ON PLANAR CAYLEY DIAGRAMS

A thesis presented

by

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We, the dissertation committee for the above candidate for the Ph. D. degree, hereby recommend acceptance of the dissertation.

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Herbert Weisinger, Dean

Abstract

If a group G has a Cayley diagram $\Gamma \subset E^2$, then the word problem for G can be solved by inspection.

Maschke [8] determined all finite groups on three or fewer independent generators with a planar Cayley diagram. All regular pavings of the plane (which give rise to Cayley diagrams of some infinite groups) are classically known. A visual examination of these two classes of Cayley diagrams yield the fact that they are all point symmetric (the clockwise ordering of the edges at each vertex is the same) or weakly point symmetric (if v_1 and v_2 are any vertices in Γ, then the clockwise ordering of the edges about v_1 is the same as either the clockwise or counterclockwise ordering of the edges about v_2). However, no results have been obtained to show that every Cayley diagram $\Gamma \subset E^2$ must have at least a weakly point symmetric embedding, yet no example exists of a Cayley diagram with no such point symmetry. This paper, therefore, deals with embeddings of Cayley diagrams which are assumed to be at least weakly point symmetric.

For finite groups, it is determined directly which groups with three or more independent

generators have Cayley diagrams in E^2. It is also shown that all such groups must have a weakly point symmetric Cayley diagram.

Results for the case of infinite groups include the following extensions of two theorems of Maschke for the finite case:

No two edges of one color can be crossed by two edges of another color.

Any polygon of one color determines a component of the complement of Γ in E^2 if Γ is locally finite (every finite region of the plane contains but a finite number of vertices of Γ).

If D is a connected component of the complement of Γ in E^2, then the boundary of D $[\delta(D)]$ cannot have two consecutive edges of the same color unless all edges of $\delta(D)$ have the same color.

In addition, $\delta(D)$ is a Jordan curve under the last named conditions.

The main result of this paper, Theorem 3.3, gives a method to determine (under certain conditions) whether a group G has a planar Cayley diagram Γ with a weakly point symmetric embedding merely by looking at the presentation for G. In addition, under the conditions of this theorem, weak point symmetry is shown to imply point symmetry, and the word problem for G is reduced to finding the order γ of a certain element $(x_1x_2...x_n)$ in G.

I wish to express to Professor Elvira Rapaport Strasser my appreciation for her help and encouragement.

Section 1-Graph of a group

Let G be a group which has a finite presentation on the generators $a_1, a_2, \ldots a_n$. Each element $g \in G$ will correspond to some distinct and unique point $v \in E^3$. Certain pairs of points will be joined by oriented edges of colors c_i; each color c_i corresponding to the generator a_i. Specifically, if

$$g_j a_i = g_j', \quad g_k a_i^{-1} = g_k',$$

the points v_j and v_j' will be joined by an oriented edge of color c_i beginning at v_j and terminating at v_j'. The points v_k and v_k' will be joined by an oriented edge of color c_i beginning at v_k' and terminating at v_k. The edge of color c_i is <u>positively directed</u> from v_j to v_j', is <u>negatively directed</u> from v_k to v_k'.

This system of points and edges is called the <u>Cayley diagram</u> (in short, <u>graph</u>)of the group G. Call the points v_j the <u>vertices</u> of the Cayley diagram $\Gamma \subset E^3$.

If $w(a_1, a_2, \ldots a_n)$ is any element in G, $g_j \in G$, and $g_j \cdot w(a_1, a_2, \ldots a_n) = g_k$, then v_j is joined to v_k by a path $\overline{w(a_1, a_2, \ldots a_n)} \subset \Gamma$ (corresponding to $w(a_1, a_2, \ldots a_n) \in G$) of oriented edges beginning at v_j and terminating at v_k, by regarding multiplication on the right in G as the sucession of edges in Γ. That is

$$g_j \cdot a_i \cdot a_k = g_j' \Leftrightarrow v_j \cdot \overline{a}_i \cdot \overline{a}_k \equiv v_j \cdot \overline{a_i a_k} = v_j' \quad .$$

Obviously, $w(a_1, a_2, \ldots a_n)$ is a relator (equals 1 in G) iff $\overline{w(a_1, a_2, \ldots a_n)}$ is a closed path in Γ. Call a closed path a <u>cycle</u>.

A cycle is a Jordan Curve if it has no multiple points.

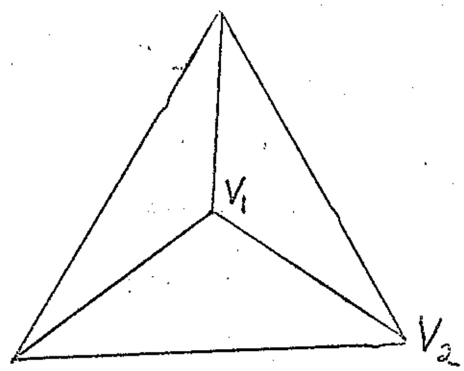
It will be assumed throughout that the generators, $\langle a_i \rangle$, are independent, that is, it is not possible to express any one by the remaining generators.

The edges $\overline{a_i}$ in Γ will be assumed to meet only at the vertices $v_j \in \Gamma$. Of course, this immediately restricts the space in which Γ may be embedded. For example, it is impossible under this restriction to connect each of five points to the other four in E^2, whereas it is simple to do so in E^3. Some graphs, such as the one consisting of a single point, may be embedded in any space $T \neq \emptyset$.

A Cayley diagram $\Gamma \subset E^2$ is locally finite if every finite region of E^2 contains but a finite number of vertices of Γ. Compactifying E^2 to the sphere, a finite region is simply any region not containing the point at infinity. Every local graph (some subset of Γ at some $v \in \Gamma$) considered will be assumed to lie in a finite region of E^2. Obviously, the graph of a finite group is locally finite.

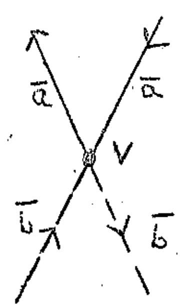
If $\Gamma \subset E^2$ is a Cayley diagram of the group $G \cong F/R$, and $w \in F$ is 1 in G, and $\overline{w}$ is a Jordan curve, then $\overline{w}$ separates E^2 into two open connected components, one finite, one infinite. If either of these components contains no vertices or edges of Γ then it will be called a disk, and $\overline{w}$ bounds (determines) this disk. If $\overline{w}$ is a Jordan curve, let $\overline{wo}$ denote the finite component of $E^2-\overline{w}$. If $\overline{w}^{o}$ is a disk, it is called a <u>finite disk</u>.

Call a vertex <u>interior</u> if it is not on the boundary of some infinite component of the complement of Γ in E^2. For example, in the following, v_1 is interior while v_2 is not.



Let O_{v_i} be the clockwise ordering of the edges about the vertex $v_i \in \Gamma$, so that $-O_{v_i}$ is the counterclockwise ordering about v_i.

Use the arrows $\longrightarrow$, $\longleftarrow$ to denote that a positively directed edge is leaving or coming into a vertex. For example, if $v \in \Gamma$ looks like this:



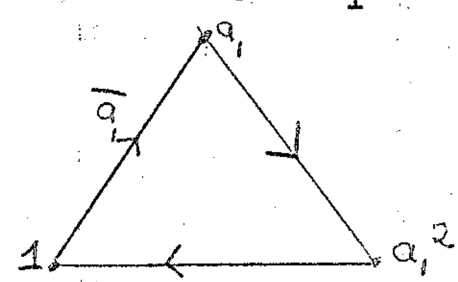
then $O_v = \begin{Bmatrix} a \rightarrow \\ a \leftarrow \\ b \rightarrow \\ b \leftarrow \end{Bmatrix} = \begin{Bmatrix} a \leftarrow \\ b \rightarrow \\ b \leftarrow \\ a \rightarrow \end{Bmatrix}$ etc.

and $-O_v = \left\{ \begin{matrix} b \leftarrow \\ b \rightarrow \\ a \leftarrow \\ a \rightarrow \end{matrix} \right\}$ etc.

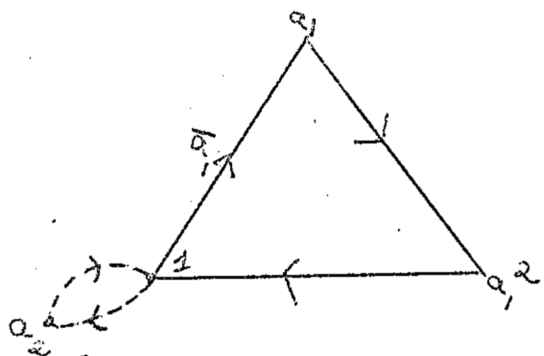
If $O_{v_i} = O_{v_j}$ for all $v \in \Gamma$, then Γ is said to be *point symmetric*. If $O_{v_i} = \pm O_{v_j}$ for all $v \in \Gamma$, then Γ is said to be *weakly point symmetric*. Obviously, point symmetry implies weak point symmetry.

If the order of any generator, say a_i is 2 $(o(a_i)=2)$, the edges of color c_i will be undirected so as to avoid "digons." That is, there will be but one edge of color c_i at each vertex. This approach, while not necessary, will permit point symmetric or weakly point symmetric embeddings in E^2 in certain cases where the existence of digons would not. Consider for example the group G presented by $G = \langle a_1, a_2; a_1^3, a_2^2, a_1 a_2 a_1^{-1} a_2 \rangle$. This group has no point symmetric or weakly point symmetric embedding in E^2 if the edges of color c_2 (single dotted lines) corresponding to a_2 are directed. To see this, assume that there is at least a weakly point symmetric embedding.

At the vertex $v \in \Gamma$ corresponding to 1 in G, there must be a triangle corresponding to $\overline{a_1^3}$;

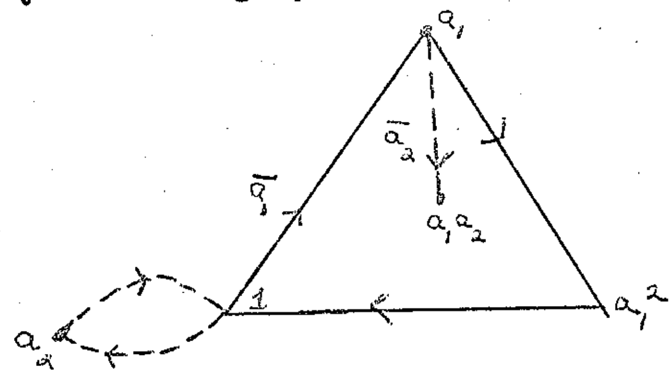


Without loss of generality, assume the edge $\overline{a_2}$ from 1 enters the infinite region of the plane determined by the triangle, and that the ordering of the edges about 1 is as follows;

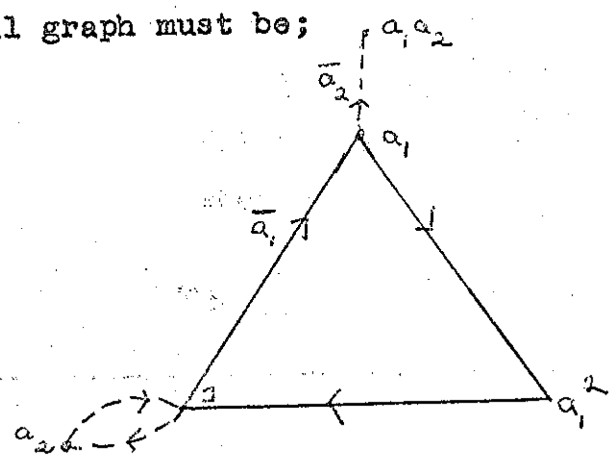


Obviously $a_2 \neq a_1$, $a_2 \neq a_1^2$ since a_1 and a_2 are independent of each other.

If the edge $\overline{a_2}$ from a_1 enters the finite region determined by the triangle, the local graph at a_1 is;



and since $a_1 a_2 a_1^{-1} a_2 = 1$, we must have $a_2 a_1 a_2 a_1^{-1} = 1$. So there must be an edge $\overline{a_1}$ from $a_1 a_2$ to a_2. But this cannot be accomplished in a planar fashion. Therefore, the local graph must be;

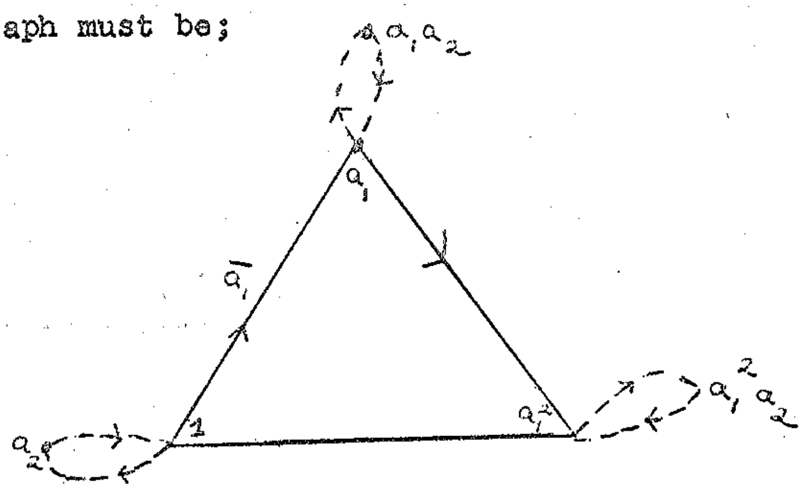


$O_1 = \left\{ \begin{array}{l} a_1 \rightarrow \\ a_1 \leftarrow \\ a_2 \rightarrow \\ a_2 \leftarrow \end{array} \right\}$ and we insist that $O_{a_1} = \pm O_1$.

Since $O_{a_1} = \left\{ \begin{array}{ll} a_1 & \rightarrow \\ a_1 & \leftarrow \\ a_2 & ? \\ a_2 & ? \end{array} \right\}$, in order that $O_{a_1} = \pm O_1$,

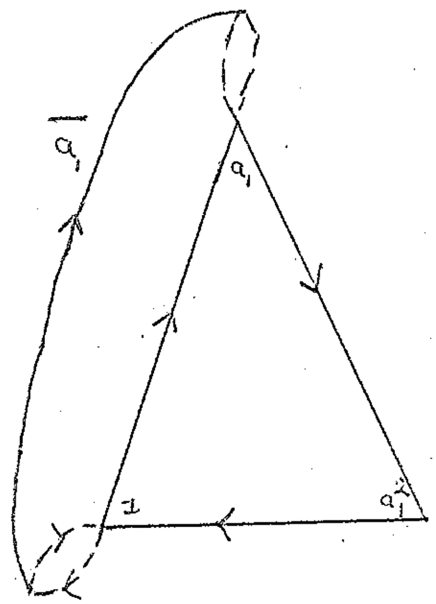
the complete ordering about a_1 must be $\left\{ \begin{array}{l} a_1 \rightarrow \\ a_1 \leftarrow \\ a_2 \rightarrow \\ a_2 \leftarrow \end{array} \right\}$.

