non-compact.

In connection with Question C, in [18] we give an example of a complete metric g_o on $R^2 \times S^1 \times S^1$ with everywhere non-positive sectional curvature and with points of all negative sectional curvature such that no complete metric "C^2-nearby" g_o has everywhere negative sectional curvature.

Chapter 7: Metric Deformations on Compact 3-Manifolds

In light of the non-existence of positive at first order variations of sectional curvature, two natural questions that come to mind are the following:

Question I: Is the sectional curvature conjecture true under weaker hypotheses? Explicitly, for instance, given (M^n, g_o) compact, $n \geq 3$, with $K_{g_o} \geq 0$ and $Ric_{g_o} > 0$. Does M admit a metric g with $K_g > 0$?

Question II: Are there results in dimension 3 which might hold given the especially nice algebraic properties of the curvature tensor in dimension 3. For instance, one such property is that $Ric_g \equiv 0$ implies $K_g \equiv 0$. Secondly, if x,y,z are orthonormal vectors diagonalizing

$$Ric = Ric_g : M_p \to M_p$$

and

$$v = a_1 x + a_2 y + a_3 z$$

$$w = b_1 x + b_2 y + b_3 z$$

then

$$Ric(x,x) = K(x,y) + K(x,z)$$

$$Ric(y,y) = K(x,y) + K(y,z)$$

$$Ric(z,z) = K(x,z) + K(y,z)$$

$$Ric(x,y) = Ric(x,z) = Ric(y,z) = 0$$

and

$$K(v,w) = (a_1b_2-a_2b_1)^2K(x,y) + (a_1b_3-a_3b_1)^2K(x,z) + (a_2b_3-a_3b_2)^2K(y,z) \quad \text{if} \quad \|v\wedge w\|^2 = 1.$$

Hence, one checks easily that

1) $1/2\, c\, g \leq \mathrm{Ric} \leq c\, g$, $c > 0$, at p implies $K(x,y) \geq 0$, $K(x,z) \geq 0$, and $K(y,z) \geq 0$ and hence $K \geq 0$ at p.

2) $1/2\, c\, g < \mathrm{Ric} \leq c\, g$, $c > 0$ implies $K(x,y)$, $K(x,z)$, $K(y,z) > 0$ and hence $K > 0$ at p.

These relations do not hold for $n > 3$.

In considering Question I, we are led to consider

<u>Question I'</u>: Given (M,g_o) as in Question I, does there exist a symmetric two tensor h not in $\mathrm{Im}\delta^*$ such that

(*) $(\Sigma(h))(x,y) = \mathrm{Ric}_{g_o}(x,x)\mathrm{Ric}_{g_o}(y,y) - (\mathrm{Ric}_{g_o}(x,y))^2$?

Notice that if (*) holds, $g(t) := g_o + t\, h$, and $\mathrm{Ric}_{g_o} > 0$, $K_{g_o} \geq 0$, then $g(t)$ is a positive variation at first order. Hence for $(S^2 \times S^2, g_{can})$ for which $\mathrm{Ric}_{g_{can}} > 0$, $K_{g_{can}} \geq 0$, the general existence of such a tensor h would imply the existence of a positive variation at first order for $(S^2 \times S^2, g_{can})$. But this is impossible by a lemma of Berger, [4]. Hence the answer to Question I' is NO in general if $\dim M \geq 4$.

In considering Question I for (M, g_o) compact, the most obvious thing to try is to compute K' for the deformation

$$g(t) = g_o + t\ \mathrm{Ric}_{g_o}$$

Surprisingly enough, the computation of K' for this deformation leads to an answer of YES for Question I for dim $M = 3$ only and answers Question I' for dim $M = 3$. This constitutes part 1 of Chapter 7.

In part 2, we compute Ric' for $g(t) = g_o + t\ \mathrm{Ric}_{g_o}$. In trying to understand the global significance of $K' = 0$ when $\mathrm{Ric}_{g_o} \geq 0$, $K_{g_o} \geq 0$ (instead of $\mathrm{Ric}_{g_o} > 0$, as in part 1) we are led to consider the following:

a) What do D*D R and D*D Ric mean?

b) The study of Ricci-product-like manifolds.

Part I: Deformation of Sectional Curvature by the Ricci Tensor on 3-Manifolds

Let (M,g_o) be a Riemannian manifold. We will use $\langle,\rangle$ to denote g_o in this paragraph.

Recall that in deforming curvature, we want to use tensors h not in $\mathrm{Im}\delta^*$.

In [10], Bourguignon showed that for any g_o, $\mathrm{Ric}_{g_o} \notin \mathrm{Im}\delta^*$ unless $\mathrm{Ric}_{g_o} \equiv 0$. Hence, it is not unreasonable to consider the deformation $g(t) = g_o + t\,\mathrm{Ric}_{g_o}$.

Let $Z_{g_o} := K_{g_o}^{-1}(0)$ and suppose $K_{g_o} \geq 0$. We will say g(t) is a <u>positive deformation</u> of g_o if and only if $\sigma \in Z_{g_o}$ implies $K'(\sigma) > 0$. Note that if M is compact, $K_{g_o} \geq 0$, and g(t) is a positive variation, then we can find an $s > 0$ with $K_{g(s)} > 0$ on $G_2(M)$.

We will see that for a 3-manifold (M^3,g_o) that $\Sigma(\mathrm{Ric})$ has a particularly simple formula and that $g(t) = g_o + t(-\mathrm{Ric})$ is a positive deformation if $K \geq 0$ and $\mathrm{Ric} > 0$. Hence a compact 3-manifold that is $\frac{1}{2}$ positively Ricci pinched admits a metric of positive sectional curvature.

<u>Remark</u>: It is false that $\mathrm{Ric} > 0$ implies <u>algebraically</u> for 3-manifolds that $K > 0$. Using the local convex deformation on the 3-sphere flattened at the North pole of Chapter 6, we can easily given an example of a compact 3-manifold

with Ric > 0 but for which there is some negative sectional curvature. S.T. Yau has informed us orally that he also has a counterexample.

We would like to thank J. P. Bourguignon for suggesting the use of the second Bianchi identity to study $\Sigma(\mathrm{Ric})$.

Define a (4,0) tensor R by $R(x,y,z,w) := (R(x,y)z,w)$. Then

$$(*)\quad (D_X D_Y R)(S,T,U,V) - (D_Y D_X R)(S,T,U,V) - (D_{[X,Y]}R)(S,T,U,V)$$
$$= - R(R(X,Y)S,T,U,V) - R(S,R(X,Y)T,U,V)$$
$$- R(S,T,R(X,Y)U,V) - R(S,T,U,R(X,Y)V), \text{ for}$$

X,Y,S,T,U,V vector fields.

Let $e_1,\ldots,e_n$ be an orthonormal basis at p. Given x,y orthonormal vectors at p, extend $x,y,e_1,\ldots,e_n$ to local vector fields $X,Y,E_1,\ldots,E_n$. Then

$$(\Sigma(\mathrm{Ric}))(x,y) = \sum_i \tfrac{1}{2}\{D_X D_Y R(E_i,X,Y,E_i) + D_X D_Y R(E_i,X,Y,E_i)$$
$$- D_X D_X R(E_i,Y,Y,E_i) - D_Y D_Y R(E_i,X,X,E_i)\}$$
$$+ \mathrm{Ric}(R(x,y)y,x)$$

The idea is to use (*) and the second Bianchi identity to write this as $\sum_i D_{E_i} D_{E_i} R(X,Y,X,Y)$ + curvature terms.

Start by writing the second term $D_X D_Y R(E_i,X,Y,E_i) = D_Y D_X R(E_i,X,Y,E_i)$ + curvature terms. Then use the second

Bianchi identity on $D_Y R(E_i, X, \cdot, \cdot)$ and $D_X R(\cdot, \cdot, Y, E_i)$ to get $- D_X D_{E_i} R(X,Y,Y,E_i) - D_Y D_{E_i} R(E_i,X,X,Y)$ + curvature terms.

Then, writing this using (*) as

$$- D_{E_i} D_X R(X,Y,Y,E_i) - D_{E_i} D_Y R(E_i,X,X,Y) + \text{CURVATURE terms},$$

use the Bianchi identity on $D_X R(\cdot, \cdot, Y, E_i)$ to get

$$\text{Ric}(x,y) = -\frac{1}{2} \sum_i D_{E_i} D_{E_i} R(X,Y,Y,X) + \text{curvature terms}.$$

Definition: Define $D^*DR(x,y,y,x) := \sum_i DDR(e_i, e_i, x,y,y,x)$ where $e_1, \ldots, e_n$ is an orthonormal basis at p. Then we have

Proposition 1: For $g(t) = g_o + t\,h$, where $h = -\text{Ric}_{g_o}$, $(\Sigma h)(x,y) = \frac{1}{2} D^*DR(x,y,y,x) + \text{Curv}(x,y)$ where for an orthonormal basis $e_1, \ldots, e_n$ at p,

$$\text{Curv}(x,y) = \sum_i \{\langle R(x,e_i)x,\ R(y,e_i)y\rangle + \langle R(y,e_i)x, R(y,e_i)x\rangle$$
$$- 2\langle R(x,e_i)y,\ R(y,e_i)x\rangle + \langle R(x,y)x,\ R(y,e_i)e_i\rangle$$
$$+ \langle R(y,x)y,\ R(x,e_i)e_i\rangle\}.$$

Remarks: (1) Although $\langle R(y,e_i)x,\ R(y,e_i)x\rangle$ is not symmetric in x and y, $\sum_i \langle R(y,e_i)x,\ R(y,e_i)x\rangle$ is symmetric in x and y.

(2) If (M^n, g_o) has constant curvature k, then

$$\text{Curv}(x,y) = -k^2(n-1)g_o(x,y).$$

Now suppose $K_{g_o} \geq 0$. Then lemma 2.2 shows if $e_1, \ldots, e_n$ are orthonormal vectors and $K(e_1,e_2) = 0$ then $\langle R(e_1,e_2)e_1,e_j\rangle = 0$ for $1 \leq j \leq n$. Hence

$$(**) \qquad K(x,y) = 0 \text{ implies } R(x,y)y = R(y,x)x = 0.$$

It is not hard to see that if we make a good extension of e_i,x,y to E_i,X,Y then

$$D\,D\,R(e_i,e_i,x,y,y,x) = e_i(E_i(R(X,Y,Y,X)))$$

$$- 2\langle R(x,y)y,\ R(x,e_i)e_i\rangle - 2\langle R(y,y)x,\ R(y,e_i)e_i\rangle.$$

Hence, by (**), if $K(x,y) = 0$, then

$DDR(e_i,e_i,x,y,y,x) = e_i(E_i(R(X,Y,Y,X)))$. Since $K \geq 0$, $e_i(E_i(R(X,Y,Y,X))) \geq 0$. Thus,

<u>Lemma 2</u>: If $K \geq 0$, then $K(x,y) = 0$ implies $D^*DR(x,y,y,x) \geq 0$.

<u>Remark</u>: Since $R(X,Y,Y,X) = (\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2)K(X,Y)$ it is not necessarily true that $K \geq A$ implies $R(X,Y,Y,X) \geq A$. Hence, if $K \geq A$ and $K(x,y) = A$, we cannot conclude that

$$e_i(E_i(R(X,Y,Y,X))) \geq 0.$$

Suppose (M^3,g_o) is a 3-manifold with $K_{g_o} \geq 0$. For orthonormal vectors x and y,

$$\mathrm{Curv}(x,y) = \mathrm{Ric}(x,x)\mathrm{Ric}(y,y) - (\mathrm{Ric}(x,y))^2 - 2(K(x,y))^2$$

$$- 2K(x,y)\mathrm{Ric}(z,z)$$

where z is any unit vector orthogonal to x and y. Thus, if $K(x,y) = 0$,

$$K'(x,y) = \frac{1}{2} D^*DR(x,y,y,x) + Ric(x,x)Ric(y,y) - (Ric(x,y))^2.$$

Hence, if $Ric > 0$, then $g(t) = g_o + t(-Ric)$ is a positive deformation by the generalized Cauchy-Schwarz inequality for positive operators. In light of our introductory remarks,

Theorem 3: Let (M^3, g_o) be a compact 3-manifold with $K \geq 0$ and $Ric > 0$. Then M admits a metric of everywhere positive sectional curvature.

