ON TOEPLITZ OPERATORS ON QUARTER PLANE WITH MATRIX VALUED SYMBOLS

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ABSTRACT

Our object in this work is to show, using results of Douglas and Howe [1] and Gokhberg and Kreĭn [1], that Toeplitz operators on the quarter plane with matrix valued continuous symbols and which are Fredholm form a dense open subset of the set of Toeplitz operators with continuous matrix valued symbols whose determinants are nonvanishing and homotopic to a constant.

CHAPTER I

ON TOEPLITZ OPERATORS ON QUARTER PLANE WITH MATRIX VALUED SYMBOLS

§0. Almost since the beginning of this century various linear problems have arisen in various contexts. Solution of linear integral equation is one of them. They can be classified into two different types, namely the Voltera type and the Fredholm type. Wiener-Hopf integral equations belong to the later kind. An equation of the type

$$f(x) + \int_0^\infty k(x-t)f(t)dt = g(x),$$

the well known Wiener-Hopf equation, has been studied in various contexts by various authors. Although the name is derived from an attempt by Hopf and Wiener to solve certain problems in radiative equilibrium (cf. Hopf [1]), its orgin can be traced to Hilbert and even Riemann (cf. Plemelh [1]). The problem of Riemann was to find n functions which are holomorphic inside and outside a simple closed curve C, so that the boundary values $f_1^-(z), f_2^-(z), \ldots, f_n^-(z)$ of the exterior functions and the boundary values $f_1^+(z), f_2^+(z), \ldots, f_n^+(z)$ of the interior functions are related, for each z in the boundary C, in the following manner:

$$f_k^-(z) = \sum_{j=1}^{n} \varphi_{kj}(z) f_j^+(z) \quad (0.1)$$

$$f_k^+(z) = \sum_{j=1} \varphi'_{kj}(z) f_j^-(z) \quad (0.1')$$

where the coefficients φ_{ij} are constants which change their values from section to section of the boundary C.

Now if $L_n^2(\mathbb{T})$ denotes the space of norm square-integrable measurable functions from the circle group $\mathbb{T}(=\{z : |z|=1\})$ to the n-dimensional Hilbert space $\mathbb{C}^n$, and $H_n^2(\mathbb{Z}_+)$ is the corresponding Hardy space of functions in $L_n^2(\mathbb{T})$ with Fourier transform (series) supported on the semigroup $\mathbb{Z}_+$ of non-negative integers, then it is clear that the space $H_n^2(\mathbb{Z}_+)$ is a closed subspace of $L_n^2(\mathbb{T})$ and that we can define a projection operator P from $L_n^2(\mathbb{T})$ onto $H_n^2(\mathbb{Z}_+)$. Let φ be an essentially bounded function from the circle group $\mathbb{T}$ into the algebra M_n of endomorphisms of $\mathbb{C}^n$, i.e. the algebra of nxn matrices with complex entries. We define the Toeplitz operator T_φ with symbol φ on $H_n^2(\mathbb{Z}_+)$ by

$$T_\varphi f = P(\varphi f) \quad (0.2)$$

for every f in $H_n^2(\mathbb{Z})$.

Now it is clear that solving (0.1) is equivalent to finding a non trivial element f of the kernel of the Toeplitz operator T_φ with symbol φ when the simple closed curved C is the unit circle $\mathbb{T}$. Moreover, Devinatz [1] has shown how to identify Toeplitz operators T_φ with matrix valued Wiener-Hopf operators, so the study of the two classes may be consolidated. (Also see Rosenblum [1]).

The study of solutions of Wiener-Hopf equation (0.0) in the scalar case has been studied in detail (c.f. Paley and Wiener [1]) for certain classes of kernel k (also see Rapaport [1], [2]). It is interesting to note that the Toeplitz and Wiener-Hopf operators also arise in many other contexts like stochastic processes (c.f. Grenander and Szegö [1]), and prediction theory (c.f. Wiener and Masani [1], [2]). Of late their study has been found interesting in examining convergence of certain difference schemes for solving partial differential equations (Osher [1]).

Gokhberg and Kreĭn [1] have studied in great detail the solution of systems of integral equations of the Wiener-Hopf type

$$\chi_p(t) - \sum_{q=1}^{n} \int_0^\infty k_{pq}(t-s)\chi_q(s)ds = f_p(t) \quad p = 1,\ldots\ldots,n \quad (0.3$$

Methods employed in the study of equations (0.3) depend heavily on the technique of factorization. In their work Gokhberg and Kreĭn have obtained through such considerations results on the Fredholmness of the Wiener-Hopf operator defined by (0.3) and have computed its index. Similar results have also been obtained by Douglas [1]. We recall that if $\mathcal{L}(\mathcal{H})$ is the algebra of bounded operators on the Hilbert space $\mathcal{H}$ and $\mathcal{LC}(\mathcal{H})$ is the compact operators on $\mathcal{H}$ then $\mathcal{LC}(\mathcal{H})$ forms a closed ideal of $\mathcal{L}(\mathcal{H})$.