The same argument holds at the vertex a_1^2, and so the local graph must be;

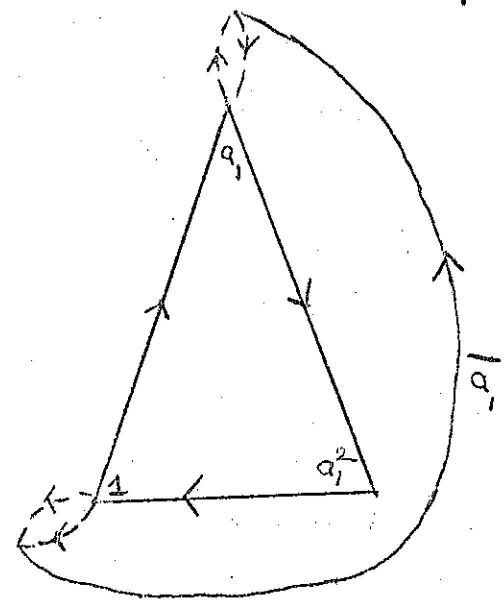


Since $a_2 a_1 a_2 a_1^{-1} = 1$, there must be an edge $\overline{a_1}$ from $a_1 a_2$ to a_2. This can be accomplished in one of two ways;

a

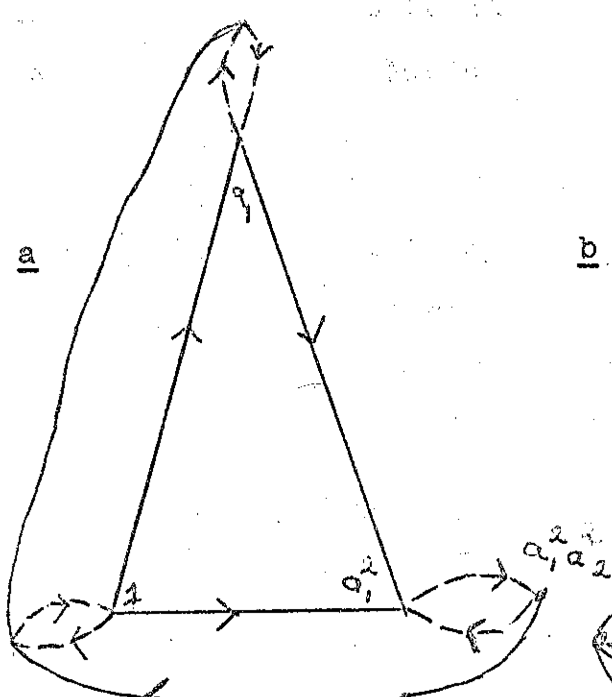


b

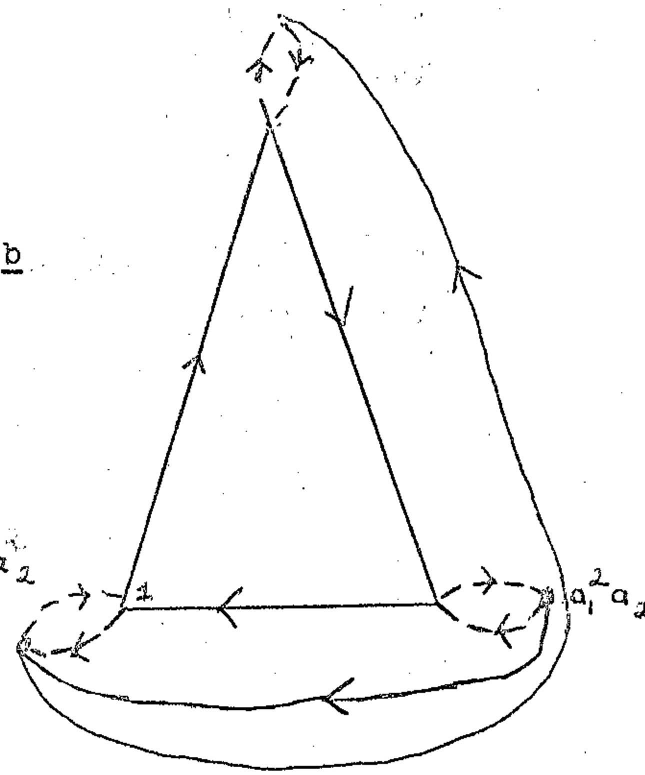


There must also be an edge $\overline{a_1}^{-1}$ from a_2 to $a_1^2a_2$.
Therefore, the two choices for the local graph are;

a



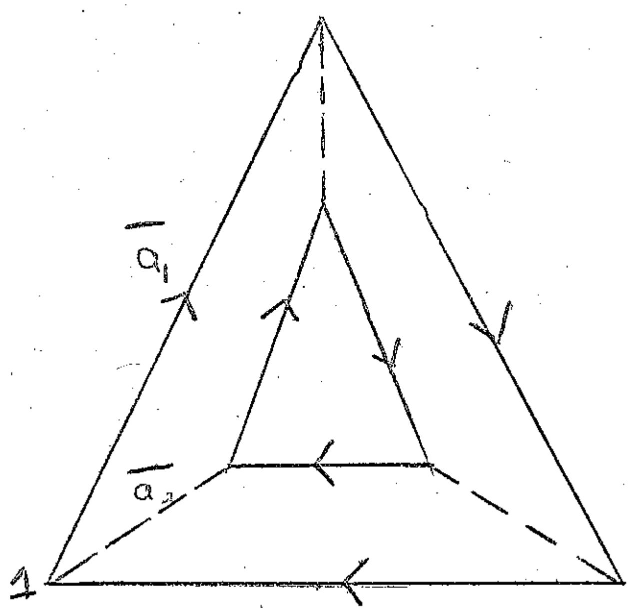
b



So that in either case, $0_{v_{a_2}} = \left\{ \begin{matrix} a_1 \leftarrow \\ a_1 \rightarrow \\ a_2 \rightarrow \\ a_2 \leftarrow \end{matrix} \right\}$

and $0_{v_{a_2}} \neq \pm 0_{v_1}$, and Γ is not even weakly point symmetric.

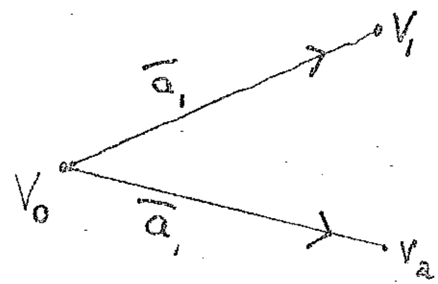
However, if the edges corresponding to a_2 are undirected, a weakly point symmetric graph is easily obtained;



Section II-Some Properties of Finite Planar Groups

Proposition 2.1. If $G = \langle a_1, a_2, \dots a_n; R_1, \dots R_n \rangle$, and G has a planar Cayley diagram Γ, then at most 2 edges of any one color can meet at ant vertex. In addition, they must have opposite orientation. (This property, while included in most definitions of Cayley diagrams, is easily obtained from the defition given above.)

Proof. Assume that more than 2 edges of color c_i meet at some vertex $v_o \in \Gamma$. At least two must have the same orientation. Without loss of generality, assume that they are directed away from v_o. Therefore, we have;



But this implies $v_o \overline{a_1} = v_1$, $v_o \overline{a_1} = v_2$. That is,

$g_o a_1 = g_1$, $g_o a_1 = g_2$.

Therefore, $g_1 = g_2 \Rightarrow v_1 = v_2$, a contradiction.

Q.E.D.

Proposition 2.2. If $G = \langle a_1, a_2, \dots a_n; R_1, R_2, \dots R_q \rangle$ admits a Cayley diagram $\Gamma \in E^2$, and for some $i \in (1, 2, \dots n)$ $\exists$ m, $2 < m < \infty$, such that $a_i^m = 1$, then the path $\overline{a_i^m}$ from any vertex $v_o \in \Gamma$ is a Jordan Curve, if m is assumed minimal.

Proof. Assume m minimal. Pick any $v_0 \in \Gamma$. Let $v_1 = v_0\overline{a_i^m}$. Therefore, in G, $g_1 = g_0 a_i^m = g_0$. Therefore, $v_0 = v_1$, and $v_0\overline{a_i^m}$ is a cycle. If this cycle has a double point, say v_2, then there must be at least 3 edges of color c_i at v_2, which contradicts Proposition 2.1.

Q.E.D.

Proposition 2.3. [8, P. 159] If G is a finite group, and G has a planar Cayley diagram Γ, then two edges of one color are not crossed by two edges of another color.

Proposition 2.4. [8, P. 159] If $G = \langle a_1, a_2, \ldots a_n; R_1 \ldots R_m \rangle$ is a finite group, and G has a Cayley diagram $\Gamma \subset E^2$, then for any $v_0 \in \Gamma$ and any a_i such that $k = o(a_i) > 2$, the cycle $v_0\overline{a_i^k}$ determines a disk.

(These propositions will be proven more generally in section III to include the case that $o(G) = \infty$.)

A visual examination of Maschke's (finite planar) Cayley diagrams yield the following facts;

1. If $G = \langle a_1, a_2; R_1 \ldots R_m \rangle$ and $\Gamma \subset E^2$, and $o(a_1) \geq 3$, $o(a_2) \geq 3$, then:

   a. $O_{v_i} = + O_{v_j}$ all $v \in \Gamma$.

b. The disks at each vertex $v \in \Gamma$ are those determined by precisely; $\overline{a_1^{o(a_1)}}$, $\overline{a_2^{o(a_2)}}$, $\overline{(a_1 a_2^{\pm 1})^{\gamma}}$, $\overline{(a_2^{\pm 1} a_2)^{\gamma}}$, some γ, $1 < \gamma < \infty$.

c. There exists at least one interior vertex.

2. If $G \cong \langle a_1, a_2; R_1 \ldots R_m \rangle$, $\Gamma \subset E^2$, $o(a_1) \geq 3$, $o(a_2) = 2$, then;

a. $O_{v_i} = \pm O_{v_j}$, all v .

b. The disks at each vertex $v \in \Gamma$ are those determined by precisely; either;

$\overline{a_1^{o(a_1)}}$, $\overline{(a_1 a_2)^{\gamma}}$, $\overline{(a_2 a_1)^{\gamma}}$, $1 < \gamma < \infty$, or

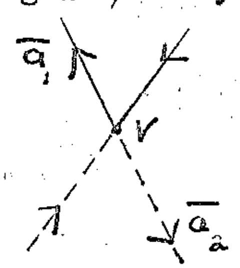
$\overline{a_1^{o(a_1)}}$, $\overline{[a_1 a_2]^{\gamma}}$, $\overline{[a_2 a_1^{-1}]^{\gamma}}$, $1 \leq \gamma < \infty$.

c. There exists at least one interior vertex.

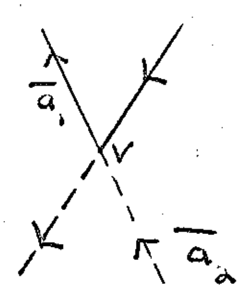
1c and 2c will be proven directly in propositions 2.5 and 2.6. We first need the

Lemma 2.1. If G is a finite group on the generators a_1 and a_2, $o(a_1) = n \geq 3$, $o(a_2) = m \geq 3$, and if G has a Cayley diagram $\Gamma \subset E^2$, then;

a $O_v = \pm$



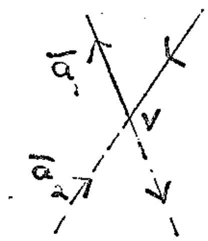
or b $O_v = \pm$



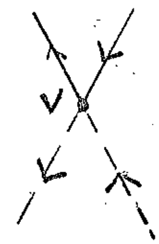
for all $v \in \Gamma$.

<u>Proof.</u> Since there are 4 edges at each vertex, there are 3! 6 possible orderings of the edges about each vertex;

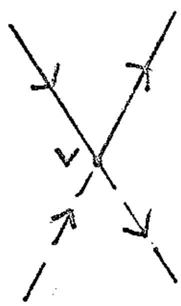
<u>1</u>



<u>2</u>



<u>3</u>



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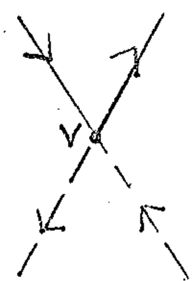
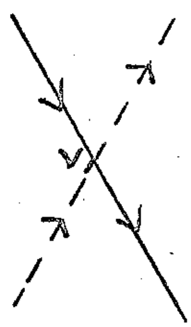
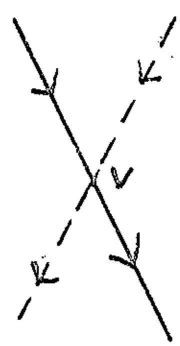


<u>5</u>



<u>6</u>



<u>5</u> and <u>6</u> are impossible(Prop 2.3)

<u>1</u> and <u>4</u> are case <u>a</u> above.

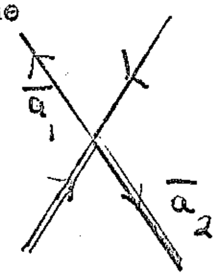
<u>2</u> and <u>3</u> are case <u>b</u> above.

Q.E.D.

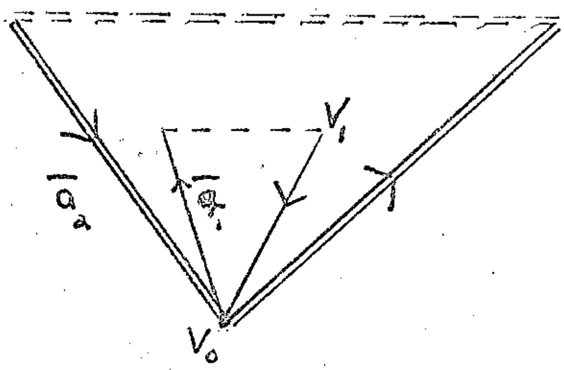
<u>Proposition 2.5.</u> Let G be a finite group on the generators a_1 and a_2 such that $o(a_1) = n \geq 3$, $o(a_2) = m \geq 3$. Assume G has a Cayley diagram $\Gamma \subset E^2$. Then Γ has at least one interior vertex.

<u>Proof.</u> Pick any $v_0 \in \Gamma$. Without loss of generality, (Lemma 2.1) assume

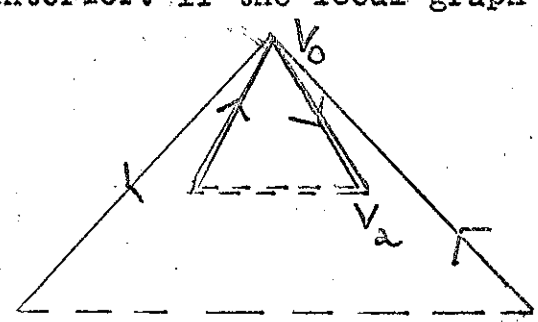
O_{v_0} =



Since G is finite, $n, m < \infty$, and $\overline{a_1^n}$, $\overline{a_2^m}$, describe Jordan curves at v_0 (Prop2.2). If the local graph at v_0 is;

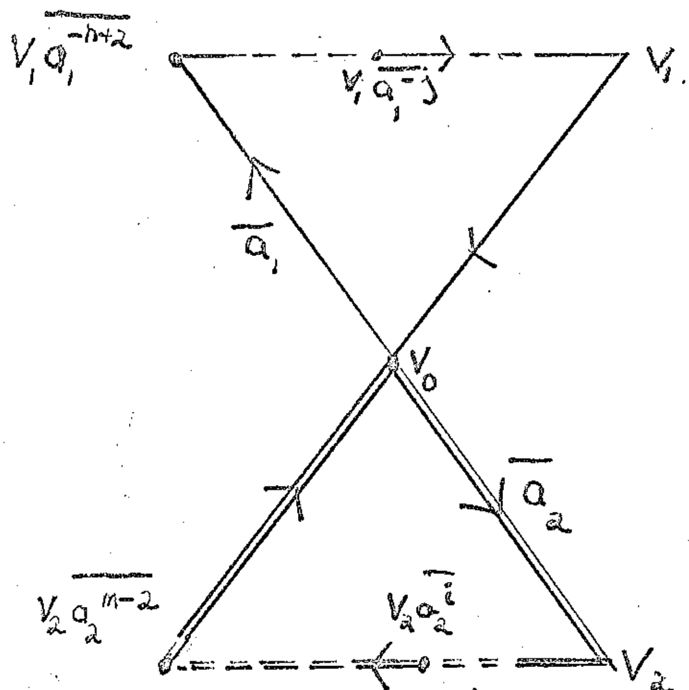


then v_1 is interior. If the local graph at v_0 is;



then v_2 is interior.

So assume the local graph at v_0 is;



Since $o(G) < \infty$, and $a_1 \neq a_2^{\pm 1}$, we have;

$$(a_1a_2)^{\gamma} = (a_2a_1)^{\gamma} = 1, \quad 1 < \gamma < \infty .$$

So, $v_1\overline{(a_1a_2)}$ v_1 $v_2\overline{(a_1a_2)}$.

Assume that the path $\overline{(a_1a_2)^{\gamma-1}}$ from v_2 meets one of the vertices $v_2\overline{a_2^{i}}$, $1 \leq i \leq m-2$. Since all vertices of the form $v_2\overline{a_2^{i}}$ already have two edges of color c_2, it must do so with an edge of color c_1(Prop 2.1). So, $(a_1a_2)^{\delta} a_1 = a_2^{i}$. Since $a_1 \neq a_2^{i}$, we must have $1 < \delta < \gamma < \infty$.

But, $v_2\overline{(a_1a_2)^{\delta}a_1} = v_2\overline{a_2^{i}}$, so

$v_1\overline{(a_1a_2)(a_1a_2)^{\delta}a_1} = v_2\overline{a_2^{i}}$ since $v_2 = v_1\overline{a_1a_2}$.