Remarks: (1) The proof of Theorem 3 does not generalize to $n \geq 4$. The lemma of Berger (see [4], implies that for the canonical metric g_{can} on $S^2 \times S^2$, if $g(t)$ is any deformation with $K' \geq 0$ on $Z_{g_{can}}$, then $K' \equiv 0$ on $Z_{g_{can}}$. Hence even though $Ric_{g_{can}} > 0$, the deformation $g(t) = g_o + t(-Ric_{g_{can}})$ cannot be a positive deformation. In fact, for the "mixed" 2-planes which are the zeroes of $K_{g_{can}}$, Curv = 0.

(2) Consider (M^3, g_o) with $K_{g_o} \leq 0$ and $Ric_{g_o} < 0$. Then $K(x,y) = 0$ implies $K'(x,y) = -\frac{1}{2} D^*DR(x,y,y,x) - Ric(x,x)Ric(y,y) + (Ric(x,y))^2$.

At a zero of K, the curvature term will be negative, but in this case $D^*D\,R(x,y,y,x) \leq 0$, so we cannot be sure that $K'(x,y) < 0$. This contrasts with the situation in Chapter 5 for local convex deformations where $g(t) = e^{-2t\rho^5} g_o$ worked for $Ric \geq 0$ and $g(t) = e^{2t\rho^5} g_o$ worked for $Ric \leq 0$.

In [6], p. 73, the eigenvectors of Ric for 3-manifolds are related to the sectional curvature. Using this, it is not hard to see

Lemma 4: If Ric is positively $\frac{1}{2}$ pinched, then $K \geq 0$. If Ric is positively $\frac{1}{2} + \delta$ pinched for any $\delta > 0$, then $K > 0$. Hence, by Lemma 4 and Theorem 3 we have

Theorem 5: Let (M^3, g_o) be a compact positively $\frac{1}{2}$ - Ricci pinched 3-manifold. Then M admits a metric of everywhere positive sectional curvature.

Remark: By the methods of Chapter 5, to conclude that M admits a metric of positive sectional curvature, we would need Ric $\frac{1}{2}$ - pinched and at some point, for all vectors, the pinching not attained. Thus, Theorem 4 gives a strenthening of the results using local deformation, but Ric cannot be used to prove the analogous theorem for negative $\frac{1}{2}$ Ricci pinching.

Suppose (M^n, g_o) is a Riemannian manifold with $K_{g_o} \geq 0$ and that Z_{g_o} is characterized as follows:

(***) if $p \in \pi(Z_{g_o})$, then there exists $n \in M_p$ such that for $\sigma = \{x,y\}$ in $\pi^{-1}(p)$, then $K(\sigma) = 0$ iff $\langle x,n\rangle = 0$ and $\langle y,n\rangle = 0$.

Let σ be a zero 2-plane with orthonormal basis $\{x,y\}$. Choose an orthonormal basis $e_1 = x$, $e_2 = y$, $e_3, \ldots, e_m = n$ for M_p. Using (**) it is not hard to see that

<u>Lemma 6</u>: Suppose $K \geq 0$ and $u, v, w, \tilde{p}$ are orthonormal vectors in M_p. Suppose $K(u,w) = K(v,w) = K(u+v,w) = 0$. Then

$$\langle R(u,w)v, \tilde{p}\rangle = -\langle R(v,w)u, \tilde{p}\rangle .$$

For our 2-plane σ in Z_{g_o}, we have

$$\mathrm{Curv}(x,y) = \sum_i \{\langle R(x,e_i)x, R(y,e_i)y\rangle + \langle R(y,e_i)x, R(y,e_i)x\rangle - 2\langle R(x,e_i)y, R(y,e_i)x\rangle\}.$$

Now

$$\sum_i \langle R(x,e_i)x, R(y,e_i)y\rangle = \sum_{i=1}^{m-1} \langle R(x,e_i)x, R(y,e_i)y\rangle + \langle R(x,n)x, R(y,n)y\rangle .$$

Now using the stronger form of Lemma 2.2, we have $\langle R(x,e_i)x, n\rangle = 0$ for $1 \leq i \leq n-1$ since $K(x,e_i) = 0$ for $1 \leq i \leq n-1$. Hence

$$\sum_{i=1}^{m-1} \langle R(x,e_i)x, R(y,e_i)y\rangle = \sum_{i=1}^{m-1} \langle R(x,e_i)x,n\rangle\langle R(y,e_i)y,n\rangle = 0.$$

(Obviously $\langle R(x,e_i)x,e_j\rangle = 0$ for $1 \le i,j \le m-1$ by formula (15) of [22], p. 93, as well as by the strong form of (**).) Hence

$$\sum_{i=1}^{m} \langle R(x,e_i)x, R(y,e_i)y\rangle = \langle R(x,n)x, R(y,n)y\rangle = \mathrm{Ric}(x,x)\mathrm{Ric}(y,y)$$

since $K(x,e_i) = K(y,e_i) = 0$ for $1 \le i \le m-1$. A similar analysis of the other two terms of the formula for $\mathrm{Curv}(x,y)$ above, using Lemma 6 on the third term shows that

<u>Lemma 7</u>: $K(x,y) = 0$, x and y orthonormal implies

$$\mathrm{Curv}(x,y) = \mathrm{Ric}(x,x)\mathrm{Ric}(y,y) - (\mathrm{Ric}(x,y))^2 + 5\sum_i \|R(y,e_i)x\|^4.$$

Notice that for $x \in (n)^{\perp}$, $\mathrm{Ric}(x,x) = K(x,n) > 0$ by assumption (***). Hence $\mathrm{Ric}|_{(n)^{\perp}}$ is a positive operator so $\mathrm{Curv}(x,y) > 0$ for σ in Z_{g_o} with orthonormal basis $\{x,y\}$. Hence

<u>Proposition 8</u>: Suppose $K_{g_o} \ge 0$ and Z_{g_o} satisfies the algebraic condition (***) above. Then

$$g(t) = g_o + t(-\mathrm{Ric}_{g_o})$$

is a positive variation on Z_{g_o}.

In particular, suppose $\tilde{M}^{n-1} \underset{i}{\subset} (N^n, g_o)$ is an embedded submanifold so that the zero set of K_{g_0} is precisely the "tangential" 2-planes formed by vectors in $i_*(\tilde{T}M)$. Then if M is compact, there exists a metric for M with everywhere positive sectional curvature.

Part 2: More Metric Deformations on Compact 3-Manifolds: How to Understand D*D Ric

Let h be a tensor. Let $\{e_i\}$ be an orthonormal basis at p. Define

$$D^*Dh(v_1,v_2,\ldots) = \sum_i DDh(e_i,e_i;v_1,v_2,\ldots)$$
$$= \sum_i (D_{e_i}D_{e_i}h)(v_1,v_2,\ldots).$$

Suppose X is a local unit vector field on M^3 with $D_X X \equiv 0$. Choose y,z in M_p so that $\{X_p,y,z\}$ are orthonormal. Let

$$\|dX^\flat\|^2 := ((dX^\flat)(y,z))^2$$

$$(dX^\flat)(u,v) := \langle D_uX,v\rangle - \langle D_vX,u\rangle$$

$$(\delta^*X^\flat)(u,v) := \langle D_uX,v\rangle + \langle D_vX,u\rangle$$

$$\|DX\|^2 := (\langle D_yX,y\rangle)^2 + (\langle D_zX,z\rangle)^2$$
$$+ (\langle D_yX,z\rangle)^2 + (D_zX,y\rangle)^2$$

Note the non-standard omission of the factor of 1/2. These definitions make sense in light of the standard definitions because $\|X\| = 1$ and $D_XX \equiv 0$ implies

$$(dX^\flat)(X,.) \equiv 0 \quad \text{and}$$
$$(\delta^*X^\flat)(X,\cdot) \equiv 0\ .$$

As usual, we write $(DX)(v) := D_v X$ and for any extension of w to a local vector field W,

$$(D\,D\,X)(v,w) := D_v D_W X - D_{D_v W} X.$$

Where there is no danger of confusion, we will write $\langle , \rangle$ for g_o, R for R_{g_o}, Ric for Ric_{g_o}, τ for τ_{g_o} (scalar curvature function of M).

We have seen that for a compact 3-manifold (M, g_o) with $\mathrm{Ric}_{g_o} > 0$ and $K_{g_o} \geq 0$, perturbing the metric by the Ricci tensor of g_o will give M a metric of positive sectional curvature. The natural questions to ask then are:

(I) How does this proof fail if $\mathrm{Ric}_{g_o} \geq 0$ only?

(II) What happens to the Ricci curvature when the metric is perturbed by the Ricci tensor?

Recall that if $g(t) = g_o + t(-\mathrm{Ric}_{g_o})$ and $K(\sigma) = 0$ then

$$K'(\sigma) = 1/2(D^*D\,\mathrm{Ric})(x,y,y,x) + \mathrm{Ric}(x,x)\mathrm{Ric}(y,y)$$

where x,y are a g_o-orthonormal basis for σ diagonalizing Ric restricted to σ. We saw that if $K \geq 0$, then $K(x,y) = 0$ implied

$$(D^*D\,R)(x,y,y,x) \geq 0.$$

Thus, if $\mathrm{Ric}(x,x) = 0$ and $K(x,y) = 0$, then $K'(\sigma) = 0$ iff

$$(D^*D\,R)(x,y,y,x) = 0.$$

Even for $Ric_{g_0} \geq 0$, this variation is thus still a non-negative sectional curvature variation. Below we deal with the case in which this variation vanishes identically on the zero 2-planes (for $Ric_{g_0} \geq 0$) and show that this can happen only if (M, g_0) is locally isometric to a product. If this condition is not satisfied, then $g(t) = g_0 + t(Ric_{g_0})$ is a non-negative variation for M that is positive on some zero 2-planes in $G_2(M)$.

If $\{x,y,z\}$ are a g_0-orthonormal basis for M_p, it is not hard to see that

$$(1) \quad (D^*D\ R)(x,y,y,x) + (D^*D\ R)(x,z,z,x) = (D^*D\ Ric)(x,x).$$

While we do not understand $D^*D\ R$ very well, equation (1) suggests that there is some connection between (I) and (II). In the case where $Ric' = 0$ on the zero set of Ric_{g_0}, we will see that this tensor has a nice expression. Here we only remark that

<u>Lemma 1</u>: Let X be a vector field in M. Then

$$(DD\ Ric)(v,v,X,X) = \delta^*(d(Ric(X,X))(v,v) - 4(D_v Ric)(D_v X, X)$$
$$- 2\ Ric(D_v X, D_v X) - 2\ Ric((DDX)(v,v),X)$$

so that

$$(D^*D\ Ric)(X,X) = \Delta(Ric(X,X)) - 4\ tr\{z \to (D_z Ric)(D_z X, X)\}$$
$$- 2\ tr\{z \to Ric((DX)(z),(DX)(z))\}$$
$$- 2\ tr\{z \to Ric((DDX)(z,z),X)\}$$

which does not seem to be very enlightening.