The quotient algebra $\mathcal{L}(\mathcal{H})/\mathcal{LC}(\mathcal{H})$ is called the Calkin algebra and an operator T on $\mathcal{L}(\mathcal{H})$ is called Fredholm if $\pi(T)$ is invertible in $\mathcal{L}(\mathcal{H})/\mathcal{LC}(\mathcal{H})$, where π is the natural map from $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H})/\mathcal{LC}(\mathcal{H})$. There is also an alternative way of defining a Fredholm operator. An operator T in $\mathcal{L}(\mathcal{H})$ is Fredholm if it has a closed range and has its kernel finite dimensional. It is easy to see that these two definitions are equivalent (c.f. Douglas [4]). It can be shown that dim ker $T^* < \infty$ also. So dim ker T- dim ker T^* is an integer when T is Fredholm. It is interesting to note that the integer $j(T) = \dim\ker T - \dim\ker T^*$ has certain important properties. We recall that if T is Fredholm then T+K is also Fredholm when K is in $\mathcal{LC}(\mathcal{H})$ and $j(T+K) = j(T)$. So while investigating the solution space of certain equation of the type

$$Tf = 0$$

it is important to know that the solution space has no particular significance where as the index is invariant under compact perturbation. So in general we are interested in finding the index $j(T)$ of T rather than the dimension of its kernel. In case T is invertible it is obvious that it is Fredholm and its index is zero. But it is also true that there are Fredholm operators of index zero that are not invertible. But in certain cases they coincide. In particular, if T_φ is Fredholm and of index zero then it is invertible in the scalar case (c.f. Douglas [4]). Douglas in [1] showed that a Toeplitz operator T_φ with continous matrix valued symbol φ is Fredholm if and only if det φ does not vanish anywhere on the unit circle $\mathbb{T}$. Further more, the index $j(T_\varphi)$ of the operator is related to the winding number $i_t(\det \varphi, 0)$ of the image of $\mathbb{T}$ by det φ with respect to the origin in the following way

$$j(T_\varphi) = - i_t(\det \varphi, 0). \quad (0.5)$$

This result was obtained by Gokhberg and Kreĭn [1] for a dense subset of $C_n(\mathbb{T})$ and in full generality by Douglas [2].

So it is clear that in order that T_φ be invertible it is necessary that $\det \varphi \neq 0$ on $\mathbb{T}$ and det φ be homotopic to a constant. We have said before that in scalar case i.e. when $n = 1$, this is also a sufficient condition. But it is not when $n \geq 2$ as can easily be shown. In fact, we can have φ with $\det \varphi \equiv 1$ yet have dim ker $T_\varphi = m$ for any m in $\mathbb{Z}_+$. For example when $n = 2$ let φ be defined by

$$\varphi = \begin{pmatrix} \chi_m & 0 \\ 0 & \chi_{-m} \end{pmatrix} \quad (0.6)$$

where $\chi_m(e^{i\theta}) = e^{im\theta}$ and $\chi_{-m}(e^{i\theta}) = e^{-im\theta}$

Then ker T_φ contains $f_1 = [{0 \atop 1}], f_2 = [{0 \atop \chi}] \dots . f_m = [{0 \atop \chi_{m-1}}]$. But in their celebrated work Gokhberg and Kreĭn [1] have shown a very important relation between Fredholm Toeplitz operators of index zero and invertible Toeplitz operators. In fact they have shown that the left (or right) invertible Toeplitz operators form a dense open subset of the class of Fredholm Toeplitz operators with continuous symbol and of index zero. [We shall give another proof of this in our work].

There have been several attempts at generalizing the above results to the several variable case. One generalization of this to the "half plane" case has been studied by Gokhberg and Goldenstein ([1], [2]), and more recently by Coburn, Douglas, Singer and Schaeffer ([1], [2]). In this case it has been shown that the operators can be represented as ordinary Toeplitz operators involving a parameter. A significantly different situation is encountered in the so-called "quarter plane" case. Precisely let $\mathbb{T}^2$ denote the torus group $\mathbb{T} \times \mathbb{T}$ and $L^2_n(\mathbb{T}^2)$ the space of norm square integrable measurable functions from the torus group $\mathbb{T}^2$ to the n-dimensional Hilbert space $\mathbb{C}^n$. The Fourier transform of a function in $L^2_n(\mathbb{T}^2)$ is a $\mathbb{C}^n$-valued function on $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. Again let $H^2_n(\mathbb{Z}^2_+)$ denote the subspace of functions in $L^2_n(\mathbb{T}^2)$ with Fourier transform supported on $\mathbb{Z}^2_+$ and P_2 the projection from $L^2_n(\mathbb{T}^2)$ onto $H^2_n(\mathbb{Z}^2_+)$. If φ is a continuous function from $\mathbb{T}^2$ to M_n, then the Toeplitz operator W_φ on $H^2_n(\mathbb{T}^2)$ is defined by

$$W_\varphi f = P_2(\varphi f) \text{ for } f \text{ in } H^2_n(\mathbb{Z}^2_+).$$

