REAL FORMS IN HERMITIAN SYMMETRIC SPACES

AND

REAL ALGEBRAIC VARIETIES

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Abstract

We consider a real algebraic manifold U to be a compact connected complex manifold, together with a projective embedding invariant by complex conjugation. The map induced on U is an anti-holomorphic involution σ, and the real points U^+ are the intersection with real projective space. Conversely, if we start with an algebraic U, and such a map σ with fixed points U^+, we can find an embedding realizing σ and U^+ via complex conjugation.

From σ, we induce "complex conjugations" on

1) the field of meromorphic functions $\mathfrak{m}(U)$
2) the cohomology over $\mathbb{C}$ (with its Hodge decomposition)
3) Jacobians,

and when U is a quotient $\Gamma\backslash X$ of a symmetric domain by a discontinuous group, on

4) the algebra of holomorphic automorphic forms.

These induced conjugations can all be described by "$\omega \to \overline{\sigma^*\omega}$". If also $U^+(\sigma) \neq \emptyset$, we can lift σ to an anti-holomorphic involution (called complex conjugation, and thought of as a "real form") of

5) the universal cover X.

In fact, we can find a different lift for each component of U^+, so that the various fixed spaces X^+ give the local

isometry types of the various components. The existence of conjugations in 1) or 4) implies the existence of a $\sigma : U \to U$. With an additional assumption, we conjecture the same is true of 3). If a conjugation in 5) normalizes Γ, it induces a conjugation of U.

We classify complex conjugations in all symmetric domains X. We obtain all conjugations (up to conjugacy by holomorphic automorphisms) from certain elementary conjugations in the irreducible factors. We determine these elementary ones for non-exceptional irreducible X, and their associated fixed spaces X^+. For $X = \mathfrak{so}(p,2)$ their number is $\frac{p+1}{2}$ (resp. $\frac{p+2}{2}$) for p odd (resp. even); otherwise the number ranges among 1, 2 and 3 depending on type and dimension. The X^+ are described as "standard" types, except for two of them which are associated to concrete Lie algebras whose types are not yet determined. The known X^+ are either simple ($\mathfrak{so}(p,q)$, $\mathfrak{so}(n,\mathbb{C})$, or $\mathfrak{sp}(p/2,q/2)$), semi-simple with two factors ($\mathfrak{so}(k,1) \times \mathfrak{so}(p-k,1)$), or reductive with $\mathbb{R}^1$ as flat part ($\mathbb{R}^1 \times \mathfrak{sl}_n(\mathbb{R})$ or $\mathbb{R}^1 \times \mathfrak{sl}_p(\mathbb{C})$).

Preface

A compact quotient space $U = \Gamma\backslash X$ of a symmetric domain $X = G/K$ with respect to a discrete subgroup $\Gamma \subset G$ is a projective algebraic variety. Often the polarization of U is unique. One has been able to employ the tools of algebraic geometry in the study of the arithmetic of those Γ which are "arithmetically" defined; in fact, these arithmetic studies have been so successful precisely because of this approach. (Examples of such Γ are discontinuous groups associated to quarternion algebras, and to alternating hermitian forms.)

In contrast to the above situation, when a discontinuous group Γ' is defined arithmetically in a Lie group G' whose symmetric space $X' = G'/K'$ has no invariant complex structure, the arithmetic study of Γ' looks quite difficult. The typical example here has been a discontinuous subgroup of the orthogonal group of a real quadratic form.

With the hope of narrowing this gap, I have started the investigation of the locally symmetric space $U' = \Gamma'\backslash X'$ as a real projective algebraic variety. Of course, U' might have infinitely many embeddings in real projective space, for example by means of spherical functions. However, I do not discuss this possibility, since there are "too many" spherical functions. (One could define Γ' -automorphic forms on X' as certain "invariant" spherical functions, but it appears that there would be too many of these.) Instead I will explore the possibility of realizing U' as (one of the components of) a real section $U \cap \mathbb{P}^N(\mathbb{R})$ of a locally symmetric domain U, projectively embedded "over $\mathbb{R}$". (The notion of Γ'-automorphic form on X' should then probably involve directly the "complexifications" X and Γ of X' and Γ'.)

In order to classify the above U', I want to investigate the topology-geometry of the $\mathbb{R}$-points of certain complex projective varieties. If U is a smooth compact quotient $\Gamma\backslash X$ of a symmetric domain, then U is automatically algebraic (Kodaira, [9]). Sometimes U is a real algebraic variety, that is definable over $\mathbb{R}$, and one should be able to make a list of the X and Γ which give a real quotient. A submanifold U^+ of U is the locus of $\mathbb{R}$-points ($= U \cap \mathbb{P}^N(\mathbb{R})$ for some embedding $U \to \mathbb{P}^N(\mathbb{C})$) if and only if U^+ is the fixed-point-set of an anti-holomorphic involution of U. We call such an involution a "complex conjugation". The restrictive condition $U^+ \neq \emptyset$ allows the complex conjugation of U to be lifted to a complex conjugation on the symmetric domain (universal cover) X.

In this paper I classify complex conjugations, and their fixed spaces X^+, in all symmetric domains which as hermitian symmetric spaces contain no "exceptional" factor. One should think of these conjugations as "real forms" of X. X^+ is always the homogeneous symmetric space of a reductive group. Each component of U^+ is a quotient $\Gamma^+\backslash X^+$ of some X^+ by an "invariant" subgroup $\Gamma^+ \subset \Gamma$, (cf. [10]). The real form X^+ should vary as a Riemannian space with the component of U^+. The list of all X^+ gives a classification of possible local-isometry types of components U' of the $\mathbb{R}$-points U^+ of real algebraic varieties $U = \Gamma\backslash X$.

A typical example of such real forms is the embedded symmetric space X^+ of the group $SO(Q)$ of an indefinite quadratic form in the hermitian symmetric space X of the group $SU(Q)$ of the same form. Here the complex conjugation of X fixing X^+ is actually induced by complex conjugation of the matrices in $SU(Q)$. (The other cases are not quite so nice.) This example will be important in the study of the number theory of quadratic forms and related hermitian forms obtained by quadratic imaginary extensions of a base field. Our other real forms should define, and aid in studying analogous number theoretic situations. The study of discontinuous groups which act on hermitian symmetric spaces should extend (restrict) to the number theoretic study of discontinuous groups acting on real forms of hermitian symmetric spaces. The list of X^+ appears to show that some symmetric spaces Y cannot be real forms of any hermitian symmetric space X. For the discontinuous groups which act naturally on such Y, the number theory should be much more difficult. I would like to return to all these considerations in a later paper. In fact, in this paper, I study in detail only the problem on the level of the symmetric space X. I view the application to the quotient $U = \Gamma\backslash X$ as only a (very difficult) corollary. (cf. §1.6).

My case-by-case study for the "classical" symmetric domains X uses essentially the fact that the Lie groups involved have standard representations as matrix groups. The classification of the real forms can then be carried out, using essentially only linear algebra. There might exist a more Lie group-theoretical solution which would extend to the spaces of exceptional type; this is clearly desirable. For the applications to the quotient $\Gamma\backslash X$, one will want to regard the isometry group of X as the automorphism group of an "algebra with involution", in the sense of Weil [16]. This will allow a systematic discussion of the group Γ. Work on this project has begun.

The importance of the study of the manifold U' should lie not only in its possible application to number theory. The algebrao-geometric investigation of U' should itself be quite interesting. In the beginning of algebraic geometry, Newton and Descartes studied real planar curves, and in those days it was not even acceptable to speak of the complex points (called geometric today) of the curves. [Even then, if a curve was defined over $\mathbb{Q}$, it was felt that only the <u>rational</u> points truly "belonged" to the curve.] Although Hilbert and Petrowski [19] later paid some attention to real algebraic varieties, the field has not until quite recently come close to being central in modern mathematics. The varieties U' obtained by our classification will supply concrete and computable examples in real algebraic geometry, which has recently attracted the attention of some geometers (Thom, Atiyah, Malgrange, Hironaka).

Specifically, what Hilbert proposed in his Paris lectures of 1900 (Question 16) was a study of real algebraic hypersurfaces - number of connected components, and their spacial relationship in the ambient projective space. Although our manifolds U^+ supply examples of his problem only when U is a curve ([23], [24]), it still makes sense to ask for the number of components of U^+, and how this number relates to the way U is defined as an algebraic variety. For the special cases of $U = \Gamma\backslash X$ which arise from quaternion algebras, and for families of abelian varieties over such U, Stephen Kudla and Allan Adler have computed the number of complex conjugations of U, and number of components corresponding to each, and the various lifts of these to a fibred variety V over U, and the fixed subvariety of V for each lift.

Formulas for all these in terms of the arithmetic properties of the quaternion algebra come out of number theoretical investigations of quaternion algebras, and appear in their dissertations [20], [21].

Detailed explanations, definitions, and the program of my paper are given in the Introduction.

Finally, I must give thanks to those who made my paper possible. I thank the professors and students of Haverford College and SUNY at Stony Brook, for influencing my mathematical perspective and insight. Out of that collection, I must single out a fellow graduate student, Allan Adler, and above all else, my thesis advisor, Professor Michio Kuga, whose faith in me has never waned. Specifically, it was John Millson who first came to Professor Kuga with the observation that the symmetric space of the Lorentz group SO(3,1) was a real form of the symmetric space of SU(3,1). I had helpful conversations with Professors Helgason and B. Kostant of M. I. T, and it was a letter of A. Borel which sent me to the correct place in Cartan, and which made the explicit computations eventually possible. Also, thanks are due to Professor Henry Laufer of Stony Brook for reading my thesis and being on my examination committee. Finally, I want to thank Mrs. Virginia LaLumia for typing the manuscript.

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<u>Fields of Interest</u>

Lie groups and Symmetric Spaces

Real and Complex Algebraic Manifolds

Complex Manifolds (Deformation Theory, Moduli)

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Introduction

In part I, we study a complex manifold U which has the structure of a projective algebraic variety, definable over $\mathbb{R}$. This is equivalent with supposing that U has an anti-holomorphic involution σ. We call such an involution a complex conjugation. Generally speaking, such a map induces complex conjugations, suitably defined, on structures associated to U. These conjugations are involutive endomorphisms of the underlying structure "over $\mathbb{R}$" of a given structure "over $\mathbb{C}$", and the involutions are required to take the structure "over $\mathbb{C}$" into its conjugate structure. We study induced complex conjugations on

1) the field of rational (meromorphic) functions $\mathfrak{M}(U)$
2) the cohomology ring over $\mathbb{C}$, with its Hodge decomposition
3) Jacobian varieties

and, when U is a quotient of a symmetric domain by a discontinuous group,

4) the algebra of automorphic forms, and
5) the universal cover $\mathfrak{X}$.

In fact, starting with complex conjugations on 1) or 4), we can induce a complex conjugation on U. We conjecture that, under suitable assumptions, the same is true of 3).

Finally, we give the condition which added to 5) is equivalent with the existence of a complex conjugation on U. It is the classification of complex conjugations in symmetric domains which is the concern of the rest of the paper . We think of these as "real forms", and for each we determine the "real points", as a Riemannian symmetric space. When the conjugation arises from some map σ on some quotient U, this space of fixed points is locally isometric to one of the components of U^+, the real points of U with respect to σ.

In part II, we construct for each irreducible hermitian symmetric space (not compact, not flat, and not exceptional), a <u>standard</u> complex conjugation. We do this via the Lie algebra $\mathfrak{g}$ of the isometry group of the symmetric space X; this is a standard technique, after E. Cartan. Also, by Cartan, the fixed points X^+ can be determined as a Riemannian space from the corresponding sub-Lie algebra, $\mathfrak{g}^+$.

1) $\mathfrak{so}(p,1) \approx \mathfrak{g}^+ \subset \mathfrak{so}(p,2)$

2) $\mathbb{R} \times \mathfrak{sl}_n(\mathbb{R}) \approx \mathfrak{g}^+ \subset \mathfrak{sp}_n(\mathbb{R})$

3) $\mathfrak{so}(n,\mathbb{C}) \approx \mathfrak{g}^+ \subset \mathfrak{so}^*(2n)$

4) $\mathfrak{so}(p,q) \approx \mathfrak{g}^+ \subset \mathfrak{su}(p,q)$.

We note that all $\mathfrak{g}^+$ are simple, except for 2), which is reductive.

In part III, we define an equivalence relation on complex conjugations, and for the spaces of part I, we determine the set of equivalence classes. Denote by C the set of all conjugations of a symmetric domain X, and $C(x_o)$ the subset fixing a base point x_o. If G^h denotes the group of holomorphic automorphisms (not just the identity component) and K^h the isotropy group at x_o, then G^h acts by conjugation on C, inducing an action of K^h on $C(x_o)$. We call a set of representatives for C/G^h a set of elementary complex conjugations of the space X. We prove this is bijective to $C(x_o)/K^h$, which is actually what we compute. Let $Gal = \{1,\sigma_o\}$ be the group generated by our standard σ_o from part I for irreducible X, and for reducible X, the appropriate product of standard conjugations. Gal acts by conjugation on G^h and K^h. We identify the sets C and $C(x_o)$ with the 1-cocycles $Z^1(Gal,G^h)$ and $Z^1(Gal,K^h)$; the quotients identify $C/G^h \longleftrightarrow H^1(Gal,G^h)$ and $C(x_o)/K^h \longleftrightarrow H^1(Gal,K^h)$. We show that the map $\iota : H^1(Gal,K^h) \to H^1(Gal,G^h)$ defined by the inclusion $K^h \subset G^h$ is a bijection. In point of fact, the surjectivity is trivial and the injectivity eventually follows from the computation of $H^1(Gal,K^h)$, but we give anyhow an a priori proof. (If $\sigma_1,\sigma_2 \in C(x_o)$ are not K^h-equivalent, they turn out to have non-isometric real sections X_1^+ and X_2^+, and this prevents σ_1 and σ_2 from being G^h-equivalent.) At the end of part III, we list a set of elementary conjugations for irreducible X, and we give the corresponding fixed-point-sets X^+ by the Lie algebras of their isometry groups.

In part IV, we compare our results against known isomorphisms between the types 1) → 4) of part II in low dimensions. We check the computed cardinalities of C/G^h, and also match up the corresponding complex conjugations by their X^+.

In part V, we show how to obtain C/G^h for reducible symmetric domains (exceptional ones included) in terms of that for the irreducible factors. We have shown that the cardinality is finite, at least when there is no exceptional factor.