Proof of Lemma 1: Extend v to a local vector field V with $D_V V \equiv 0$.

$$(D_v \mathrm{Ric})(X,X) = v(\mathrm{Ric}(X,X)) - 2\,\mathrm{Ric}(D_v X, X).$$

Thus

$$v((D_v \mathrm{Ric})(X,X)) = v\cdot V(\mathrm{Ric}(X,X)) - 2v(\mathrm{Ric}(D_v X,X))$$

$$= v\cdot V(\mathrm{Ric}(X,X)) - 2(D_v \mathrm{Ric})(D_v X,X) - 2\,\mathrm{Ric}(D_v D_v X,X)$$

$$- 2\,\mathrm{Ric}(D_v X, D_v X)$$

Then

$$(DD\,\mathrm{Ric})(v,v;X,X) = v((D_v \mathrm{Ric})(X,X)) - 2(D_v \mathrm{Ric})(D_v X,X)$$

so substituting the expression derived for $v((D_v \mathrm{Ric})(X,X))$, we are done.

Q. E. D.

We want to compute the first derivative of Ric^t at $t = 0$ for the variation $g(t) = g_o + t(-\mathrm{Ric}_{g_o})$. We will need the following well-known formula which is just the Second Bianchi Identity rephrased and in the notation and sign conventions of [6] is just

$$"\delta' \mathrm{Ric} = -\frac{1}{2}\, d\tau".$$

Lemma 2: Let $\{e_i\}$ be an orthonormal basis for M_p. Then

$$\sum_i (D_{e_i} \mathrm{Ric})(e_i, w) = \frac{1}{2}\, w(\tau)\ .$$

<u>Proof</u>: Step 1: $(D_X\mathrm{Ric})(y,z) = \sum_i (D_X R)(e_i,y,z,e_i)$

Step 2: The Second Bianchi Identity and Step 1 gives

$$2\sum_i (D_{e_i}\mathrm{Ric})(e_i,w) = \sum_i (D_w\mathrm{Ric})(e_i,e_i)$$

Step 3: Let c be the geodesic through $p := \pi(v)$ with $c(0) = w$. Extend the e_i at p to the parallel fields E_i along c with $E_i|_p = e_i$. Then $\{E_i\}$ are orthonormal along c and

$$\sum_i (D_w\mathrm{Ric})(e_i,e_i) = \sum_i w(\mathrm{Ric}(E_i,E_i)) - 2\sum_i \mathrm{Ric}(D_wE_i|_p,e_i)$$

$$= w(\sum_i \mathrm{Ric}(E_i,E_i)) \quad \text{since} \quad D_wE_i|_p = 0$$

$$= w(\tau).$$

Q. E. D.

In [2], Berger showed that for $g(t) = g_o + t\,h$ that

$$\mathrm{Ric}' = -\frac{1}{2} D^*D\,h + \mathrm{Ric}\otimes h - R\otimes h - \delta^*\delta' h - \frac{1}{2}\mathrm{Hess}(\mathrm{tr}\,h)$$

where for an orthonormal basis $\{e_i\}$ at p,

$$\mathrm{Ric}\otimes h(x,y) = \sum_i \mathrm{Ric}(x,e_i)h(y,e_i)$$

$$\mathrm{Ric}\otimes h(x,y) = \sum_{i,j} R(x,e_i,e_j,y)h(e_i,e_j)\ .$$

Since $h = -\mathrm{Ric}$, by Lemma 1; $\delta' h = \frac{1}{2}\,d\tau$ so that

$$\delta^*\delta' h + \frac{1}{2}\,\mathrm{Hess}(\mathrm{tr}\ h) = \frac{1}{2}\,\delta^*(d\tau) - \frac{1}{2}\,\delta^*(d\tau) = 0$$

and thus for $g(t) = g_o + t(-\mathrm{Ric}_{g_o})$,

$$\mathrm{Ric}' = 1/2\ D^*D\ \mathrm{Ric} - \mathrm{Ric}\otimes\mathrm{Ric} + R\otimes\mathrm{Ric}.$$

From now on, all manifolds will be 3-dimensional unless explicitly exempted from this convention! Recall that in 3-dimensions, if $\{x,y,z\}$ are g_o-orthonormal vectors at p then

(2) $K(x,y) = 1/2(\mathrm{Ric}(x,x) + \mathrm{Ric}(y,y) - \mathrm{Ric}(z,z)).$

Then

$$\begin{aligned}\mathrm{Ric}'(x,x) &= \frac{1}{2}\,D^*D\ \mathrm{Ric}(x,x) - (\mathrm{Ric}(x,x))^2 - (\mathrm{Ric}(x,y))^2\\ &\quad - (\mathrm{Ric}(x,z))^2 + R(x,y,y,x)\mathrm{Ric}(y,y)\\ &\quad + R(x,z,z,x)\mathrm{Ric}(z,z) + 2\ R(x,y,z,x)\mathrm{Ric}(y,z)\\ &= \frac{1}{2}\,D^*D\ \mathrm{Ric}(x,x) - (\mathrm{Ric}(x,x))^2 - (\mathrm{Ric}(x,y))^2 - (\mathrm{Ric}(x,z))^2\\ &\quad + K(x,y)\mathrm{Ric}(y,y) + K(x,z)\mathrm{Ric}(z,z) + 2\ (\mathrm{Ric}(y,z))^2\end{aligned}$$

Substituting (2) for $K(x,y)$ and $K(x,z)$ we obtain

<u>Proposition 3:</u> For (M,g_o), let $g(t) = g_o + t(-\mathrm{Ric}_{g_o})$ and let $\{x,y,z\}$ be g_o-orthonormal vectors at p so that y and z diagonalize $\mathrm{Ric} : x^{\perp} \otimes x^{\perp} \to \mathbb{R}$. Then

$$\mathrm{Ric}'(x,x) = \frac{1}{2}(D^*D\,\mathrm{Ric})(x,x) - (\mathrm{Ric}(x,x))^2 - (\mathrm{Ric}(x,y))^2 - (\mathrm{Ric}(x,z))^2 + \frac{1}{2}\mathrm{Ric}(x,x)(\mathrm{Ric}(y,y) + \mathrm{Ric}(z,z)) + \frac{1}{2}(\mathrm{Ric}(y,y) - \mathrm{Ric}(z,z))^2.$$

Now suppose $\mathrm{Ric}_{g_0} \geq 0$. Then

<u>Lemma 4</u>: $\mathrm{Ric}(x,x) = 0$ implies $\mathrm{Ric}(x,v) = 0$ for all v.

<u>Proof</u>: Just consider $f(t) := \mathrm{Ric}(x+tv, x+tv) \geq 0$ which has 0 as a minimum, so $f'(0) = 0$.

Q. E. D.

<u>Lemma 5</u>: If $\tilde{X} = E_1, E_2$, and E_3 are a good extension of x, y, z in a neighborhood of p, then

$$(D^*D\,\mathrm{Ric})(x,x) = \Delta(\mathrm{Ric}(\tilde{X},\tilde{X})) - 2\sum_i \mathrm{Ric}(x, R(E_i|_p, x)E_i|_p).$$

Thus, if $\mathrm{Ric}(x,x) = 0$, then

$$(D^*D\,\mathrm{Ric})(x,x) = \frac{1}{2}\Delta(\mathrm{Ric}(\tilde{X},\tilde{X})) \geq 0.$$

<u>Proof</u>: Using Lemma 1 applied to $\tilde{X}$, since $D\tilde{X}|_p = 0$ we have

$$(D^*D\,\mathrm{Ric})(x,x) = \Delta(\mathrm{Ric}(\tilde{X},\tilde{X})) - 2\sum_i \mathrm{Ric}(D_{E_i|_p} D_{E_i}\tilde{X}, x)$$

But $D_{E_i|_p} D_{E_i}\tilde{X} = D_{E_i|_p} D_{\tilde{X}} E_i$ using $[\tilde{X}, E_i] \equiv 0$ near p,

$$= D_{E_i|_p} D_{\tilde{X}} E_i - D_x D_{E_i} E_i - D_{[E_i,\tilde{X}]} E_i$$

$$= R(E_i|_p, x)E_i|_p \quad \text{since} \quad D_{E_i}E_i \equiv 0 \text{ near } p.$$

Q. E. D.

<u>Definition:</u> Given x in M_p, let $l_1(x)$ and $l_2(x)$ be the eigenvalues of the Ricci tensor Ric_{g_o} restricted to the g_o-orthogonal complement of x in M_p. Let $\lambda_1(p) \le \lambda_2(p) \le \lambda_3(p)$ be the eigenvalues of Ric_{g_o} on M_p. Thus, we have

<u>Proposition 3':</u> Suppose $Ric_{g_o} \ge 0$. If x is a g_o-unit vector with $Ric_{g_o}(x,x) = 0$, then

$$Ric'(x,x) = 1/2(D^*D\, Ric)(x,x) + 1/2(l_1(x)-l_2(x))^2 \ge 0.$$

Examples: (1) In [25], p. 29, Hernandez gives an example of a metric on $RP(3)$ such that at each point p, there is a unit vector $z(p)$ so that the Ricci tensor is given by

$$Ric_{g_o}(x,x) = (\langle x, z(p)\rangle)^2.$$

Then $0 \le Ric_{g_o} \le 1$, $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$ and the zeroes of Ric_{g_o} are precisely the vectors orthogonal to $z(p)$ in M_p. It is not difficult to see that $\lambda_1(p) = \lambda_2(p) = 0$ and $\lambda_3(p) > 0$ implies that there is a 2-plane $\sigma \subset M_p$ with $K_{g_o}(\sigma) < 0$. (We will see this below.) However, for x with $Ric_{g_o}(x,x) = 0$, $Ric'(x,x) \ge 1/2$ by Proposition 3', so we can perturb this metric by the Ricci tensor to get a metric of positive Ricci curvature. Note that since $b_1(RP(3)) = 0$, if X is a global vector field with $Ric(X,X) \equiv 0$, then $DX \equiv 0$ on RP(3) is impossible.

(2) Let g_2 be any metric on S^2 with $K_{g_2} > 0$. Let g_1 be the usual metric on S^1. Form

$(M,g_o) = (S^1 \times S^2, g_1 \times g_2)$. Then g_o is a metric for M with $\tau_{g_o} > 0$. Let X be the global unit field defined on M by: if $\frac{\partial}{\partial t}$ is a unit field on S^1 trivializing TS^1, let

$$X|_{(p,q)} = (\frac{\partial}{\partial t}|_p, 0_q) \text{ using } M|_{(p,q)} \cong S^1|_p \times S^2|_q .$$

Then X is a global g_o-unit field with $DX \equiv 0$ so the integral curves of X are geodesics and $Ric(X,X) \equiv 0$. Furthermore, if $v \in M_p$

$$Ric(v,v) = 0 \text{ iff } v = t\, X_p \text{ for some } t \text{ in } R.$$

Since $l_1(X) = l_2(X) = 1/2$ on M, and $DX \equiv 0$ implies $(D^*D\, Ric)(X,X) \equiv 0$ by Lemma 1, we have $Ric'(X,X) \equiv 0$. Thus in this case the Ricci variation vanishes identically on the zero set of Ric_{g_o}. Since $b_1(M) = 1$, M does not admit a metric g with $Ric_g \geq 0$, and $Ric_{g|_p} > 0$ for some p in M, let alone a metric with everywhere positive Ricci curvature by the Bochner-Lichnerowicz harmonic theory. Also, since $\pi_1(M) = Z$, by the Myers-Synge Theorem and Theorem 5.1, M does not admit a metric with non-negative Ricci curvature and all Ricci curvatures positive at some point. Thus the fact that $\mathrm{Ric}'(X,X) \equiv 0$ here would seem to provide an "operational verification" of the classical topological restrictions of homology and homotopy on the sign of the curvature of a metric on M. In fact, it follows from our computations below that not only is $\mathrm{Ric}'(X,X) \equiv 0$ for this metric on M, but also $\mathrm{Ric}^t(X,X) \equiv \mathrm{Ric}(X,X) \equiv 0$ for all t for which g(t) is a metric.