It was shown by Douglas and Howe ([1]) that W_φ on $H^2_n(\mathbb{Z}^2_+)$

is Fredholm if and only if the operators $T_{\varphi(\cdot,w)}$ and $T_{\varphi(z,\cdot)}$ on $H_n^2(\mathbb{Z}_+)$ are invertible for each z and w in $\mathbb{T}$. They also showed, in case $n = 1$, that the collection of invertible Toeplitz operators on $H_n^2(\mathbb{Z}_+)$ is a proper dense open subset of the collection of Fredholm Toeplitz operators. In particular they showed there exist Toeplitz operators W_φ on the quarter plane which are Fredholm operators of index zero which are not invertible. Their result on Toeplitz operators on the quarter plane in the case $n = 1$ yields that the Fredholm alternative holds if the symbol φ is homotopic to a constant and in the class of non-vanishing functions on the torus. If the Fourier series of φ is absolutely convergent, then Strang [1] has obtained an explicit operator which is inverse of W_φ modulo the ideal of compact operators. Simonenko [1] also obtains a similar result in the quarter plane. It is also clear that for $n > 1$, i.e. in the matrix quarter plane case, not every operator whose symbol is homotopic to a identity, is a Fredholm operator. Douglas and Howe also conjectured that a refinement of their argument for $n = 1$ could show that the generic case is a Fredholm operator and even in all probability invertible. The recent work of Douglas, Coburn and Singer [1] shows that the last statement is false. In fact they showed that in case of $n = 2$ we can get Toeplitz operators which are Fredholm of arbitrary index. So all we have been able to show here is that Fredholm Toeplitz operators form a dense open subset of the class of Toeplitz operator with determinant of its symbol homotopic to a constant and non vanishing continuous on the torus $\mathbb{T}^2$.

We shall obtain the above mentioned result in two steps. First we shall find a necessary and sufficient condition for the invertibility of a Toeplitz operator T_φ, where the symbol φ is a matrix valued continuous function on $\mathbb{T}$ is of a certain form and has determinant homotopic to a constant. Now let

$\mathfrak{F}_0 = \{W_\varphi$: det φ homotopic to a constant and φ continuous on $\mathbb{T}^2$ and det φ non vanishing on $\mathbb{T}^2\}$ and

$\mathfrak{F}_1 = \{W_\varphi : T_{\varphi(\cdot,w)}$ is invertible for all w in $\mathbb{T}\}$

$\mathfrak{F}_2 = \{W_\varphi : T_{\varphi(z,\cdot)}$ is invertible for each z in $\mathbb{T}\}$.

Then using the result of Gokhberg and Kreĭn [1], of which we shall give an independent proof, we shall show that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are dense open subsets of $\mathfrak{F}_0$. Then $\mathfrak{F}_1 \cap \mathfrak{F}_2$ is a dense subset of $\mathfrak{F}_0$, consisting of all Toeplitz operator W_φ on $H^2_n(\mathbb{Z}^2_+)$ for which $T_{\varphi(\cdot,w)}$ and $T_{\varphi(z,\cdot)}$ are invertible on $H^2_n(\mathbb{Z}_+)$ for each z and w in $\mathbb{T}$. Hence $\mathfrak{F}_1 \cap \mathfrak{F}_2$ consists of all Toeplitz operators which are Fredholm by the results of Douglas and Howe [1]. Because of symmetry it is sufficent to prove $\mathfrak{F}_1$ is a dense open subset of $\mathfrak{F}_0$. We organize our work as follows. First we shall obtain a criterion of invertibility for a Toeplitz operator T_φ on $H^2_n(\mathbb{Z}_+)$ when the symbol φ is of the form $\frac{1}{Q}\psi$, where the $i,j\underline{^{th}}$ entry ψ_{ij} of ψ is analytic, i.e. each ψ_{ij} is continuous on $\mathbb{T}$ and has analytic extension to

$\mathbb{D} = \{z : |z| < 1\}$ and Q is a scalar valued analytic trigonometric polynomial. Then we use this criterion to give a different proof of the result of Gokhberg and Kreĭn [1]. In the next section we shall use these results to show density of $\mathfrak{J}_1$ in $\mathfrak{J}_0$. The way to do that will be to first approximate every φ continuous and det φ homotopic to a constant by a trigonometric polynomial ψ_1. Then we shall show, using the criterion of invertibility which we mentioned, that there is a function ψ from $\mathbb{T}^2$ to M_n with all its entries trigonometric polynomials such that $T_{\psi(\cdot,w)}$ is invertible for all w in $\mathbb{T}$ and ψ approximates ψ_1.

Before beginning with the proof of the results the author wishes to express his sincere thanks to Professor R.G. Douglas who first suggested this problem and without whose constant encouragement this might not have been completed. The author also wishes to thank Professor R. Howe, who made two important suggestions regarding the proof of the main theorem, and his fellow graduate student, Mr. Allan R. Adler for many helpful suggestions. The idea of proof of the first lemma was derived from the paper of Douglas [3] on the invertibility of a class of Toeplitz operators on the quarter plane.