Part I. Generalized Complex Conjugations

1.0. Let V be a complex vector space, and $\sigma : V \to V$ an anti-complex linear (conjugate linear) involution of V. Denote by J the endomorphism of V with square -1 given by multiplication by the complex number i. We have $\sigma \circ J = -J \circ \sigma$. As a real vector space, V decomposes into $V^+ \oplus V^-$, the +1 and -1 eigenspaces of σ. If $v \in V^+$, then $Jv \in V^-$ since $\sigma(Jv) = -J\sigma v = -Jv$. Conversely, $v \in V^-$ implies $Jv \in V^+$, so that J defines an isomorphism (over $\mathbb{R}$) $V^+ \rightleftarrows V^-$. If $\dim_{\mathbb{C}} V = n < \infty$, then $\dim_{\mathbb{R}} V^+ = \dim_{\mathbb{R}} V^- = n$. Moreover, V is naturally (via J) the complexification $V^+ \otimes_{\mathbb{R}} \mathbb{C}$, obtained by extending the scalars from $\mathbb{R}$ to $\mathbb{C}$. We can look at our involution σ either as $(+1) \oplus (-1)$ in the decomposition $V^+ \oplus V^-$, or as $(+1) \otimes$ (complex conjugation) on $V^+ \otimes \mathbb{C}$. Now conversely, if we are given an $\mathbb{R}$-subspace of V, say W, such that the complex structure J gives isomorphisms $V = W \oplus J(W) = W \otimes_{\mathbb{R}} \mathbb{C}$, then we can define a conjugate linear involution σ by $(+1) \oplus (-1)$ or $(+1) \otimes$ (complex conjugation). We have just seen that every σ arises in this way.

In case V is an algebra over $\mathbb{C}$ (for example a field) in which the scalars $\mathbb{C}$ are an embedded sub-field, and σ is a conjugate linear algebra-involution preserving the subfield $\mathbb{C}$, then we can say more. The subspace V^+ fixed by σ is a sub-$\mathbb{R}$-algebra (containing the scalars $\mathbb{R}$ as a subfield), $V = V^+ \oplus V^-$ (as subspaces), and V is again the complexification $V^+ \otimes_{\mathbb{R}} \mathbb{C}$, as $\mathbb{R}$-algebras.

The involution σ is complex conjugation on the subfield $\mathbb{C}$, and is $(+1) \otimes$ (complex conjugation) on $V^+ \otimes \mathbb{C}$, this time as algebra morphisms (over $\mathbb{R}$).

Definitions

(1.0.1) We call such an involution a complex conjugation of V, either of V as a $\mathbb{C}$-vector space, or as a $\mathbb{C}$-algebra if the conditions of the second paragraph are satisfied.

(1.0.2) Let V be a complex vector bundle over a space U, and $\sigma' : V \to V$ an involution which anti-commutes with the complex structure J. That is, if $\sigma : U \to U$ is the involution induced by σ', then we have $\sigma' \circ J_u = -J_{\sigma(u)} \circ \sigma'$ for each $u \in U$. We call such a σ' a complex conjugation of V, and such a σ a complex conjugation of the space U. (See Atiyah, [1]).

(1.0.3) Let U be a complex manifold, and $\sigma : U \to U$ a differentiable involution whose differential σ_* is a complex conjugation of the (complex) tangent bundle of U. Then we call σ a complex conjugation of (the complex manifold) U. If, moreover, U happens to be a complex torus, then we say σ is a complex conjugation (in the sense of complex tori) if it is also a group homomorphism. Denote by U^+ or $\mathrm{Fix}(\sigma)$ (or $U^+(\sigma)$ if need be) the real submanifold of fixed points of σ.

1.1 Let U be a connected compact complex manifold, and $\sigma : U \to U$ a complex conjugation. Denote by $\mathfrak{m}(U)$ the field of meromorphic functions on U. The scalars for this $\mathbb{C}$-algebra are of course the subfield $\mathbb{C}$ of constant functions. The mapping $\mathfrak{m}(U) \ni f \to \overline{f\circ\sigma} = f^\sigma$ is a complex conjugation of the algebra $\mathfrak{m}(U)$, in the sense of (1.0). In case U is a projective algebraic variety (that is, U can be embedded holomorphically in some projective space $\mathbb{P}^N(\mathbb{C})$), then a complex conjugation of $\mathfrak{m}(U)$ insures that U has some model "defined over $\mathbb{R}$". This means that complex conjugation in some $\mathbb{P}^{N'}(\mathbb{C})$ preserves the submanifold U, and in fact some embedding exists with $U^+ = U \cap \mathbb{P}^{N'}(\mathbb{R})$, i.e. , we have a commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\ \varphi\ } & \mathbb{P}^{N'}(\mathbb{C}) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle c} \;=\; \text{complex conjugation} \\ U & \xrightarrow{\ \varphi\ } & \mathbb{P}^{N'}(\mathbb{C}). \end{array}$$

We state this well-known fact for completeness:

Proposition (1.1.0)

For U a projective variety (smooth), the following are equivalent:

1) U has a complex conjugation σ, with fixed points U^+.

2) $\mathfrak{M}(U)$ has a complex conjugation.

3) U has some model defined over $\mathbb{R}$, with real points U^+.

<u>Proof</u>: We give a proof later for the equivalence of 1) and 3) for a special case of U, the only case in which we will be interested. See also Weil's article [15].

1.2. The conjugation $\sigma : U \to U$ induces an involution (R-linear) of the cohomology ring $H^*(U,R)$ for any ring R. First let $R = \mathbb{C}$ and suppose U is compact Kähler. The $\mathbb{C}$-vector spaces $H^p(U,\mathbb{C})$ decompose (Hodge) as $\bigoplus_{a+b=p} H^{a,b}(U)$, where the $H^{a,b}(U)$ are the spaces of complex harmonic forms of type (a,b). We study the induced map σ^* on cohomology by composing it with complex conjugation (of a form). Let $x \in H^p(U,\mathbb{C})$ be represented by $\sum_{a+b=p} \omega^{a,b}$, and define x^σ to be the class of the form $\Sigma\ \overline{\sigma^*\omega^{a,b}} = \overline{\sigma^*\Sigma\omega^{a,b}}$. The map $x \to x^\sigma$ thus defined on $H^*(U,\mathbb{C})$ is the complex conjugate of σ^*. Since the pull-back σ^* commutes with complex conjugation (of a form), the map $x \to x^\sigma$ is a complex conjugation of $H^*(U,\mathbb{C})$ (as an algebra over $\mathbb{C}$, in the sense of (1.0)). Each of the $\mathbb{C}$-vector spaces $H^{a,b}(U)$ is preserved, and receives a <u>complex conjugation</u>. The elements of $H^{a,b}(U)$ fixed by the conjugation are forms ω, with $\sigma^*\omega = \bar{\omega} \in H^{b,a}(U)$. Since $\sigma^*(\bar{\omega}) = \omega$, and $H^{a,b}(U)^+$ spans $H^{a,b}(U)$ over $\mathbb{C}$, we see that σ^* preserves the sum $H^{a,b}(U) \oplus H^{b,a}(U)$ (if $a \neq b$) and has a matrix representation

$$\sigma^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad \text{Therefore, on } \sum_{a>b} H^{a,b}(U) \oplus H^{b,a}(U),$$

the map σ^* has a similar matrix representation. We have the

<u>Proposition</u> (1.2.0) Let U be a compact (connected) Kähler manifold of complex dimension n, and $\sigma : U \to U$ a complex conjugation. The map σ^* on $H^*(U,\mathbb{C})$ preserves the subspaces $\bigoplus_{i=0}^{n} H^{i,i}(U)$ and $\bigoplus_{\substack{0<a+b<2n \\ a>b}} H^{a,b}(U) \oplus H^{b,a}(U)$,

and has zero trace on the second subspace.

1.2.1. Suppose now that U is a Hodge manifold, so that by (1.1.0) we may assume we have a holomorphic projective embedding φ, and a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \varphi\ } & \mathbb{P}^N(\mathbb{C}) \\ \Big| \sigma & & \Big| c \\ U & \xrightarrow{\ \varphi\ } & \mathbb{P}^N(\mathbb{C}). \end{array}$$

Denote by $\eta \in H^{1,1}(\mathbb{P}^N(\mathbb{C}))$ the (1,1) form associated to the standard Kähler metric, and $\omega = \varphi^*\eta \in H^{1,1}(U)$ the restriction of η to U (equal to the (1,1) form associated to the induced metric on U). We want to show that $\sigma^*\omega = -\omega$.

First let $j : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ be inclusion, so that $j \circ c = c \circ j$ if c = complex conjugation. Since

$c : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is orientation <u>reversing</u>, and $\dim_{\mathbb{C}} H^{1,1}(\mathbb{P}^N(\mathbb{C})) = 1$, we have $c^*(j^*\eta) = -j^*\eta$ and therefore, $c^*\eta = -\eta$. (Consequently, $c : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ is orientation reversing if and only if N is odd.)

<u>Proposition</u>.(1.2.2) $\sigma^*\omega = -\omega$.

<u>Proof</u>. $\sigma^*\omega = \sigma^*(\varphi^*\eta) = \varphi^*(c^*\eta) = \varphi^*(-\eta) = -\varphi^*(\eta) = -\omega$.

Denote by t_i the trace $tr_{\mathbb{C}}(\sigma^* : H^{i,i}(U) \to H^{i,i}(U))$. We have $t_o = +1$ automatically, and the proposition says $t_n = (-1)^n$ since ω^n generates $H^{n,n}(U)$. We generalize these facts in the

<u>Proposition</u> (1.2.3). $t_i = (-1)^n\, t_{n-i}$, $(i \neq n/2)$

<u>Proof</u>. Denote by $L : H^*(U,\mathbb{C}) \to H^*(U,\mathbb{C})$ the Lefschetz operator $\xi \to L(\xi) = \omega \wedge \xi$. It is known (Weil [17], p. 75) that $L^{n-2i} : H^{i,i}(U) \to H^{n-i,n-i}(U)$ is an isomorphism. Since $\sigma^*\omega = -\omega$, we have that σ^* and L anti-commute as endomorphisms of the $\mathbb{C}$-vector space $H^*(U,\mathbb{C})$. Therefore, $\sigma^* \circ L^{n-2i} = (-1)^{n-2i} L^{n-2i} \circ \sigma^* = (-1)^n L^{n-2i} \circ \sigma^*$. The trace statement follows immediately.

<u>Remark</u> (1.2.4). For k such that $i < n-i-k < n-i$, the map L^{n-2i-k} is <u>injective</u> $H^{i,i}(U) \to H^{n-i-k,n-i-k}(U)$. The trace of σ^* on the image of L^{n-2i-k} in $H^{n-i-k,n-i-k}(U)$ is $(-1)^{n-k} t_i$, so that in some cases we may not have to compute all t_i for $i \leq n/2$; some of these may come automatically

from "smaller" t_i.

Proposition (1.2.5) The Lefschetz number $L(\sigma) = \sum_{i=0}^{n} t_i$ is equal to the euler number $E(U^+)$ of the fixed-point-set of σ. Specifically, if n is

odd: $L(\sigma) = 0$.

even: $L(\sigma) = 2(\sum_{i=0}^{n/2-1} t_i) + t_{n/2}$.

Proof. It may happen that $U^+ = \emptyset$; $E(\emptyset) = 0$. For odd n, we have $L(\sigma) = \sum_{i=0}^{n/2-1} t_i + t_{n-i} = \Sigma\, t_i - t_i = 0$. In case $U^+ = \emptyset$, the proposition follows. If $u \in U^+$, then by looking at the differential σ_* at $u \in U$, the discussion of (1.0) gives that $\dim_{\mathbb{R}} U^+ = \dim_{\mathbb{C}} U$ which is odd. The euler number of the odd-dimensional manifold U^+ is zero. For even n, σ is orientation preserving and an isometry (restriction of the isometry $c : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$), and the Lefschetz fixed-point formula gives $L(\sigma) = E(U^+)$. (Kuga, [9]). The formula for $L(\sigma)$ results from (1.2.3).

1.3. The map σ^* on $H^*(U,\mathbb{C})$ restricts to the subrings $H^*(U,\mathbb{R})$ and $H^*(U,Z)$. For $x \in H^*(U,\mathbb{R})$, $\sigma^* x = \overline{\sigma^* x}$, so that the restriction of our conjugation $x \to x^\sigma$ to $H^*(U,\mathbb{R})$ is σ^*. For p odd, consider (following Weil, [17]) the quotients $\mathcal{J}_p(U) = H^p(U,\mathbb{R})/H^p(U,Z)$. These are tori, and σ^* induces involutions σ_p on $\mathcal{J}_p(U)$, in the sense of real Lie groups. Moreover, since p is odd, the operator

C of Weil ([17] p. 82) defines a complex structure on the R-vector spaces $H^p(U,\mathbb{R})$, and thus the $\mathcal{J}_p(U)$ are complex tori (in fact, abelian varieties). Since $\sigma^*\omega$ has type (b,a) whenever ω has type (a,b), $\sigma^* : H^p(U,\mathbb{R}) \to H^p(U,\mathbb{R})$ anti-commutes with the complex structure C, and is thus a complex conjugation, in the sense of (1.0). The induced map σ_p is thus a <u>complex conjugation</u> of the complex torus $\mathcal{J}_p(U)$.

The system $\{\sigma_p^* : H^p(U,\mathbb{R}) \to H^p(U,\mathbb{R})\}_{\text{odd } p}$ is <u>compatible</u>, in the sense that $\sigma_{p+2}^* \circ L = (-1)L\circ\sigma_p^*$. (1.3.0.)

<u>Conjecture</u> (1.3.1) Let U be a compact Kähler manifold, and suppose that (all of) its Jacobians $\mathcal{J}_p(U)$ admit complex conjugations σ_p, which are <u>compatible</u> in the sense of (1.3.0). Then U itself has a complex conjugation σ such that σ_p^* on $H^p(U,\mathbb{R})$ induces σ_p for odd p.

<u>Remark.</u> (1.3.2) The Jacobians given in the form

$$\mathcal{J}_p(U)' = H_+^p(U)\backslash H^p(U,\mathbb{C})/H^p(U,\mathbb{Z}), \text{ where } H_+^p(U) = \bigoplus_{\substack{a>b\\ a+b=p}} H^{a,b}(U)$$

can also be given complex conjugations. (Note that $\mathcal{J}_p(U) \neq \mathcal{J}_p(U)'$). Our map $x \to x^\sigma = \overline{\sigma^* x}$ on $H^p(U,\mathbb{C})$ preserves $H^p(U,\mathbb{Z})$ since σ^* itself preserves $H^p(U,\mathbb{Z})$ and the bar has no effect. If $x \in H^{a,b}(U)$, $\overline{\sigma^* x} \in H^{a,b}(U)$, so that the space H_+^p is preserved. Thus $\mathcal{J}_p(U)'$ has a complex conjugation, σ_p'. The maps $x \to x^\sigma$ on $H^p(U,\mathbb{C})$ are compatible (1.3.0), and we can make the obvious conjecture (1.3.1)' about the $\mathcal{J}_p(U)'$. The following would be a corollary of either (1.3.1) or (1.3.1)'.

<u>Conjecture</u> (1.3.1)* Let α be a complex conjugation of the algebra $H^*(U,\mathbb{C})$, preserving the Hodge decomposition, and preserving the sub-rings $H^*(U,\mathbb{Z})$ and $H^*(U,\mathbb{R})$, and equivalently anti-commuting with the operator L or sending to its negative the class of the (1,1) form associated to the Kähler metric. Then $\alpha = \overline{\sigma^*}$ for some complex conjugation σ of U.

<u>1.3.3</u> The conjugation $\sigma : U \to U$ induces a conjugation on the Picard variety of U, [6]. If ξ is a holomorphic line bundle on U with zero Chern class, then $\overline{\sigma^*\xi}$ is also holomorphic, with zero Chern class. $\xi \to \xi^\sigma = \overline{\sigma^*\xi}$ is a complex conjugation, in the sense of complex tori.