If we could find a non-negative Ricci variation with $\mathrm{Ric}'(X,X)|_p > 0$ for some p on M, since

$\lambda_1(X) = \lambda_2(X) = \frac{\tau}{2} \geq \epsilon > 0$ on M, it is not inconceivable

that we could get a metric g with $\mathrm{Ric}_g \geq 0$ and $\mathrm{Ric}_g|_p > 0$ and hence using Theorem 5.1 produce a metric with positive Ricci curvature on M. So it might be that the topology is imposing conditions even on the existence of non-negative variations that are positive at some zero direction. However, as we will see below, it appears to be quite difficult to improve the amount of positive curvature using a non-negative variation.

Considering Proposition 3' again, we see that for $Ric_{g_o} \geq 0$,

$$g(t) = g_o + t(-Ric_{g_o})$$

is a non-negative variation on the zero directions for Ric_{g_o}, and if for all zero directions x, $(D^*D\, Ric)(X,X) > 0$, or $l_1(x) \neq l_2(x)$, then the variation is a positive variation. The second condition suggests that we should algebraically study the Ricci tensor in 3 dimensions and in particular we should explore the implication that $l_1(x) = l_2(x)$ for all zero directions x for Ric_{g_o}.

Example (2) above has already shown that given (M,g_o) with non-negative Ricci curvature and positive scalar curvature, we cannot always produce a metric with positive Ricci curvature. Hence, the analogue of the sectional curvature deformation theorem of Part 1 is false. But, in light of example (2), a natural question to ask is whether the analogous theorem fails to be true only when (M,g_o) is locally isometrically a product manifold.

Since Ric is a symmetric 2-tensor on M_p, we can find orthonormal vectors $\{x,y,z\}$ diagonalizing Ric on M_p so that $Ric(x,x) \leq Ric(y,y) \leq Ric(z,z)$. Also,

$$(3) \qquad \begin{cases} Ric(x,x) = K(x,y) + K(x,z) \\ Ric(y,y) = K(x,y) + K(y,z) \\ Ric(z,z) = K(x,z) + K(y,z) \end{cases} .$$

From (3) it follows that

<u>Lemma 4</u>: Suppose $Ric(x,x) = 0$ and $Ric(y,y) = Ric(z,z) \neq 0$. Then $Ric(y,y) = Ric(z,z) = l_1(x) = l_2(x) = K(y,z) = \frac{\tau(p)}{2}$.

$Ric(v,v) = 0$ iff $v \in Rx$, so $K(x,y) = K(x,z) = 0$.

$v \perp x$ implies $Ric(v,v) = K(y,z) = \tau(p)/2$.

Furthermore, $K(\sigma) = 0$ iff $x \in \sigma$. Finally, $K(\sigma) \geq 0$ for all $\sigma \subset M_p$.

Let us assume in (3) that $Ric(x,x) = 0$. Then we have the following possibilities.

<u>Case (i)</u>: $Ric(y,y) > 0$, $Ric(z,z) > 0$.

Then $Ric(v,v) = 0$ iff $v \in Rx$

$$K(x,y) = -K(x,z)$$

$$K(y,z) > 0.$$

If $Ric(y,y) = Ric(z,z)$, then $K(x,y) = K(x,z) = 0$.

<u>Case (ii)</u>: $Ric(x,x) = Ric(y,y) = 0$, $Ric(z,z) > 0$.

Then $Ric(v,v) \neq 0$ iff $v \in Rz - \{0_p\}$

$Ric(v,v) = 0$ iff $v = ty + sx$, s,t in R

$K(x,z) = K(y,z) > 0$ implies $K(x,y) < 0$.

Thus, there may be 2 zero directions for Ric in M_p iff there are negative sectional curvatures in M_p.

<u>Case (iii)</u>: $Ric(x,x) = Ric(y,y) = Ric(z,z) = 0$ implies

$$K(\sigma) = 0 \text{ for all } \sigma \subset M_p .$$

<u>Definition</u>: (M,g_o) is <u>Ricci product-like</u> iff

(I) $|\tau_{g_o}| > 0$

(II) at all points p in M, one of the eigenvalues of Ric_{g_o} on M_p is zero and the other two eigenvalues are equal (and hence non-zero by (I)).

By connectivity of $M, \tau_{g_o} > 0$ on M, or $\tau_{g_o} < 0$ on M.

A natural question raised by the example of $S^1 \times S^1$ above which motivated this definition is whether (M,g_o) Ricci product-like implies (M,g_o) is locally isometrically a product. At this point we only remark that if (x,y,z) are the usual coordinates on R^3, and if we take $f : R^3 \to R$ to be a smooth function depending only on z, for

$$g = e^{2f}\langle,\rangle_{can}, \text{ we have}$$

$$Ric(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = -2f''(z)e^{-2f} \quad \text{and}$$

$$Ric(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = Ric(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = -(f''(z) + (f'(z))^2)e^{-2f}.$$

Thus, if we put f(x,y,z) = z, we get a metric g with

$$Ric(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = 0, \; Ric(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = Ric(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) < 0.$$

However, in the case of non-negative Ricci curvature, it is not difficult to see that using the warped product construction of Bishop, O'Neill, [9], p. 23, it is not possible to construct from (R,g_{can}) and (R^2,g_{can}) by

In order to even consider this question, we need to understand D*D Ric. As a first step, writing $\langle , \rangle$ for g_o in the computations below

<u>Lemma 7</u>: (Bourguignon, Ehrlich): Let (M, g_o) be Ricci product-like. Let X be a local unit vector field defined on an open set $U \subset M$ so that $Ric(X,X) = 0$ on U. Then

(1) $D_X X \equiv 0$ so the integral curves of X are geodesics,

(2) $\operatorname{div} X = \operatorname{tr}\{v \to D_v X\} = -X(\tau)/\tau$, and

(3) $(dX^\flat)(X,\cdot) = 0$, $(\delta^* X^\flat)(X,\cdot) = 0$.

<u>Remarks</u>: (A) In fact on an n-manifold if Ric_{g_o} diagonalizes globally as

$$\left(\begin{array}{c|cccc} 0 & & 0 & & \\ \hline & \tau/n-1 & & & 0 \\ & & \tau/n-1 & & \\ & & & \ddots & \\ & 0 & & & \tau/n-1 \end{array}\right)\left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} n-1$$

$$\underbrace{\hphantom{\tau/n-1 \quad \tau/n-1 \quad \ddots \quad \tau/n-1}}_{n-1}$$

then (1) $\operatorname{div} X = -\frac{n-1}{2\tau} X(\tau)$

(2) $D_X X = -\frac{n-3}{2\tau}(\operatorname{grad} \tau - X(\tau)X)$

where X is a local unit field with $Ric(X,X) = 0$.

(B) In example (2) above of the product metric on $S^1 \times S^2$, property (1) of Lemma 7 held since $DX \equiv 0$.

<u>Proof of Lemma 7</u>: For p in U, we can find local fields Y and Z so that {X,Y,Z} are orthonormal and diagonalize Ric (this uses Lemma 4). Thus

$$(*)\qquad \begin{cases} \mathrm{Ric}(X,\cdot)^{\#} = 0 \\ \mathrm{Ric}(Y,\cdot)^{\#} = \frac{\tau}{2}Y \\ \mathrm{Ric}(Z,\cdot)^{\#} = \frac{\tau}{2}Z \end{cases}$$

Let $I = D_X(\mathrm{Ric}(X,\cdot)^{\#}) + D_Y(\mathrm{Ric}(Y,\cdot)^{\#}) + D_Z(\mathrm{Ric}(Z,\cdot)^{\#})$,

$II = (D_X\mathrm{Ric})(X,\cdot)^{\#} + (D_Y\mathrm{Ric})(Y,\cdot)^{\#} + (D_Z\mathrm{Ric})(Z,\cdot)^{\#}$, and

$III = \mathrm{Ric}(D_XX,\cdot)^{\#} + \mathrm{Ric}(D_YY,\cdot)^{\#} + \mathrm{Ric}(D_ZZ,\cdot)^{\#}$.

By definition of the derivative of a tensor field,

$$I = II + III.$$

But $I = D_Y(\frac{\tau}{2}Y) + D_Z(\frac{\tau}{2}Z)$ by (*)

$= \frac{1}{2}(Y(\tau)Y + Z(\tau)Z) + \frac{\tau}{2}(D_YY + D_ZZ)$, and by

Lemma 2,

$$II = \frac{1}{2}\,d\tau = \frac{X(\tau)}{2}X + \frac{Y(\tau)}{2}Y + \frac{Z(\tau)}{2}Z$$

so $\frac{\tau}{2}(D_YY + D_ZZ) = \frac{X(\tau)}{2}X + III.$ (**)

Now

$$\mathrm{Ric}(D_X X,\cdot)^{\#} = \mathrm{Ric}(\langle D_X X,Y\rangle Y + \langle D_X X,Z\rangle Z,\cdot)^{\#}$$

(since $\langle D_X X,X\rangle = \frac{1}{2}X(1) = 0$)

$$= \langle D_X X,y\rangle \mathrm{Ric}(Y,\cdot)^{\#} + \langle D_X X,Z\rangle \mathrm{Ric}(Z,\cdot)^{\#}$$

$$= \frac{\tau}{2}(\langle D_X X,Y\rangle Y + \langle D_X X,Z\rangle Z) = \frac{\tau}{2} D_X X.$$

Similarly,

$$\mathrm{Ric}(D_Y Y,\cdot)^{\#} = \frac{\tau}{2}\langle D_Y Y,Z\rangle Z$$

and

$$\mathrm{Ric}(D_Z Z,\cdot)^{\#} = \frac{\tau}{2}\langle D_Z Z,Y\rangle Y.$$

Also,

$$D_Y Y = \langle D_Y Y,X\rangle X + \langle D_Y Y,Z\rangle Z$$

and

$$D_Z Z = \langle D_Z Z,X\rangle X + \langle D_Z Z,Y\rangle Y.$$

Substituting in (**) we obtain

$$\frac{\tau}{2}\langle D_Y Y+D_Z Z,X\rangle X + \frac{\tau}{2}(\langle D_Y Y,Z\rangle Z + \langle D_Z Z,Y\rangle Y)$$

$$= \frac{X(\tau)}{2}X + \frac{\tau}{2}D_X X + \frac{\tau}{2}(\langle D_Y Y,Z\rangle Z + \langle D_Z Z,Y\rangle Y)$$

and hence

$$-\frac{\tau}{2}\,\mathrm{div}\,X\cdot X = \frac{X(\tau)}{2}\cdot X+\frac{\tau}{2}D_X X.$$ Since $\langle D_X X,X\rangle = 0$

and $|\tau| > 0$, this yields $D_X X \equiv 0$ and $\mathrm{div}\,X = \frac{-X(\tau)}{\tau}$.

Q. E. D.

On a disk U for which the nullity foliation is trivial, there ar[e] exactly two choices for a local unit field X with $Ric(X,X) = 0$, namely X and -X.