§1. We begin with the following lemma.

Lemma 1. Let g in $L^2(\mathbb{T})$ have the Fourier expansion

$\sum_{n=-\infty}^{\infty} a_n\chi_n$, where $\chi_n(e^{i\theta}) = e^{in\theta}$ for $e^{i\theta}$ in $\mathbb{T}$. If $P(\frac{g}{\chi-\lambda}) = 0$, for some λ in $\mathbb{D} = \{\mu : |\mu| < 1\}$ then $a_n = 0$ for all $n \geq 1$.

Proof: Since $|\lambda| < 1$ the Fourier expansion of $(\chi-\lambda)^{-1}$ is $\sum_{n=0}^{\infty} \lambda^n\chi_{-n-1}$. So if $\sum_{n=-\infty}^{\infty} C_n\chi_n$ is the Fourier expansion of $(\chi-\lambda)^{-1}g$ then

$$C_n = \sum_{r=0}^{\infty} \lambda^r a_{r+n+1} \quad (1.1.1)$$

Now if $P((\chi-\lambda)^{-1}g) = 0$, then $C_n = 0$ for all $n \geq 0$. But from (1.1.1) we get

$$C_n - \lambda C_{n+1} = a_{n+1} \quad (1.1.2)$$

Equation (1.1.2) immediately gives us $a_{n+1} = 0$ for all $n \geq 0$ i.e. $a_n = 0$ for all $n \geq 1$.

Lemma 2. If g in $L^2(\mathbb{T})$ has Fourier expansion $\sum_{n=-\infty}^{\infty} a_n\chi_n$ and $P(\frac{g}{(\chi-\lambda_1)\ldots(\chi-\lambda_m)}) = 0$, with $\lambda_i \in \mathbb{D}$ for $i = 1,\ldots,m$, and $m \in \mathbb{Z}_+$, then $a_n = 0$ for all $n \geq m$.

Proof: Assuming this to be true for m we shall prove it for m + 1. Since the case m = 1 is the previous lemma, the result follows by induction. Let us assume that

$$P(\frac{g}{(\chi-\lambda_1)\ldots(\chi-\lambda_m)}) = 0$$

implies $a_n = 0$ for all $n \geq m$. Now let $h = g(\chi-\lambda_{m+1})^{-1}$ with $|\lambda_{m+1}| < 1$. Hence if $\sum_{n=-\infty}^{\infty} a'_n \chi_n$ is the Fourier expansion of h, then $P(\frac{h}{(\chi-\lambda_1)\cdots(\chi-\lambda_m)}) = 0$ would imply $a'_n = 0$ for all $n \geq m$. But

$$a'_n = \sum_{r=0}^{\infty} \lambda^r a_{n+r+1}. \quad (1.2.1)$$

And as in lemma 1, this implies $a_n = 0$ for all $n \geq m + 1$. But $(\chi-\lambda_1)^{-1}(\chi-\lambda_2)^{-1}\cdots(\chi-\lambda_m)^{-1} h = (\chi-\lambda_1)^{-1}(\chi-\lambda_2)^{-1}\cdots(\chi-\lambda_{m+1})^{-1}g$ and hence is true for $m + 1$.

<u>Corollary</u>: If f is $H^2(\mathbb{Z}_+)$ and

$$P(\frac{f}{(\chi-\lambda_1)\cdots(\chi-\lambda_m)}) = g \text{ for } |\lambda_i| < 1$$

for $i = 1,\ldots,m$ where g is in $H^2(\mathbb{Z}_+)$, then

$$f = a_0 + a_1\chi + \cdots + a_{m-1}\chi_{m-1} + (\chi-\lambda_1)\cdots(\chi-\lambda_m)g, \quad (1.3.1)$$

where $a_0,\ldots,a_{m-1}$ are complex numbers.

We shall use the above corollary to obtain the necessary and sufficient condition for invertibility of a Toeplitz operator T_φ on $H^2_n(\mathbb{Z}_+)$, when the symbol φ is of the following form: Let $\varphi = \frac{\psi}{Q}$, where ψ is a continuous function from $\mathbb{T}$ to the algebra M_n of $n \times n$ complex matrices such that each of the ij<u>th</u> entry ψ_{ij} of ψ has analytic extension to $\mathbb{D}$ and Q is a scalar valued analytic trigonometric polynomial with all its zeros inside $\mathbb{D}$. Let us further suppose that det φ is homotopic to a constant. Then T_φ on $H^2_n(\mathbb{Z}_+)$ is Fredholm and has index zero. Now we shall find out when such a T_φ is invertible. As it is extremely complicated to write down, without previous inkling as to how it was obtained, the necessary and sufficient condition for invertibility of T_φ we shall write the condition at the end of our proof.

Because $\varphi = \frac{\psi}{Q}$, $\det \varphi = \frac{\det \psi}{(Q)^n}$.