<u>1.3.4</u> Let U = E be an elliptic curve and σ a complex conjugation. Since we can identify E and $\mathcal{J}_1(E)$, E has another conjugation, which is a homomorphism (as a real Lie group). With this information it is not hard to show that $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with τ either

1) purely imaginary and $|\tau| \geq 1$
2) $\tau = e^{i\theta}$, $\pi/3 \leq \theta \leq \pi/2$

or 3) $Re(\tau) = 1/2$ and τ "above" the unit circle.

Conversely, each E with such a τ has a complex conjugation.

Geometrically, the elliptic curves defined over $\mathbb{R}$ are thus precisely the points of (upper half-plane)/$SL_2(\mathbb{Z})$ invariant by the involution induced from $z \to -\bar{z}$ in the upper half-plane. One would like to know how much of this is true for higher-dimensional abelian varieties.

<u>1.4</u> Now suppose U is a smooth quotient (and compact) $\pi : X \to U$ of a bounded symmetric domain by a group Γ of holomorphic automorphisms acting freely and properly discontinuously. Then π is the universal covering of U and Γ is the fundamental group. Denote by K the canonical line bundle of U, and K^r the tensor power $K \otimes \overset{(r)}{\ldots} \otimes K$ for $r \geq 1$, and $K^0 = U \times \mathbb{C}$, the trivial line bundle. Let $A_r(U) = H^0(U,K^r)$ be the $\mathbb{C}$-vector space of holomorphic sections of the line bundle K^r, $r \geq 0$. Then $A(U) = \bigoplus_{r=0}^{\infty} A_r(U)$ is the $\mathbb{C}$-algebra (1.0) of holomorphic automorphic forms on U [8]. The algebra A(U) can be identified with the classically defined algebra $A(X,\Gamma) = \bigoplus_{r=0}^{\infty} A_r(X,\Gamma)$ ([8], [13], [14]) of "automorphic forms on X with respect to Γ". For that, let $\omega \in A_r(U)$, so that $\pi^*\omega$ is a holomorphic section of the rth power of the canonical bundle of X. Since we have global coordinates $z_1,\ldots,z_n$ in $X \subset \mathbb{C}^n$, the form $\pi^*\omega$ is expressable as $f(dz_1\wedge\ldots\wedge dz_n)^r$ with some (global) holomorphic function $f : X \to \mathbb{C}$. Since

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & X \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & U & \end{array}$$

commutes for all $\gamma \in \Gamma$, $\pi^*\omega$ is Γ-invariant, or equivalently $f(\gamma x) = f(x) \cdot J_\gamma(x)^{-r}$ for all $x \in X$, where $J_\gamma(x)$ is the complex Jacobian (determinant) at $x \in X$. This is the classical definition for an automorphic form f of weight r, with respect to Γ, i.e., $f \in A_r(X,\Gamma)$. The mapping $\omega \to f$ is the desired isomorphism.

We wish to show how $A(U)$ and $A(X,\Gamma)$ can be given complex conjugations, assuming U has one. Thus, if $\sigma : U \to U$ is the conjugation and $\omega \in A_r(U)$, then $\overline{\sigma^*\omega} \in A_r(U)$ also for any r. The mapping $A(U) \ni \omega \to \overline{\sigma^*\omega} = \omega^\sigma$ is the complex conjugation (in the sense of algebra over $\mathbb{C}$, as in 1.0). Now the isomorphism $A(U) \to A(X,\Gamma)$ of course gives a complex conjugation in $A(X,\Gamma)$ automatically. We examine this one explicitly.

The composition $X \xrightarrow{\pi} U \xrightarrow{\sigma} U$ lifts over $\pi : X \to U$ since π is the universal cover, i.e., we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\sigma}_0} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ U & \xrightarrow{\sigma} & U \end{array},$$

The map $\tilde{\sigma}_o$ (not unique) is automatically anti-holomorphic since π is a local holomorphic isomorphism, however it need not be an involution, and cannot be unless $U^+ \neq \emptyset$. (The anti-holomorphic <u>involution</u> $\tilde{\sigma}_o$ would have a fixed point, as we will show later, implying that σ must have had one.) Since $\tilde{\sigma}_o^{\,2}$ covers the identity on U, we have $\tilde{\sigma}_o^{\,2} = \gamma_o \in \Gamma$, and therefore, $\tilde{\sigma}_o$ is a diffeomorphism. Suppose $\tilde{\sigma}_1$ was another lift; then $\tilde{\sigma}_o^{-1} \circ \tilde{\sigma}_1$ covers the identity, thus is some $\gamma_1 \in \Gamma$.

Now let $f \in A_r(X,\Gamma)$, and denote $\eta = f(dz_1 \wedge \ldots \wedge dz_n)^r$ the corresponding section of the rth power of the canonical bundle. $\overline{\tilde{\sigma}_o{}^*\eta}$ is also a holomorphic section of that bundle, and therefore equals $f^\sigma(dz_1 \wedge \ldots \wedge dz_n)^r$ for some $f^\sigma : X \to C$. Since $\tilde{\sigma}_o^{\,2} \in \Gamma$ and η is Γ-invariant, the mapping $f \to f^\sigma$ is an involution, say σ_r, obviously conjugate linear, of $A_r(X,\Gamma)$. The sum $\bigoplus_r \sigma_r : A(X,\Gamma) \to A(X,\Gamma)$ is a complex conjugation, in the sense of algebra over $\mathbb{C}$, (1.0). We need to know that

1) $\oplus\, \sigma_r$ is independent of the choice of lift $\tilde{\sigma}_o$

2) $\oplus\, \sigma_r$ is the conjugation induced on $A(X,\Gamma)$ from the isomorphism $A(U) \to A(X,\Gamma)$ given above. The first statement follows because two lifts differ by an element of Γ, and the second statement follows because, for any lift, <u>say</u> $\tilde{\sigma}_o$, <u>we have</u> $\underline{\sigma \circ \pi} = \pi \circ \tilde{\sigma}_o$: if $\eta = f(dz_1 \wedge \ldots \wedge dz_n)^r = \pi^*\omega$, then $\overline{\tilde{\sigma}_o{}^*\eta} = \overline{\tilde{\sigma}_o{}^*\pi^*\omega} = \overline{\pi^*\sigma^*\omega} = \pi^*(\overline{\sigma^*\omega})$.

It is no trivial matter to construct a complex conjugation in $A(X,\Gamma)$; in fact, we have the

Proposition (1.4.0). For $U = \Gamma\backslash X$, the following are equivalent:

1) U has a complex conjugation, with fixed points U^+.

2) $A(X,\Gamma)$ has a complex conjugation.

3) U has a model as a projective variety, which is defined over $\mathbb{R}$ with real points U^+.

Proof: Let $\sigma : U \to U$ be a conjugation; we have seen how 1) $\Rightarrow$ 2). It is a theorem of Kodaira ([9], p. 41) that the canonical bundle K of U is ample, in other words, for some $m \in N$, any basis $\{\varphi_o,\ldots,\varphi_N\}$ of $A_m(X,\Gamma)$ gives a holomorphic embedding $\varphi : U \to \mathbb{P}^N(\mathbb{C})$: for $u \in U$, let $x \in X$ be such that $\pi(x) = u$. Define $\varphi(u)$ to be the class of $(\varphi_o(x),\ldots,\varphi_N(x))$ in $\mathbb{P}^N(\mathbb{C})$. Now, by (1.0), we can find a basis of $A_m(X,\Gamma)$ invariant by σ_m, so we may assume $\{\varphi_o,\ldots,\varphi_N\}$ has that property. Let $F = \Sigma\, a_{i_o,\ldots,i_N} X_o^{i_o},\ldots,X_N^{i_N}$ be a homogeneous polynomial vanishing on the image of φ: we have an identity $\Sigma\, a_{i_o,\ldots,i_N}\varphi_o^{i_o}\ldots\varphi_N^{i_N} = 0$. Applying the conjugate linear algebra morphism $\oplus\sigma_r$, and using $\sigma_m(\varphi_i) = \varphi_i$ for all i, we get $\Sigma\, \bar{a}_{i_o,\ldots,i_N}\varphi_o^{i_o}\ldots\varphi_N^{i_N} = 0$. Then applying complex conjugation in $\mathbb{P}^N(\mathbb{C})$, we see that F vanishes also on the conjugate of $\varphi(U)$, in other words, the map $c : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ preserves the image $\varphi(U)$, inducing a complex conjugation on U.

To get 1) ⇒ 3) we have to show that the following commutes:

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & \mathbb{P}^N(\mathbb{C}) \\ \downarrow \sigma & & \downarrow c \\ U & \xrightarrow{\varphi} & \mathbb{P}^N(\mathbb{C}) \end{array}$$

We need that for all i, $\varphi_i \circ \tilde{\sigma}(x)$ differs from $\overline{\varphi_i(x)}$ by the <u>same</u> complex valued function. Calculate the pull-back $\overline{\tilde{\sigma}_o^*\varphi_i(dz_1\wedge\ldots\wedge dz_n)^m} = (\overline{\varphi_i \circ \tilde{\sigma}_o})(\overline{\tilde{\sigma}_o{}^* dz_1\wedge\ldots\wedge dz_n})^m = \overline{\varphi_i(dz_1\wedge\ldots\wedge dz_n)^m}$, by the invariance. Also, $\overline{\tilde{\sigma}_o^* dz_1\wedge\ldots\wedge dz_n}$ is a holomorphic n-form $g(x)dz_1\wedge\ldots\wedge dz_n$, with $g : X \to \mathbb{C}$ holomorphic. Thus, conjugating both sides, we get $(\varphi_i \circ \tilde{\sigma}_o)\overline{g(x)}^m(\overline{dz_1\wedge\ldots\wedge dz_n})^m = \overline{\varphi_i}(\overline{dz_1\wedge\ldots\wedge dz_n})^m$, which is what we needed: $\overline{\varphi_i} = (\varphi_i \circ \tilde{\sigma}_o) \cdot \bar{g}^m$.

Finally, 3) ⇒ 1) because the map c induces on U a complex conjugation. The fixed points are clearly $U \cap \mathbb{P}^N(\mathbb{R})$ = the real points U^+.

<u>1.5</u>. We have seen how a complex conjugation σ of $U = \Gamma\backslash X$ gives rise to anti-holomorphic automorphisms (lifts $\tilde{\sigma}$) of X.

<u>Proposition</u> (1.5.0). A conjugation $\sigma : U \to U$ lifts to a <u>complex conjugation</u> of X if and only if σ has a fixed point. Moreover, for each $u_o \in U^+$(= Fix (σ)), and any given $x_o \in X$ lying above it ($\pi(x_o) = u_o$), we can find a conjugation of X fixing x_o.

Proof: First suppose some lift $\tilde{\sigma}$ is an involution. Then $\{1,\tilde{\sigma}\}$ is a compact group of isometries (of the Bergmann metric) as we shall show shortly, so $\{1,\tilde{\sigma}\}$ has a common fixed point. [Actually, we can give an easier argument: Pick any $x \in X$. Then there is a unique geodesic segment from x to $\tilde{\sigma}(x)$, and this must be preserved by the isometry $\tilde{\sigma}$. The mid-point of the segment will be fixed.] Now if $\tilde{\sigma}(x_o) = x_o$, then $\sigma(\pi(x_o)) = \pi(x_o)$.

Now, conversely, suppose σ fixes some u_o. Lift σ to some $\tilde{\sigma}$, and pick x_o lying above u_o. $\tilde{\sigma}$ must permute the fibre $\pi^{-1}(u_o)$, and Γ acts transitively (and simply) on this fibre. Therefore, we have a $\gamma \in \Gamma$, with $\tilde{\sigma}(x_o) = \gamma x_o$. Now $\gamma^{-1}\circ\tilde{\sigma}$ is an anti-holomorphic lift of σ (since it covers $id \circ \sigma = \sigma$). Clearly $\gamma^{-1}\circ\tilde{\sigma}(x_o) = x_o$, so we need only show $(\gamma^{-1}\circ\tilde{\sigma})^2 = 1$. But $(\gamma^{-1}\circ\tilde{\sigma})^2$ is a deck transformation (it covers $1\circ\sigma\circ 1\circ\sigma = 1$), and has x_o as a fixed point. This implies $(\gamma^{-1}\circ\tilde{\sigma})^2 = 1$.

Remark (1.5.1) Recalling (1.4.0) we see that the pair (U,σ) gives rise to an algebraic variety, defined over $\mathbb{R}$, which has real points ($U^+ \neq \emptyset$) if and only if σ lifts to a complex conjugation of its universal cover.

Theorem (1.5.2) (Structure Theorem on $(X,\tilde{\sigma})$)

Let $\tilde{\sigma} : X \to X$ be a complex conjugation of the complex manifold X (1.0.3). Then

1) $\tilde{\sigma}$ is an isometry of the Bergmann metric.

2) The set of fixed points $X^+(\tilde{\sigma})$ is a connected totally geodesic sub-Riemannian symmetric space; $\dim_R X^+ = \dim_C X$.

3) X^+ is "holomorphically dense" in the sense that a holomorphic or anti-holomorphic automorphism of X is determined by its restriction to X^+.

Proof: 1) From [7], for example, it follows that the Bergmann metric is the invariant metric defined by a Killing form (see the discussion in 2.0, p. 24). Therefore, not only is the Bergmann metric invariant under holomorphic automorphisms of X, but it is also invariant under automorphisms induced by automorphisms of the group of holomorphic automorphisms. The function $f \to \tilde{\sigma} f \tilde{\sigma}$ defines an automorphism of the group of holomorphic automorphisms, and this automorphism induces $\tilde{\sigma}$ on X. Therefore, we have the statement of 1).

2) Let $x_o \in X^+(\tilde{\sigma})$, and v a tangent vector to X^+ at x_o, and c the unique geodesic emanating from x_o with $\dot{c} = v$. Since $\tilde{\sigma}$ is an isometry, and $\tilde{\sigma}_*(v) = v$, we have

$\tilde{\sigma}\circ c = c$ globally. Therefore, $c \subset X^+$, i.e., X^+ is totally geodesic. Let $\mathrm{Exp}^+_{x_0}$ be the restriction of the full exponential map of X at x_0 to the tangent space to X^+; then $\mathrm{Exp}^+_{x_0}(X^+_{x_0}) \subset X^+$. Now let $x \in X^+$ be arbitrary, and consider the unique geodesic segment from x_0 to x. Since this geodesic is contained in X^+, its tangent vector at x_0 is fixed. This shows x is in the image of $\mathrm{Exp}^+_{x_0}$, so X^+ is connected. X^+ is a globally sub-Riemannian symmetric space because $\tilde{\sigma}$ is an isometry. For the dimension statement we use (1.0) where V is the $\mathbb{C}$-vector space X_{x_0}. $\dim_{\mathbb{R}} X^+_{x_0} = \dim_{\mathbb{C}} X_{x_0}$, and the fact that $\mathrm{Exp}^+_{x_0}$ is a diffeomorphism gives the result.

3) Let f_1, f_2 be two holomorphic or two anti-holomorphic diffeomorphisms which agree on $X^+(\tilde{\sigma})$. Then $f_1 f_2^{-1}$ is holomorphic, and therefore an isometry ([17],p. 62, Cor. 2), and is the identity on X^+. Its differential is the identity on the linearly dense (over $\mathbb{C}$) subspace $X^+_{x_0}$, and therefore, on all of X_{x_0}. An isometry whose differential is the identity somewhere must be the identity, so $f_1 = f_2$.