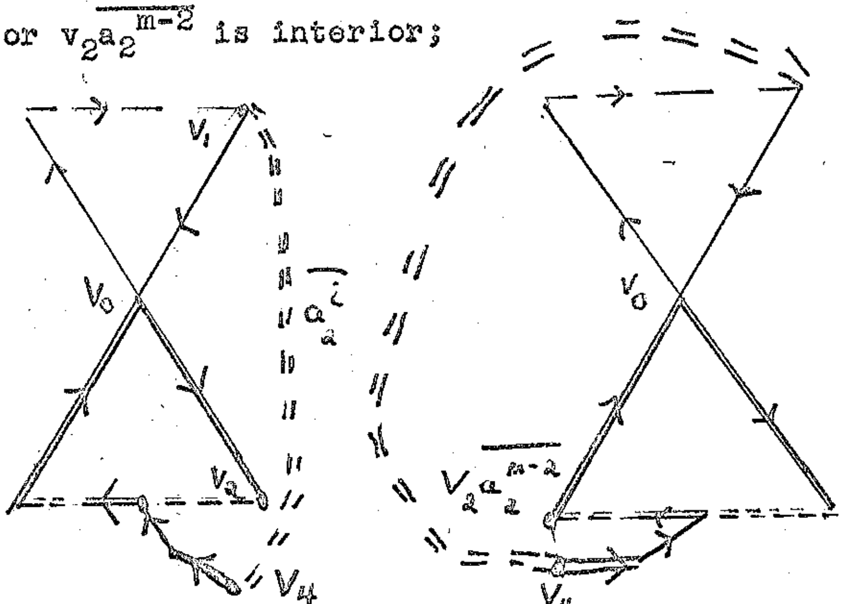
Therefore, $v_1\overline{(a_1a_2)^{\delta}a_1a_2a_1} = v_2\overline{a_2^{i}} \Longrightarrow$

$$v_1\overline{(a_1a_2)^{\delta}a_1} = v_2\overline{(a_2^{i}a_1^{-1}a_2^{-1})}.$$

Since $(a_1a_2)^{\delta}a_1 = a_2^{i}$, we have;

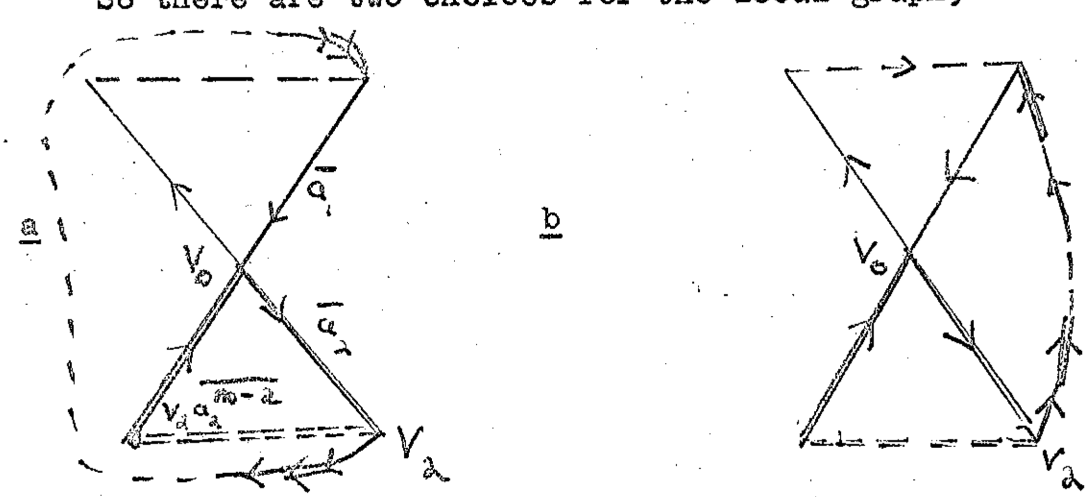
$$v_1\overline{a_2^{i}} = v_2\overline{a_2^{i}a_1^{-1}a_2^{-1}} \cong v_4.$$

But, since $v_1\overline{a_2^{i}} = v_4$, there is a path of color c_2 from v_1 to v_4. As above, this implies that either v_2 or $v_2\overline{a_2^{m-2}}$ is interior;



An identical argument shows that the path $\overline{(a_1a_2)^{\delta-1}}$ from v_2 does not meet any of the vertices $v_1\overline{a_1^{-j}}$, $1 \leq j \leq n-2$.

So there are two choices for the local graph;

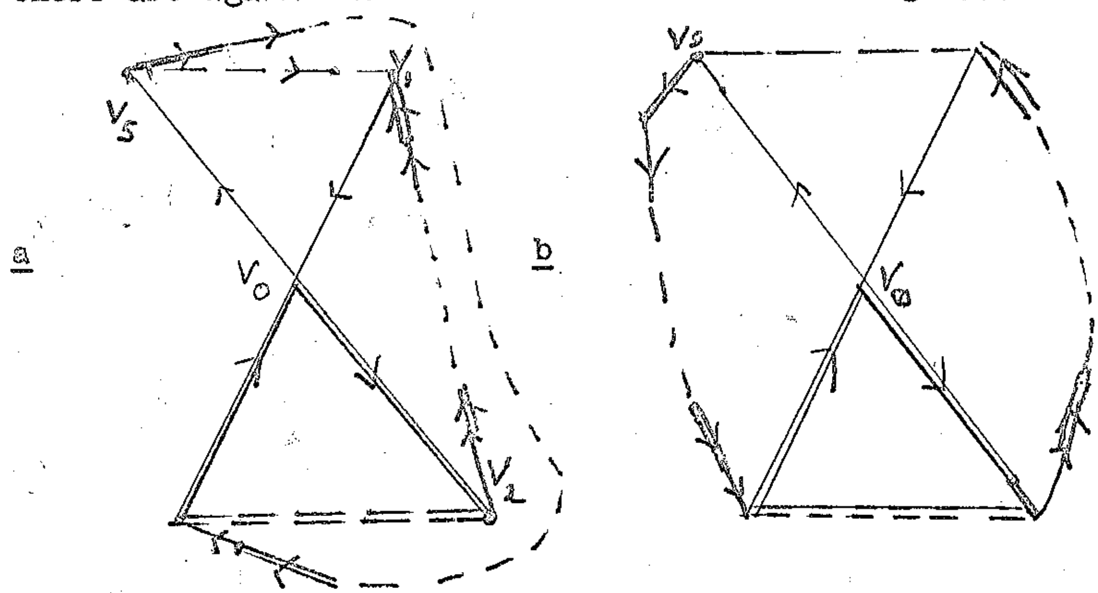


If $\underline{a}$ is the case, then $v_2\overline{a_2^{m-2}}$ is interior.

So assume that $\underline{b}$ is the case.

By considering the path $\overline{(a_1^{-1}a_2^{-1})^{\gamma}}\ \overline{(a_2a_1)^{-\gamma}}$

from $v_5 = v_0\overline{a_1}$, and arguing identically to the above, there are again but two choices for the local graph;

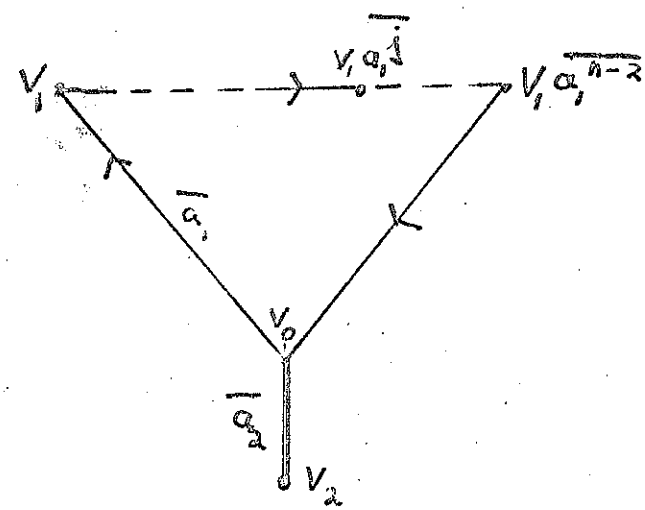


In case $\underline{a}$, v_1 is interior. In case $\underline{b}$, v_0 is interior.

Q.E.D.

Proposition 2.6. Let G be a finite group on the generators a_1 and a_2 such that $o(a_1) = n \geq 3$, $o(a_2) = 2$. If G has a Cayley diagram $\Gamma \subset E^2$, then Γ has an interior vertex.

Proof. Pick any $v_0 \in \Gamma$. Without loss of generality, assume that the local graph at v_0 contains;



(If necessary, replace a_1 by a_1^{-1} in the following argument)

Since $o(G) < \infty$, $(a_1a_2)^\gamma = (a_2a_1)^\gamma = 1$, some γ, $1 < \gamma < \infty$, and $(a_2a_1^{-1})^\xi = (a_1^{-1}a_2)^\xi = 1$, some ξ, $1 < \xi < \infty$.

Therefore, $v_2\overline{(a_2a_1)^\gamma} = v_2$, and so $v_1\overline{(a_2a_1)^{\gamma-1}} = v_2$.

If the path $v_1\overline{(a_2a_1)^{\gamma-1}}$ meets any of the vertices $v_1\overline{a_1^j}$, $0 \le j \le n-2$, it must be with an edge of color c_2, so it must be the case that $v_1\overline{a_1^j} = v_1\overline{(a_2a_1)^\delta a_2}$, $1 < \delta < \gamma$.

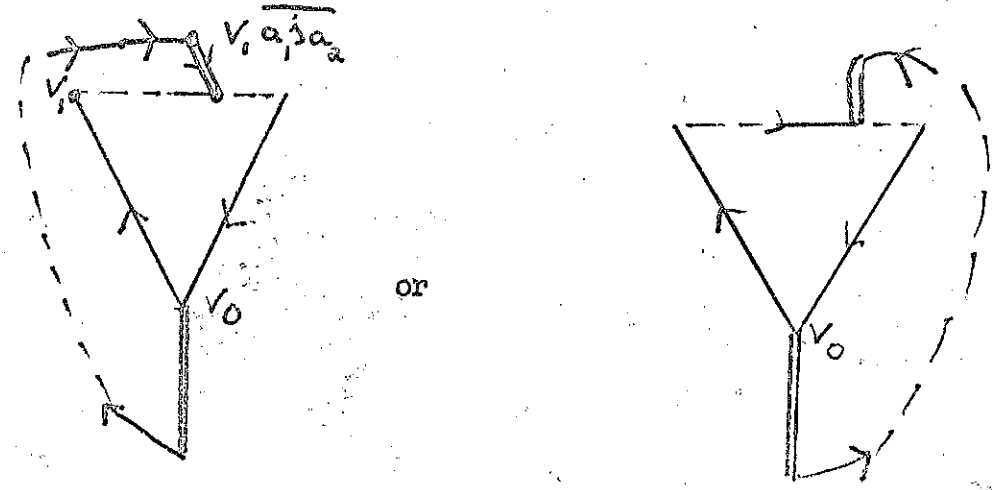
So, $a_1^j = (a_2a_1)^\delta a_2$. But, since $v_2\overline{a_2a_1} = v_1$,

$v_1\overline{a_1^j} = v_2\overline{(a_2a_1)^\delta a_2} = v_2\overline{(a_2a_1)(a_2a_1)^\delta a_2}$ so that,

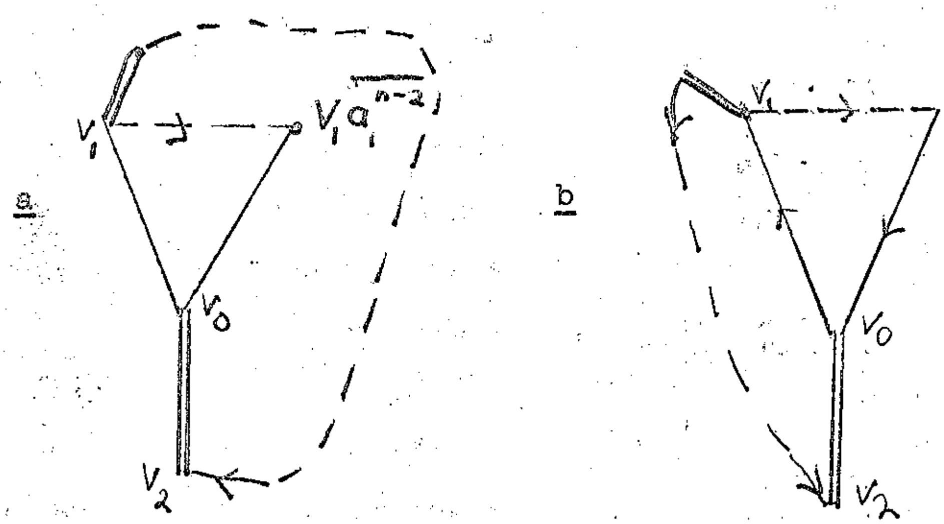
$v_1\overline{a_1^j} = v_2\overline{(a_2a_1)^\delta a_2a_1a_2} = v_2\overline{a_1^ja_1a_2}$.

Therefore, since $a_2 = a_2^{-1}$, $v_1\overline{a_1^j a_2} = v_2\overline{a_1^{j+1}}$.

Since $j+1 > 0$, $v_1\overline{a_1^ja_2} \neq v_2$. Also, $v_1\overline{a_1^ja_2}$ cannot be a a vertex on the c_2 polygon at v_0 as $a_2 \neq a_1^k$, so there are only 2 choices for the local graph at v_0, both of which contain an interior vertex;

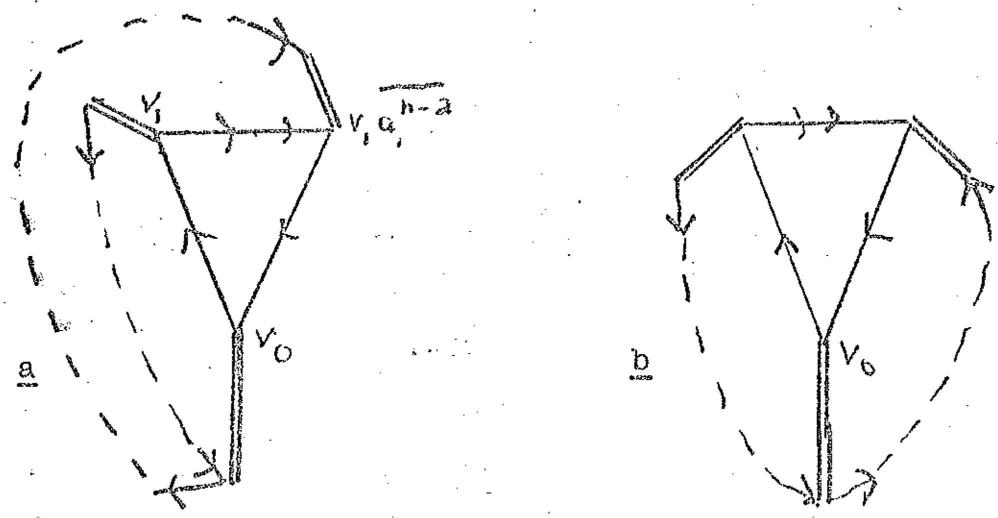


If the path $v_1\overline{(a_2a_1)^{-1}}$ does not meet any of the vertices $v_1\overline{a_1^j}$, 0 j n-2, we have two possible cases;



In case a, $v_1\overline{a_1^{n-2}}$ is interior.

Assume case b. An argument identical to the above yields 2 possibilities for the path $v_2\overline{(a_2a_1^{-1})^{\xi}}$;



In case $\underline{a}$, v_1 is interior. In case $\underline{b}$, v_0 is interior.

Q.E.D.

For Proposition 2.7 we need;

Lemma 2.2. [7, P.69]. Let Γ be the graph of a group G on the generators $a_1, a_2, a_3, \ldots$. If the edges of Γ corresponding to a_1 are deleted, then Γ decomposes into disjoint, isomorphic connected subgraphs. The vertices of a subgraph consist of elements in a left coset of H, the subgroup of G generated by $a_2, a_3, \ldots$. (Note; "disjoint" only if independence of generators is assumed.)

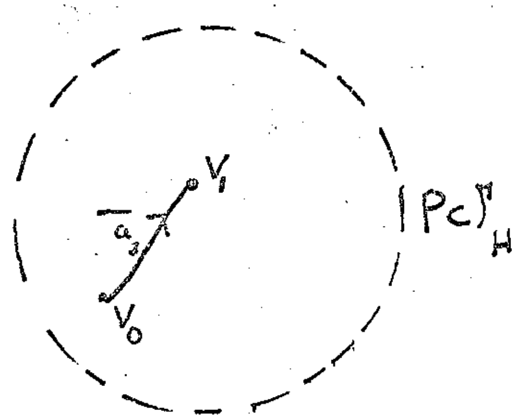
Proposition 2.7. Let G be a finite group on the generators a_1, a_2, a_3. Assume G has a Cayley diagram

$\Gamma \subset E^2$. Then $a_1^{\,2} = a_2^{\,2} = a_3^{\,2} = 1$.

Proof. Assume $a_1^{\,n} = 1 \Rightarrow n \geq 3$. ($n < \infty$ since G is finite). As in lemma 2.2, delete the edges of corresponding to a_3. Therefore Γ decomposes into disjoint subgraphs, the vertices of each subgraph consisting of the elements of some left coset of H, the subgroup of G generated by a_1 and a_2. Pick one subgraph, say that of H. Call it Γ_H. Since a_1, a_2 and a_3 are independent in G, a_1 and a_2 are independent in H. By proposition 2.5 or 2.6, Γ_H contains some interior vertex v_0.

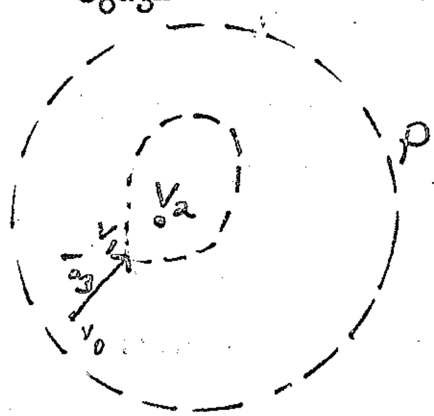
Restore the edges in Γ corresponding to a_3, and consider the vertex $v_1 = v_0\overline{a}_3$.