As index of $\det \varphi(\mathbb{T})$ = number of zeros of $\det \psi$ - number of zeros of $(Q)^n$ inside $\mathbb{T}$, i.e. in $\mathbb{D}$, $\det \varphi$ homotopic to a constant implies $\det \psi$ has as many zeros in $\mathbb{D}$ as $(Q)^n$ has in $\mathbb{D}$, that is $\det \psi$ has mn zeros in $\mathbb{D}$, (by abuse of notation by $\det \psi$ we shall mean the determinant of extension of ψ to $\mathbb{D}$) if Q has m zeros inside $\mathbb{D}$. Since T_φ is Fredholm and of index zero (See Douglas [2]) it is sufficient to look at its kernel to check its invertibility. If f is in ker T_φ, then

$$P(\varphi f) = 0 \quad (1.4.1)$$

Now if $Q = (\chi-\lambda_1)\ldots(\chi-\lambda_m)$ with $|\lambda_i| < 1$ for $i = 1,\ldots,m.$, then by (1.3.1) in the corollary

$$\sum_{j=1}^{n} \psi_{ij} f_j = a_{0i} + a_{ri}\chi + \ldots + a_{m-1,i}\chi_{m-1} \quad (1.4.2)$$

where $f_j(t)$ is the $j^{\underline{th}}$ component of $f(t)$ in $\mathbb{C}^n$.

So we can write in fact

$$\sum_{j=1}^{n} \psi_{ij}(z) f_j(z) = \sum_{r=0}^{m-1} a_{ri} z^r \quad (1.4.2')$$

for $|z| \leq 1$. As mentioned before we write $\psi_{ij}(z)$ and $f_j(z)$ for $|z| < 1$ to mean values of extensions of ψ_{ij} and f_j to $\mathbb{D}$, respectively. Thus each f_j is expressible as a quotient of two analytic functions in $\mathbb{D}$

$$f_j = \frac{F_j}{\det\psi}, \quad (1.4.3)$$

where

$$F_j(z) = \sum_{j=1}^{n} \left(\sum_{r=0}^{m-1} \alpha_{ri} z^r \right) \Phi_{ij}(z)$$

and Φ_{ij} is the cofactor of ψ_{ij} in det ψ.

Now in order that each f_j be analytic we require that each F_j has zeros at the points in $\mathbb{D}$ at which det ψ has zeros and with at least the same multiplicities. Let α_k $k = 1,\ldots,mn$ denote the zeros of det ψ and suppose they are distinct. Then we should have

$$F_j(\alpha_k) = 0 \text{ for } 1 \le k \le mn \text{ and } 1 \le j \le n \quad (1.4.5)$$

But to determine such F_j's we need mn arbitrary constants α_{ri} $i = 1,\ldots,n$ and $r = 0,1,\ldots, m-1$.

Thus we have mn^2 equations

$$\sum_{i=1}^{n} \left(\sum_{r=0}^{m-1} \alpha_{ri}\alpha_k^r \right) \Phi_{ij}(\alpha_k) = 0 \quad \begin{matrix} k = 1,\ldots,mn \\ j = 1,\ldots,n \end{matrix} \quad (1.4.6)$$

with mn unknown α_{ri}. But we know from linear algebra (cf. Greub [1] p. 35) that necessary and sufficient condition that system of equations have a non trivial solution is that the mn x mn^2 matrix

$$\begin{bmatrix} \Phi_{11}(\alpha_k), \ldots, \Phi_{n1}(\alpha_k), \ldots, \alpha_k^{m-1}\ \Phi_{n1}(\alpha_k) \\ \Phi_{12}(\alpha_k), \ldots, \Phi_{n2}(\alpha_k), \ldots, \alpha_k^{m-1}\ \Phi_{n2}(\alpha_k) \\ \Phi_{13}(\alpha_k), \ldots, \Phi_{n3}(\alpha_k), \ldots, \alpha_k^{m-1}\ \Phi_{n3}(\alpha_k) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \Phi_{1j}(\alpha_k), \ldots, \Phi_{nj}(\alpha_k), \ldots, \alpha_k^{m-1}\ \Phi_{nj}(\alpha_k) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \Phi_{1n}(\alpha_k), \ldots, \Phi_{nn}(\alpha_k), \ldots, \alpha_k^{m-1}\ \Phi_{nn}(\alpha_k) \end{bmatrix} \qquad 1.4.7$$

$k = 1, \ldots, mn$

has rank $< mn$. So we get.

Main Lemma:

A necessary and sufficient condition that T_φ, as described before, is invertible is that the rank of the matrix (1.4.7) is maximal.

Corollary: If the rank of the matrix (1.4.7) is $p < mn$ then the dimension of the null space of T_φ is exactly $mn - p$.

Remark 1 We can from the above analysis also compute the null space of T_φ, i.e. we can find $mn - p$ linearly independent vectors in $H_n^2(\mathbb{Z}_+)$ which span ker T_φ. That is because when $mn - p > 0$

there are only mp-p linearly independent solutions to the equation (1.4.6) and with those solutions we can find, by substituting them in (1.4.4), the F_j's and hence f_j's.