<u>1.6.</u> (Some final remarks) We have seen how the quotient $U = \Gamma\backslash X$ can be defined over $\mathbb{R}$ if and only if it admits a complex conjugation σ, and that such a σ lifts to a complex conjugation $\tilde{\sigma}$ of X if and only if σ has fixed points.

<u>Proposition</u> (1.6.0) A complex conjugation $\tilde{\sigma}$ of X lifts (or induces) a conjugation σ of $U = \Gamma\backslash X$ if and only if $\tilde{\sigma}$ normalizes Γ.

<u>Proof</u>: If $\tilde{\sigma}$ is a lift, then $\tilde{\sigma}\gamma\tilde{\sigma}$ is a deck transformation, so $\tilde{\sigma}$ normalizes Γ. If $\tilde{\sigma}\gamma\tilde{\sigma} = \gamma^{\tilde{\sigma}} \in \Gamma$, then $\tilde{\sigma}\gamma(x) = \gamma^{\tilde{\sigma}}\tilde{\sigma}(x)$ so $\tilde{\sigma}$ takes $\Gamma \cdot x$ into $\Gamma \cdot \tilde{\sigma}(x)$, and $\tilde{\sigma}$ induces a $\sigma : U \to U$.

We can thus eliminate explicit mention of the map σ on U, that is, U has some model defined over $\mathbb{R}$ (with $\mathbb{R}$-points) if and only if the universal cover X has a complex conjugation which normalizes the fundamental group Γ of U. We want to take the point of view that if we can determine "all" complex conjugations of all bounded symmetric domains X, then we can find manifolds U definable over $\mathbb{R}$ (with $\mathbb{R}$-points) if we can find discrete groups Γ (giving compact smooth quotient) and which are normalized by a complex conjugation of X. We begin, therefore, a detailed study of the complex conjugations of X. Since the map $\pi : X \to U$ is a local isometry, we will always want to determine X^+, and think of it as the local-isometry type of one of the components of U^+.

Part II. The Standard Conjugation on X

2.0. We assume the reader is familiar with the basic facts about the symmetric spaces X of (non-compact) hermitian type. These can be found in the books by Helgason or Wolf, the lecture notes of Borel, or the papers of E. Cartan, ([7], [18], [3], [4,5]).

X factors into a product of irreducible symmetric spaces, each hermitian. According to the classification of Cartan, each one of these factors falls into one of four "classical" types, or is one of two "exceptional" spaces. We assume from now on that neither of these exceptional spaces occurs in the decomposition of our X. Our aim is to construct a standard complex conjugation σ_o on each X; clearly, it suffices to do that for X irreducible. For each σ_o, we will determine the set of "real" points X^+, as a global Riemannian symmetric space.

So now assume X irreducible, and recall that the Lie algebra $\mathfrak{G}$ of its isometry group G has a Cartan decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$, where $\mathfrak{K}$ is a sub-algebra, and $\mathfrak{P}$ a vector subspace. $\mathfrak{K}$ is the Lie algebra of the isotropy subgroup K of G for a point $x_o \in X$; $\mathfrak{P}$ is the orthogonal complement of $\mathfrak{K}$ with respect to the Killing form. This is the eigenspace decomposition of $\mathfrak{G}$ for the Cartan involution (the geodesic symmetry at the point x_o) when it acts as an inner automorphism of G. Since our X is hermitian, there is an element Z in the center of $\mathfrak{K}$ (unique except for its negative for our irreducible X) such that $(\mathrm{ad}Z)|_{\mathfrak{p}} = J$ defines a complex structure on the subspace $\mathfrak{p}$, therefore on the tangent space X_{x_o}, and therefore, a G_o-invariant almost complex structure (integrable) on X. In the same way the Killing form of $\mathfrak{G}$, restricted to $\mathfrak{p}$, defines a G-invariant metric on the manifold X. [It is known [7] that X is (holomorphically) isometric to a bounded symmetric domain in complex number space, with its Bergmann metric [17], and holomorphic structure as an open subset of the ambiant complex vector space. Conversely, ([7] also) every symmetric domain is holomorphically isometric to one of the spaces X of non-compact-hermitian type. It suffices therefore, for our study of symmetric domains, to use the language of symmetric spaces.] The group G^h of holomorphic automorphisms of X (holomorphicity with respect either to the complex structure just defined, or to that defined on X as a domain in a complex vector space) is then a subgroup of G. The isotropy group $K^h = G^h \cap K$ is then

$\{k \in K | \mathrm{Ad}(k)(Z) = Z\} = \{k \in K | \mathrm{Ad}(k) \circ J = J \circ \mathrm{Ad}(k)|_{\mathfrak{p}}\}$.

The complement of K^h in K is then $\{k \in K | \mathrm{Ad}(k)(Z) = -Z\} = \{k \in K | \mathrm{Ad}(k) \circ J = -J \circ \mathrm{Ad}(k)|_{\mathfrak{p}}\}$, so that $K - K^h$ is precisely the set of anti-holomorphic automorphisms of X fixing x_o. Finally, the identity components G_o^h and G_o are equal, so that G is a non-compact Lie group with a finite number of connected components, half of these components being made up of holomorphic isometries of X, and half being anti-holomorphic isometries of X.

The correspondence $g \to Ad(g)$ defines an isomorphism from G onto the automorphism group $Aut(\mathfrak{g})$. Therefore, to construct a complex conjugation σ_0 on X , fixing x_0, it is sufficient to construct an automorphism of $\mathfrak{g}$, of order 2, preserving the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and sending Z to its negative (or equivalently anti-commuting with the endomorphism J on $\mathfrak{p}$).

For the problem of describing X^+, let $\mathfrak{g}^+ = \mathfrak{k}^+ + \mathfrak{p}^+$ be the subalgebra of $\mathfrak{g}$ fixed by the involutive automorphism of $\mathfrak{g}$ just described. Since global Riemannian symmetric spaces with no compact factors are determined up to isometry by the Lie algebra of their isometry groups, we can determine X^+ as follows. Let $\mathfrak{g}(X^+) = \mathfrak{k}(X^+) + \mathfrak{p}(X^+)$ be a Cartan decomposition of this Lie algebra. Then a homomorphism of Lie algebras $F : \mathfrak{g}(X^+) \to \mathfrak{g}^+$ preserving the Cartan decompositions, and which defines an isomorphism $\mathfrak{p}(X^+) \rightleftarrows \mathfrak{p}^+$, corresponds to an isometry between X^+ and the symmetric space associated to $\mathfrak{g}(X^+)$. We will describe X^+ by giving a $\mathfrak{g}(X^+)$ and an F.

There are four cases to consider. (The notation for the "type" is Helgason's.)

(2.1) <u>Case 1</u>. $X = BDI(p,2)$, $p \geq 3$. $\mathfrak{g} = so(p,2)$

$$\mathfrak{k} = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in so(p),\ B \in so(2)\}$$

$$\mathfrak{p} = \{i\begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \mid X \in \mathbb{R}(p,2)\}$$

(We use the notation $\mathbb{R}(m,n)$ or $\mathbb{C}(m,n)$ for all matrices over $\mathbb{R}$ or $\mathbb{C}$ with m rows and n columns.) Define σ_o by

$$\sigma_o : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \to \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix},$$

$$\sigma_o : i\begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} = i\begin{pmatrix} 0 & (X_1,X_2) \\ -{}^t(X_1,X_2) & 0 \end{pmatrix} \to i\begin{pmatrix} 0 & (X_2,X_1) \\ -{}^t(X_2,X_1) & 0 \end{pmatrix}.$$

The center of $\mathfrak{k}$ is defined by setting $A = 0$, so that $\sigma_o(Z) = -Z$.

$$\mathfrak{k}^+ = \{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in so(p+2) \mid A \in so(p)\}$$

$$\mathfrak{p}^+ = \{i\begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \in \mathfrak{p} \mid X = (x,x),\ x \in \mathbb{R}(p,1)\}.$$

Let $\underline{\mathfrak{g}} = \underline{\mathfrak{k}} + \underline{\mathfrak{p}}$ be the Lie algebra of type $BDI(p,1)$:

$$\underline{\mathfrak{k}} = \{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in so(p+1) \mid A \in so(p)\}$$

$$\underline{\mathfrak{p}} = \{i\begin{pmatrix} 0 & x \\ -{}^tx & 0 \end{pmatrix} \mid x \in \mathbb{R}(p,1)\}.$$ Define $F : \underline{\mathfrak{g}} \to \mathfrak{g}^+$

by setting

$$F\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{k}^+$$

$$F : i\begin{pmatrix} 0 & x \\ -{}^tx & 0 \end{pmatrix} \to \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & (x,x) \\ -{}^t(x,x) & 0 \end{pmatrix} \in \mathfrak{p}^+.$$

F is in fact an isomorphism, and we have that X^+ is of type $BDI(p,1)$.

(2.2) Case 2. $X = CI$ (Siegel upper half-space).

$\mathfrak{G} = \mathfrak{sp}(n,\mathbb{R})$, $n \geq 1$.

$$\mathfrak{K} = \mathfrak{sp}(n) \cap \mathfrak{so}(2n) = \{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \mid X_1 \in \mathfrak{so}(n), \ X_2 \text{ symmetric} \}$$

$$\mathfrak{P} = \{ i \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} \mid Z_1, Z_2 \in u(n); \ Z_1, Z_2 \text{ symmetric} \}$$

[Note: $\mathfrak{u}(n)$ + symmetric $\Leftrightarrow$ $\mathfrak{u}(n)$ + purely imaginary]. Define σ_o by

$$\sigma_o : \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \to \begin{pmatrix} X_1 & -X_2 \\ X_2 & X_1 \end{pmatrix}$$

$$\sigma_o : i\begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} \to i\begin{pmatrix} Z_1 & -Z_2 \\ -Z_2 & -Z_1 \end{pmatrix}.$$ The center of $\mathfrak{K}$ is

generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so that $\sigma_o(Z) = -Z$.

Remark. When $Sp(n,\mathbb{R})$ acts on the Siegel upper half-space X by linear fractional transformations ($X = \{x \in \mathbb{C}(n,n) \mid {}^t x = x, \ Im(x) > 0\}$), the complex conjugation σ_o is described by $\sigma_o(x) = -\bar{x}$. Then, of course, X^+ = "generalized Y-axis". Moreover, σ_o arises from the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_{2n}(\mathbb{R})$.
The subalgebra $\mathfrak{G}^+ = \mathfrak{K}^+ + \mathfrak{P}^+$ is

$$\mathfrak{K}^+ = \{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \in \mathbb{C}(2n,2n) \mid X \in \mathfrak{so}(n) \}$$

$$\mathfrak{P}^+ = \{ i\begin{pmatrix} Z & 0 \\ 0 & -Z \end{pmatrix} \in \mathbb{C}(2n,2n) \mid Z \in u(n), \text{ symmetric, pure imaginary} \}$$

We claim X^+ has type $\mathfrak{Gl}(n,\mathbb{R})$:

$$\mathfrak{gl}(n,\mathbb{R}) = \{A \in \mathbb{R}(n,n)\}$$

$$= \{\frac{A - {}^tA}{2} + i(\frac{A + {}^tA}{2i}) \mid A \in \mathbb{R}(n,n)\}$$

Define $F : \mathfrak{gl}(n,\mathbb{R}) \to \mathfrak{g}^+$ by

$$F(A) = \begin{pmatrix} \frac{A - {}^tA}{2} & 0 \\ 0 & \frac{A - {}^tA}{2} \end{pmatrix} + i\begin{pmatrix} \frac{A + {}^tA}{2i} & 0 \\ 0 & -\frac{A + {}^tA}{2i} \end{pmatrix}.$$

$\mathfrak{gl}(n,\mathbb{R})$ is reductive, and not semi-simple: $X^+ = \mathbb{R}^1 \times (AI)$ $= \mathbb{R}^1 \times [SL(n,\mathbb{R})/SO(n)]$.

(2.3) <u>Case 3.</u> $X = DIII$, $\mathfrak{g} = \mathfrak{so}^*(2n)$, $n > 2$.

$$\mathfrak{K} = \mathfrak{so}(2n) \cap \mathfrak{sp}(n) = \{\begin{pmatrix} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{pmatrix} \mid Z_1 \in \mathfrak{so}(n),\ Z_2 \text{ symmetric}\}$$

$\mathfrak{P} = \{i\begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \mid X_1, X_2 \in \mathfrak{so}(n)\}$. Define σ_o by

$$\sigma_o : \begin{pmatrix} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{pmatrix} \to \begin{pmatrix} Z_1 & -Z_2 \\ Z_2 & Z_1 \end{pmatrix}$$

$\sigma_o : i\begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \to i\begin{pmatrix} X_1 & -X_2 \\ -X_2 & -X_1 \end{pmatrix}$. Both $\mathfrak{K}$, and the restriction of σ_o to $\mathfrak{K}$ are the same as in Case 2, in particular, $\sigma_o(Z) = -Z$. Also, we note that $\sigma_o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_{2n}(\mathbb{C})$.

$\mathfrak{g}^+ = \mathfrak{K}^+ + \mathfrak{P}^+$ where

$$\mathfrak{K}^+ = \{\begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} \mid Z \in \mathfrak{so}(n)\}$$

$\mathfrak{P}^+ = \{i\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \mid X \in \mathfrak{so}(n)\}$. We give an

$F : \mathfrak{so}(n,\mathbb{C}) \approx \mathfrak{g}^+$, as follows:

$\mathfrak{so}(n,\mathbb{C}) = \{A + Bi \mid A,B \in \mathfrak{so}(n)\}$; Define F by

$F(A + Bi) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + i\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$. X^+ is therefore of type $\mathfrak{so}(n,\mathbb{C})$, and therefore,

$$X^+ = \mathfrak{b}_n \quad \text{for n odd,}$$

$$X^+ = \mathfrak{d}_n \quad \text{for n even.}$$

(2.4) Case 4. $X = AIII$, $\mathfrak{g} = \mathfrak{su}(p,q)$, $p \geq q \geq 1$.

$$\mathfrak{k} = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{su}(p+q) \mid A \in \mathfrak{u}(p), B \in \mathfrak{u}(q), Tr(A)+Tr(B) = 0\}$$

$\mathfrak{p} = \{i\begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix} \mid Z \in \mathbb{C}(p,q)\}$. Define σ_o by

$$\sigma_o : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \to \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{B} \end{pmatrix}$$

$\sigma_o : i\begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix} \to i\begin{pmatrix} 0 & \bar{Z} \\ -{}^tZ & 0 \end{pmatrix}$. Since the center of $\mathfrak{k}$ consists of certain purely imaginary diagonal matrices, $\sigma_o(Z) = -Z$.

$\mathfrak{g}^+ = \mathfrak{k}^+ + \mathfrak{p}^+$ where:

$$\mathfrak{k}^+ = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{so}(p),\ B \in \mathfrak{so}(q),\ Tr(A)+Tr(B) = 0\}$$

$$= \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{so}(p),\ B \in \mathfrak{so}(q)\},$$

$$\mathfrak{p}^+ = \{i\begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \mid X \in \mathbb{R}(p,q)\}.$$

$\mathfrak{g}^+$ is precisely the copy of BDI(p,q) given in Helgason; X^+ is of type BDI(p,q).