It then follows that if (M,g_o) is Ricci product-like, that we can find a local unit vector field X on M satisfying (1) - (3) of Lemma 7 and such that locally

$$Ric_{g_o} = \frac{\tau_{g_o}}{2}(g_o - X^\flat \otimes X^\flat), \text{ where } \flat = \flat_{g_o} .$$

At this point we remark that in 3 dimensions, but not in higher dimensions, the notions of nullity and Ric_{g_o} being Ricci product-like coincide. Explicitly, fixing a metric g_o for M

<u>Definition</u>: x in M_p is a <u>nullity vector</u> iff $R(x,y)z = 0$ for all y and z in M_p. Define the <u>nullity space at</u> p by

$$Null(p) := \{x \text{ in } M_p : R(x,y)z = 0 \text{ for all } y,z \text{ in } M_p\}$$

and the nullity at p by $n(p) := \dim Null(p)$. Define the <u>conullity space at</u> p by

$$CoNull(p) := \{R(x,y)z \ ; \ x,y,z \text{ in } M_p\}.$$

Then $M_p = Null(p) \oplus CoNull(p)$. These concepts were defined by Chern and Kuiper [16], and have been subsequently studied by Maltz [28], Gray [21], Rosenthal [31] and Abe [1], among others.

In particular, if the nullity $n(p)$ is constant on M, then the nullity subspace defines an integrable distribution and hence, a foliation with complete, totally geodesic leaves. We will thus refer to the nullity and conullity distributions below. In our case, (1) of Lemma 7 expresses the fact that the nullity distribution is totally geodesic. It is, of course, integrable being one dimensional.

In [31], for complete simply connected manifolds N^m with a globally constant nullity n satisfying $0 < n \leq m-3$ a splitting theorem for N as

$$N^m = B^n \times C^{m-n}$$

with B flat is obtained under additional hypotheses on the curvature tensor. In particular, this result applies to constant 1 dimensional nullity if dim N > 3, but not for dim N = 3.

It is not difficult to see that

Lemma 8: (M^3, g_o) has constant nullity 1 iff g_o is Ricci product-like for M.

Thus, asking whether (M^3, g_o) with constant nullity 1 is locally or globally a Riemannian product is equivalent to asking whether (M, g_o) with g_o Ricci-product-like for M is in fact locally or globally a Riemannian product. Here we only state the next theorem. Details will appear elsewhere.

<u>Theorem 9</u>: (Bourguignon, Ehrlich) Let (M^3, g_o) be a complete simply connected 3-manifold of constant nullity 1 such that the conullity distribution is integrable. Then M^3 admits a global parallel field of length 1 and thus splits isometrically as $(M, g_o) = (R, g_1) \times (M^2, g_2)$ where both factors are complete and $M^2 = S^2$ or R^2. In particular, M is non-compact.

<u>Corollary 10</u>: S^3 does not admit a codimension 1 foliation which arises from an <u>integrable</u> conullity distribution of a curvature tensor defined by a metric on S^3.

<u>Problem</u>: Does S^3 admit a Ricci product-like metric? If so, this would show that Ricci product-like does not imply isometric to a product.

Having digressed into the connection with the nullity in dimension 3, we return to study of Ric' for Ricci-product-like metrics. An easy calculation shows that

<u>Lemma 11</u>: Let $g(t) = g_o + t(-Ric_{g_o})$ where

$$Ric_{g_o} = \frac{\tau}{2}(g_o - X^\flat \otimes X^\flat) \text{ as above.}$$

Then $Ric'(X,X) = 1/2(D^*D\ Ric)(X,X) = \frac{\tau}{2}\|D\ X\|^2$.

As a corollary we obtain

<u>Proposition 12</u>: Let (M,g_o) be Ricci product-like. Either

(1) (M,g_o) is locally isometric to a product, or

(2) if $\tau_{g_o} > 0$, M admits a non-negative Ricci variation that is positive on the zero direction for some p in M, or

(3) if $\tau_{g_o} < 0$, M admits a non-positive Ricci variation that is negative on the zero direction for some p in M.

In light of Aubin's Theorem, we thus ask

<u>Question</u>: If (M,g_o) is Ricci-product-like, is it true that either

(1) (M,g_o) is locally a product, or

(2) M admits a metric of positive Ricci curvature if $\tau_{g_o} > 0$, or

(3) M admits a metric of negative Ricci curvature if $\tau_{g_o} < 0$.

We do not know whether this is true or false.

<u>Remark</u>: Observe that if (M,g_o) is Ricci product-like with X as above, and Y and Z are local vector fields so that $\{X,Y,Z\}$ are g_o-orthonormal, then the variation $g(t) = g_o+t(-Ric_{g_o})$ is the same as the variation

$$g(t) = (1 - t\,\tfrac{\tau}{2})g_o + t\,\tfrac{\tau}{2}\,X^\flat \otimes X^\flat .$$

Then $g(t)(X,X) = 1$, $g(t)(Y,Y) = g(t)(Z,Z) = 1 - t\,\frac{\tau}{2}$, and $\{X,Y,Z\}$ are $g(t)$ orthogonal.

A conformal variation amounts to declaring $\{(1+t\,f)X,\ (1+t\,f)Y,\ (1+t\,f)Z\}$ to be an orthonormal basis at time t for some function $f : M \to R$. Here, we are studying in a particular case the "next simplest" variation where we declare

$$\{X, (1+t\,f)Y,\ (1+t\,f)Z\}$$

to be orthonormal at time t.

We now want to compute $Ric^t(v,v)$ for (M,g_o) Ricci product-like and for

$$g(t) = g_o + t(-Ric_{g_o}) = (1-t\,\tfrac{\tau}{2})g_o + \tfrac{t\tau}{2}\,X^\flat \otimes X^\flat$$

We give a number of computational lemmas, reminding the reader of the various factors of 1/2 ommited as given in the "Notational Conventions" section. Fix local fields Y,Z so that $\{X,Y,Z\}$ are g_o-orthonormal. We write $\langle , \rangle$ for g_o below.

<u>Lemma 13</u>: $$(dX^\flat)(Y,D_YX) + (dX^\flat)(Z,D_ZX) = ((dX^\flat)(Y,Z))^2 = \|dX^\flat\|^2$$

<u>Proof</u>:

$$(dX^\flat)(Y,D_YX) = \langle D_YX, D_YX\rangle - \langle D_{D_YX}X, Y\rangle$$

Now use $D_Y X = \langle D_Y X, Y\rangle Y + \langle D_Y X, Z\rangle Z$ to get

$$(dX^\flat)(Y, D_Y X) = (\langle D_Y X, Y\rangle)^2 + (\langle D_Y X, Z\rangle)^2$$
$$- \langle D_Y X, Y\rangle\langle D_Y X, Y\rangle - \langle D_Y X, Z\rangle\langle D_Z X, Y\rangle$$
$$= (\langle D_Y X, Z\rangle)^2 - \langle D_Y X, Z\rangle\langle D_Z X, Y\rangle$$

Similarly,

$$(dX^\flat)(Z, D_Z X) = (\langle D_Z X, Y\rangle)^2 - \langle D_Z X, Y\rangle\langle D_Y X, Z\rangle$$

so that $(dX^\flat)(Y, D_Y X) + (dX^\flat)(Z, D_Z X) =$

$$(\langle D_Z X, Y\rangle - \langle D_Y X, Z\rangle)^2 = ((dX^\flat)(Y,Z))^2 = \|dX^\flat\|^2.$$

Q. E. D.

<u>Lemma 14</u>: (Bochner Lemma): With the above assumptions on g_o, Ric_{g_o}, and X, then

$$- X(\mathrm{div}\ X) = \|D\ X\|^2 - \|dX^\flat\|^2.$$

<u>Proof</u>: Modify the argument of [15] p. 121, the $dX^\flat$ in our case coming from the fact that in [15], X was locally the gradient of a function so $\langle D_Y X, Z\rangle = \langle D_Z X, Y\rangle$ which does not hold in our case unless the <u>conullity distribution is integrable</u> and Theorem 9 applies.

<u>Theorem 15</u>:

$$\delta^*(d\tau)(x,x) = -X(\tau)\mathrm{div}\ X + \tau\|DX\|^2 - \tau\|dX^\flat\|^2$$

<u>Proof</u>: $X(\tau) = -\tau \operatorname{div} X$ by Lemma 7, so

$$\delta^*(d\tau)(X,X) = X(X(\tau)) - (D_X X)(\tau)$$

$$= X(X(\tau)) = -X(\tau)\operatorname{div} X - \tau\, X(\operatorname{div} X)$$

so done by Lemma 14.

Q. E. D.

For the variation $g(t) = g_o + t(-\mathrm{Ric}_{g_o})$, we now sketch the computation of $\mathrm{Ric}^t(X,X)$.

Now $\mathrm{Ric}^t(X,X) = (1 - \frac{t\tau}{2})^{-1}(\langle R^t(x,y,y)x\rangle_t + \langle R^t(x,z)z,x\rangle_t)$.

Using the standard formula expressing the relation between the metric and its Levi-Civita connection, for any v,w in M_p extending w arbitrarily to a local field W,

<u>Lemma 16</u>: $2(1 - \frac{t\tau}{2})(D_v^t W - D_v W) =$

$$- \frac{t^2\tau}{4} X|_p(\tau)(\langle v,w\rangle - \langle v,X|_p\rangle\langle w,X|_p\rangle)X|_p - \frac{t^2\tau^2}{4}(\delta^* X^\flat)(v,w)\, X|_p$$

$$- \frac{t}{2}[v(\tau)(w - \langle w,X|_p\rangle X|_p) + w(\tau)(v - \langle v,X|_p\rangle X|_p) - (\langle v,w\rangle - \langle v,X|_p\rangle\langle w,X|_p\rangle)\operatorname{grad}\tau$$

$$+ \frac{t\tau}{2}[(\delta^* X^\flat)(v,w)\, X|_p + \langle w,X|_p\rangle(dX^\flat)(v,\cdot)^\# + \langle v,X|_p\rangle(dX^\flat)(w,\cdot)^\#].$$

In particular,

<u>Lemma 17</u>: For $v \in M_p$,

$$2(1 - \frac{t\tau}{2})D^t_v X = 2(1 - \frac{t\tau}{2})D_v X - \frac{t}{2} X_{|p}(\tau)(v - \langle v, X_{|p}\rangle X_{|p})$$

$$+ \frac{t\tau}{2}(dX^\flat)(v,\cdot)^\#.$$

Now

$$\langle R^t(y,x)x,y\rangle_t = \langle D^t_Y D^t_X X, Y\rangle_t - \langle D^t_X D^t_Y X, Y\rangle_t$$

$$- \langle D^t_{[Y,X]}X, Y\rangle_t$$

$$= \langle D^t_Y X, D^t_X Y\rangle_t - X \cdot \langle D^t_Y X, Y\rangle_t - \langle D^t_{[Y,X]}X, Y\rangle_t$$

since $D^t_X X \equiv 0$.