Remark 2. The lemma can be generalized to the case when det ψ has multiple zeros in $\mathbb{D}$.

If $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are the zeros of det ψ with multiplicities $k_1, \ldots, k_\ell$ respectively in $\mathbb{D}$, then we replace (1.4.5) by

$$F_j^{(s)}(\alpha_i) = 0 \text{ for } 0 \le s \le k_i - 1. \quad (1.45')$$
$$1 \le j \le n$$

where $F_j^{(s)}$ denotes the $s^{\underline{th}}$ derivative of F_j and we get the corresponding condition of invertibility of T_φ as maximality of the rank of a matrix of the type (1.4.7) whose entries will be of the form

$$\frac{d^s}{dz^s}(z^r \Phi_{ij}(z))_{z=\alpha_k} \quad (1.4.7')$$

whose explicit form as in (1.4.7) will be more complicated to write down. We can also compute the null space of T_φ exactly as described above.

We observe that the above analysis also gives a method of finding the inverse of T_φ.

If $T_\varphi f = g$ and φ is of the form we have described then by corollary to lemma (1.2) we get

$$\sum_{j=1}^{n} \psi_{ij}(z) f_j(z) = \sum_{r=0}^{m-1} a_{ri} z^r + Qg_i \quad (1.51).$$

Thus $$f_j = \frac{F_j}{\det \psi} \quad (1.52)$$

where

$$F_j(z) = \sum_{i=1}^{n} \left(\sum_{r=0}^{m-1} a_{ri} z^r + Qg_i \right) \Phi_{ij}(z). \quad (1.53)$$

We can solve for a_{ri} to satisfy the conditions

$$F_j^{(s)}(\alpha_k) = 0 \quad 0 \leq s \leq k_i \quad k = 1, \ldots, \ell$$
$$j = 1, \ldots, n$$

which is possible when the condition of maximality of rank of (1.4.7) or (1.4.7') is satisfied. The explicit form of T_φ^{-1} is too complicated to be written down here but probably can be used to find an estimate of $\|T_\varphi^{-1}\|$. Note that it is in all possibilities not necessary to find out exact zeros of det ψ inside $\mathbb{D}$ to obtain an estimate of $\|T_\varphi^{-1}\|$.

§2. Now with this acomplished we shall use it to prove the result of Gokhberg and Kreĭn.

<u>Theorem 1</u> Let φ be a continuous map from the circle group $\mathbb{T}$ to the group, $Gl(n,\mathbb{C})$, of the nxn invertible matrices. Let det φ be homotopic to a constant. Then for $\epsilon > 0$ we can find another continuous map θ from $\mathbb{T}$ to $Gl(n,\mathbb{C})$ such that

$\|\theta - \varphi\| < \epsilon$ and T_θ is invertible.

<u>Proof</u>:

Let $J_{n,m} = \{T_\theta$: θ is a continuous map from $\mathbb{T}$ to M_n with its Fourier transform supported on $[-m,m]\}$

Let $\mathcal{J}_{n,m} = \{T_\theta \in J_{n,m}$: det θ does not vanish at any point on $\mathbb{T}$ and is homotopic to a constant$\}$.

Obviously $J_{n,m}$ is a vector space of dimension $n^2(2m+1)$ over $\mathbb{C}$. Hence we can identify $J_{n,m}$ with $\mathbb{C}^{n^2(2m+1)}$ with obvious identification. Now as determinant is a continuous function in the Fourier coefficients of a trigonometric polynomial θ and so is index, $\mathcal{J}_{n,m}$ is an open subset of $J_{n,m}$. For $\epsilon > 0$ we can find a trigonometric polynomial θ_0 in $\mathcal{J}_{n,m_0}$ for some m_0 such that $\|\varphi-\theta_0\| < \epsilon/2$. We know that $\mathcal{J}_{n,m_1} \subset \mathcal{J}_{n,m_2}$ if $m_1 \leq m_2$. So θ_0 is in $\mathcal{J}_{n,m}$ for all $m > m_0$. Consider the function 1 from $\mathbb{T}$ to $Gl(n,\mathbb{C})$ which takes every point of $\mathbb{T}$ to the identity matrix in $Gl(n,\mathbb{C})$. So 1 is in $\mathcal{J}_{n,m}$ for all $m > 0$. Moreover θ_0 and 1 are in same connected component U of $\mathcal{J}_{n,m}$ if m is large enough. We observe that the Toeplitz operator with symbol 1 is obviously invertible. By the main lemma, if we write $\theta_0 = \frac{\psi_0}{\chi_{m_0}}$, we shall have θ_0 not invertible if all the $(2m+1)n \times (2m+1)n$ submatrices formed from the $(2m+1)n \times (2m+1)n^2$ matrix (1.4.7) or (1.4.7') have their determinants zero. Let M_i $i = 1,2,\ldots,\binom{(2m+1)n^2}{(2m+1)n}$ be the determinants of the submatrices of (1.4.7) or (1.4.7'). Now if we write a for the coefficients $a_1,\ldots,a_{n^2(2m+1)}$ in ψ, then each M_i is locally an analytic function of a because each M_i is a function of Φ_{ij} evaluated at the zeros α_k's of det ψ in $\mathbb{D}$ and of a_k. But the determinant of ψ is a polynomial in a. Hence each of the zeros of det ψ represents an analytic function locally when the zeros are distinct. Let θ in U have the fom $\frac{\psi}{\chi_m}$. Let $\det \psi = \sum_{i=}^{nm} A_i(a) z^i$ (2.1)

where $a = (a_1, \ldots, a_{n^2(2m+1)})$.