Part III. A Complete Set of Elementary Conjugations for X.

3.0. Consider for the moment again an arbitrary bounded symmetric domain X, and suppose it has on it a complex conjugation σ_o, which we have called standard. If X is one of the irreducible domains considered in Part 2, (resp. a product of such domains), then we take σ_o to be the conjugation already constructed (resp. a product of these). We will use σ_o to find all other conjugations and will classify them according to the following

Definition (3.0.1) Let C be the set of all complex conjugations of X, $\sigma_1, \sigma_2 \in C$. For any subgroup $H \subset G$, we say σ_1 and σ_2 are H-equivalent (denoted $\sigma_1 \underset{H}{\sim} \sigma_2$) if $\sigma_1 = h\, \sigma_2\, h^{-1}$ for some $h \in H$. For any point $x \in X$, let C(x) be the subset of C of conjugations fixing x. Denote the quotients with respect to H-equivalence by C/H and C(x)/H. Any set of representatives in C for C/G^h will be a complete set of elementary conjugations.

Remark: (3.0.2) Let $G^h(\sigma_o)$ be the subgroup of G consisting of holomorphic or anti-holomorphic automorphisms. Then $\sigma_1 = g\, \sigma_2\, g^{-1} = (g\sigma_2)\sigma_2(\sigma_2 g^{-1})$ for $g \in G^h(\sigma_o)$ implies $C/G^h = C/G^h(\sigma_o)$. We will eventually show (Cor. (5.0.4)) that the notions of G^h-equivalence and G-equivalence are identical for arbitrary X. This is of course, obvious for X irreducible (since $G^h(\sigma_o) = G$), and, in fact, the general proof uses this fact. We do not, however, ever get the stronger statement that an isometry between two real forms X_1^+ and X_2^+ should extend to an isometry (holomorphic or otherwise) of the domain X. We can only conclude that the isometry $X_1^+ \rightleftarrows X_2^+$ can be replaced by another one which is the restriction of a holomorphic automorphism of X.

Now let $x_o \in X$ be a base point, the same base point as in Part 2 for those spaces already discussed, and assume $\sigma_o(x_o) = x_o$. The (maximal compact) isotropy group at x_o is K. We use the language of Galois cohomology to discuss the relationship of $C(x_o)/K^h$ to C/G^h.

Denote by Gal the group $\{1,\sigma_o\}$, and define an action of Gal on G^h and K^h (in fact, on all of G) by $g \to \sigma_o g \sigma_o$. If we call the invariant subgroups G_+^h and K_+^h, and the fixed points of σ_o, X_o^+, we have $X_o^+ = (G^h/K^h)^{Gal} = G_+^h/K_+^h$. To each $\sigma \in C$ we can associate some $h \in G^h$ by noting $\sigma = h\sigma_o$ for some unique such h; moreover, $h \in K^h$ if and only if $\sigma \in C(x_o)$. Since $\sigma^2 = 1$, we see that an arbitrary $h \in G^h$ is associated to some $\sigma \in C$ if and only if $\sigma_o h \sigma_o = h^{-1}$. Now if we have $\sigma_1 = h_1\sigma_o$ and $\sigma_2 = h_2\sigma_o$, then $\sigma_1 \underset{G}{\sim}{}^h \sigma_2$ if and only if $h_1\sigma_o = gh_2\sigma_o g^{-1}$ for some $g \in G^h$, or equivalently, $h_1 = gh_2\sigma_o g^{-1}\sigma_o$. Moreover, if $\sigma_1, \sigma_2 \in C(x_o)$, then $\sigma_1 \underset{K}{\sim}{}^h \sigma_2$ if and only if $h_1 = kh_2\sigma_o k^{-1}\sigma_o$ for some $k \in K^h$.

We translate these notions by the following

Proposition (3.0.3): The association $C \ni \sigma \to h \in G^h$ (defined by $\sigma = h\sigma_o$) gives bijections

1) $C \longleftrightarrow Z^1(\mathrm{Gal}, G^h)$ = 1-cocycles of Gal in G^h

2) $C(x_o) \longleftrightarrow Z^1(\mathrm{Gal}, K^h)$ = 1-cocycles of Gal in K^h

Moreover, G^h-equivalence on C, and K^h-equivalence on $C(x_o)$ correspond exactly (via the bijections above) with cohomological equivalence in G^h and K^h: We have bijections

1)' $C/G^h \longleftrightarrow H^1(\mathrm{Gal}, G^h)$

2)' $C(x_o)/K^h \longleftrightarrow H^1(\mathrm{Gal}, K^h)$.

Theorem (3.0.4): The natural map $H^1(\mathrm{Gal}, K^h) \xrightarrow{\iota} H^1(\mathrm{Gal}, G^h)$, induced from the inclusion $K^h \subset G^h$, is a bijection.

Proof. (surjectivity) Let $h \in G^h$ be a 1-cocycle of Gal in G^h, and $\sigma = h\sigma_o$ the conjugation associated to it. Some point $x \in X$ is fixed by the map σ, and since G^h acts transitively on $X = G^h/K^h$, we can find some $f \in G^h$ with $f(x) = x_o$. Then $f\sigma f^{-1}(x_o) = f\sigma(x) = f(x) = x_o$, so that $f\sigma f^{-1} \in C(x_o)$. This new conjugation is G^h-equivalent to σ, so the 1-cocycle h is cohomologous (in G^h) to the 1-cocycle associated to $f\sigma f^{-1}$, and this last 1-cocycle is in $Z^1(\mathrm{Gal}, K^h)$. This proves ι onto.

(injectivity) According to Serre ([12], Cohomologie Galoisienne, Cor. 1, p. I-65), the kernel of ι is bijective to the quotient of X_o^+ by G_+^h; but we have $X_o^+ = G_+^h/K_+^h$ so this kernel is precisely the trivial element of $H^1(\mathrm{Gal},K^h)$. This fact alone does not give injectivity of ι since the cohomology sets are not groups. However, we do have that no conjugation $\sigma \in C(x_o)$ is G^h-equivalent to the standard σ_o unless they were K^h-equivalent, using the second part of the above Proposition. Since we were arbitrary in calling σ_o standard, the same statement is true of any two $\sigma_1,\sigma_2 \in C(x_o) : \sigma_1 \underset{G^h}{\sim} \sigma_2 \Rightarrow \sigma_1 \underset{G^h}{\sim} \sigma_2$. Translating again to the cohomology sets, we get the injectivity of ι .

Remark. We can use this theorem to compute C/G^h, and obtain a complete set of elementary complex conjugations, at least for the irreducible spaces X of Part 2, using the standard σ_o constructed there. The four cases are far from independent; look at the compacts $\mathfrak{K}$ and the restrictions of σ_o to them.

(3.1) Case 1. $X = BDI(p,2)$, $p \geq 3$. $G = \mathrm{Isom}(X) = O(p,2)/\{\pm 1\}$ (see the appendix of the article by Baily and Borel [2]).

p odd: In this case $\det(-1) = -1$, so we can identify G with the subgroup $SO(p,2)$. Then $K = \{g \in O(p)\times O(2) \mid \det(g) = +1\}$, and $K^h = K_o = SO(p)\times SO(2)$. Recall the Lie algebra $\mathfrak{K} = \mathfrak{so}(p)\times\mathfrak{so}(2)$, and that the standard σ_o constructed in (2.1) was $(\mathrm{id})\times(-\mathrm{id})$. This implies that the action of σ_o, acting as an inner automorphism in G, is $(\mathrm{id})\times(\mathrm{inversion})$ on $K^h = SO(p)\times SO(2)$. In fact, we can see that σ_o is the matrix

$$(-1)\cdot\left(\begin{array}{c|cc} 1 & & \\ \hline & 0 & 1 \\ & 1 & 0 \end{array}\right) \in K - K^h .$$

We have to compute the cohomology set $H^1(\mathrm{Gal},K^h) = H^1(\{1,\sigma_o\},SO(p)\times SO(2))$. Since the action of Gal on K^h is a product of its actions on $SO(p)$ and $SO(2)$, we get $H^1(\mathrm{Gal},K^h) = H^1(\mathrm{Gal},SO(p))\times H^1(\mathrm{Gal},SO(2))$, the action being trivial on $SO(p)$, and $\sigma_o(\lambda) = \lambda^{-1}$ for $\lambda \in SO(2)$.

i) $Z^1(\mathrm{Gal},SO(p)) = \{A \in SO(p) \mid A = A^{-1}\}$, and $A_1,A_2 \in Z^1(\mathrm{Gal},SO(p))$ are cohomologous if and only if there is an $\Theta \in SO(p)$ with

$A_1 = \Theta A_2(\Theta^{-1})^{\sigma_o} = \Theta A_2\Theta^{-1}$. Moreover, such an Θ exists in $SO(p)$ if and only if A_1 and A_2 are conjugate in $O(p)$, for if $A_1 = \Theta' A_2\Theta'^{-1}$ and $\det(\Theta') = -1$, then $A_1 = \Theta A_2\Theta^{-1}$ with $\Theta = (-1)\cdot\Theta' \in SO(p)$. Finally, any $A \in SO(p)$ with $A^2 = 1$ is conjugate (in $SO(p)$) to one of the matrices

$$I_{k,p-k} = \left(\begin{array}{c|c} -1_k & 0 \\ \hline 0 & 1_{p-k} \end{array}\right), \text{ and } 0 \le k \text{ even} < p.$$

Therefore, $H^1(\mathrm{Gal},SO(p)) = \{I_{k,p-k}; k \text{ even}, 0 \le k < p\}$

ii) $Z^1(\mathrm{Gal},SO(2)) = \{\lambda \in SO(2) \mid \lambda^{\sigma_o} = \lambda^{-1}\} = SO(2)$. Now we show that all $\lambda \in SO(2)$ are cohomologous to the identity: We have to find some $\eta \in SO(2)$ with $\lambda = \eta 1(\eta^{-1})^{\sigma_o} = \eta^2$, and of course λ does have a square root in $SO(2)$. Therefore, $H^1(\mathrm{Gal},SO(2)) = \{1\}$.

Finally, we can identify the cohomology set $H^1(\mathrm{Gal},K^h)$ with $\{I_{k,p-k}; k \text{ even}\}$. The corresponding elementary complex conjugations are:

$$\left\{ (-1)\cdot\left(\begin{array}{cc|cc} -1_k & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \hline 0 & 1_{p-k} & & \\ \hline \multicolumn{2}{c|}{0} & 0 & 1 \\ & & 1 & 0 \end{array}\right) \;\middle|\; k \text{ even} \right\}$$

, and we denote these conjugations by σ_k, $0 \le k$ even $< p$. We defer computation of their fixed-point-sets X_k^+ until we study the case p even.

<u>p even</u>: Now we have $\det(-1) = +1$; $-1 \in O(p,2)_o$, so the group $G = O(p,2)/\{\pm 1\}$ has four connected components, two holomorphic and two anti-holomorphic. Also, $K = [O(p)\times O(2)]/\{\pm 1\}$ has four components, the subgroup K^h being $[O(p)\times SO(2)]/\{\pm 1\}$. As when p was odd, $\mathcal{K}$ and σ_o have the form $\sigma_o = (\mathrm{id})\times(-\mathrm{id})$: $\mathfrak{so}(p)\times\mathfrak{so}(2) \to \mathfrak{so}(p)\times\mathfrak{so}(2)$. The action of Gal (i.e., σ_o) on K^h is actually induced from the action $\sigma_o = (\mathrm{id})\times(\mathrm{inversion})$: $O(p)\times SO(2) \to O(p)\times SO(2)$, because σ_o has the same matrix:

$$(-1)\cdot\left(\begin{array}{c|cc} 1 & & \\ \hline & 0 & 1 \\ & 1 & 0 \end{array}\right).$$

We have $Z^1(\mathrm{Gal},K^h) = Z^1(\mathrm{Gal},[O(p)\times SO(2)]/\{\pm 1\}) =$
$\{\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix} \in O(p)\times SO(2) \mid \begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}^{\sigma_o} = \begin{pmatrix} A & 0 \\ 0 & B^{-1}\end{pmatrix} = \pm\begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1}\end{pmatrix}\}/\{\pm 1\}$.

Since the condition $B^{-1} = -B^{-1}$ is absurd; we need only consider $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in O(p)\times SO(2)$ with $A = A^{-1}$, and no requirements on B.

By the same method as in ii), of the case p odd, we have that $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is cohomologous to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Recalling that the action on the factor O(p) is trivial, the same reasoning of part i) of p odd gives that $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ is cohomologous to one of the matrices

$$\left(\begin{array}{c|c|c} -1_k & & \\ \hline & +1_{p-k} & \\ \hline & & 1_2 \end{array}\right) \quad , \ 0 \le k \le p.$$

But since we are in the quotient $[O(p)\times SO(2)]/\{\pm 1\}$, we need only consider those matrices for $k \le p/2$: If $k \ge p/2$,

$$(-1)\cdot\left(\begin{array}{c|c} I_{k,p-k} & \\ \hline & 1_2 \end{array}\right) = \left(\begin{array}{c|c} I_{p-k,k} & \\ \hline & -1_2 \end{array}\right) \quad \text{and}$$

$p-k \le p/2$. Again we dispose of the "-1_2" by the method of part ii) of the case p odd, as above.