Since a_3 is not expressable in terms of a_1 and a_2, $v_1 \notin \Gamma_H$. Since v_0 is interior, v_1 must be in a finite component of the complement of $\Gamma_H \subset E^2$, and this component is determined by some cycle P in Γ_H;

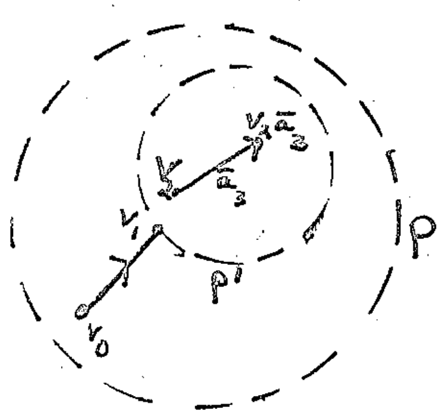


But if $v_0 \in \Gamma \sim g_0 \in G$, then $g_0a_3 \in g_0a_3H$, and so $v_0\overline{a_3} \in \Gamma_{g_0a_3H}$. Therefore, $v_1 \in \Gamma_{g_0a_3H}$, the graph of the coset g_0a_3H. Since $g_0 \in H$, $H = Hg_0^{-1}$. Therefore, since $a_3 \in H$, $g_0a_3 \in H$ so that $g_0a_3H \neq H$.

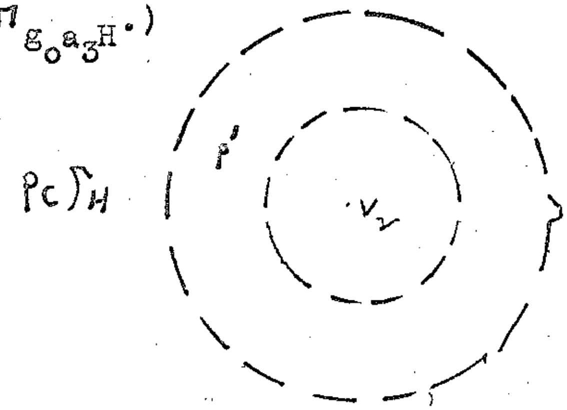
Therefore, since $v_1 \in \Gamma_{g_0a_3H} \neq \Gamma_H$, $\Gamma_{g_0a_3H}$ must be entirely contained in the finite region of E^2 determined by P since the graphs of the cosets are disjoint. In addition, $\Gamma_{g_0a_3H}$ has some interior vertex v_2;



Therefore, v_2 must be in the finite region of E^2 determined by some cycle $P' \in \Gamma_{g_0a_3H}$, where P' is in the finite open region of E^2 determined by P;

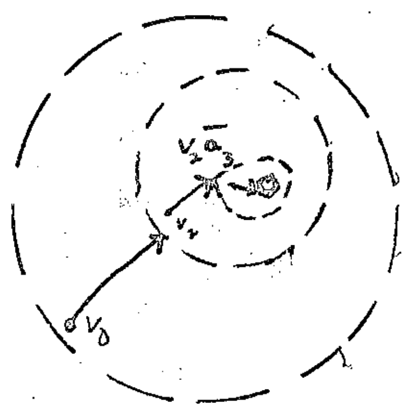


(Since Γ_H and $\Gamma_{g_0a_3H}$ are disjoint, $P \cap P' = \emptyset =$ $\Gamma_H \cap P'$. Also, $P' \subset \mathring{P}$ because both Γ_H and Γ_{g_0H} are connected. In addition, the independence of a_1, a_2, a_3 imply $v_2\overline{a_3} \notin \Gamma_{g_0a_3H}$.)



Therefore, since $v_2\overline{a_3} \notin \Gamma_{g_0a_3H}$, $v_2\overline{a_3} \notin P'$, and so $v_2\overline{a_3} \in \Gamma_H$.

So $v_2\overline{a_3}$ must begin at v_2 and terminate at some third coset graph which also has some interior vertex v_3. This process can continue indefinitely;

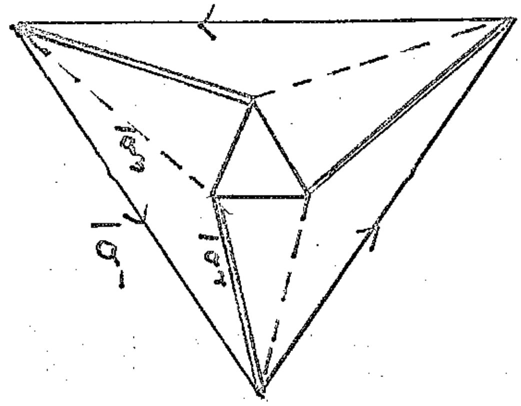


But this is impossible since $o(G) < \infty$.

Q.E.D.

The necessity of insisting on the independence of generators is obvious in the following example where $a_3 = a_1a_2$. In this case the proposition fails.

$$G = \langle a_1, a_2, a_3;\ a_1^3, a_2^2, a_3^2, a_1a_2a_3^{-1} \rangle$$



Lemma 2.3 Let $G = \langle a_1, a_2, a_3;\ a_1^2, a_2^2, a_3^2 \rangle$. Assume that G has a Cayley diagram Γ embedded in E^2. Then, for any $v_1, v_2 \in \Gamma$, $O_{v_1} = O_{v_2}$ or $O_{v_1} = -O_{v_2}$.

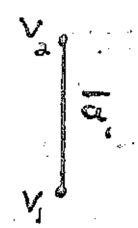
Proof. The proof is obvious as there are only two possible orderings of the edges at each vertex;

Q.E.D.

Lemma 2.4. Let $G=\langle a_1,a_2,a_3;a_1^2,a_2^2,a_3^2,(a_1a_2)^{\gamma}\ldots\rangle, \gamma<\infty$. Assume G has a planar Cayley diagram Γ. Then, $\overline{(a_1a_2)^{\gamma}}$ bounds a disk at every vertex $v\in\Gamma$ if and only if $O_{v_1}=-O_{v_2}$ for every $v_1,v_2\in v\overline{(a_1a_2)}$ such that v_1 and v_2 are one edge apart.

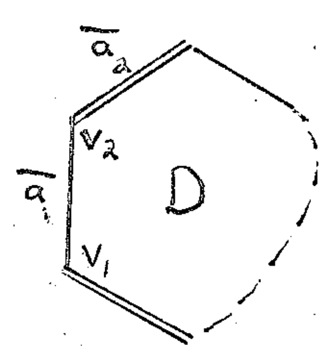
$\Rightarrow$ Proof. Since $a_1\neq a_2$, $\gamma\neq 1$. Therefore, $1<\gamma<\infty$.

Assume that $\overline{(a_1a_2)^{\gamma}}$ determines a disk at every $v\in\Gamma$, and that $v_1,v_2\in\Gamma$, v_1 and v_2 are one edge apart. Without loss of generality, assume that the edge is of color c_1, i.e. a single solid edge.

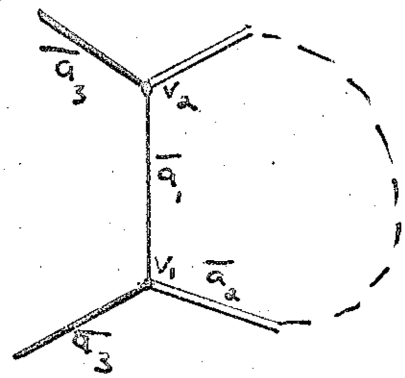


By assumption, the path $\overline{(a_1a_2)^{\gamma}}$ from v_1 bounds a disk. Without loss of generality, assume that the disk is a finite disk. The proof is identical if the disk is the infinite region determined by $\overline{(a_1a_2)^{\gamma}}$.

Therefore, the local graph at v_1 is;



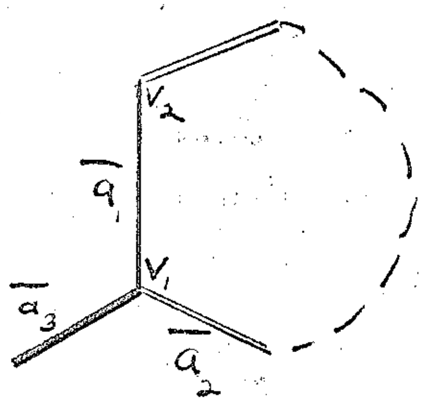
Since D is a disk, the edges corresponding to a_3 at v_1 and v_2 must enter the infinite region of E^2 determined by $\overline{(a_1a_2)\gamma}$. Therefore the subgraph is;



and inspection shows that $O_{v_1} = -O_{v_2}$.

$\Leftarrow$ Assume that $O_{v_1} = -O_{v_2}$ for all v_1, v_2 on a path $\overline{(a_1a_2)\gamma}$ such that v_1 and v_2 are one edge apart.

Pick any vertex v_1 on any path $\overline{(a_1a_2)\gamma}$. Assume that the edge $\overline{a_3}$ at v_1 enters the infinite region of the plane determined by $\overline{(a_1a_2)\gamma}$.(The proof is identical if $\overline{a_3}$ enters the finite region.)

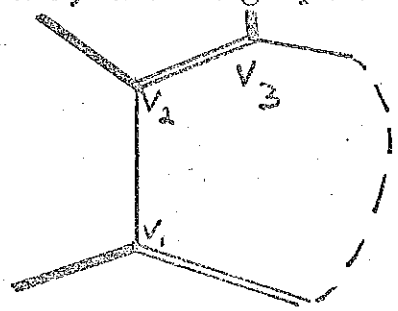


Since $O_{v_1} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$, and since v_1 and v_2 are one

edge apart, $O_{v_2} = \begin{Bmatrix} a_1 \\ a_3 \\ a_2 \end{Bmatrix}$ by assumption.

Similarly, $O_{v_3} = -O_{v_2} = O_{v_1}$.

Therefore, the subgraph is;



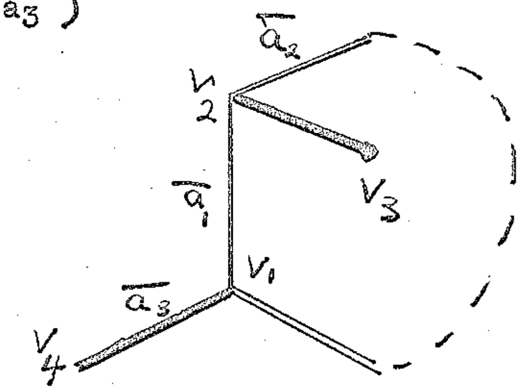
A similar argument for v_3, v_4 etc. shows that every $\overline{a_3}$ edge meeting the path $\overline{(a_1a_2)}\delta$ must enter the infinite region of E^2 determined by that path. Therefore the finite region must be a disk.

Q.E.D.

<u>Theorem 2.1.</u> Let $G = \langle a_1, a_2, a_3;\ a_1^2, a_2^2, a_3^2, (a_1a_2)^i, (a_2a_3)^j, (a_3a_1)^k \rangle$, $2 \leq i, j, k, < \infty$, i, j, k, minimal, [3, P.36] Assume that G has a Cayley diagram $\Gamma \subset E^2$. Then, the disks in Γ are determined precisely by $\overline{(a_1a_2)^i}$, $\overline{(a_2a_3)^j}$, $\overline{(a_3a_1)^k}$. Also, $O_{v_1} = -O_{v_2}$ for v_1, v_2 one edge apart, and Γ has at least one interior vertex.

Proof. The proof is symmetric on a_1, a_2 and a_3. Since the a_i's are independent, $i, j, k \geq 2$. Consider the path $\overline{(a_1a_2)^I}$ from some vertex $v_1 \in \Gamma$. Let v_2 be one $\overline{a_1}$ edge away from v_1. By lemma 2.3, $O_{v_1} = \pm O_{v_2}$, so assume $O_{v_1} = O_{v_2}$. In addition, without loss of generality, assume that $O_{v_1} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$ and therefore,

$O_{v_2} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$ and the local graph at v_1 is;



The vertex $v_3 = v_2\overline{a_3} \notin v_1\overline{(a_1a_2)^I}$ since the a_i's are independent. Likewise for $v_4 = v_1\overline{a_3}$.

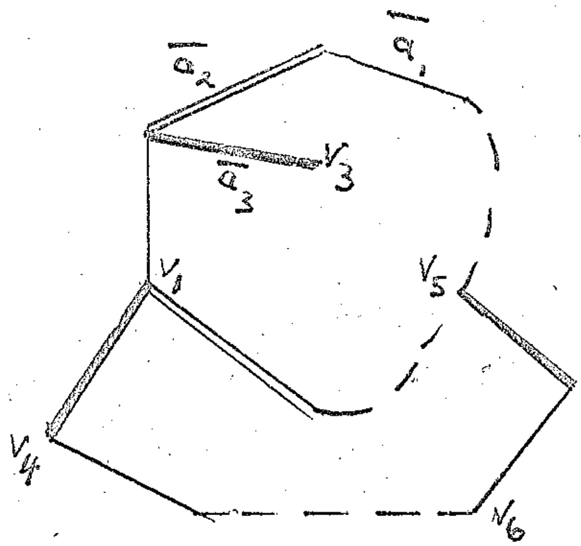
Since $(a_3a_1)^k = 1$, and $k \geq 2$, we have;

$$v_3 = v_3\overline{(a_3a_1)^k} = v_3\overline{(a_3a_1a_3)\cdot(a_1a_3)^{k-2}\,a_1} = v_4\overline{(a_1a_3)^{k-2}a_1}.$$

$(k-2 \geq 0,$ since $k \geq 2.)$

Therefore, the path $\overline{(a_1a_3)^{k-2}a_1}$ from v_4 must meet the path $v_1\overline{(a_1a_2)^I}$ at some $v \neq v_1$, and it must do so with an $\overline{a_3}$ edge since $o(a_1) = 2$, and every vertex on $v_1\overline{(a_1a_2)^I}$

already has an $\overline{a_1}$ edge. Let v_5 be the first vertex on $v_4\overline{(a_1a_3)^{k-2}\,a_1}$ such that $v_5 \in v_1\overline{(a_1a_2)^i}$. Therefore the local graph is;



Let $v_6 = v_5\overline{a_3a_1} = v_5\overline{a_3^{-1}a_1^{-1}}$.

For some $q < k-2$, we have; $v_1\overline{(a_3a_1)^q a_3} = v_5$, and for some $w(a_1a_2) \in G$, $v_1\overline{w(a_1a_2)} = v_5$ by inspection.
If $q = 0$, $v_1\overline{a_3} = v_5 = v_1\overline{w(a_1a_2)} \Rightarrow a_3 = w(a_1a_2)$, which is impossible since the a_i's are independent. Therefore, $0 < q < k-2$, and $v_4 \neq v_5$, and $(a_3a_1)^q a_3 = w(a_1a_2)$.

Since $v_1\overline{(a_3a_1)^q a_3} = v_5$, $v_3\overline{a_3a_1(a_3a_1)^q a_3} = v_5$ (because $v_1 = v_3\overline{a_3a_1}$).

Therefore, $v_5 = v_3\overline{(a_3a_1)^q a_3a_1a_3}$, so that $v_5\overline{a_3a_1} = v_3\overline{(a_3a_1)^q a_3}$, i.e. $v_6 = v_3\overline{(a_3a_1)^q a_3}$.

But, since $(a_3a_1)^q a_3 = w(a_1,a_2)$, we have;

$$v_6 = v_3(\overline{w(a_1,a_2)}).$$

However, this is impossible since no path consisting of only $\overline{a_1}$ and $\overline{a_2}$ edges can cross the path $\overline{(a_1a_2)^i}$ which also consists of only $\overline{a_1}$ and $\overline{a_3}$ edges. Therefore it must be that $O_{v_1} = -O_{v_2}$.

Then, by lemma 2.4, $v_1\overline{(a_1a_2)^i}$ determines a disk. Similarly, the paths $\overline{(a_2a_3)^j}$ and $\overline{(a_3a_1)^k}$ determine disks at every $v \in \Gamma$.

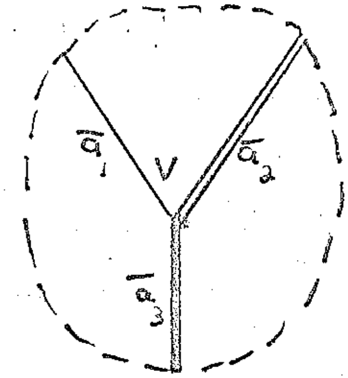
In addition, since every vertex $v \in \Gamma$ has but three concurring edges, there can be at most three disks at v. Therefore those disks at v are precisely the ones determined above.

It will now be shown that Γ has an interior vertex. At each vertex $v \in \Gamma$, there are two choices;

<u>1.</u> Every disk is finite.

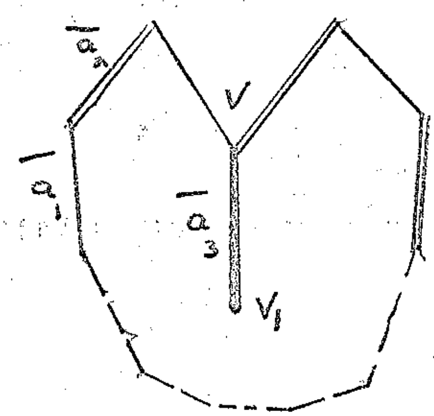
<u>2.</u> There exists an infinite disk at v.