Lemma 18:

(1) $$D^t_Y X = D_Y X - \frac{t}{4(1-\frac{t\tau}{2})} X(\tau)Y + \frac{t\tau}{4(1-\frac{t\tau}{2})}(dX^\flat)(Y,\cdot)^\#$$

(2) $$\langle D^t_Y, D^t_X Y\rangle_t = (1 - \frac{t\tau}{2})\langle D_Y X, D_X Y\rangle$$

$$- \frac{t}{4} X(\tau)\langle [Y,X], Y\rangle + \frac{t\tau}{4}(dX^\flat)(Y, D_Y X)$$

$$+ \frac{t\tau}{4}(dX^\flat)(Y, D_X Y) + \frac{t^2\tau^2}{16(1-\frac{t\tau}{2})^2}\|(dX^\flat)(Y,\cdot)^\#\|^2$$

$$+ \frac{t^2}{16(1-\frac{t\tau}{2})^2}(X(\tau))^2$$

(3) $$\langle D^t_Y X, Y\rangle_t = (1-\frac{t\tau}{2})\langle D_Y X, Y\rangle - \frac{t}{4} X(\tau)$$

(4) $$X \cdot \langle D^t_Y X, Y\rangle_t = (1 - \frac{t\tau}{2})X \cdot \langle D_Y X, Y\rangle$$

$$- \frac{t}{4}\, \delta^*(d\tau)(X,X) + \frac{t}{2} X(\tau)\langle D_Y X, Y\rangle$$

(5) $$\langle D^t_{[Y,X]}X,Y\rangle_t = (1 - \tfrac{t\tau}{2})\langle D_{[Y,X]}X,Y\rangle$$

$$- \tfrac{t}{4}\, X(\tau)\langle [Y,X],Y\rangle + \tfrac{t\tau}{4}(dX^b)(D_Y X,Y)$$

$$- \tfrac{t\tau}{4}(dX^b)(D_X Y,Y)$$

Corollary 19: $$\langle R^t(x,y)y,x\rangle_t = (1 - \tfrac{t\tau}{2})\langle R(x,y)y,x\rangle$$

$$+ \tfrac{t\tau}{2}(dX^b)(y,D_y X) + \frac{t\ X(\tau)}{2}\langle D_y X,y\rangle + \tfrac{t}{4}\ \delta^*(d\tau)(X,X)$$

$$+ \frac{t^2\tau^2}{16(1-\frac{t\tau}{2})^2}\|(dX^b)(y,\cdot)^\#\|^2 + \frac{t^2(X(\tau))^2}{16(1-\frac{t\tau}{2})^2}$$

and a similar expression of course holds for $\langle R^t(x,z)z,x\rangle_t$.

Now $\|(dX^b)(y,\cdot)^\#\|^2 = \|(dX^b)(z,\cdot)^\#\|^2 = ((dX^b)(y,z))^2 = \|dX^b\|^2$

so that

$$\langle R^t(x,y)y,x\rangle_t + \langle R^t(x,z)z,x\rangle_t = (1-\tfrac{t\tau}{2})\mathrm{Ric}(x,x)$$

$$+ \tfrac{t}{2}[\tau(dX^b)(y,D_y X) + \tau(dX^b)(z,D_z X) + X(\tau)\,\mathrm{div}\, X + \delta^*(d\tau)(X,X)]$$

$$+ \frac{t^2(X(\tau))^2}{8(1-\frac{t\tau}{2})^2} + \frac{t^2\tau^2}{8(1-\frac{t\tau}{2})^2}\|dX^b\|^2 .$$

Using Lemma 15, we finally obtain

Proposition 20: With everything as above,

$$\mathrm{Ric}^t(X,X) = \frac{t\tau}{2(1-\frac{t\tau}{2})}\|DX\|^2 + \frac{t^2}{8(1-\frac{t\tau}{2})^2}[(X(\tau))^2+\tau^2\|dX^b\|^2]$$

Hence, if (M,g_o) is Ricci product-like, on the zero direction X for $Ric_{g_o} = \tau/2(g_o - X^\flat \otimes X^\flat)$ we can certainly find a $t_o > 0$ such that if $\|DX\|_{|p} > 0$, then for all t with $0 < t \leq t_o$ we have either

(i) if $\tau_{g_o} > 0$, then $Ric^t(X,X) \geq 0$ and $Ric^t(X_{|p},X_{|p}) > 0$,

or

(ii) if $\tau_{g_o} < 0$, then $Ric^t(X,X) \leq 0$ and $Ric^t(X_{|p},X_{|p}) < 0$.

However, it is necessary to determine whether we can find such a $t_o > 0$ without introducing any negative Ricci curvature. We distinguish 3 cases. Let $\lambda_1^t,\lambda_2^t,\lambda_3^t$ be the eigenfunctions of Ric^t.

Case (A): $\|DX\|_{|p} > 0$.

Then since $\lambda_2,\lambda_3 \geq \epsilon > 0$ on M by compactness, it is clear we can find a $t_o(p) > 0$ so that

$\lambda_1^t(p),\lambda_2^t(p),\lambda_3^t(p) > 0$ for $0 < t \leq t_o(p)$.

Case (B): $\|DX\|_{|p} = 0$ but $DX \equiv 0$ in an open neighborhood of p. Then we can find a $t_o(p) > 0$ so that $Ric^t(x,v) = 0$ for all v in M_p and for all t with $0 \leq t \leq t_o(p)$. Then $\lambda_1^t(p) = 0$ for $0 \leq t \leq t_o(p)$ and these points are no problem.

Case (C): $\|DX\|_{|p} = 0$, but we do not know that $DX \equiv 0$ in a neighborhood of p.

These points are the "bad" points. In order to see what happens here, we must look at the map $t \to Ric^t$ in a neighborhood of the zero direction X_p in M_p. We use a method similar to the method used in [12] for the sectional curvature. Let Y and Z be local fields near p so that $\{X,Y,Z\}$ are g_0-orthonormal. Define a neighborhood of X_p by

$$v_{a,b} := X_p + a\, Y_p + b\, Z_p, \quad a,b \text{ in } R.$$

We have a function

$$H(t,a,b) := Ric^t(v_{a,b}, v_{a,b})$$

and we want to know if we can find a $t_0(p) > 0$ so that $H(t,a,b) \geq 0$ for all t with $0 \leq t \leq t_0(p)$, and <u>for all</u> a,b. A necessary condition is that if we consider the 3 x 3 matrix defined by the second order terms in t, a and b in $H(t,a,b)$ that this matrix be positive definite.

We need the following computation, the proof of which is laborious but straightforward so omitted in detail.

<u>Lemma 21</u>: Suppose $DX_{|p} = 0$. Let $v \in M_p$. Then

$$Ric^t(v,v) = ((\langle v,z\rangle)^2 + (\langle v,y\rangle)^2)\left[\frac{\tau(p)}{2}\left(1 - \frac{t\tau}{2}\right) + t\left(\frac{\Delta\tau(p)}{2} - \frac{t}{4} z(\tau)y(\tau)\right)\right]$$

$$+\ t\langle v,x\rangle\langle v,z\rangle\left[\frac{\tau(p)}{2}\, y((dX^\flat)(Y,Z)) + \frac{1}{2}\,\delta*(d\tau)(z,x)\right]$$

$$+\ t\langle v,x\rangle\langle v,y\rangle\left[\frac{\tau}{2}\, z((dX^\flat)(Y,Z)) + \frac{1}{2}\,\delta*(d\tau)(y,x)\right].$$

Idea of the proof of Lemma 21: As a first step, we need

<u>Lemma 22</u>: If $D X_p = 0$, then for all v in M_p,

$$\langle R^t(v,x)x,v\rangle_t = 0.$$

<u>Proof</u>: Extend v to a local vector field V.

$$\langle R^t(v,x)x,v\rangle_t = X \cdot \langle D^t_V X, V\rangle_t - \langle D^t_V X, D^t_V X\rangle_t$$

$$- \langle D^t_{[v,x]} X, V\rangle_t \,.$$

Since $D X_{|p} = 0$, $\langle D^t_V X, D^t_V X\rangle_{t|p} = \langle D^t_{[V,X]} X, V\rangle_{t|p} = 0$

$$\langle D^t_V X, V\rangle_t = \langle D_V X, V\rangle - \frac{t}{4(1-\frac{t\tau}{2})} X(\tau)(\langle v,v\rangle - (\langle v,x\rangle)^2)$$

$$+ \frac{t\tau}{4(1-\frac{t\tau}{2})} (dX^\flat)(V,V)$$

Since $(dX^\flat)(V,V) \equiv 0$ near p, $X_{|p}(\tau) = 0$ by Lemma 7, and by Lemma 15, $\delta^*(d\tau)(X_{|p}, X_{|p}) = 0$, we have

$$X \cdot \langle D^t_V X, V\rangle_t = X\langle D_V X, V\rangle - \frac{t}{4(1-\frac{t\tau}{2})} \delta^*(d\tau)(x,x)$$

$$\cdot (\langle v,v\rangle - (\langle v,x\rangle)^2)$$

$$= X \cdot \langle D_V X, V\rangle .$$

Hence, $\langle R^t(v,x)x,v\rangle_t = \langle R(v,x)x,v\rangle = 0$.

Q. E. D.

By Lemma 22, $Ric^t(v,v) = \dfrac{\langle R^t(y,v)v,y\rangle_t}{(1 - \frac{t\tau}{2})} + \dfrac{\langle R^t(z,v)v,z\rangle_t}{(1 - \frac{t\tau}{2})}$.

Now $\langle R^t(y,v)v,y\rangle_t = 2\langle v,x\rangle_t\langle v,z\rangle_t\langle R^t(y,x)z,y\rangle_t$

$$+ (\langle v,z\rangle_t)^2\langle R^t(y,z)z,y\rangle_t$$

and $\langle R^t(y,x)z,y\rangle_t$, etc. can be calculated from previous lemmas.

Q. E. D.

Let $c(t) = \frac{\tau}{2}\, z((dX^\flat)(Y,Z)) + \frac{1}{4}\, \delta*(d\tau)(y,x)$

and

$$d(t) = \frac{\tau}{2}\, y((dX^\flat)(Y,Z)) + \frac{1}{4}\, \delta*(d\tau)(z,x)$$

<u>Proposition 23</u>: The matrix of second order terms in $H(t,a,b)$ is given by

	t	a	b
t	0	c(t)	d(t)
a	c(t)	$\tau/2$	0
b	d(t)	0	$\tau/2$

Thus the 2 x 2 minor in t and a has determinant $-(c(t))^2$. Hence the matrix is <u>not</u> positive semi-definite unless $c(t) = d(t) = 0$. Hence, for any small $t > 0$ we are introducing negative Ricci curvatures with $Ric^t(v_{a,b},v_{a,b})$ in a neighborhood of X_p where p is of type (C).

<u>Remarks</u>: (1) By Lemma 21 for points p of type (C), we can define a function $S_1(M,g_o)|_p \xrightarrow{t} R$ on the g_o-unit vectors at p as follows: given a unit vector v in M_p, choose $t(v) > 0$ so that $0 \le t \le t(v)$ implies $Ric^t(v,v) \ge 0$. The difficulty which is made explicit by Proposition 23 is that this function $v \to t(v)$ need not be continuous or even semi-continuous so we <u>cannot</u> use the compactness of $S_1(M,g_o)|_p$ to choose a $t_o(p) > 0$ so that $t(v) \ge t_o(p)$ for all v.

(2) Even if we assume that $X(\tau) = 0$ or even τ is constant, we still have the directional derivatives

$$y((dX^\flat)(Y,Z)) \quad \text{and} \quad z((dX^\flat)(Y,Z))$$

in Proposition 23 in the direction of the <u>conullity</u> distribution. Since in our hypothesis there is rigidity only in the X direction, there is no reason to believe that these terms should vanish. It would appear that even in 3-dimensions, the curvature tensor being a second order derivative of the metric is not strong enough to control D X which is a first order derivative of the metric. In order to get a positive semi-definite matrix in Proposition 23 we must assume that DX = 0 near p, but in this case we have a local product and the variation does nothing to increase the curvature in the X direction.

(3) Even if $\|DX\| > 0$ in $M - \{p_o\}$, but $DX|_{p_o} = 0$, we can not find a $t > 0$ so that $Ric^t|_{p_o} > 0$ by Proposition 23, even though for each q in $M - \{p_o\}$ we can find a $t(q)$ so that

$$0 < t \le t(q) \quad \text{implies} \quad Ric^t|_q > 0.$$

We can even make the map $q \to t(q)$ continuous in $M - \{p_o\}$ near p_o. Evidently, $t(q) \to 0$ as $q \to p_o$.