The result is $H^1(\mathrm{Gal},K^h) = \left\{\left(\begin{array}{c|c} I_{k,p-k} & 0 \\ \hline 0 & 1_2 \end{array}\right) \middle| \, 0 \le k \le p/2\right\}$,

and the elementary conjugations are

$$\left\{\sigma_k = \left(\begin{array}{c|c} I_{k,p-k} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right) \in [O(p)\times O(2)]/\{\pm 1\} \,\middle|\, 0 \le k \le p/2\right\}.$$

We give the fixed-point-sets X_k^+ of σ_k for $0 \leq k < p$. The effect of σ_k on the Lie algebra $\mathfrak{g} = \mathfrak{so}(p,2)$ is as follows:

$$\mathfrak{k} = \left\{ \left(\begin{array}{cc|c} A_1 & A_2 & \\ A_3 & A_4 & 0 \\ \hline & 0 & B \end{array} \right) \in \mathfrak{so}(p+2) \;\middle|\; \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathfrak{so}(p), B \in \mathfrak{so}(2) \right\}$$

$$\mathfrak{p} = \left\{ i \begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \middle| X = \begin{pmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \end{pmatrix} \in \mathbb{R}(p,2) \right\}$$

$$\sigma_k : \left(\begin{array}{cc|c} A_1 & A_2 & \\ A_3 & A_4 & 0 \\ \hline & 0 & B \end{array} \right) \to \left(\begin{array}{cc|c} A_1 & -A_2 & \\ -A_3 & A_4 & 0 \\ \hline & 0 & -B \end{array} \right)$$

$$\sigma_k : \; i \begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \to i \begin{pmatrix} 0 & \begin{pmatrix} -X_{21} & -X_{11} \\ X_{22} & X_{12} \end{pmatrix} \\ -{}^t\begin{pmatrix} -X_{21} & -X_{11} \\ X_{22} & X_{12} \end{pmatrix} & 0 \end{pmatrix} .$$

Then $\mathfrak{g}_k^+ = \mathfrak{k}_k^+ + \mathfrak{p}_k^+$ where:

$$\mathfrak{k}_k^+ = \left\{ \left(\begin{array}{cc|c} A_1 & 0 & \\ 0 & A_4 & 0 \\ \hline & 0 & 0 \end{array} \right) \in \mathfrak{so}(p+2) \right\} = \mathfrak{so}(k) \times \mathfrak{so}(p-k) \times \{0\}$$

$$\mathfrak{p}_k^+ = \left\{ i \begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \middle| X = \begin{pmatrix} X_1 & -X_1 \\ X_2 & X_2 \end{pmatrix} \in \mathbb{R}(p,2), \begin{array}{l} X_1 \in \mathbb{R}(k,1) \\ X_2 \in \mathbb{R}(p-k,1) \end{array} \right\}.$$

We define an isomorphism $F : \mathfrak{so}(k,1) \times \mathfrak{so}(p-k,1) \rightleftarrows \mathfrak{g}_k^+$, as follows.

$$\left(\begin{pmatrix} A & iX \\ -{}^t iX & 0 \end{pmatrix}, \begin{pmatrix} B & iY \\ -{}^t iY & 0 \end{pmatrix}\right) \in \mathfrak{so}(k,1)\times \mathfrak{so}(p-k,1)$$

$$\Big\downarrow F$$

$$\left(\begin{array}{c|c} \begin{matrix} A & 0 \\ 0 & B \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) + \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & \begin{pmatrix} X & -X \\ Y & Y \end{pmatrix} \\ -{}^t\begin{pmatrix} X & -X \\ Y & Y \end{pmatrix} & 0 \end{pmatrix} \in \mathfrak{g}_k^+ .$$

It is easy to check that F is an isomorphism, and thus we get an isometry $X_k^+ \approx BDI(k,1)\times BDI(p-k,1)$. (Of course, when k=0, this is the same F we gave on p 26 for the standard conjugation σ_o.)

(3.2) Case 2. $X = CI$ (Siegel upper half-space).
$\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$, $n \geq 1$. The holomorphic isometry group is $Sp(n,\mathbb{R})$ divided by its center: $G^h = Sp(n,\mathbb{R})/\{\pm 1\}$. Recall that any maximal compact of $Sp(n,\mathbb{R})$ is isomorphic to $U(n)$, and we embed $U(n)$ into $Sp(n,\mathbb{R})$ as usual by

$$U(n) \ni A + Bi \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(n,\mathbb{R}).$$

The effect of the standard σ_o on $\mathfrak{K}$ (2.2) is:

$$\mathfrak{K} = \mathfrak{sp}(n) \cap \mathfrak{so}(2n) = \left\{\begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \,\middle|\, \begin{matrix} X_1 \in \mathfrak{so}(n) \\ X_2 \text{ symmetric} \end{matrix}\right\}$$

$\sigma_o: \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \to \begin{pmatrix} X_1 & -X_2 \\ X_2 & X_1 \end{pmatrix}$. Then the effect of σ_o on matrices $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ of $Sp(n,\mathbb{R})$ is $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$.

Pulling σ_o back (by the above embedding) to U(n), we have an isomorphic action of Gal on U(n):

$$U(n) \ni U \xrightarrow{\sigma_o} \bar{U}.$$

Now, since $G = Sp(n,R)/\{\pm 1\}$, we should consider the induced action on the quotient $U(n)/\{\pm 1\}$. Then a set of elementary conjugations for X is given by $H^1(\mathrm{Gal}, U(n)/\{\pm 1\})$, with the above action.

The result is the following

Theorem (3.2.1.):

1) If n is odd, $H^1(\mathrm{Gal}, U(n)/\pm 1) = \{1\}$

2) If n is even, $H^1(\mathrm{Gal}, U(n)/\pm 1)$ has two elements represented by the identity, and the matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. The 1-cocycles $Z^1(\mathrm{Gal}, U(n)/\{\pm 1\}) = \{A \in U(n) \mid \bar{A} = \pm A^{-1}\}/\{\pm 1\}$

Case 1. $\bar{A} = A^{-1}$. There exists a matrix $P \in U(n)$, and a diagonal matrix

$$\Lambda = \begin{pmatrix} \Lambda_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \Lambda_k \end{pmatrix}$$

, where the Λ_i are scalar matrices $\lambda_i \cdot 1$, and $\lambda_i \neq \lambda_j$ for $i \neq j$, and such that $A = P^{-1} \Lambda P$. Denote by M, $\bar{P}P^{-1}$; then $\bar{A} = A^{-1}$ implies $M\Lambda = \Lambda M$. Moreover, $P^{-1}\Lambda P = \bar{P}^{-1}\bar{P}P^{-1}\Lambda P = \bar{P}^{-1}(M\Lambda)P$, and therefore, $M\Lambda$ is a 1-cocycle (with $\overline{M\Lambda} = (M\Lambda)^{-1}$) cohomologous to A. Now the condition $M\Lambda = \Lambda M$ (plus $\lambda_i \neq \lambda_j$) implies that M is a matrix of blocks,

$$M = \begin{pmatrix} \boxed{M_1} & & \\ & \ddots & \\ & & \boxed{M_k} \end{pmatrix}$$, where the blocks are the same size as the Λ_i.

<u>Assume k = 1.</u> Then $A = P^{-1}\lambda P = \lambda$ for some scalar matrix $\lambda \cdot 1$, $\lambda \in U(1)$. Recall the computation of $H^1(\mathrm{Gal}, SO(2))$ with the action given by inversion in SO(2), (3.1, p odd, ii)). By the same method we get that the 1-cocycle $A = \lambda$ is cohomologous to the identity. These statements apply also to the case $n = 1$, since k is automatically one there also, namely, that 1-cocycles $A \in U(1)/\{\pm 1\}$ with $\bar{A} = A^{-1}$ are cohomologous to 1. We want to use induction on n to prove that 1-cocycles $A \in U(n)/\{\pm 1\}$ with $\bar{A} = A^{-1}$ are cohomologous to 1. So assume this statement true for $m < n$.

<u>Arbitrary k</u>. Now $\overline{M\Lambda} = (M\Lambda)^{-1}$ implies that $\overline{M_i\Lambda_i} = (M_i\Lambda_i)^{-1}$ for each i, and that $M_i\Lambda_i \in Z^1(\mathrm{Gal}, U(m_i))$ for some $m_i < n$. The induction gives us that $M_i\Lambda_i$ is cohomologous to 1 (in $U(m_i)$) and therefore, $M\Lambda$ is cohomologous to 1 in U(n), and therefore, our A also.

<u>Remark</u>. So far we have proved that $H^1(\mathrm{Gal}, U(n))$ is trivial for <u>all</u> n, since $Z^1(\mathrm{Gal}, U(n)) = \{A \in U(n) \mid \bar{A} = A^{-1}\}$.

Case 2: $\bar{A} = -A^{-1}$. Taking determinants (for example) implies that this condition is absurd if n is odd, so that when n is odd, the only 1-cocycles are those in Case 1. Therefore, we have the first half of the theorem: $H^1(\mathrm{Gal}, U(n)/\{\pm 1\}) = \{1\}$ for n odd. We proceed with a method analogous to the one in the first case. Our A is conjugate by some $P \in U(n)$ to a matrix Λ of the form

$$\Lambda = \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_k \end{pmatrix},$$

where the Λ_i are blocks of the form $\Lambda_i = \lambda_i J = \lambda_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\lambda_i \neq \pm \lambda_j$ for $i \neq j$. Denote again $M = \bar{P}P^{-1}$ $(A = P^{-1}\Lambda P)$; since $\bar{\Lambda} = -\Lambda^{-1}$ we get again $M\Lambda = \Lambda M$. Just as before we have A cohomologous to $M\Lambda$, and therefore, $\overline{M\Lambda} = -(M\Lambda)^{-1}$. The condition $M\Lambda = \Lambda M$ (plus $\lambda_i \neq \pm \lambda_j$, for $i \neq j$) implies that M is a matrix of blocks

$$M = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_k \end{pmatrix},$$

where M_i is the same size as Λ_i.

Assume $k = 1$: Then $A = P^{-1}\lambda\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}P$, with a scalar $\lambda \in U(1)$. $\bar{A} = -A^{-1}$ implies that $\frac{1}{\lambda} A = P^{-1}JP$ is purely real, namely, $\frac{1}{\lambda} A \in O(n)$. Now A and $\frac{1}{\lambda} A$ are cohomologous in $U(n)$, (the factor $\frac{1}{\lambda} \in U(1)$ can be eliminated just as in Case 1 ($k=1$)). Therefore, we can work with $B = \frac{1}{\lambda} A \in O(n)$. Since A and B are cohomologous, $\bar{B} = -B^{-1} = B$ (since B is real.) $B^2 = -1$ implies the existence of some $\Theta \in O(n)$ with

$B = \Theta^{-1}J\Theta = \Theta^{-1}J\bar{\Theta}$, and the last equality says B and J are cohomologous, and therefore A also. These remarks apply to the case $n = 2$, since k is automatically one there also, and we have that 1-cocycles $A \in U(2)/\{\pm 1\}$ with $\bar{A} = -A^{-1}$ are cohomologous to J. We use induction on (even) n to show 1-cocycles A with $\bar{A} = -A^{-1}$ are cohomologous to J. Assume this for (even) $m < n$.

Arbitrary k: $\overline{M\Lambda} = -(M\Lambda)^{-1}$ implies that $\overline{M_i\Lambda_i} = -(M_i\Lambda_i)^{-1}$ for each i, so that $M_i\Lambda_i \in Z^1(\mathrm{Gal}, U(m_i)/\{\pm 1\})$ for some even $m_i < n$. The induction gives that $M_i\Lambda_i$ is cohomologous to J in $\frac{U(m_i)}{\{\pm 1\}}$, therefore $M\Lambda$ is cohomologous to a block matrix

$$\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}$$

, where the J's are blocks of size m_i.

This last matrix is real, so that it is conjugate (also cohomologous) by some $\Theta \in O(n)$ to $J \in U(n)$.

We conclude the proof by showing J cannot be cohomologous to 1. If so, we would have $J = P^{-1} 1 \bar{P} = P^{-1}\bar{P}$ for some $P \in U(n)$. But $P^{-1}\bar{P}$ is symmetric and J is not, so this is impossible.

The theorem says that there is only one conjugation on $X = CI(n) = Sp(n,\mathbb{R})/U(n)$ for n odd (up to G-equivalence). In particular, the fixed space of any anti-holomorphic involution is isometric to the generalized "Y-axis" = $\mathbb{R}^1 \times [SL_n(\mathbb{R})/SO(n)]$. For even n, there are two possible fixed-point-sets, associated to the standard σ_o and to σ_1, which is the composition of σ_o and the matrix in $Sp(n,\mathbb{R})$ associated to $J \in U(n)$:

$$\sigma_1 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \cdot \sigma_o = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

It is easy to write down the fixed Lie algebra $\mathfrak{g}_1^+$ of σ_1, as a sub-algebra of $\mathfrak{sp}(n,\mathbb{R})$. However I have not yet identified its type, except for n = 2. (This was an accident - see Part 4.)

$$\mathfrak{k}_1^+ = \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2 & X_1 \end{pmatrix} \,\middle|\, \begin{matrix} X_1 \in \mathfrak{so}(n), \\ X_2 \text{ symmetric} \end{matrix} \quad X_1 = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},\ X_2 = \begin{pmatrix} C & D \\ D & -C \end{pmatrix} \right\}$$

$$\mathfrak{p}_1^+ = \left\{ i \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} \,\middle|\, Z_1, Z_2 \in \mathfrak{u}(n), \text{ symmetric}; \right.$$

$$\left. Z_1 = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix},\ Z_2 = \begin{pmatrix} R & S \\ S & -R \end{pmatrix} \right\}$$

(3.3) Case 3. $X = \mathrm{DIII}$, $n > 2$. According to Cartan, [5], Theórème H, p. 152 the group G^h is connected in all cases. However, this is contradictory to his earlier paper [4], p. 459, in which he shows that when $n = 4$, X is isometric to $\mathrm{BDI}(6,2)$ (at least their compact duals, but this is equivalent). He does not in fact state explicitly that the spaces are isomorphic as hermitian symmetric spaces, although this is automatic. For if $F : X \to \mathrm{BDI}(6,2)$ is an isometry, then it is either holomorphic or anti-holomorphic (for the same reason that isometries of an irreducible hermitian symmetric space onto itself are holomorphic or anti-holomorphic). Then (if necessary) composing with an anti-holomorphic map on either side (we have constructed these) produces a holomorphic isometry. K. Morita ([11], p. 195) actually constructs a holomorphic equivalence. We come back to this in Part 4.

So now $n = 3$ or $n > 4$, and G^h is connected. $G^h = SO^*(2n)/\{\pm 1\}$ and $K^h = [SO^*(2n) \cap Sp(n)]/\{\pm 1\}$ which is precisely the same group (contained also in $Sp(n,\mathbb{R})/\{\pm 1\}$) as K^h for the Siegel upper half-space. Moreover, the action of our standard σ_0 is exactly the same, and we have already computed $H^1(\mathrm{Gal},K^h)$. It is given by the theorem in the last section.

We give here the fixed Lie algebra $\mathfrak{g}_1^+$ for the non-standard conjugation σ_1, when n is even.

$\sigma_1 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \cdot \sigma_o = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$, as before.

$$\mathfrak{k}_1^+ = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{pmatrix} \middle| \begin{array}{l} Z_1 \in \mathfrak{so}(n), \\ Z_2 \text{ symmetric} \end{array} \quad Z_1 = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix},\ Z_2 = \begin{pmatrix} R & S \\ S & -R \end{pmatrix} \right\}$$

$$\mathfrak{p}_1^+ = \left\{ i \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \middle| X_1, X_2 \in \mathfrak{so}(n),\ X_1 = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},\ X_2 = \begin{pmatrix} C & D \\ D & -C \end{pmatrix} \right\}.$$

(3.4) <u>Case 4.</u> $X = AIII(p,q)$, $p \geq q \geq 1$. $\mathfrak{g} = \mathfrak{su}(p,q)$.

According to Cartan ([5], Théorème H), the group G^h is connected except when $p = q \geq 2$, so eliminating that case for the moment we have $G^h = U(p,q)/U(1)$, where by $U(1)$ I mean the scalar matrices $\lambda \cdot 1_{p+q}, \lambda \in \mathbb{C}^*, |\lambda| = 1$. Then $K^h = [U(p) \times U(q)]/U(1)$, and the action of σ_o is defined by complex conjugation of a matrix $U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in U(p) \times U(q)$.

We have to compute $H^1(\mathrm{Gal}, [U(p) \times U(q)]/U(1))$ with this action. The 1-cocycles $Z^1(\mathrm{Gal}, K^h)$ are
$\{U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in U(p) \times U(q) \mid \bar{U} = \lambda U^{-1}$ for some $\lambda \in U(1)\}/U(1)$.
First we observe that $\bar{U} = \lambda U^{-1}$ implies $\lambda = \pm 1$:
$U = \bar{\lambda} \bar{U}^{-1}$ so that $\bar{U} = \lambda(\bar{\lambda}\bar{U}^{-1})^{-1} = \lambda^2 \bar{U}$.