If <u>1.</u> is the case, the local graph at v is;



and v is interior.

If 2. is the case, without loss of generality assume that the infinite disk at v is determined by $\overline{(a_1a_2)^i}$. Therefore, the local graph at v is;



($v_1 = v\overline{a_3} \notin v(a_1a_2)^i$ since the a_i's are independent.) Therefore, v_1 is interior.

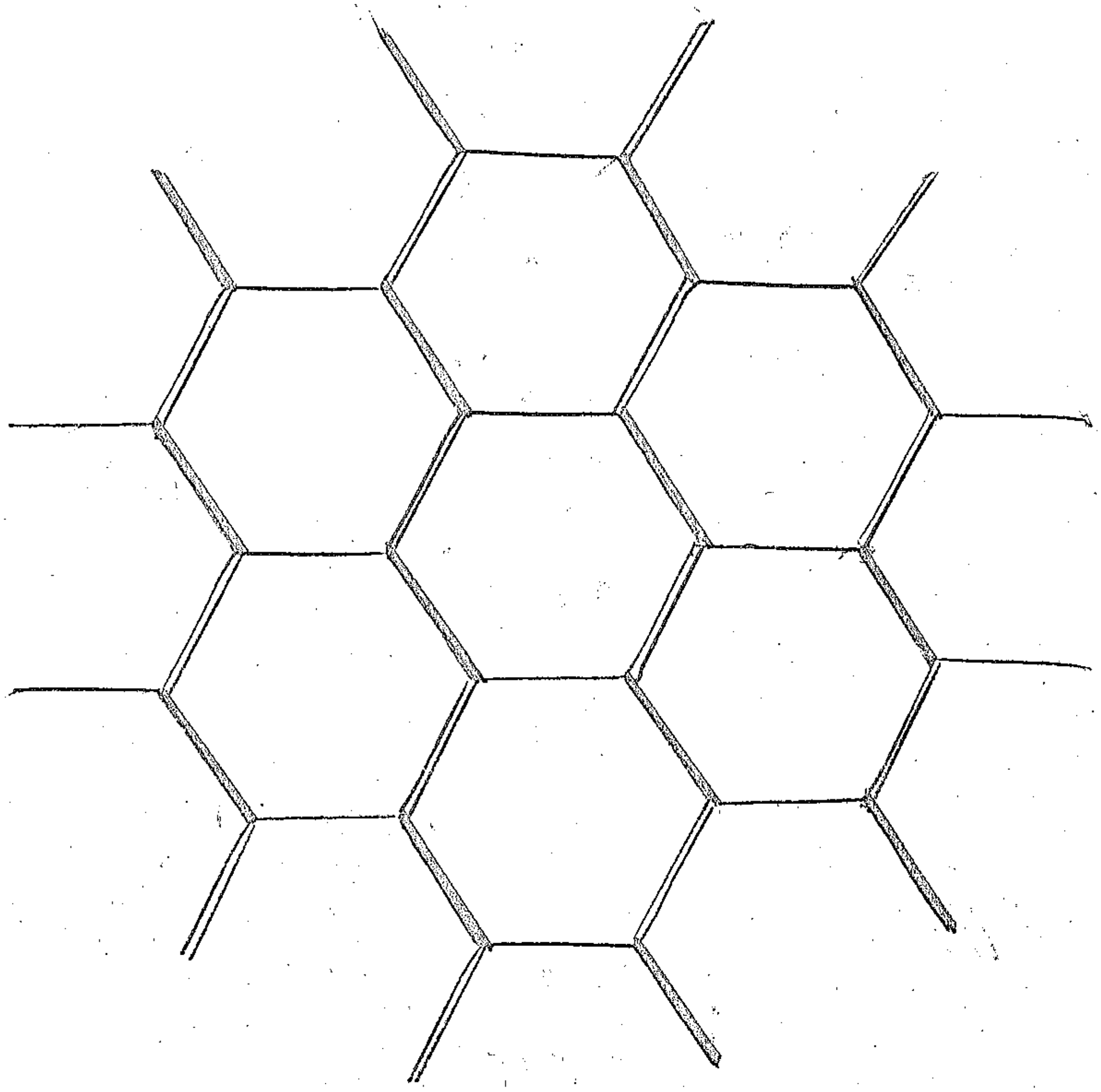
Q.E.D.

Corollary 2.1. Let $G = \langle a_1, a_2, a_3; R_1, R_2, \ldots R_m \rangle$. Assume G is a finite group, and the G has a Cayley diagram $\Gamma \subset E^2$. Then, $O_{v_1} = -O_{v_2}$ for v_1, v_2 one edge apart, and the disks are determined precisely by $\overline{(a_1a_2)^i}$, $\overline{(a_2a_3)^j}$ and $\overline{(a_3a_1)^k}$, some i, j, k such that $1 \angle i, j, k \angle \infty$.

Proof. Since G is finite, for some i, j, k $(a_1a_2)^i = (a_2a_3)^j = (a_3a_1)^k = 1$. Since the a_i's are independent, $i, j, k > 1$. By proposition 2.7, $a_1^2 = a_2^2 = a_3^2 = 1$. Therefore, the result follows immediately from theorem 2.1.

Q.E.D.

Note: The conditions $a_1^2 = a_2^2 = a_3^2 = 1$ are not sufficient to guarantee the conclusions of theorem 2.1 for in the following embedding of the Cayley diagram of the group $G = \langle a_1, a_2, a_3; a_1^2, a_2^2, a_3^2, (a_1a_2a_3)^2 \rangle$, $O_{v_i} = +O_{v_j}$ for all v_i, $v_j \in \Gamma$;



Where $a_1 \approx$

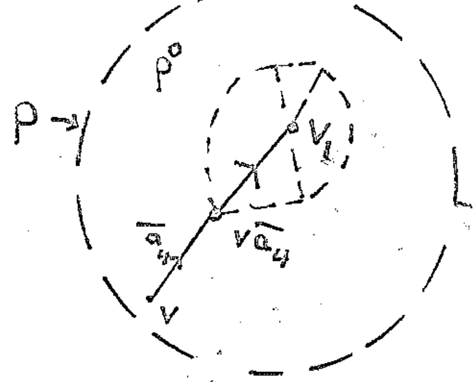
$a_2 \approx$

$a_3 \approx$

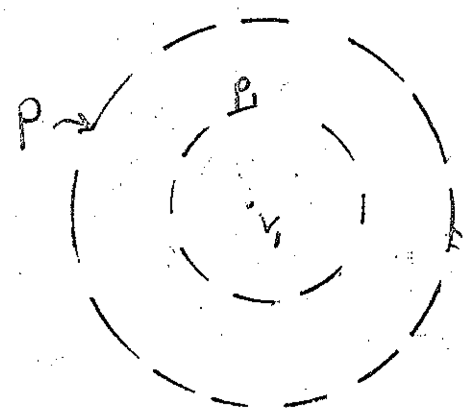
The following was proven in private communication using different means by Arthur White.

Corollary 2.2. There are no 4 generator finite groups with a Cayley diagram $\subset E^2$.

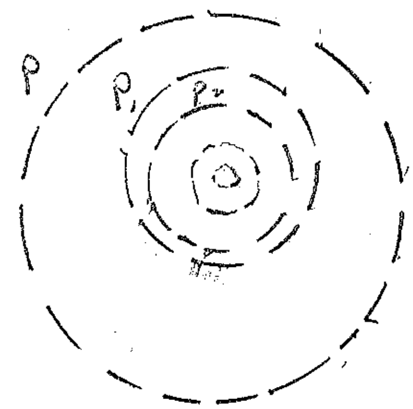
Proof. Assume $G = \langle a_1, a_2, a_3, a_4; R_1, \ldots R_m \rangle$ is a finite group which has a Cayley diagram $\Gamma \subset E^2$. As in lemma 2.2, delete the edges in Γ corresponding to a_4. Γ decomposes into disjoint connected isomorphic subgraphs, the vertices of each subgraph corresponding to the elements of a left coset of H, the subgroup of G generated by a_1, a_2 and a_3. Pick one such subgraph, say Γ_H itself. H is finite since G is finite, so $a_1^2 = a_2^2 = a_3^2 = 1$ by proposition 2.7, and $(a_1a_2)^i = 1 = (a_2a_3)^j = (a_3a_1)^k$ by corollary 2.1. By theorem 2.1 Γ_H has some interior vertex v(as do all $\Gamma_{gH}, g \in G$). Restore the $\overline{a_4}$ edge at v. $v\overline{a_4} \notin \Gamma_H$ since the a_i's are independent. Since v is interior $\subset \Gamma_H$, $v\overline{a_4}$ is in some bounded component of the complement of $\Gamma_H \subset E^2$ determined by some cycle $P \subset \Gamma_H$. But, $v\overline{a_4} \in \Gamma_{ga_4H}$ where $g \in G$ corresponds to $v \in \Gamma$, and $g \in H$. Since the Γ_{gH} are disjoint, $\Gamma_{ga_4H} \subset P^o$, the finite component of the complement of $\Gamma \subset E^2$ determined by P. In addition, Γ_{ga_4H} has some interior vertex v_1, and $v_1\overline{a_4} \in \Gamma_{g'H}$. As above, $v_1\overline{a_4} \notin \Gamma_{ga_4H}$;



Since v_1 is interior in Γ_{ga_4H} ,it is contained in the interior, P_1^o, of some cycle $P_1 \subset \Gamma_{ga_4H}$. Since the coset graphs are disjoint, $P_1 \cap P = \emptyset$, and since $v_1 \in P^o$, $P_1 \subset P^o$;



In addition, there can be no vertices of Γ_H in P_1^o since Γ_H is connected. Therefore, $v_1\overline{a_4} \notin \Gamma_H$. This process can continue indefinitely;



and this implies that there are an infinite number of vertices in P^{o}. But this is impossible since G was assumed to be a finite group.

Q.E.D.

Corollary 2.3. If the group G has 4 or more generators, and G is a finite group, then G has no Cayley diagram $\subset E^{2}$.

Proof. The result follows immediately from Corollary 2.2.

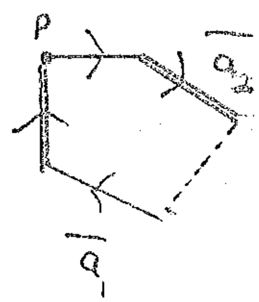
Q.E.D.

Section III-Some properties of infinite planar groups.

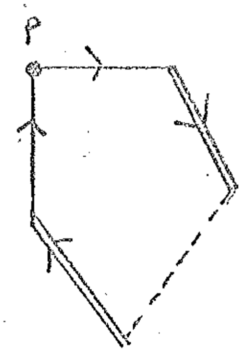
Lemma 3.1. Let $G = \langle a_1, a_2 \ldots ; a_1^n, a_2^m \ldots R_q \rangle$, $n, m, < \infty$. Assume that G has a Cayley diagram $\Gamma \subset E^2$. If the path $(a_1 a_2)^r a_1^{\epsilon}$ $1 \leq r < \infty$, $\epsilon = 0, 1$, from any vertex $v \in \Gamma$ meets itself at some point $p \in \Gamma$, then $(a_1 a_2)^{\gamma} = 1$, some $1 < \gamma < \infty$.

Proof. Note that γ must be greater than 1 since a_1 and a_2 are independent of each other.

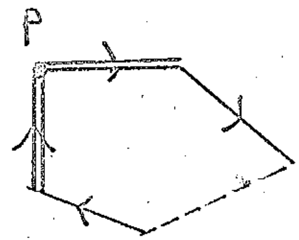
One of the following four case must hold;

1. 

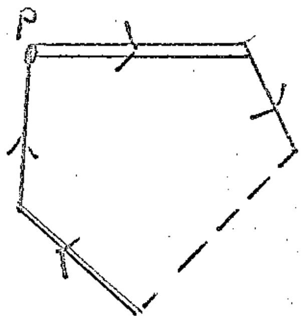
Immediately implies $(a_1 a_2)^{\gamma} = 1$ by inspection

2. 

$\Rightarrow (a_1 a_2)^k a_1 = 1 \Rightarrow (a_1 a_2)^k = a_1^{-1} \Rightarrow$ $(a_1 a_2)$ has finite order since a_1 does.

3. 

$\Rightarrow (a_2a_1)^s a_2 = 1 \Rightarrow (a_2a_1) = a_2^{-1}$ $\Rightarrow (a_2a_1)$ has finite order since a_2 does $\Rightarrow (a_1a_2)$ has finite order.

4. 

$\Rightarrow (a_2a_1)^\gamma = 1$ by inspection $\Rightarrow$ $(a_1a_2)^\gamma = 1$.

If $n=2$, or $m=2$, represent the edges corresponding to a_1 and a_2 in diagrams 1-4 by undirected lines. The proof is the same, but with the following considerations;

a. If $n=2$, case 2 is impossible, but each of cases 1, 3 and 4 still yield $(a_1a_2)^\gamma = 1$.

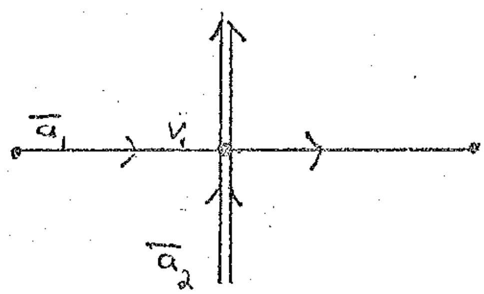
b. If $m=2$, case 3 is impossible, but each of cases 1, 2 and 4 yield $(a_1a_2)^\gamma = 1$.

Q.E.D.

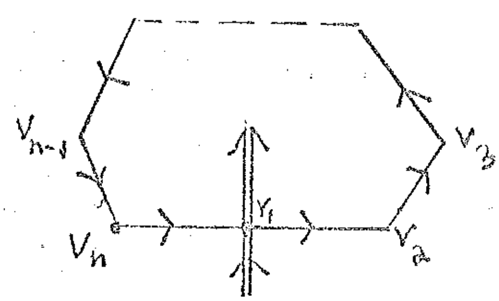
We now extend the two results of Maschke, mentioned in Section II, to the case where $o(G)=\infty$;

Lemma 3.2. Let $G = \langle a_1, a_2 \ldots ; a_1^n \ldots \rangle$ $3 \leq n < \infty$. If G has a locally finite Cayley diagram $\Gamma \subset E^2$, then the two $\overline{a_1}$ edges at any vertex $v \in \Gamma$ are not crossed by 2 edges of any other color.

Proof. If all other generators have order two, then the result is vacuously true, so assume $o(a_2) \geq 3$. Assume that at some vertex v_1, the two $\overline{a_1}$ edges are crossed by two $\overline{a_2}$ edges. Without loss of generality, assume that the local graph at v_1 is;

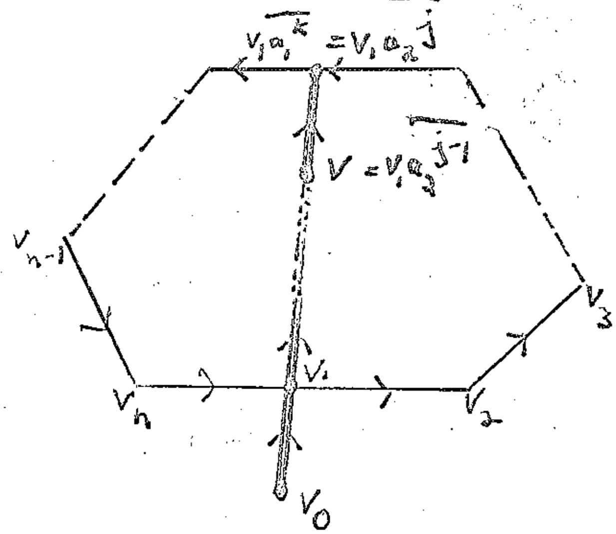


Since $a_1^n = 1$, $3 \leq n < \infty$, the path $\overline{a_1}^n$ from v_1 determines a polygon P of n sides;



Since a_1 and a_2 are independent, $a_2 \notin (a_1)$, and so $v_1\overline{a_2} \in P^o$.

Since Γ is locally finite, $v_1\overline{a_2^{\,j}} \in P$, for some j, $1<j<\infty$.
Therefore, for some k, $1<k<\infty$, $v_1\overline{a_2^{\,j}} = v_1\overline{a_1^{\,k}}$;



Therefore, $a_1^{\,k} = a_2^{\,j}$. But, since $v_0 = v_1\overline{a_2^{-1}}$,

$v_0\overline{a_2^{\,j}} = v_1\overline{a_2^{\,j-1}} = v$. ($j-1>0$ since $j>1$.).