It is clear that the coefficients A_i in (2.1) are polynomials in a. Let D(a) denote the discriminant of the polynomial (2.1). The set of points at which the discriminant D does not vanish are the points where (2.1) has distinct zeros. But as D is a polynomial in A_i's it is a polynomial in a. Hence the set of points S(D) in U at which the discriminant D vanishes in a subvariety of U. Hence $U_1 = U \setminus S(D)$ is a dense open subset of U and as U is connected U_1 is connected (c.f. Gunning and Rossi [1]). Because U_1 is dense open in U we can find in the neighborhood of 1 a polynomial θ_1 in U_1 such that T_{θ_1} is invertible. Now if we write $M_i(\theta)$ for the minors of the matrix (1.4.7) corresponding to the trigonometric polynomial θ then all M_i cannot be identically zero in U_1. That is because θ_1 is in U_1 and for invertibility we need at least one of the minors $M_i(\theta_1)$ is non zero. Now for every a in U_1 there is a neighborhood U_a of a where each M_i is an analytic function. Recall (cf. Gunning and Rossi [1] p. 86) that if U_1 is a domain in $\mathbb{C}^n$, a subset V of U_1 is called a subvariety if for every z in U_1 there are a neighborhood U and functions $M_1, \ldots, M_t$ holomorphic in U_2, such that

$$V \cap U_z = \{\theta \in U_z \ ; M_1(\theta) = M_2(\theta) .. = M_t(\theta) = 0\},$$

and that a subvariety V of U_1 is a closed, no where dense subset of U_1. If U_1 is connected, then $U_1 - V$ is connected.

But by the main lemma of section 1 V consists of those trigonometric polynomials θ in U_1 for which the corresponding Toeplitz operator T_θ is not invertible. Hence we can find a θ in $U_1 \setminus V$ such that $\|\theta-\theta_0\| < \epsilon/2$ and T_θ is invertible. This is true because U_1 is dense open in U and $U_1 \setminus V$ is dense open in U_1. Hence $U_1 \setminus V$ is dense open in U and θ_0 is in U. So we get $\|\theta-\varphi\| < \epsilon$ and T_θ invertible.

Theorem 2: Given a continuous function θ from $\mathbb{T}^2$ to the group $Gl(n,\mathbb{C})$ of n x n invertible matrices such that det φ is homotopic to a constant and given $\epsilon > 0$, there exists a continuous function θ from $\mathbb{T}^2$ to M_n such that the Toeplitz operator $T_{(\cdot,w)}$ on $H^2_n(\mathbb{Z}_+)$ is invertible for each w in $\mathbb{T}$ and