<u>Case 1</u>: $\bar{U} = U^{-1}$. Then we have $\bar{A} = A^{-1}$ and $\bar{B} = B^{-1}$ with $A \in U(p)$, $B \in U(q)$. By the theorem in (3.2), A and B are cohomologous, as elements of $Z^1(\mathrm{Gal}, U(p))$ and $Z^1(\mathrm{Gal}, U(q))$, to the identity. This means U itself is cohomologous to 1.

Case 2. $\bar{U} = -U^{-1}$. As before (by determinants) we must have both p and q even, because $\bar{A} = -A^{-1}$, $\bar{B} = -B^{-1}$. In that case then by the same theorem, A and B are cohomologous to J-matrices of size p, and q, and so U is cohomologous to $\left(\begin{array}{c|c} J & 0 \\ \hline 0 & J \end{array}\right)$.

We have proved the following

Theorem (3.4.1): The space $X = AIII(p,q)$ with $p = q = 1$ or $p > q \geq 1$ has:

1) one elementary conjugation σ_0 if p and q are not both even,

2) two elementary conjugations σ_0 and $\sigma_1 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \circ \sigma_0$ if p and q are both even.

We give the fixed space X_1^+ of σ_1 above after studying the case $X = AIII(p,p)$, $p \geq 2$, since it occurs there, too. The group G^h is not connected here (Cartan [5], Théorème H); it has two connected components, G_0^h being $U(p,p)/U(1)$ as before. We give a generator T for $G^h - G_0^h$, by giving an automorphism of the Lie algebra $\mathfrak{su}(p,p)$ of G^h:

Recall $\mathfrak{g} = \mathfrak{su}(p,p) = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{k} = \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} | A,B \in \mathfrak{u}(p),\ Tr(A+B) = 0\}$$

$$\mathfrak{p} = \{i\begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix}\ Z \in \mathbb{C}(p,p)\}.$$

Define T by

$$T : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \to \begin{pmatrix} \bar{B} & 0 \\ 0 & \bar{A} \end{pmatrix}$$

$T : i\begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix} \to i\begin{pmatrix} 0 & -{}^tZ \\ \bar{Z} & 0 \end{pmatrix}$. First T represents a holomorphic isometry of X, since the center of $\mathfrak{K}$ is the set of diagonal matrices

$$\mathbb{R} \cdot \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and T is the identity on such elements. Next, T fixes the base point x_o corresponding to $\mathfrak{K}$, so that if we had $T \in G_o^h$, we would have $T \in K_o^h$. But this is impossible because of the restriction of T to $\mathfrak{K}$, and hence its effect on K_o^h as an inner automorphism of G^h. Therefore, we do have $T \in K^h - K_o^h$. Finally, it is evident that T and σ_o commute on $\mathfrak{G}$, and therefore, $T^{\sigma_o} = T$ when σ_o acts on K^h.
(Remark: When $p = 1$, we can define a holomorphic isometry by the above formulas for T; it is the Cartan involution, so we would have $T \in K_o^h$.)

We can now give the elementary conjugations of AIII(p,p), $p > 1$. The 1-cocycles of Gal in K^h are

$$\{U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in U(p)\times U(p) \mid \bar{U} = \lambda U^{-1} \text{ for some } \lambda \in U(1)\}/U(1)$$

$$\cup \ \{UT = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\circ T \mid A,B \in U(p),\ \bar{U}T = \lambda TU^{-1} \text{ for some } \lambda \in U(1)\}/U(1).$$

By the above discussion for $p \neq q$, we can reduce the first set to just the identity matrix if p is odd, or to the identity and $\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$ if p is even. The second set consists of elements UT such that $U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $B = \bar{\lambda}A^{-1}$.

<u>Lemma 1</u>: The 1-cocycle $\begin{pmatrix} A & 0 \\ 0 & \bar{\lambda}A^{-1} \end{pmatrix} \circ T$ is cohomologous to $\begin{pmatrix} 1 & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \circ T$.

<u>Proof</u>. Denote by b_2 any $\sqrt{A}$ in the group U(p), and $b_1 = b_2^{-1}$. Then we have

$$\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \bar{\lambda} \end{pmatrix} T \begin{pmatrix} \bar{b}_1 & 0 \\ 0 & \bar{b}_2 \end{pmatrix} T = \begin{pmatrix} b_1^{-1} & 0 \\ 0 & \bar{\lambda} b_2^{-1} \end{pmatrix} \begin{pmatrix} b_2 & 0 \\ 0 & b_1 \end{pmatrix}$$

$$= \begin{pmatrix} b_1^{-1} b_2 & 0 \\ 0 & \bar{\lambda}(b_1^{-1} b_2)^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \bar{\lambda}A^{-1} \end{pmatrix}.$$ Multiplying by T on the right gives the lemma.

<u>Lemma 2</u>: $\begin{pmatrix} 1 & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \cdot T$ is cohomologous to T.

<u>Proof</u>. Denote $c_1 = e^{\frac{2\pi i \theta}{4}}$, if $\bar{\lambda} = e^{2\pi i \theta}$, and $c_2 = c_1^{-1}$.

$$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}^{-1} T \begin{pmatrix} \bar{c}_1 & 0 \\ 0 & \bar{c}_2 \end{pmatrix} T = \begin{pmatrix} c_1^{-1} c_2 & 0 \\ 0 & (c_1^{-1} c_2)^{-1} \end{pmatrix} = \begin{pmatrix} e^{-\pi i \theta} & 0 \\ 0 & e^{\pi i \theta} \end{pmatrix}.$$

The last matrix multiplied by $e^{\pi i \theta}$ represents the same element of K_o^h. This gives $\begin{pmatrix} 1 & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$, and then multiplying by T on the right as before gives the lemma.

<u>Lemma 3</u>: The 1-cocycle T is not cohomologous to any 1-cocycle U in K_o^h.

<u>Proof</u>. If so, we would have some $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \in U(p) \times U(p)$ with either

1) $T = \epsilon_1 \begin{pmatrix} d_1^{-1} & 0 \\ 0 & d_2^{-1} \end{pmatrix} U \begin{pmatrix} \bar{d}_1 & 0 \\ 0 & \bar{d}_2 \end{pmatrix}$,

or 2) $T = \epsilon_2 T \begin{pmatrix} d_1^{-1} & 0 \\ 0 & d_2^{-1} \end{pmatrix} U \begin{pmatrix} \bar{d}_1 & 0 \\ 0 & \bar{d}_2 \end{pmatrix} T$ for some $\epsilon_1, \epsilon_2 \in U(1)$.

But both right hand sides are contained in K_o^h, and T is not.

We have proved the following

<u>Theorem</u> (3.4.2): The space $X = AIII(p,p)$ with $p \geq 2$ has

1) two elementary conjugations, σ_o and $T \circ \sigma_o = \sigma_2$ if p is odd,

2) three elementary conjugations; $\sigma_o, \sigma_1 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \circ \sigma_o$, and $\sigma_2 = T\sigma_o$, if p is even.

<u>Theorem</u> (3.4.3): The fixed spaces X^+ of $\sigma_o, \sigma_1 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \sigma_o$, and $\sigma_2 = T\sigma_o$ are respectively:

i) $X_o^+ = BDI(p,q) = \mathfrak{so}(p,q))$

ii) $X_1^+ = CII(p/2,q/2) = \mathfrak{sp}(p/2,q/2))$

iii) $X_2^+ = \mathfrak{a}_{p-1} \times \mathbb{R}^1 = \mathbb{R}^1 \times \mathfrak{sl}(p,\mathbb{C})$

<u>Proof</u>. We have already done i).

ii) We give an isomorphism between $\mathfrak{g}_1^+$ and the form of CII given in Helgason (p. 351):

$\mathfrak{k}_1^+ = \{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in u(p) \times u(q) \mid Tr(A)+Tr(B)=0, A = \begin{pmatrix} a_1 & a_2 \\ -\bar{a}_2 & \bar{a}_1 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ -\bar{b}_2 & \bar{b}_1 \end{pmatrix} \}$

$\mathfrak{p}_1^+ = \{ i \begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix} \mid Z = \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in \mathbb{C}(p,q) \}$. Define

$F : \mathcal{G}_1^+ \to \mathfrak{sp}(p/2,q/2)$ by

$$F : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \to \left(\begin{array}{cc|cc} a_1 & 0 & a_2 & 0 \\ 0 & b_1 & 0 & b_2 \\ \hline -\bar{a}_2 & 0 & \bar{a}_1 & 0 \\ 0 & -\bar{b}_2 & 0 & \bar{b}_1 \end{array}\right),$$

$$F : i\begin{pmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{pmatrix} \to i\left(\begin{array}{cc|cc} 0 & Z_1 & 0 & Z_2 \\ -{}^t\bar{Z}_1 & 0 & {}^tZ_2 & 0 \\ \hline 0 & -\bar{Z}_2 & 0 & \bar{Z}_1 \\ -{}^t\bar{Z}_2 & 0 & -{}^tZ_1 & 0 \end{array}\right).$$

It is straightforward, although tedious, to check that F is an isomorphism.

iii) First $\mathcal{G}_2^+$ for $\sigma_2 = T\sigma_o$ is given by:

$$\mathcal{K}_2^+ = \{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} | A \in \mathfrak{su}(p)\}$$

$\mathcal{P}_2^+ = \{i\begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix} | Z \in \mathbb{C}(p,p),\ Z = -{}^t\bar{Z}\}$. Define an

$F : \mathcal{G}_2^+ \to \mathcal{G}\ell_p(\mathbb{C})$ by

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + i\begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix} \overset{F}{\longmapsto} A + iZ \in \mathbb{C}(p,p) = \mathcal{G}\ell_p(\mathbb{C}).$$

F is a Lie homomorphism, injective, and sends $\mathcal{P}_2^+$ onto the quotient $\mathcal{G}\ell_p(\mathbb{C})/u(p)$, and $\mathcal{K}_2^+$ isomorphically onto $\mathfrak{su}(p) \subset u(p)$. An inverse to F is described (on the image of $F \subsetneq \mathcal{G}\ell(p,\mathbb{C})$) by

$$\mathfrak{gl}_p(\mathbb{C}) \ni B \to \begin{pmatrix} \frac{B-{}^t\bar{B}}{2} & 0 \\ 0 & \frac{B-{}^t\bar{B}}{2} \end{pmatrix} + i \begin{pmatrix} 0 & \frac{B+{}^t\bar{B}}{2i} \\ \frac{B+{}^t\bar{B}}{2i} & 0 \end{pmatrix} .$$

F induces an isometry $X_2^+ \approx GL_p(\mathbb{C})/U(p) = \mathbb{R}^1 \times [SL_p(\mathbb{C})/SU(p)]$.

We remark that the subalgebra of codimension one in $\mathfrak{g}_2^+$ defined by $tr(Z) = 0$ is isomorphic to

$\mathfrak{sl}_p(\mathbb{C})$ by $F^{-1}\mathfrak{sl}_p(\mathbb{C})$.

Real Forms of Irreducible Hermitian Symmetric Spaces.

X	Number of real forms	X^+
$\mathfrak{so}(p,2)(p\neq 2)$		$\mathfrak{so}(k,1)\times\mathfrak{so}(p-k,1)$
p odd	$\frac{p+1}{2}$	$0 \leq k < p$ (k even)
p even	$\frac{p+2}{2}$	$0 \leq k \leq p/2$
$\mathfrak{sp}(n,\mathbb{R})$		
n odd	1	$\mathbb{R} \times \mathfrak{sl}(n,\mathbb{R})$
n even	2	$\mathbb{R} \times \mathfrak{sl}(n,\mathbb{R})$, ? (cf. §2.2)
$\mathfrak{so}^*(2n)(n=3$ or $n>4)$		
n odd	1	$\mathfrak{so}(n,\mathbb{C})$
n even	2	$\mathfrak{so}(n,\mathbb{C})$, ? (cf. §2.3)
$\mathfrak{su}(p,q)(p\geq q\geq 1)$		
$p=q=1$ or $p>q$		
p,q not both even	1	$\mathfrak{so}(p,q)$
p,q both even	2	$\mathfrak{so}(p,q)$, $\mathfrak{sp}(p/2,q/2)$
$p=q \geq 2$		
p odd	2	$\mathfrak{so}(p,p)$, $\mathbb{R}\times\mathfrak{sl}p(\mathbb{C})$
p even	3	$\mathfrak{so}(p,p)$, $\mathbb{R}\times\mathfrak{sl}_p(\mathbb{C})$, $\mathfrak{sp}(p/2,p/2$

Part IV. Isomorphisms.

4.0. It is known (Helgason, [7], p. 351-353), that in low dimensions there are isomorphisms between certain Lie algebras of "different" type. This implies certain holomorphic equivalences between hermitian symmetric spaces of "different" types. We can use these equivalences to check our computations of the number of elementary conjugations, and of possible fixed spaces X^+. We can also use them to determine the type of one of the X^+.

(4.1) $\quad BDI(3,2) = CI(2) = Sp(2,\mathbb{R})/U(2)$

We counted two elementary conjugations for BDI(3,2):

$$\sigma_0 = \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & 0 & 1 \\ & & & 1 & 0 \end{array}\right), \quad \sigma_1 = \left(\begin{array}{ccc|cc} -1 & & & & \\ & -1 & & & \\ & & +1 & & \\ \hline & & & 0 & 1 \\ & & & 1 & 0 \end{array}\right).$$

Their fixed spaces were

$$X_0^+ = BDI(3,1) \text{ and } X_1^+ = BDI(2,1) \times BDI(1,1) = BDI(2,1) \times \mathbb{R}^1$$

We also counted two conjugations for CI(2):

$$\sigma_0(x \to -\bar{x})$$

$$\sigma_1 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \circ \sigma_0 .$$

Their fixed spaces were $X_0^+ = \text{Y-axis} = [SL_2(\mathbb{R})/SO(2)] \times \mathbb{R}^1$,

and $\quad X_1^+ = \text{unknown}.$

We can conclude that the unknown X_1^+ must be BDI(3,1), since

$(x \to -\bar{x})$ must correspond to $\left(\begin{array}{ccc|cc} -1 & & & & \\ & -1 & & & \\ & & +1 & & \\ \hline & & & 0 & 1 \\ & & & 1 & 0 \end{array}\right)$ in $O(3,2)$.

(4.2) AIII(2,2) = BDI(4,2). We counted <u>three</u> conjugations for each of these spaces:

AIII(2,2) : σ_o, $\begin{pmatrix} J & \\ & J \end{pmatrix} \circ \sigma_o$, $T\sigma_o$ with fixed X^+ of type BDI(2,2) = BDI(2,1) × BDI(2,1), CII(1,1), and $\mathbb{R}^1 \times \mathbb{G}_1$ respectively.