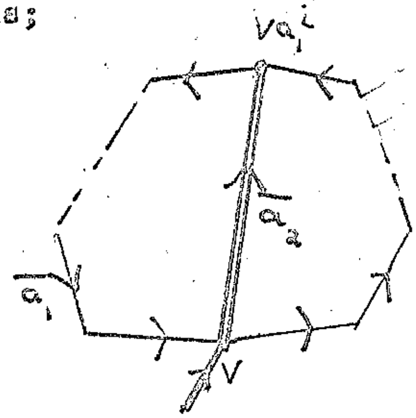
Therefore , since $a_1^{\,k} = a_2^{\,j}$, we have $v_0\overline{a_1^{\,k}} = v = v_0\overline{a_2^{\,j}}$.
So there must be a path of all $\overline{a_1}$ edges from v_0 to v, and this would require 4 $\overline{a_1}$ edges concurring at some vertex $v_i \in P$. But this is impossible by Proposition 2.1.

Q.E.D.

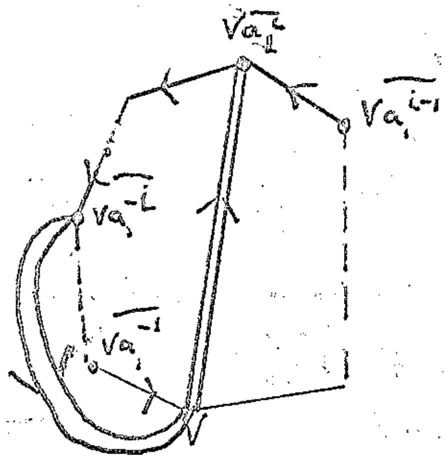
It was not necessary in this lemma to use the full strength of the assumption that the generators of G are independent. Only $a_2 \neq a_1^{\,i}$ ($|i|>1$) was needed.
As seen in the following corollary, even this assumption is stronger than it need be;

<u>Corollary 3.1</u>. The above result can be strengthened to insist only that $a_1^{\pm 1} \neq a_2$.

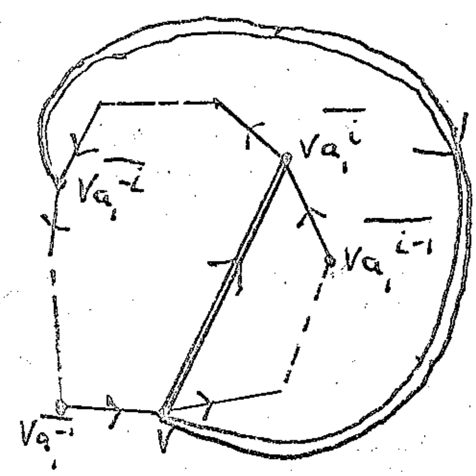
<u>Proof</u>. Assume $a_2 = a_1^i$, $2 \leq i < n$, and that two $\overline{a_2}$ edges cross two $\overline{a_1}$ edges at some vertex v. The local graph at v is;



There must consequently be an $\overline{a_2}$ edge positively directed from $\overline{va_1^{-1}}$ to v;



or

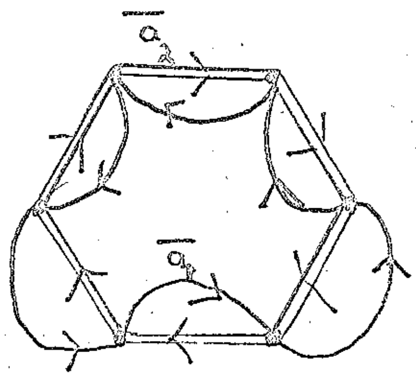


In either case, it is impossible to draw the required $\overline{a_2}$ edge from $\overline{va_1^{-1}}$ to $\overline{va_1^{i-1}}$ without contradicting Proposition 2.1.

Q.E.D.

Note: The result fails in the following example where $a_1 = a_2^{-1}$.

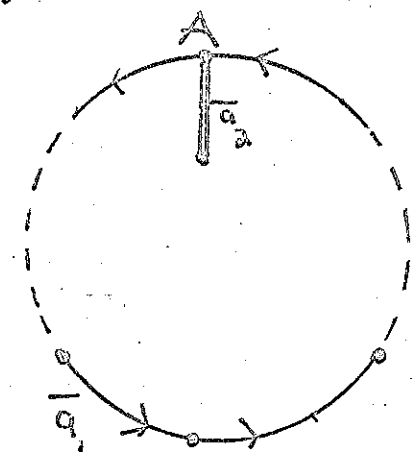
$$G = \langle a_1, a_2;\ a_1^6, a_1a_2 \rangle$$



Theorem 3.1 Let $G = \langle a_1, a_2, a_3 \ldots ; a_1^n \ldots R_q \rangle$, $n < \infty$. If G has a locally finite planar Cayley diagram Γ then any polygon P determined by the relation a_1^n determines a disk.

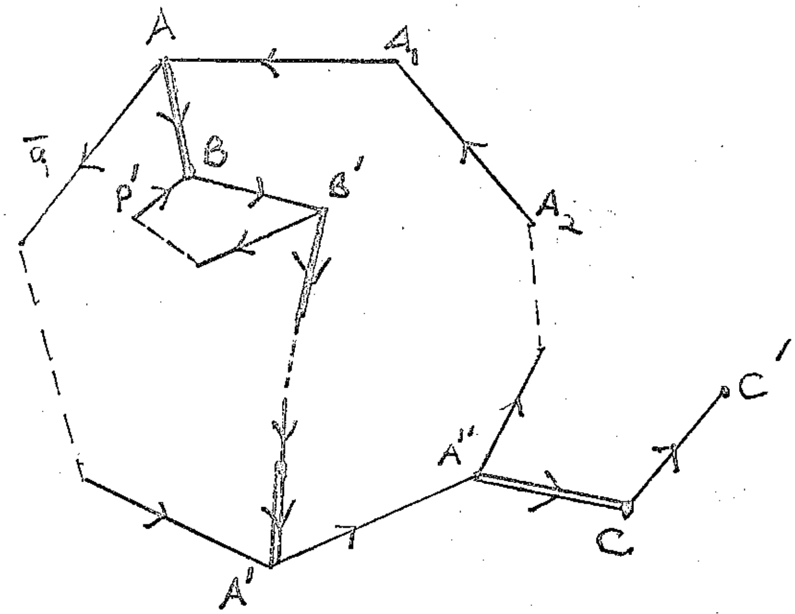
Proof. If $n = 0, 1, 2,$ the theorem is obviously true since a path $\overline{a_1^n}$ from any vertex does not even determine a polygon. Let P be a polygon determined by $\overline{a_1^n}$, $3 \le n < \infty$, and assume that the finite region determined by P (P^o) is not a disk. Therefore, P^o must contain some vertex $v \in \Gamma$, since if it contained only an edge $\overline{a_i} \in \Gamma$ then $a_i \in (a_1)$. It will be shown that $P \cup P^{o}$ contains every vertex of Γ, and therefore the exterior of P is a disk. An identical proof would show that if the exterior of P were not a disk, then P^{o} would be a disk.

Since there exists some $v \in P^{o}$, and since Γ is connected, there must be an edge from some vertex A of P to P^{o}. This edge cannot be an $\overline{a_1}$ edge by Proposition 2.1, so assume that it is an $\overline{a_2}$ edge;



If $a_2^2 = 1$, we can consider the $\overline{a_2}$ edge from A to be positively directed into P^{o}. If $a_2^2 \neq 1$, then Lemma 3.2 implies that both $\overline{a_2}$ edges concurring at A lead to P^{o}. In any event there is an $\overline{a_2}$ edge positively directed from A to P^{o}. Say $A\overline{a_2} = B$. B must belong to some $\overline{a_1}$ polygon P', and P' must lie entirely within P^{o}, for if not, there would be more than 2 $\overline{a_1}$ edges at some point. Consider the path $\overline{(a_2a_1)}^{\lambda}$ applied to the point A. This path will either stay inside $P^{o} \cup P$, or will cross P.

Assume the latter. In order to cross P, it must first meet P, and it must do so with an $\overline{a_2}$ edge by Proposition 2.1. So we have the subgraph;

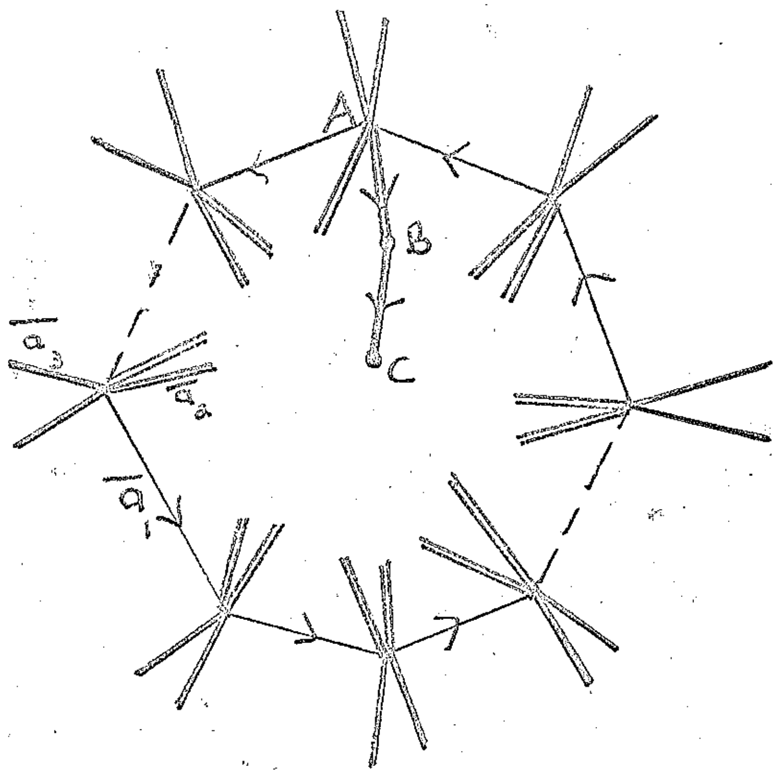


$C \notin P$ since $a_2 \neq a_1^{i}$. $C' \notin P$ by Proposition 2.1. Since $B \in P^o$, $C \in$ exterior P, $B \neq C$, so that $A'' \neq A$. Therefore, for some k, $A\overline{(a_2a_1)^k} = A''$, $k \neq 1$ since a_1 and a_2 are independent. Also, $A''\overline{a_1^j} = A$, some j, $0 < j < n$. Therefore, $A\overline{(a_2a_1)^k \; a_1^j} = A''\overline{a_1^j} = A$, so that $(a_2a_1)^k = a_1^{-j}$, or $(a_2a_1)^k a_1^j = 1$. Now, since the polygon P' lies entirely within P^o, $B' \in P^o$. But, $B'\overline{(a_2a_1)^k} = C'$ Since $(a_2a_1)^k a_1^j = 1$, we have $C'\overline{a_1^j} = B'$. This implies the existence of an $\overline{a_1}$ path from C' to B'. But this is impossible by Proposition 2.1.

Therefore, the path $\overline{(a_2a_1)^\lambda}$ from A must remain in $P^o \cup P$. But since Γ is locally finite, $P^o \cup P$ contains but a finite number of vertices. By Lemma 3.2, the path $A\overline{a_2}^\alpha$ $\alpha = 1, 2, \ldots$ can never cross P, and therefore the path $A\overline{a_2}^\alpha$ must meet itself, so a_2 has finite order. Also, since the path $A\overline{(a_2a_1)^\lambda}$ stays in $P \cup P^o$, it too must meet itself, so by Lemma 3.1, $(a_2a_1)^\gamma = 1$, some $\gamma > 1$. Therefore, $A\overline{(a_2a_1)}^{-1}a_2 = A_1$, and so there is a negatively directed $\overline{a_2}$ edge from A_1 to P^o, and Lemma 3.2 implies that there is a positively directed $\overline{a_2}$ edge from A_1 to P^o. Similarly, both $\overline{a_2}$ edges from A_2 lead into P^o, and so on for every $v \in P$. Therefore, there is no connection by $\overline{a_2}$ edges from P to the exterior of P.

Now assume that there is some vertex $p \in \Gamma$ such that $p \in$ exterior of P. Since Γ is connected, there must be some edge leading from P to the exterior of P. This edge, as shown above can neither be an $\overline{a_1}$ edge nor an $\overline{a_2}$ edge. Therefore it must be an edge corresponding to some third generator of G, say a_3. By the above argument, if any $\overline{a_3}$ edge led from P to P^o, they all would. Therefore all $\overline{a_3}$ edges meeting P must lead to the exterior of P.

Let A be any vertex of P. We have the following subgraph;

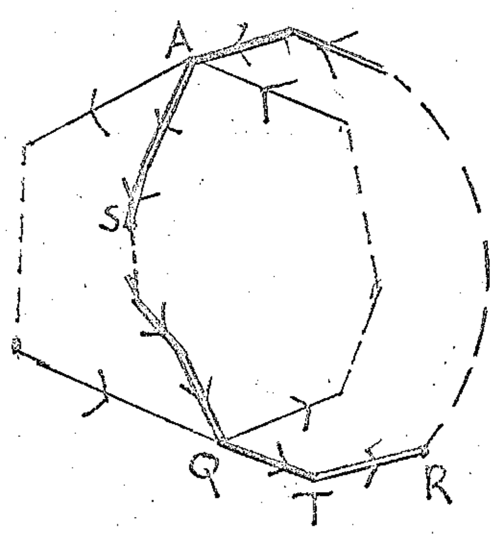


$B \in P^o$ since $a_2 \neq a_1^{1}$, and $C \in P^o$ since every $\overline{a_3}$ edge meeting P leads to the exterior of P. An argument identical to the one above shows;

1. a_3 has finite order

2. (a_2a_3) has finite order.

Therefore, we have the subgraph;

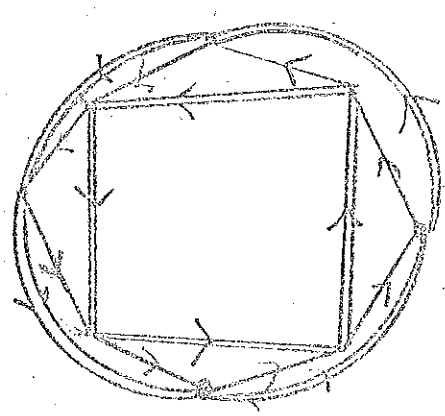


The path $\overline{AQRA}$ is $A\overline{(a_2a_3)^{\delta}}$ where $\delta = o(a_2a_3)$. $R \notin P$, since an $\overline{a_2}$ edge meeting P leads to P^o. $T \notin P$ since $a_3 \notin (a_1)$.

Therefore, $\overline{AQ} = \overline{(a_2a_3)^q a_2}$, $q < \delta$, and $\overline{QA} = \overline{a_1^{k}}$, $a < n$. But, $S\overline{(a_2a_3)^q a_2 a_1^k} = [S\overline{(a_2a_3)^q a_2}]\overline{a_1^k} = R\overline{a_1^k} = S$, since $\overline{AQ}\ \overline{QA}$ is a cycle and so $(a_2a_3)^q a_2 \cdot a_1^k = 1$. Therefore, there must be an $\overline{a_1}$ path from R to S. But this is impossible since not more than 2 edges of any one color can meet at any vertex by Proposition 2.1

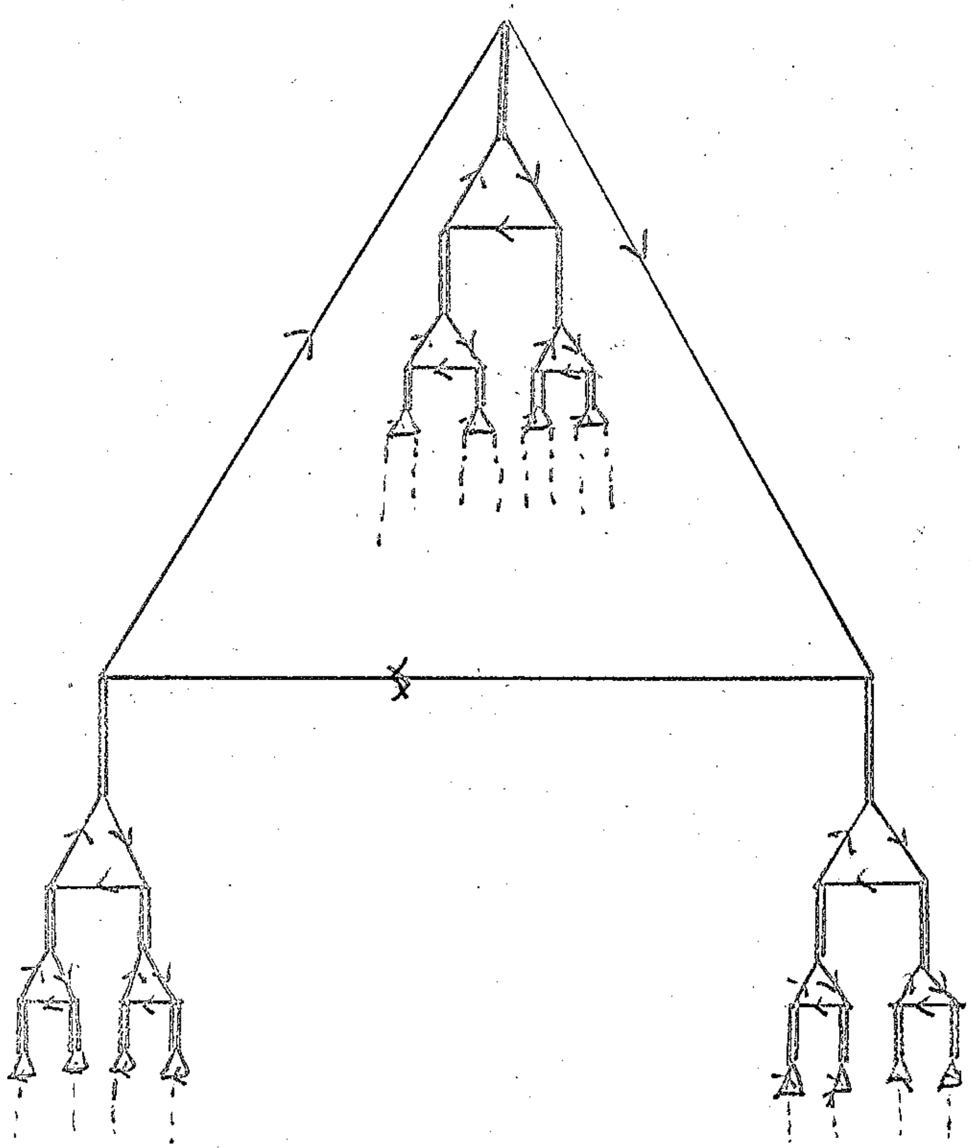
Q.E.D.