$$\|W_\varphi - W_\theta\| < \epsilon$$

Proof: Let $J_{n,m}$ and $\mathcal{J}_{n,m}$ be as before. Now because φ is continuous on $\mathbb{T}^2$ we can for $\epsilon > 0$ find a trigonometric polynomial θ_0 such that $\|\varphi-\theta_0\| < \epsilon/2$ and det θ_0 from vanishing and homotopic to a constant. Let θ_0 have its Fourier transform supported on $[-m_1^0, m_1^0] \times [-m_2^0, m_2^0]$. We can also assume that $\theta(\cdot,1)$ is in U_1, where U_1 is as defined in Theorem 1. Now for each trigometric polynomial θ, let us write E_θ for the map $E_\theta : \mathbb{T} \to \mathcal{J}_{n,m}$ defined by $E_\theta(w) = T_{\theta(\cdot,w)}$ for every w in $\mathbb{T}$. It is obvious that E_θ is a continuous map. We further know that choosing m large enough $T_{\theta_0(\cdot,1)}$ and T_{θ_1} can be in the same connected component U_1 of $\mathcal{J}_{n,m}$ where θ_1 is chosen as in Theorem 1. Now as E_θ is continuous for each θ we see that $\{T_{\theta_0}(\cdot,w) : w \in \mathbb{T}\}$ is a connected subset of $\mathcal{J}_{n,m}$, hence contained in U_1. Now we need only show that there is an E_θ for some θ arbitrarily near θ_0 such that $E_\theta(\mathbb{T})$ misses the subvariety V of U_1 because if $E_\theta(\mathbb{T})$ misses V and is contained in U_1 then each $T_{\theta(\cdot,w)}$ is invertible where V is the subvariety of U_1, as defined before, for each trigonometric polynomial θ whose Fourier transform is supported on the strip $\{[-m,m] \times \mathbb{Z}\}$ we can consider E_θ as a holomorphic map from an annular region A containing $\mathbb{T}$ to $\mathcal{J}_{n,m_0}$. Now let us denote by M the homomorphic map from U_1 to $\mathbb{C}^N$, where $N = \binom{(2m+1)n^2}{(2m+1)n}$, whose zero set is the subvariety V. Now $M \circ E_\theta$ is a holomorphic function from A to C^N. Now we hold that for every trigonometric polynomial θ_0 and $\epsilon > 0$ there is another trigonometric polyonomial θ such that $\|\theta_2 - \theta_0\| < \epsilon/4$ and the set of zeros of $M \circ E_\theta$ in $\mathbb{T}$ is finite. Otherwise we would get a limit point of zeros of $M \circ E_\theta$ in $\mathbb{T}$ for every θ in an arbitrary neighborhood of θ_0. This would mean $M \circ E_\theta \equiv 0$ which would mean $E_\theta(\mathbb{T}) \subset V$. This is impossible by the Theorem 1. Having accomplished this we further hold that for such a θ_2 there is an i such that $V^i_{\theta_2} = \{W \in \mathbb{T} : M_i E_{\theta_1}(w) = 0\}$ is finite. This is obvious, for otherwise each $M_i \circ E_{\theta_2} \equiv 0$. Because $M_i \circ E_{\theta_2}$ has a finite number of zeros in $\mathbb{T}$ we can find a θ such that $\|\theta_2 - \theta\| < \epsilon/4$ and $M_i \circ E_\theta$ has no zeros in $\mathbb{T}$. This can be proved thus if $a_1^{(2)}, \ldots, a_{(2m+1)n^2}^{(2)}$ are the coefficients in θ_1, then $M_i \circ E_{\theta_2}$ can be regarded as a function of $(2m+1)n^2 + 1$ variables including w. Now the let $w_1, \ldots, w_k$ be the zeros of $M_i \circ E_{\theta_2}$ in $\mathbb{T}$. Each of them can be regarded as a function of the variables $a_1, \ldots, a_{(2m+1)n^2}$ which we denote by a. Now we hold that in the neighborhood of $a^{(2)} = a_1^{(2)}, \ldots, a_{(2m+1)}^{(2)}$ there is a point a such that $\sup_i |a_i - a_i^{(2)}| < \epsilon/4$ and none of $w_1(a), \ldots, w_k(a)$ lie on $\mathbb{T}$. It is sufficient to prove this for one w_i. Now w_i being a holomorphic function in a neighborhood of $a^{(2)}$ it takes open sets to open sets if w_i is not a constant. Since $w_i(W)$ is open, where $W = \{a : \sup |a_i - a_i^{(2)}| < \frac{\epsilon}{4}\}$, we can find a point p in $w_i(W)$ not in $\mathbb{T}$. Now if a is a point in $w^{-1}(p) \cap W$ then we get what we want. But if w_i is a constant then we would get $M_i \circ E_\theta(w_i) = 0$. This would mean $T_{\theta(\cdot, w_i)}$ is not invertible for all θ in a neighborhoods of θ_1 which is impossible by theorem 1. So we can find a θ such that $T_{\theta(\cdot, w)}$ is invertible for every w in $\mathbb{T}$ and $\|\theta - \theta_1\| < \epsilon/4$. But $\|\theta_2 - \theta_0\| < \epsilon/4$ and $\|\varphi - \theta_0\| < \epsilon/2$. Combining these we get

$$\|\varphi - \theta\| < \epsilon.$$

<u>Corollary 1</u>: $\mathfrak{F}_1 = \{W_\varphi$: det φ does not vanish on $\mathbb{T}^2$ homotopic to a constant and $T_{\varphi(\circ, w)}$ and invertible for each w in $\mathbb{T}\}$ is a dense open subset of $\mathfrak{F}_0 = \{W_\varphi$: φ a continuous map from $\mathbb{T}^2$ to $Gl(n, C)$ and det φ homotopic to a constant$\}$.

Corollary 2: $\mathfrak{F} = \{W_\varphi : \varphi$ is continuous function from $\mathbb{T}^2$ to $Gl(n,C)$ and $\det \varphi \sim$ constant and W_φ is Fredholm$\}$ is a dense open subset of $\mathfrak{F}_0$ the set of Toeplitz operators whose symbols are invertible everywhere on $\mathbb{T}^2$ and have determinant homotopic to a constant.

Remark: We have not been able to find out if $\mathcal{J}_{n,m}$ is connected or not. Otherwise it would be easy to show the set

$$\mathcal{J}^0_{n,m} = \{T_\varphi \in \mathcal{J}_{n,m} : T_\varphi \text{ not invertible}\}$$

is a subvariety of $\mathcal{J}_{n,m}$. This may be a better and shorter way to prove the above results.

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