BDI(4,2) : $\sigma_o = \left(\begin{array}{c|cc} 1_4 & & \\ \hline & 0 & 1 \\ & 1 & 0 \end{array}\right)$, $\sigma_1 = \left(\begin{array}{cccc|cc} -1 & & & & & \\ & +1 & & & & \\ & & +1 & & & \\ & & & +1 & & \\ \hline & & & & 0 & 1 \\ & & & & 1 & 0 \end{array}\right)$

and $\sigma_2 = \left(\begin{array}{cccc|cc} -1 & & & & & \\ & -1 & & & & \\ & & +1 & & & \\ & & & +1 & & \\ \hline & & & & 0 & 1 \\ & & & & 1 & 0 \end{array}\right)$, with X^+ respectively of type

BDI(4,1), BDI(1,1) × BDI(3,1) = $\mathbb{R}^1$ × BDI(3,1), and $[\text{BDI}(2,1)]^2$.

The correspondence is as follows:

AII(2,2)		BDI(4,2)
σ_o	<——————>	σ_2
σ_1	<——————>	σ_o
σ_2	<——————>	σ_1

(4.3) AIII(3,1) = DIII(3). We found that each of these spaces has only one complex conjugation. The fixed spaces had type BDI(3,1) = $\mathfrak{so}(3,1)$ and $\mathfrak{so}(3,\mathbb{C})$, respectively. We just saw in (4.2) that BDI(3,1) was isometric to

$G_1 = SL_2(\mathbb{C})/SU(2)$. Of course, $\mathfrak{so}(3,\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$ <u>are</u> isomorphic, so our calculation checks.

(4.4) BDI(6,2) = DIII(4). Recall the article by Morita (p. 195), where this is proved. We did not yet compute the elementary conjugations for DIII(4), but we can now; we have already solved the problem for BDI(6,2). There are four elementary conjugations:

$$\sigma_o = \left(\begin{array}{c|c} I_6 & \\ \hline & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right) : X_o^+ = BDI(6,1)$$

$$\sigma_1 = \left(\begin{array}{c|c} I_{1,5} & \\ \hline & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right) : X_1^+ = BDI(1,1) \times BDI(5,1) = \mathbb{R}^1 \times [SU^*(4)/Sp(2)]$$

$$\sigma_2 = \left(\begin{array}{c|c} I_{2,4} & \\ \hline & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right) : X_2^+ = BDI(2,1) \times BDI(4,1) = CI(1) \times CII(1,1) = [SL_2(\mathbb{R})/SO(2)] \times [Sp(1,1)/Sp(1) \times Sp(1)]$$

$$\sigma_3 = \left(\begin{array}{c|c} I_{3,3} & \\ \hline & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right) : X_3^+ = BDI(3,1)^2 = [SL_2(\mathbb{C})/SU(2)]^2.$$

Part V. The General Case: Arbitrary X

5.0. We return now to the situation and problem posed at the beginning of Part III, namely, to classify the complex conjugations, as well as their fixed spaces, on arbitrary bounded symmetric domains X. We did this, via Galois cohomology, for irreducible X (not exceptional). For arbitrary X (with no exceptional factor), the problem is essentially a corollary of work already done.

As a first reduction, we note that for a product $X = X_1 \times \ldots \times X_m$ of powers X_i of distinct irreducible hermitian symmetric spaces, the isometry group $G(X)$ is the direct product of the isometry groups $G(X_i)$ of the various factors, and the same is true of the group $G^h(X)$ of holomorphic automorphisms. Let $\sigma, \tau \in \mathcal{C}(X)$ be two complex conjugations. Then σ and τ are products $\sigma = \sigma_1 \times \ldots \times \sigma_m$, $\tau = \tau_1 \times \ldots \times \tau_m$ of complex conjugations of the factors X_i, and σ and τ are conjugate by some $g = g_1 \times \ldots \times g_m \in G^h(X)$ if and only if $\sigma_i = g_i \tau_i g_i^{-1}$ for all i. Then if we denote by $H^1(X_i)$ a complete set of elementary conjugations of X_i, we clearly have $H^1(X) = H^1(X_1) \times \ldots \times H^1(X_m)$, and the classification is reduced to solving the problem for powers of an irreducible space.

Next we show how to compute $H^1(X^n)$ in terms of $H^1(X)$ of an irreducible space X. It is known (see Cartan [5], for example) that the isometry group $G(X^n)$ is a semi-direct product of its subgroup $G(X)\times\overset{(n)}{\cdots}\times G(X) = G(X)^n$ with the permutation group S_n of the n factors. These permutations are holomorphic (with respect to any of the $G(X)^n$-invariant complex structures of X^n) and the group $G^h(X^n)$ is the subgroup $G^h(X)^n \cdot S_n$ of $G(X^n)$.

Remark. We have computed $H^1(X)$ for X of classical type. For X exceptional, if X^n has any anti-holomorphic isometry, then it has one that is an involution and does not permute the factors. In particular, X itself has an anti-holomorphic involution, and $H^1(X)$ could be computed, by Galois cohomology as before.

We denote isometries of X^n contained in $G(X)^n$ by overlined small case letters: the standard conjugation of X^n will be $\bar{\sigma} = (\sigma_o,\ldots,\sigma_o)$. Every isometry is $\bar{g}\tau = (g_1,\ldots,g_n)\tau$ for some $g_i \in G(X)$, and $\tau \in S_n$. The group $Gal = \{1,\bar{\sigma}_o\}$ acts on $G(X^n)$, and on the subgroup $G^h(X^n)$, of course, with the inner automorphism

$$Int(\bar{\sigma}_o):\bar{g}\tau \to (\bar{g}\tau)^{\bar{\sigma}_o} = (\bar{\sigma}_o\bar{g}\bar{\sigma}_o)(\bar{\sigma}_o\tau\bar{\sigma}_o) = (\bar{\sigma}_o g\bar{\sigma}_o)\tau = \bar{g}^{\bar{\sigma}_o}\tau.$$

A set of elementary conjugations of X^n is in bijective correspondence with $H^1(Gal,G^h)$, with the above action, as we showed in Part III. Although we showed $H^1(Gal,K^h)\rightleftarrows H^1(Gal,G^h)$, we will not need it here. The 1-cocycles $Z^1(Gal,G^h(X^n))$ are $\{\bar{g}\tau \in G^h(X^n) \mid (\bar{g}\tau)^{\bar{\sigma}_o} = (\bar{g}\tau)^{-1} = \tau^{-1}\bar{g}^{-1}$, and $\tau^2 = 1\}$. Note that $\bar{g}\tau$ is a 1-cocycle if and only if $\bar{g}\tau\bar{\sigma}_o$ is an involution: $\bar{g}\tau\bar{\sigma}_o\bar{g}\tau\bar{\sigma}_o = 1$, an equation in the semi-direct product $G^h(X)^n \cdot S_n$. Since $G^h(X)^n$ is normal, we can divide by it, and get the equation $\tau^2 = 1$ in S_n. We can thus write all of the following:

$$Z^1 = \{\bar{g}\tau \in G^h(X^n) \mid \bar{g}^{\bar{\sigma}_o \tau} = \tau \bar{g}^{-1}\}$$

$$= \{\bar{g}\tau \in G^h(X^n) \mid \bar{g}^{\tau \bar{\sigma}_o} = \bar{g}^{\bar{\sigma}_o \tau} = \bar{g}^{-1}\}$$

$$= \{\bar{g}\tau \in G^h(X^n) \mid (\bar{g}\tau)^{\bar{\sigma}_o} = \tau \bar{g}^{-1} = \tau^{-1}\bar{g}^{-1} = (\bar{g}\tau)^{-1}\}$$

<u>Step 1</u>: $\bar{g}\tau$ is cohomologous to $\bar{h}\tau$ if and only if there is $b = (b_1, \ldots, b_n) \in G^h(X)^n$ with either

1) $\bar{g}\tau = b^{-1}\bar{h}\tau b^{\bar{\sigma}_o}$, or

2) $\bar{g}\tau = \rho^{-1} b^{-1} \bar{h}\tau b^{\bar{\sigma}_o} \rho$ for some $\rho \in S_n$.

Writing out 1), we get

$(g_1, \ldots, g_n)\tau = (b_1^{-1}, \ldots, b_n^{-1})(h_1, \ldots, h_n)\tau(b_1^{\sigma_o}, \ldots, b_n^{\sigma_o})$,

or $(g_1, \ldots, g_n) = (b_1^{-1}, \ldots, b_n^{-1})(h_1, \ldots, h_n)(b_{\tau(1)}^{\sigma_o}, \ldots, b_{\tau(n)}^{\sigma_o})$

since $\bar{\sigma}_o$ and τ commute. If $\tau(i) = i$, we get

* $\quad g_i = b_i^{-1} h_i b_i^{\sigma_o}$, with $g_i^{\sigma_o} = g_i^{-1}$. In other words, g_i and h_i are cohomologous 1-cocycles in $Z^1(\{1, \sigma_o\}, G^h(X))$. If $\tau(i) = j$ for $j \neq i$, then $\tau(j) = i$, and we get two equations

$$** \begin{cases} g_i = b_i^{-1} h_i b_{\tau(i)}^{\sigma_o} \\ g_j = b_j^{-1} h_j b_{\tau(j)}^{\sigma_o} \end{cases}$$

. Moreover, since $\bar{g}\tau$ is a 1-cocycle, $g_j = g_{\tau(i)} = (g_i^{-1})^{\sigma_o}$, so we can write

$$** \begin{cases} g_i = b_i^{-1} h_i b_{\tau(i)}^{\sigma_o} \\ (g_i^{-1})^{\sigma_o} = b_j^{-1} h_j b_{\tau(j)}^{\sigma_o} \end{cases} .$$

But now $\tau(i) = j$, and $\tau(j) = i$, so we have

$$** \begin{cases} g_i = b_i^{-1} h_i b_j^{\sigma_o} \\ (g_i^{-1})^{\sigma_o} = b_j^{-1} h_j b_i^{\sigma_o} . \end{cases}$$

In case $(h_i^{-1})^{\sigma_o} = h_j$, one sees that taking the inverse and applying σ_o to the first equation gives the second one. Furthermore, if $h_i = 1$, then the first equation is $g_i = b_i^{-1} b_j^{\sigma_o}$, which can always be satisfied since b_i and b_j are arbitrary. Then if $h = (h_1, \ldots, h_n)$ is such that $h_i = 1$ whenever $\tau(i) \neq i$, then we can say $\bar{g}\tau$ is cohomologous to $\bar{h}\tau$ if and only if (*) is satisfied for all i with $\tau(i) = i$. Therefore, we can say $\bar{g}\tau$ is cohomologous to an $\bar{h}\tau$ with $\bar{h} = (h_1, \ldots, h_n)$ and $h_i = 1$ if $\tau(i) \neq i$, and $h_i \circ \sigma_o$ = one of the elementary conjugations classified by $H^1(\{1, \sigma_o\}, G^h(X))$, if $\tau(i) = i$.

<u>Step 2</u>: If $\bar{h}' = (h_1', \ldots, h_n')$ is such that $h_i' = 1$ whenever $\tau(i) \neq i$, and $\bar{h}$ and $\bar{h}'$ differ only by a permutation ρ with $\rho(i) = i$ whenever $\tau(i) \neq i$, then $\bar{h}'\tau$ is cohomologous to $\bar{h}\tau$. To see this we use condition 2) of Step 1, with $b = 1$. We have required ρ and τ to commute, so we have $\bar{h}'\tau = \rho^{-1}\bar{h}\rho\tau = \rho^{-1}\bar{h}\tau\rho$, which is precisely 2).

<u>Step 3</u>: We note first that two involutions $\tau_1, \tau_2 \in S_n$ are S_n-conjugate if and only if they fixed the same number of objects, of course, if and only if they can be written as products of transpositions with the same number of transpositions. Now suppose $\bar{h}\tau_1$ is a 1-cocycle with $h_i = 1$ if $\tau_1(i) \neq i$. We want to define a permutation ρ which will give $\rho\tau_1\rho^{-1} = \tau_2$, if τ_2 has the same number of fixed objects as τ_1.

For all i with $\tau_1(i) = i$, define $\rho(i)$ by the condition: $i < j$ implies $\rho(i) < \rho(j)$, and $\tau_2(\rho(i)) = \rho(i)$. For the i with $\tau_1(i) \neq i$, first define $\rho(i)$ for the least of these i to be the least integer such that $\tau_2(\rho(i)) \neq \rho(i)$. Then define $\rho(\tau_1(i))$ for this i to be $\tau_2(\rho(i))$. Continue with the next i such that $\tau_1(i) \neq i$, and define the value of ρ to be the least integer not already in the image which is not fixed by τ_2. Then define $\rho(\tau_1(i))$ for this i as before. This ρ has the property that

$\rho\bar{h}\tau_1\rho^{-1} = (\rho\bar{h}\rho^{-1})\rho\tau_1\rho^{-1} = (\rho\bar{h}\rho^{-1})\tau_2 = \bar{h}^{\rho}\tau_2$, where

$\bar{h}^{\rho} = (h_{\rho(i)},\ldots,h_{\rho(n)})$ and $h_{\rho(i)} = 1$ whenever $\tau_2(\rho(i)) \neq \rho(i)$. Finally, condition 2) of Step 1 gives that $\bar{h}\tau_1$ and $\bar{h}^{\rho}\tau_2$ are cohomologous (b = 1 again).

(5.0.1.) We have shown that any anti-holomorphic involution in X^n is $G^h(X^n)$-equivalent to one of the following sort. The factors $X_1,\ldots,X_{2k}$ (first 2k factors of X^n) are permuted in successive pairs, and then the standard σ_o is applied to each of these coordinates. The factors $X_{2k+1},\ldots,X_n$ are preserved, and in each we have one of the elementary conjugations, the order of these being irrelevant. The fixed space $(X^n)^+$ of such an involution is isometric to $X^k x$ (a product of $n-2k = \ell$ fixed spaces of elementary conjugations of X).

Clearly no two conjugations with different k can be equivalent, since their fixed spaces are not isometric. We can thus give a description of $H^1(X^n)$ in terms of $H^1(X)$. If E is any set, the permutation group S_ℓ acts naturally on $E^\ell = E \times \overset{(\ell)}{\ldots} \times E$. Denote the quotient by $E^{\ell/}$. We have proved

<u>Corollary</u> (5.0.2): Let X be an irreducible hermitian symmetric space (non-compact) and $H^1(X)$ a complete set of elementary conjugations. Then a set of elementary conjugations for X^n is

$$H^1(X^n) = \sum_{\substack{0\leq \ell \leq n \\ \ell \equiv n(2)}} H^1(X)^{\ell/}$$

[$H^1(X)^{0/}$ means the set {1} where $1 \in G(X)$, and the summation means disjoint union.]

<u>Corollary</u> (5.0.3): Let X be a (non-compact) hermitian symmetric space with no exceptional factor. Then a complete set of elementary complex conjugations is a finite set. Its elements (and its order) can be written down with

knowledge of these for the irreducible factors of X. The fixed point set X^+ of a conjugation can be an arbitrary product of arbitrary powers of the irreducible factors times arbitrary fixed spaces of conjugations in these irreducible factors, so long as this candidate has $\mathbb{R}$-dimension = $\mathbb{C}$-dimension of X.

<u>Corollary</u> (5.0.4): (<u>Quasi-Witt Theorem</u>) Let X_1^+ and X_2^+ be real forms (fixed spaces of complex conjugations) of a bounded symmetric domain X. If X_1^+ and X_2^+ are isometric (as Riemannian spaces), then there exists a holomorphic isometry of X which restricts to X_1^+ to give another isometry $X_1^+ \approx X_2^+$.

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