Note: Again the full strength of the independence of generators was not needed, only the fact that $a_i \notin (a_1)$, all $i \neq 1$. However, the importance of these restrictions is apparent in the following example; $G = \langle a_1, a_2; a_1^8, a_2a_1^{-2} \rangle$.



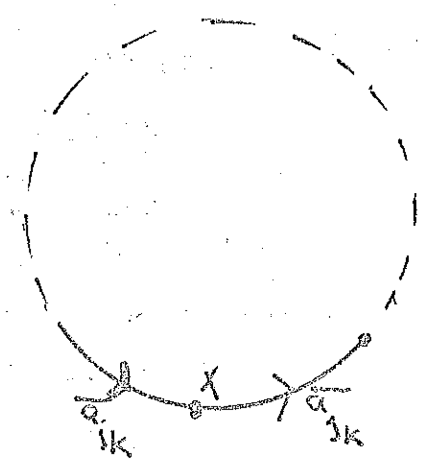
The hypothosis of local finiteness is also crucial. As in the following example, $\overline{a_1^n}$ need not determine a disk if this condition is dropped.

$$G = \langle a_1, a_2;\ a_1^3, a_2^2 \rangle$$



Lemma 3.3. Let $G = \langle a_1, a_2 \ldots a_n; a_1^{\beta_1}, a_2^{\beta_2}, \ldots a_n^{\beta_n}; R \rangle$, $\beta_i < \infty$. Assume G admits a locally finite Cayley diagram $\Gamma \subset E^2$. Then, if D is a disk in Γ, and D is determined by some word $w \in F$, then no two consequtive edges of $\bar{w}$ have the same color unless every edge of $\bar{w}$ is the same color. That is, either $w = a_i^{\beta_i}$ or $w = a_{j_1}^{\epsilon_1} a_{j_2}^{\epsilon_2} \ldots a_{j_p}^{\epsilon_p}$, where $\epsilon_k = \pm 1$, $j_k \neq j_{k+1}$, $j_1 \neq j_p$.

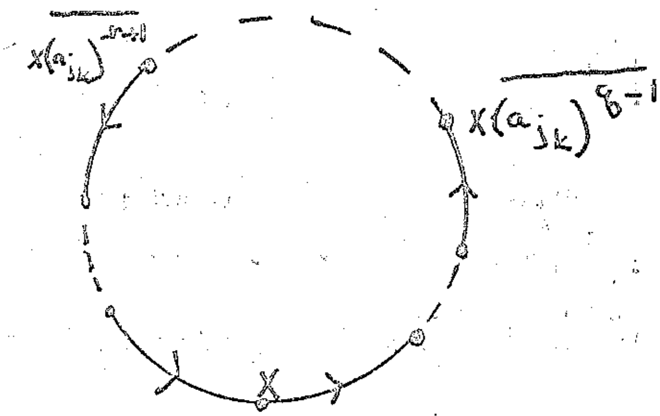
Proof. Assume that D is a disk in Γ and that D is determined by a path $\bar{w}$ having two consequtive edges of the same color. Let x be the vertex on $\bar{w}$ as shown;



If $n = 1$, the lemma is obviously true, i.e. $w = a_{j_k}^{\beta_{j_k}}$, so assume $n \geq 2$, and that $\bar{w}$ is not all of one color. Therefore, for some q, $0 < q < \beta_{j_k}$, $\overline{x(a_{j_k})^q} \not\subset \bar{w}$, and let q be the smallest such integer so that $\overline{x(a_{j_k})^{q-1}} \subset \bar{w}$.

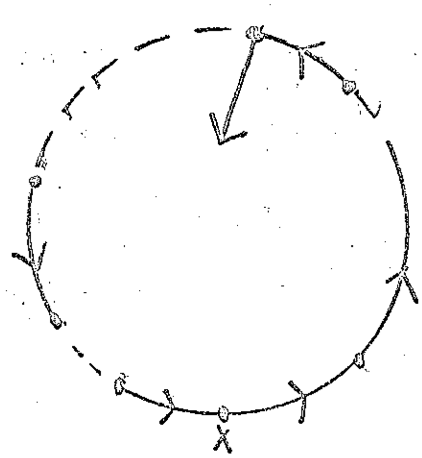
Similarly, for some r, $0 < r$ β_{j_k}, $x\overline{(a_{j_k})^{-r+1}} \in \bar{\bar{w}}$, but

$x\overline{(a_{j_1})^{-r}} \notin \bar{\bar{w}}$. So we have the subgraph;



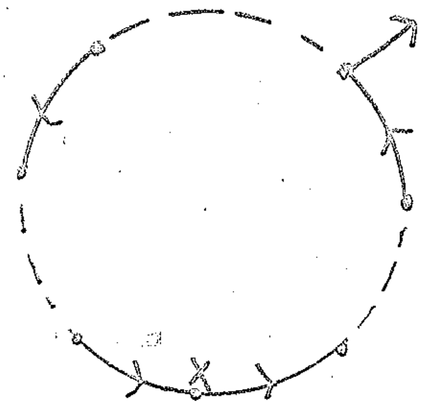
Since $x\overline{(a_{j_k})^{q}} \notin \bar{\bar{w}}$, the positively directed edge $\bar{a}_{j_k}$ from $x\overline{(a_{j_k})^{q-1}}$ must enter either the finite or infinite component of the plane determined by $\bar{\bar{w}}$. If the former is the case, the local graph at x is;

1.



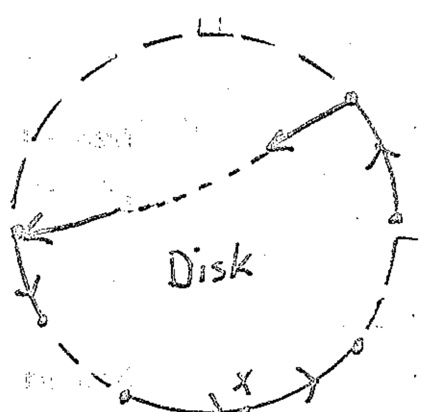
If the latter is the case, the local graph at x is;

2.



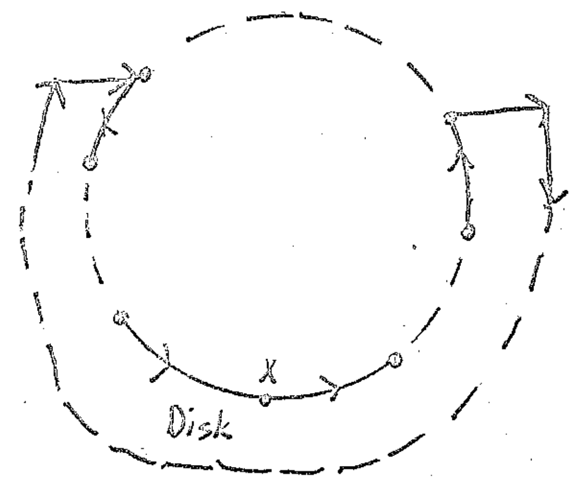
Since a_{j_k} has finite order, by Theorem 3.1, the polygon $x(a_{j_k})^{\beta_{jk}}$ must determine a disk. So diagram 1. yields;

3.



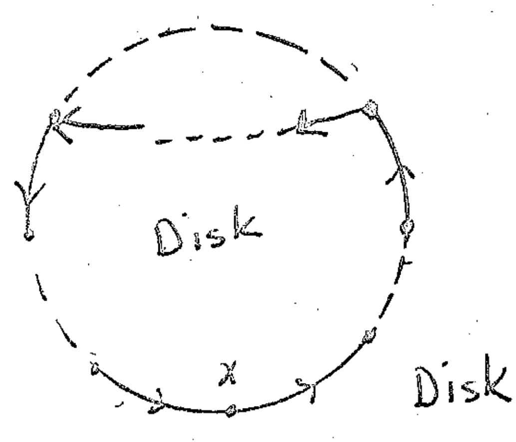
And diagram 2. yields;

4.



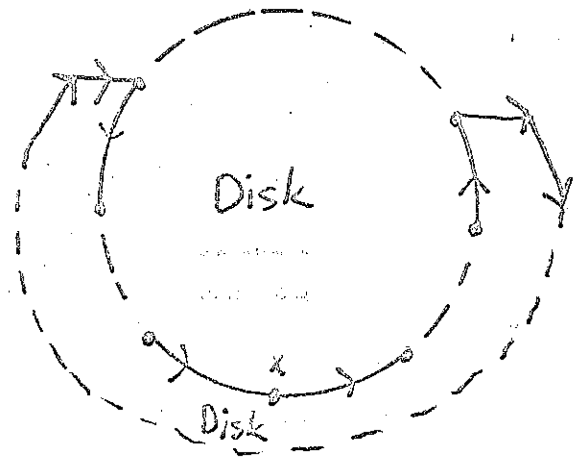
But it was assumed that $\bar{w}$ determined a disk. In diagram 3. , this disk can only be the infinite region determined by $\bar{w}$;

5.



In diagram 4.,this disk can only be the finite region determined by $\bar{w}$;

6.



But, since $n \geq 2$, some edge other than $\overline{a_{j_k}}$ must emanate from the vertex x. This is impossible, however, since in either diagram 5 or 6, the vertex x lies on a path bordering 2 disks.

Q.E.D.

Definition. Let $G = \langle a_1, a_2, \ldots, a_n; R_1, R_2, \ldots R_m \rangle$.

Assume;

1. The R_j are cyclically reduced and non-empty.
2. For every R_j, $1 \leq j \leq m$, the set of the R_j also contains R_j^{-1}, and all cyclic permutations (conjugates of the same length) of R_j and R_j^{-1}.

Such a set will be called the symmetrized set of the R_j, and will be denoted $\{R_j\}$.

If R_o is any cyclically reduced and non-empty word in F_n, the set consisting of R_o, R_o^{-1} and all of their cyclic permutations will be denoted $\{R_o\}$.

Lemma 3.4. Let $G = \langle a_1, a_2, \ldots a_n; R_1, R_2, \ldots R_m \rangle$. Assume G has a Cayley diagram $\Gamma \subset E^2$. Let $v \in \Gamma$, and suppose $v\overline{R_o}$ is a cycle which is a Jordan curve of finitely many edges. Then, $v\overline{R'}$ is a Jordan curve for any $R' \in \{R_o\}$.

Proof. Let $R_o = y_1 y_2 \ldots y_i \ldots y_t$; $y_i = a_k^{\pm 1}$, R_o a cyclically reduced and non-empty relation in G.

If $v\overline{R_o}$ is a Jordan curve, then R_o has no subrelations. Therefore, $R_o^{-1} = y_t^{-1} \ldots y_i^{-1} \ldots y_2^{-1} y_1^{-1}$ has no subrelations. Therefore, $v\overline{R_o^{-1}}$ is a Jordan curve.

Assume $v\overline{S} = v\,\overline{y_1 y_2 \ldots y_i \ldots y_n}$ is a Jordan curve, i.e., S has no subrelations.

Let $S' = y_2 y_3 \ldots y_i \ldots y_n y_1$ and assume $v\overline{S'}$ is not a Jordan curve. Therefore, S' contains some subrelation $y_p y_{p+1} \ldots y_{m-1} y_m$, $p \geq 2$, $m \leq n$, and then S contains some subrelation, contradicting the assumption.

Using S' in place of R_o in the argument shows that $S'' = y_3 y_4 \ldots y_2$ is a Jordan curve. And so on for all such conjugates of R_o. Similarly for R_o^{-1}.

Q.E.D.

Theorem 3.2. Let $G = \langle a_1, a_2, \ldots a_n;\ a_1^{\beta_1}, a_2^{\beta_2} \ldots a_n^{\beta_n}, R_1 \ldots R_n \rangle$, the β_i minimal for G, $n \geq 2$, $3 \leq \beta_i < \infty$. Assume that G has a locally finite Cayley diagram $\Gamma \subset E^2$ such that $O_{v_i} =\!= O_{v_j}$ for all $v \in \Gamma$. Assume that D is a disk in Γ and that D has boundary $v_0\bar{w}$, $v_0 \in \delta(D)$, $\bar{w}$ some cycle in Γ with a finite number of edges. Then

I. Either $w = a_i^{\beta_i}$, $i \in (1,2,\ldots n)$, or

$w = (x_1 x_2 \ldots x_n)^{\delta}$, where $x_j = a_i^{\epsilon_i}$,

$\epsilon_i = \pm 1$, $1 < \delta < \infty$, $x_q \neq x_k^{\pm 1}$, for all $q, k \in (1,2,\ldots n)$.

If the latter is the case, then;

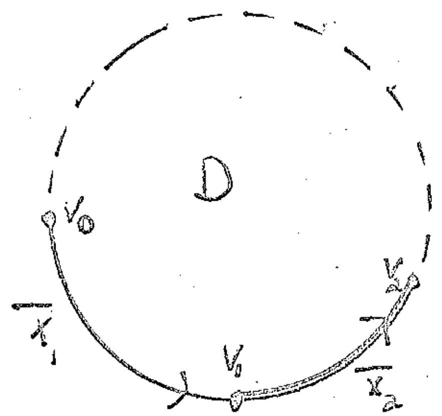
II. Every word $w' \in \{(x_1 x_2 \ldots x_n)^{\delta}\}$ determines a path $\bar{w}'$ from any vertex $v \in \Gamma$ such that $v\bar{w}'$ bounds a disk. All but at most one of these is a finite disk, and if $o(G) = \infty$, all such disks are finite.

III. $O_{v_i} =\!\!\!\!/\!= O_{v_j}$, all $v \in \Gamma$.

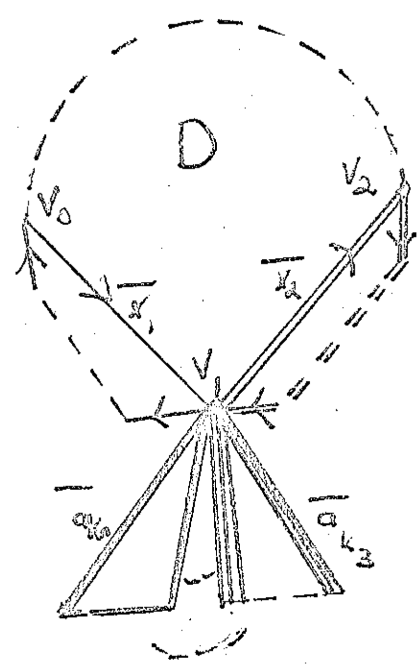
Proof. Assume that D is a finite disk. The proof is identical if D is infinite. Assume $w \neq a_i^{\beta_i}$. By Lemma 3.3, no two consecutive edges of $\bar{w}$ have the same color. Therefore;

$$w = a_{k_1}^{\epsilon_{k_1}} a_{k_2}^{\epsilon_{k_2}} z;\ k_1, k_2 \in (1,2,\ldots n),\ \epsilon_{k_i} = \pm 1,\ k_1 \neq k_2.$$

Denote $a_{k_1}^{\epsilon_{k_1}}$ by x_1. Denote $a_{k_2}^{\epsilon_{k_2}}$ by x_2, so that $w = x_1x_2z$. Therefore, the local graph at v_0 is;



Since a_{k_1} and a_{k_2} have finite order, and since Γ is locally finite, the one-color polygons with edges $\overline{x_1}$ and $\overline{x_2}$ determine disks. There must also be disks at v_1 corresponding to the other $a_i^{\beta_i}$ by Theorem 3.1;



So that the clockwise ordering of the edges about v_1 must be (since $\beta_i \geq 3$, all $i \in (1,2,\ldots n)$;

$$\begin{Bmatrix} x_1 \longrightarrow \\ x_1 \longleftarrow \\ x_2 \longrightarrow \\ x_2 \longleftarrow \\ a_{k_3}^{\epsilon_{k3}} \longrightarrow \\ a_{k_3}^{\epsilon_{k3}} \longleftarrow \\ \circ \\ \circ \\ \circ \\ a_{k_n}^{\epsilon_{kn}} \longrightarrow \\ a_{k_n}^{\epsilon_{kn}} \longleftarrow \end{Bmatrix}$$

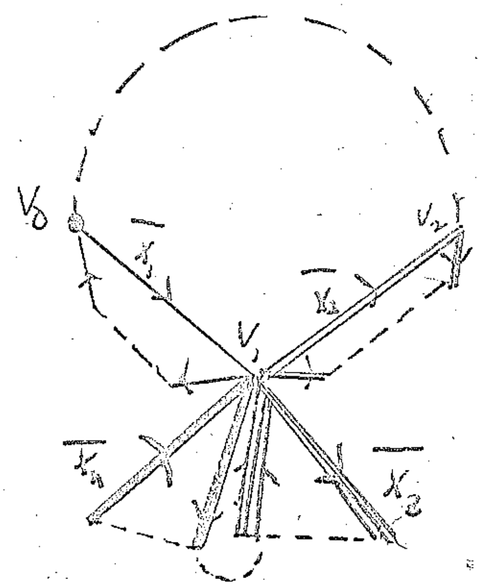
where $\epsilon_{k_p} = \pm 1$, $(k_3,k_4\ldots k_n) \in [(1,2,\ldots n) \cap (k_1,k_2)']$.