There is another "natural" tensor $*R$ defined on (M,g_o) as follows. Let e_1, e_2 and e_3 be an ordered orthonormal basis for M_p. As usual we will write τ for τ_{g_o}, R for R_{g_o}, etc. Define $* : M_p \to \Lambda^2(M_p)$ by setting

$$*e_1 = e_2 \wedge e_3$$

$$*e_2 = e_3 \wedge e_1$$

$$*e_3 = e_1 \wedge e_2$$

and extend by linearity to M_p. Define $*R$ by

$$(*R)(v,w) := \tilde{R}(*v,*w)$$

where $\tilde{R} : \Lambda^2(M_p) \otimes \Lambda^2(M_p) \to R$ is defined from g_o and R by

$$R(x\wedge y, z\wedge w) := g_o(R(x,y)z,w).$$

For instance,

$$(*R)(e_1,e_2) = R(e_2,e_3,e_3,e_1) .$$

Lemma 24: (1) $*R = -\tau/2\, g_o + \text{Ric}$

(2) if $\text{Ric} = \tau/2(g_o - X^\flat \otimes X^\flat)$ where $g_o(X,X) \equiv 1$, then

$$*R = -\tau/2\, X^\flat \otimes X^\flat$$

Thus the variation $g(t) = g_o + t{*}R_{g_o}$ for (M,g_o) Ricci product-like is the variation

$$g(t) = g_o - t\tau/2\, X^\flat \otimes X^\flat.$$

If Y and Z are local fields so that $\{X,Y,Z\}$ are g_o-orthonormal, then

$$g(t)(X,X) = 1 - t\tau/2,\ g(t)(Y,Y) = g(t)(Z,Z) = 1$$

so this deformation is only changing the length of the local orthonormal frame in the zero direction of Ric_{g_o}. We write $\langle,\rangle$ for g_o in the following formulas as usual.

Lemma 25: Let $g(t) = g_o + t\, f\, X^\flat \otimes X^\flat$ where X is a global unit vector field with $D_X X \equiv 0$ and $f : M \to R$ is a C^1 function. Then for any vector fields U and V on M,

$$D^t_u V - D_u V = \frac{-t^2 f}{2(1+t\, f)}(U(f)\langle V,X\rangle + V(f)\langle U,X\rangle - X(f)\langle U,X\rangle\langle V,X\rangle)\cdot X$$

$$+ \frac{t}{2}(U(f)\langle V,X\rangle X + V(f)\cdot\langle U,X\rangle\cdot X - \langle U,X\rangle\langle V,X\rangle\cdot \text{grad } f)$$

$$- \frac{t^2 f^2}{4(1+t\, f)}((\delta * X^\flat)(U,V) + (dX^\flat)(V,X)\cdot\langle U,X\rangle + (dX^\flat)(U,X)\cdot\langle V,X\rangle)\cdot X$$

$$+ \frac{tf}{2}((\delta * X^\flat)(U,V)\cdot X + \langle U,X\rangle\cdot(dX^\flat)(V,\cdot)^\# + \langle V,X\rangle(dX^\flat)(U,\cdot)^\#).$$

Using Lemma 25, it is possible to show that

<u>Proposition 26</u>: With everything as above,

$$\mathrm{Ric}^t(X,X) = \frac{t}{2}\Delta\tau - \frac{t\tau}{2}\|DX\|^2 + \frac{t}{4}\delta^*(d\tau)(X,X)$$

$$+ \frac{t^2}{16(1-\frac{t\tau}{2})}(\|\mathrm{grad}\ \tau\|^2 + (X(\tau))^2) + \frac{t^2\tau^2}{8}\|dX^\flat\|^2$$

Thus this variation for which

$$g(t)(X,X) = 1 - t\tau/2,\ g(t)(Y,Y) = g(t)(Z,Z) = 1$$

$$\{X,Y,Z\} \text{ remain orthogonal}$$

is not as well adapted to the "geometry" of the Ricci product-like metric g_o as is the variation $g(t) = g_o + t(-\mathrm{Ric}_{g_o})$ for which

$$g(t)(X,X) = 1,\ g(t)(Y,Y) = g(t)(Z,Z) = 1 - \frac{t\tau}{2}$$

$\{X,Y,Z\}$ remain orthogonal .

Chapter 8: A Converse in 3-dimensions to a Lemma of Berger

In [4], Berger proved the following:

Lemma: Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds. Let $(M,g) = (M_1 \times M_2, g_1 \times g_2)$ be the Riemannian product manifold. Let $g(t)$ be an deformation with $g(0) = g$. Suppose $K'(\sigma) \geq 0$ for all mixed 2-planes σ. Then $K' \equiv 0$ on all mixed 2-planes. (See [11] for a definition of mixed 2-plane).

In 3-dimensions we have seen in Chapter 7, part 1, that if a compact (M^3, g_o) satisfies $K_{g_o} \geq 0$, $Ric_{g_o} > 0$, then M admits a metric of everywhere positive sectional curvature. We remarked while discussing "rigidity" in Chapter 7, part 2, that our proof must fail for topological reasons for the product manifold $S^1 \times S^2$. Here at each point there is a zero eigenvector for the Ricci tensor and by Berger's lemma, any non-negative deformation at first order of the product metric must vanish identically on $K_{g_o}^{-1}(0)$ at first order.

It is then natural given a compact manifold (M^3, g_o) with certain curvature properties of the product metric on $S^1 \times S^2$ and such that under all perturbations any non-negative variation at first order on the zero set vanishes identically on the zero set at first order (as in Berger's

Lemma), to ask if (M^3, g_o) is locally a product.

Let $(M,g) = (S^1 \times S^2, g_1 \times g_2)$ where g_1 is the usual metric on S^1 and g_2 is any metric for S^2 with everywhere positive Gaussian curvature, then $K_g \geq 0$, $\tau_g > 0$, and there is a zero 2-plane $\sigma \subset M_p$, of the sectional curvature function $K_g : G_2(M) \to R$, for each p in M.

Proposition 1: Let g_o be any metric on $S^1 \times S^2$ with $K_{g_o} \geq 0$. Then there does not exist any point p in $S^1 \times S^2$ with

$$K_{g_o}\Big|_{\pi^{-1}(p)} > 0.$$

where $\pi : G_2(M) \to M$ is the projection map of the Grassman bundle of 2-planes in TM.

Proof: If so, then $Ric_{g_o} \geq 0$ and $Ric_{g_o|p} > 0$ so $S^1 \times S^2$ admits a metric of positive Ricci curvature by Theorem 5.1, which is impossible since $\pi_1(S^1 \times S^2) = Z$.

Q. E. D.

In light of Proposition 1, we consider the set

$\mathcal{P} := \{(M^3, g_o);$ M compact, $K_{g_o} \geq 0$, $\tau_{g_o} > 0$, and there exists a zero 2-plane $\sigma \subset M_p$ for all p in M$\}$.

($\mathcal{P}$ is for product.)

<u>Definition</u>: Let $(M^3, g_o) \in \mathcal{P}$. We say g_o is a <u>critical metric</u> for M iff the following holds: for all symmetric 2-tensors h on M, let $g_h(t) = g_o + t\,h$. Suppose $g_h(t) = g_o + t\,h$ satisfies for all σ in $G_2(M)$ that $K_{g_o}(\sigma) = 0$ implies $K'_h(\sigma) \geq 0$. Then we require that $K'_h \equiv 0$ on $K^{-1}_{g_o}(0)$.

(That is, every non-negative variation at first order on the zero set of g_o must vanish identically at first order on the zero set.)

Then on $\mathcal{P}$ we have the following converse to Berger's Lemma:

<u>THEOREM</u>: Suppose $(M^3, g_o) \in \mathcal{P}$ and g_o is a critical metric for M^3. Then M is locally isometrically a product.

First we restate several lemmas from Chapter 7.

<u>Lemma 1</u>: Let x,v,w be g_o-orthonormal vectors at p. Then

$$(D^*D\ \mathrm{Ric})(x,x) = (D^*D\ R)(v,x,x,v) + (D^*D\ R)(w,x,x,w)$$

<u>Lemma 2</u>: Given (M, g_o) with $K_{g_o} \geq 0$. Let p in M.

Then either

(1) $Ric_{g_o}(v,v) > 0$ for all $v \neq 0$ in M_p,

(2) there exists $x \neq 0$ in M_p with $Ric_{g_o}(x,x) = 0$

and

$$Ric_{g_o}(v,v) = 0 \quad \text{iff} \quad v \in Rx,$$

(3) $Ric_{g_o}(v,v) = 0$ for all v in M_p.

Proof of Theorem: Let $(M,g_o) \in \mathcal{P}$. Given any p in M, there exists a two-plane $\sigma \in G_2(M)|_p$ with $K_{g_o}(\sigma) = 0$. Choose a g_o-orthonormal basis $\{x,y\}$ for σ with $Ric_{g_o}(x,y) = 0$. Let $g(t) = g_o + t(-Ric_{g_o})$. From Chapter 7, this is a non-negative variation at first order. Hence,

$K'(\sigma) = 1/2(D^*DR)(x,y,y,x) + Ric_{g_o}(x,x)Ric_{g_o}(y,y) = 0$.

Hence, $Ric_{g_o}(x,x)Ric_{g_o}(y,y) = 0$. Since $\tau_{g_o}(p) > 0$ by hypothesis, assume, say, that $Ric_{g_o}(x,x) = 0$, $Ric_{g_o}(y,y) \neq 0$. Further, by Lemma 2, Rx can be the only zero line for Ric_{g_o} in M_p. We have thus shown in combination with Chapter 7 that $(M,g_o) \in \mathcal{P}$ implies (M,g_o) is Ricci product-like.

Now let X be a local g_o-unit field with $Ric_{g_o}(X,X) = 0$, and

$$Ric_{g_o} = \tau_{g_o}/2(g_o - X^\flat \otimes X^\flat)$$

as in Chapter 7. Then, by Chapter 7,

$$(D^*D\ Ric)(X,X) = \tau_{g_o} \|DX\|^2.$$

But now take local fields Y and Z orthonormal to X. By Lemma 1,

$$\tau_{g_o} \|DX\|^2 = (D^*D\ Ric)(X,X) = (D^*D\ R)(Y,X,X,Y) + (D^*DR)(Z,X,X,Z).$$

But $K(X,Y) = 0$ from our analysis of the curvature tensor in Chapter 7, so

$$K'(X,Y) = 1/2(D^*D\ R)(X,Y,Y,X) = 0$$

since $(M,g_o) \in \mathcal{P}$. Thus, $(D^*D\ R)(X,Y,Y,X) = 0$. Similarly, $K(X,Z) = 0$ implies that $(D^*D\ R)(X,Z,Z,X) = 0$. Hence, $\tau_{g_o} \|DX\|^2 = 0$ locally. But then since $\tau_{g_o} > 0$, this forces $DX \equiv 0$ locally, and so we are done by the DeRham Decomposition Theorem.

Q. E. D.

Chapter 9: Some Miscellaneous Results.

The Sectional Curvature Formula for Doubly Warped Product Manifolds

The purpose of this part is to sketch the derivation of the sectional curvature formula for a doubly warped product manifold

$$(M,\langle,\rangle),\ M = M_1 \times M_2$$

where the metric $\langle,\rangle$ for M is defined as follows. Let

$$g : M_1 \to R > 0$$

and

$$f : M_2 \to R > 0$$

be two smooth functions. Let

$$p_i : M \to M_i \qquad i = 1,2$$

be the projection maps associated to the Cartesian product and let

$$P_i : TM \to TM_i \qquad i = 1,2$$

be their differentials, i.e., $P_i := p_{i*}$. Let $\bar{g} = g \circ p_1$ and $\bar{f} = f \circ p_2$. Then define the metric on M by

$$\langle,\rangle := (\bar{f})^2\langle,\rangle_1 + (\bar{g})^2\langle,\rangle_2 \ .$$

In [9] this was studied in the case $f(q) \equiv 1$ for all q in M_2 in the framework of Riemannian submersions.

Identify $x \in T_pM_1$ with $\bar{x} := (x,o_q) \in T_{(p,q)}M$ for all q in M_2 and identify $v \in T_qM_2$ with $\bar{v} := (o_p, v) \in T_{(p,q)}M$ for all p in M_1. We use X,Y, and Z for vector fields on M_1 and U,V and W for vector fields on M_2 below. Note that $[\bar{X},\bar{U}] = 0$ for any such vector fields coming from different factors of the product manifold M. Define $\text{grad}_M\bar{f}$ in $C^\infty(TM)$ by

$$\langle \text{grad}_M\bar{f}, \eta\rangle = \eta(\bar{f}) \quad \text{for all } \eta \text{ in } C^\infty(TM).$$

Similarly define $\text{grad}_{M_1} g \in C^\infty(TM_1)$ using the metric $\langle,\rangle_1$ for M_1 and $\text{grad}_{M_2} f \in C^\infty(TM_2)$ using the metric $\langle,\rangle_2$ for M_2. It is then immediate that

$$(*) \qquad \begin{cases} P_1(\text{grad}_M\bar{g}) = \dfrac{1}{f^2}\, \text{grad}_{M_1} g & \text{and} \\ P_2(\text{grad}_M\bar{f}) = \dfrac{1}{g^2}\, \text{grad}_{M_2} f \,. \end{cases}$$

Let ∇ be the Levi-Civita connection for $(M,\langle,\rangle)$ and D^i be the Levi-Civita connection for $(M_i,\langle,\rangle_i)$, $i = 1,2$. Using the Koszul formula relating the Levi-Civita connection to its metric (see [22], p. 83, formula 7), we obtain

<u>Lemma 1</u>:

$$\nabla_{\bar{X}}\bar{V} = \nabla_{\bar{V}}\bar{X} = \frac{V(f)}{\bar{f}}\,\bar{X} + \frac{X(g)}{\bar{g}}\,\bar{V}$$

$$\nabla_{\bar{X}}\bar{Y} = \overline{D^1_X Y} - \frac{\langle \bar{X},\bar{Y}\rangle}{\bar{f}}\,\text{grad}_M\bar{f}$$

$$\nabla_{\bar{V}}\bar{W} = \overline{D^2_V W} - \frac{\langle \bar{V},\bar{W}\rangle}{\bar{g}}\,\text{grad}_M\bar{g} \,.$$

Of course the symmetry in the metric in M_1 and M_2 if f and g are relabelled induces symmetry in X and V in these formulas if we switch f and g. Since by (*) we see that

$$\mathrm{grad}_M \bar{f} \quad \text{and} \quad \mathrm{grad}_M \bar{g}$$

are <u>not</u> "basic" fields like $\bar{X}$ and $\bar{V}$, we must use (*) to obtain

<u>Lemma 2</u>:

$$\nabla_{\bar{X}} \mathrm{grad}_M \bar{g} = \frac{1}{\bar{f}^2} \overline{D^1_X \mathrm{grad}_{M_1} g} - \frac{X(g)}{\bar{f}} \mathrm{grad}_M \bar{f}$$

$$\nabla_{\bar{X}} \mathrm{grad}_M \bar{f} = -\frac{X(g)}{\bar{g}} \mathrm{grad}_M \bar{f} + \frac{\|\mathrm{grad}_M \bar{f}\|^2}{\bar{f}} \bar{X}$$

where we will write

$$\|\xi\|^2 := \langle \xi, \xi \rangle \quad \text{for} \quad \xi \in C^\infty(TM).$$

Define the Hessian forms $h_g^{M_1}$ and $h_f^{M_2}$ by:

for $X, Y \in C^\infty(TM_1)$, put $h_g^{M_1}(X,Y) := X(Yg) - (D^1_X Y)(g)$

and

for $V, W \in C^\infty(TM_2)$, put $h_g^{M_2}(V,W) := V(Wf) - (D^2_V W)(f)$.

Then

$$h_g^{M_1}(x,y) = \langle \nabla_{\bar{X}} \mathrm{grad}_M \bar{g}, \bar{y} \rangle$$

and the analogous formula holds for $h_f^{M_2}$.

We obtain

Lemma 3:

$$R(\bar{x},\bar{y})\bar{z} = \overline{R^1(x,y)z} + (\langle\bar{y},\bar{z}\rangle \frac{x(g)}{\bar{g}} - \langle\bar{x},\bar{z}\rangle \frac{y(g)}{\bar{g}} , \frac{\mathrm{grad}_M\bar{f}}{\bar{f}}$$

$$\frac{\|\mathrm{grad}_M\bar{f}\|^2}{\bar{f}^2} (\langle\bar{x},\bar{z}\rangle\bar{y} - \langle\bar{y},\bar{z}\rangle\bar{x})$$

$$R(\bar{x},\bar{v})\bar{y} = \frac{1}{\bar{g}} h_g^{M_1}(x,y) \cdot \bar{v} + \frac{\langle\bar{x},\bar{y}\rangle}{\bar{f}} \nabla_{\bar{V}}\mathrm{grad}_M\bar{f}$$

$$+ \frac{y(g)}{\bar{g}} \frac{v(f)}{\bar{f}} \bar{x}$$

$$R(\bar{x},\bar{y})\bar{v} = \frac{v(f)}{\bar{f}} (\frac{y(g)}{\bar{g}} \bar{x} - \frac{x(g)}{\bar{g}} \bar{y})$$

Recall that the 1-form $d\bar{f}$ on M is given by

$(d\,\bar{f})(\xi) = \xi(\bar{f})$ for any vector field ξ on M.

(Hence $(d\,\bar{f})(\bar{x}+\bar{v}) = v(f)$ for instance.)

Recall that $d\bar{f} \otimes d\bar{g}$ is defined by

$(d\bar{f} \otimes d\bar{g})(\xi,\eta) = \xi(\bar{f})\eta(\bar{g})$ for vector fields ξ,η on M

and

$$d\bar{f}\circ d\bar{g} := 1/2(d\bar{f} \otimes d\bar{g} + d\bar{g} \otimes d\bar{f}).$$

Also, the metric $\langle,\rangle_1$ for T_pM_1 induces an inner product on $\Lambda^2(T_pM_1)$ given by

$\langle x_1\wedge y_1, x_2\wedge y_2\rangle_1 := \langle x_1,x_2\rangle_1\langle y_1,y_2\rangle_1 - \langle x_1,y_2\rangle_1\langle x_2,y_1\rangle_1$, etc.

Given a 2-plane σ in $G_2(M)$, the Grassman bundle of 2-planes in TM, we can choose $x,y \in T_pM_1$ and $v,w \in T_qM_2$ so that

$$\{\bar{x}+\bar{v}, \bar{y}+\bar{w}\}$$

is a $\langle,\rangle$-orthonormal basis for σ. From the Lemmas given above

Theorem 4:

$$K(\sigma) = (f(q))^2(K_{M_1}(x,y) - \|\mathrm{grad}_M\bar{f}\|^2)\ \|x\wedge y\|_1^2$$

$$+ (g(p))^2(K_{M_2}(v,w) - \|\mathrm{grad}_M\bar{g}\|^2)\ \|v\wedge w\|_2^2$$

$$- f(q)(\langle x,x\rangle_1 h_f^{M_2}(w,w) - 2\langle x,y\rangle_1 h_f^{M_2}(v,w) + \langle y,y\rangle_1 h_f^{M_2}(v,v)))$$

$$- g(p)(\langle w,w\rangle_2 h_g^{M_1}(x,x) - 2\langle v,w\rangle_2 h_g^{M_1}(x,y) + \langle v,v\rangle_2 h_g^{M_1}(y,y))$$

$$+ \frac{2}{f(q)g(p)}(\|\bar{x}+\bar{v}\|^2(d\bar{f}\circ d\bar{g})(\bar{y}+\bar{w},\bar{y}+\bar{w}) - 2\langle\bar{x}+\bar{v},\bar{y}+\bar{w}\rangle(d\bar{f}\ d\bar{g})(\bar{x}+\bar{v},\bar{y}+\bar{w})$$

$$+ \|\bar{y}+\bar{w}\|^2(d\bar{f}\circ d\bar{g})(\bar{x}+\bar{v},\bar{x}+\bar{v})).$$

A Kazden-Warner type proof of the sectional curvature deformation theorem for $K \le 0$ in Dimension 2

Motivated by the work of Kazden and Warner as announced in [27] we consider the following

<u>Problem</u>: Given (M^2, g_o) compact with $K_{g_o} \le 0$ everywhere. Suppose there is a point p_o in M with $K_{g_o}(p_o) < 0$. Can we find a function

$$u : M \to R$$

so that if

$$g(t) = e^{2tu}\, g_o$$

then $K'(q) < 0$ for all points q in $K_{g_o}^{-1}(0)$?

By compactness, the existence of such a function implies that M admits a metric g with $K_g < 0$ everywhere. Now

$$K'(q) = -\Delta_{g_o} u(q) - 2K_{g_o}(q)u(q)$$

If we let

$$L := -\Delta_{g_o} - 2K_{g_o}\,\mathrm{Id}$$

then L is elliptic, $1 \notin \operatorname{Ker} L$ (since $K_{g_o}(p_o) < 0$), and since $K_{g_o} \le 0$, L thus has a trivial kernel. Hence if we let

$$f : M \to R$$

be a "nice" function which is negative on $K_{g_o}^{-1}(0)$, we can solve $L\dot{u} = f$ and thus obtain the required negative variation. This shows

<u>Theorem</u>: Given a compact 2-manifold (M, g_o) with $K_{g_o} \leq 0$. Suppose there exists a point of negative Gauss curvature. Then by a global conformal variation, we can produce a metric g for M with $K_g < 0$ everywhere.

An example showing the non-realizability of the local convex deformation of Chapter 3 as a bending of a piece of surface in dimension 2

We use an example of Sacksteder, [32], to show that the standard deformation on a surface M in R^3 which can produce positive Gauss curvature does not necessarily yield a new metric g for M which can be visualized as being produced by a local bending of the surface.

Let

$$S : z = x^3(1+y^2) \quad \text{for} \quad |y| < 1/2.$$

A computation shows that the line segment in S with x=0 and z=0, has K = 0. Sacksteder remarked in [32] that the second fundamental form II satisfies

$$\left\{\begin{matrix} II < 0 & \text{for} & x < 0 \\ II > 0 & \text{for} & x > 0 \end{matrix}\right\} \quad \text{which implies } K > 0$$

for $x \neq 0$.

This surface represents a metric on an open disk D of radius 1/8 centered about the origin in R^2, and hence is an open manifold (D,g_o) with $K_{g_o} \equiv 0$ on $L := \{(x,y) \in D;\ x=0\}$ and $K_{g_o} > 0$ in D - L. Let

$$L_o := \{(x,y) \in L;\ -1/1000 < y < 1/1000\}.$$

Suitably choosing a convex disk D_o, perform the standard deformation of Chapter 5 to get a new metric g for D with $K_g > 0$ in $(D-L) \cup L_o$.

Recalling the geometric meaning of positive curvature for a piece of surface in R^3, it is clear with a little thought that this metric deformation cannot be realized by physically bending the surface

$$S : z = x^3(1+y^2)$$

which represents (D,g_o) <u>in the region corresponding to D_o only</u>. This is because the bending in D_o necessary to curve L_o to make the curvature positive on D_o would rip S open since II < 0 for x < 0 and II > 0 for x > 0.

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