Denote $a_{k_p}^{\epsilon_{k_p}}$ by x_p. That is $x_p = a_{k_p}^{\pm 1}$.

So the edges at v_1 are renamed, and the clockwise ordering of the renamed edges at v_1 beginning with $x_1 \longrightarrow$ is;

$$\begin{Bmatrix} x_1 \longrightarrow \\ x_1 \longleftarrow \\ x_2 \longrightarrow \\ x_2 \longleftarrow \\ \circ \\ \circ \\ \circ \\ x_n \longrightarrow \\ x_n \longleftarrow \end{Bmatrix}$$

And the local graph at v_1 is;



It will now be shown that the next edge emanating from v_2 on the path $\bar{w}$ is $\overline{x_3^{+1}}$.

Since the $\overline{x_2}$ polygon at v_2 is a disk by Theorem 3.1, the clockwise ordering of the edges at v_2 begins;

$$\left\{\begin{array}{l} x_2 \longrightarrow \\ x_2 \longleftarrow \\ \vdots \end{array}\right\}$$

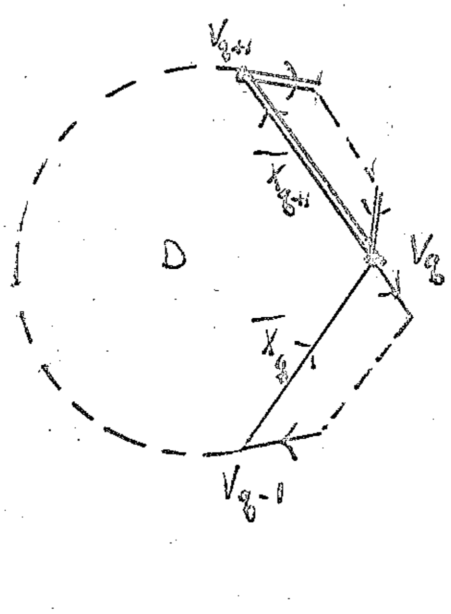
But, since $O_{v_1} = \pm O_{v_j}$, all $v \in \Gamma$, we have $O_{v_2} = \pm O_{v_1}$,

and so the complete ordering of the edges at v_2 must be;

$$\left\{\begin{array}{l} x_2 \longrightarrow \\ x_2 \longleftarrow \\ x_3 \longrightarrow \\ x_3 \longleftarrow \\ \circ \\ \circ \\ \circ \\ x_n \longrightarrow \\ x_n \longleftarrow \\ x_1 \longrightarrow \\ x_1 \longleftarrow \end{array}\right\}$$

That is, $O_{v_1} \not\simeq O_{v_2}$. But, since D is a disk, no edge can enter D from v_2, so that the next edge on $\overline{w}$ must be the edge immediately following $\{x_2 \longleftarrow\}$ in the clockwise ordering about v_2, and that edge is $\overline{x_3}$.

To extend this to the general case, assume that two consecutive edges on $\overline{w}$ are $\overline{x_q}$ and $\overline{x_{q+1}}$, $1 \leq q \leq n-2$. Therefore the subgraph at v_q is;



Therefore, the clockwise ordering at v_{q+1} beginning with $\{x_{q+1} \longrightarrow\}$ is;

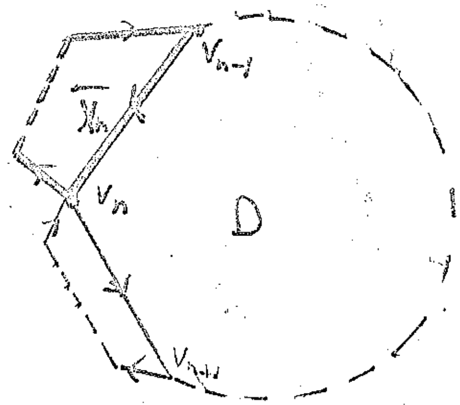
$$\left\{\begin{array}{l} x_{q+1} \longrightarrow \\ x_{q+1} \longleftarrow \\ \circ \\ \circ \\ \circ \end{array}\right\}$$

But, since $O_{v_{q+1}} = \pm O_{v_1}$, the complete clockwise ordering at v_{q+1} must be;

$$\left\{\begin{array}{l} x_{q+1} \longrightarrow \\ x_{q+1} \longleftarrow \\ x_{q+2} \longrightarrow \\ x_{q+2} \longleftarrow \\ \circ \\ \circ \\ \circ \\ x_n \longrightarrow \\ x_n \longleftarrow \\ x_1 \longrightarrow \\ x_1 \longleftarrow \\ \circ \\ \circ \\ \circ \\ x_q \longrightarrow \\ x_q \longleftarrow \end{array}\right\}$$

Therefore, as above, the next edge on the path $\bar{\bar{w}}$ must be x_{q+2}, and $O_{v_{q+1}} = +O_{v_1}$.

And so on until we arrive at $q = n-1$, i.e. $q+1 = n$, and the subgraph is;



Therefore, the clockwise ordering at v_n beginning with $v_n \rightarrow$ is;

$$\left\{\begin{matrix} v_n \rightarrow \\ v_n \leftarrow \\ \cdot \\ \cdot \\ \cdot \end{matrix}\right\}$$

But, since $O_{v_n} = \pm O_{v_1}$, as above it must be that the next edge on the path $\overline{w}$ is $\overline{x_1}$ and that $O_{v_n} = + O_{v_1}$.

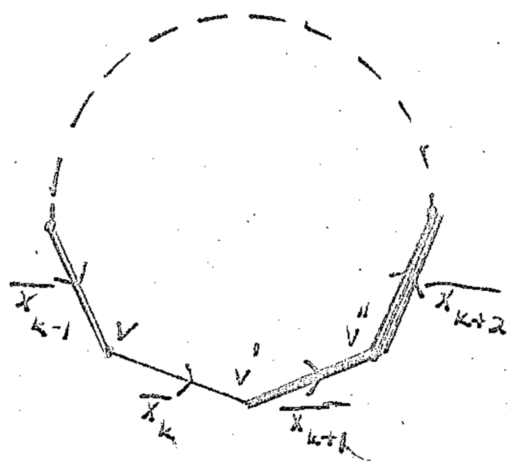
And so on. Since $\overline{w}$ has finite length, and $w = 1 \in G$, the path $v_o\overline{w}$ must terminate at v_o, and the above argument shows that it does so with the edge $\overline{x_n}$.

Therefore, for some γ, $v_o\overline{w} = v_o\overline{(x_1x_2\ldots x_n)^{\gamma}}$, and so $w = (x_1x_2\ldots x_n)^{\gamma}$. $\gamma < \infty$ since w has finite length, and $\gamma \geq 2$ since the $x_j = a_i^{\pm 1}$ are independent.

Moreover, $O_v = + O_{v_1}$ for all $v \in v_o\overline{w}$.

Since $v_o\overline{(x_1x_2\ldots x_n)^{\gamma}} = v_o\overline{w}$ is a Jordan curve, there are no proper subrelations in w. Therefore, for any $v \in \Gamma$, $v\overline{(x_1x_2 \ldots x_n)^\gamma}$ is a Jordan curve, and by Lemma 3.4, $v\,\overline{w'}$ is a Jordan curve for any $v \in \Gamma$ and any $w' \in \left\{ (x_1x_2 \ldots x_n)^\gamma \right\}$.

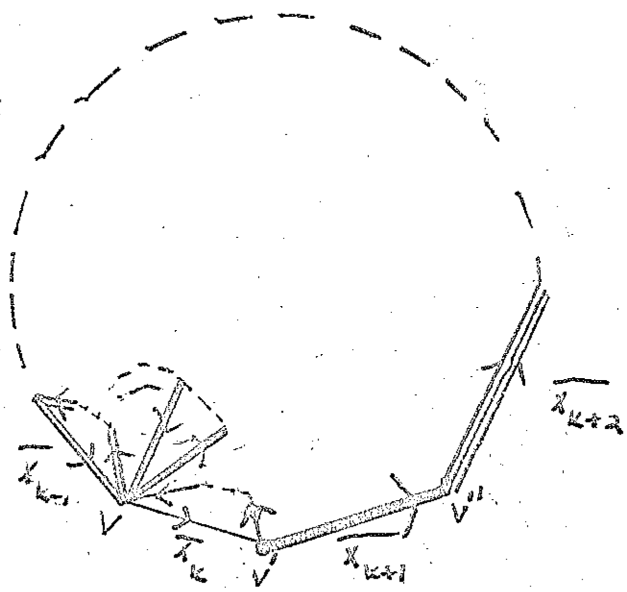
We will now show that $v\overline{w'}$ bounds a disk. Pick any $v \in \Gamma$ and any $w' = (x_kx_{k+1} \ldots x_nx_1 \ldots x_{k-1})^\gamma \in \left\{ (x_1x_2 \ldots x_n)^\gamma \right\}$. Consider the Jordan curve (and thus cycle) $v\overline{w'}$;



Since $O_{v_i} = \pm O_{v_j}$ for all $v \in \Gamma$, we have $O_{v'} = \pm O_{v_1}$, i.e.;

$$\pm O_{v'} = \left\{ \begin{array}{l} x_k \longrightarrow \\ x_k \longleftarrow \\ x_{k+1} \longrightarrow \\ x_{k+1} \longleftarrow \\ \circ \\ \circ \\ \circ \\ x_{k-1} \longrightarrow \\ x_{k-1} \longleftarrow \end{array} \right\} = O_{v_1}.$$

Assume that $O_v = -O_{v_1}$. (The argument and result are identical if we were to assume that $O_v = + O_{v_1}$.) The local graph at v is;



So there are no connections in Γ to v from the infinite open region of E^2 determined by $\overline{vw'}$.

Since $O_{v'} = \pm O_v$, and since the polygon at v' with all edges being $\overline{x_{k+1}}$ must be a disk by Theorem 3.1, the counterclockwise ordering of the edges at v' must begin;

$$\left\{\begin{array}{l} x_{k+1} \longrightarrow \\ x_{k+1} \longleftarrow \\ x_{k+2} \longrightarrow \\ \circ \\ \circ \\ \circ \end{array}\right\}$$

Therefore, $O_{v'} = + O_v = -O_{v_1}$, and similarly, there are no connections to v' from the infinite region of E^2 determined by $v\overline{w'}$.

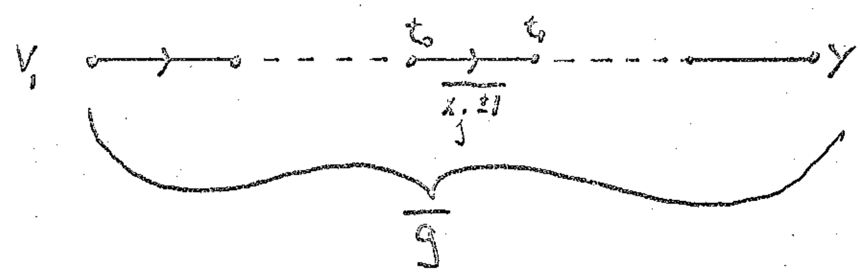
This argument repeated at each $v^i \in v\overline{w'}$ shows that there are no connections to v^i from the infinite region of E^2 determined by $v\overline{w'}$, and thus $v\overline{w'}$ determines an (infinite) disk. In addition, $O_{v^i} = + O_v$ for all $v^i \in v\overline{w'}$.

We will now prove part III, i.e. $O_{v_i} = + O_{v_j}$ for all $v \in \Gamma$.

Assume that there is some vertex $y \in \Gamma$ such that $O_y = -O_{v_1}$ (obviously, $y \neq v_1$). The vertices y and v_1 correspond to elements $\hat{y}$ and $\hat{v}_1$ in G. Therefore, there exists some $g \in G$ such that $g = \hat{v}_1^{-1} \cdot \hat{y}$, i.e. $\hat{v}_1 \cdot g = \hat{y}$.

There may be more than one way to write g as a finite word in the $\langle a_i \rangle$, but pick one and fix it. Therefore, g corresponds to some fixed path $\overline{g}$ from every vertex in Γ, and, specifically, $v_1\overline{g} = y$. Since $v_1 \neq y$, $\overline{g}$ has at least one edge.

Since $O_y = -O_{v_1}$, there must be some vertex t on the path $v_1\overline{g}$ such that $O_t = -O_{v_1}$. Let t_1 be the first such vertex, and call the vertex immediately preceding t_1 on $v_1\overline{g}$, t_o. By the choice of t_1, we have $O_{t_o} = + O_{v_1} = -O_y = -O_{t_1}$, and the subgraph is;



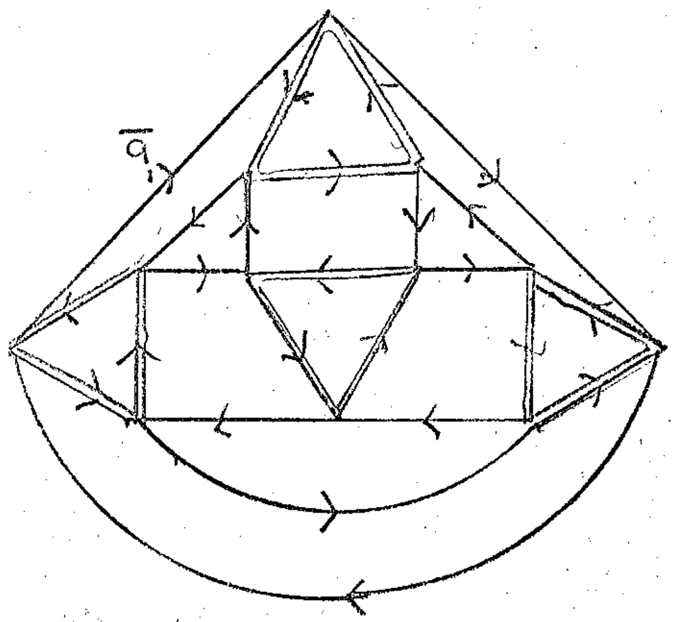
Since t_o and t_1 are one edge apart, $t_o\overline{x_j^{\pm 1}} = t_1$, some $j \in (1,2,\ldots n)$.

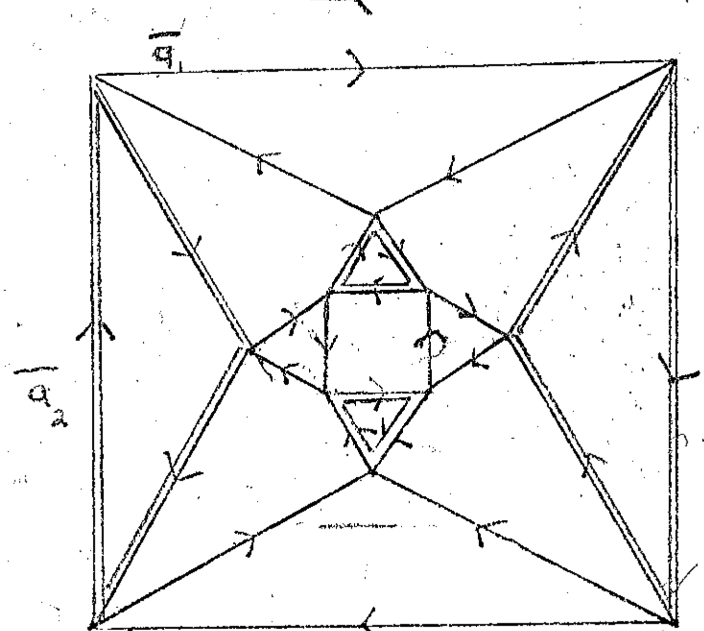
Therefore, t_o and t_1 are on the boundary of some cycle $v_k\overline{w'}$, $v_k \in \Gamma$, $w' \in \{(x_1x_2\ldots x_n)^\gamma\}$. But, it was shown above that such a path bounds some disk D' such that $O_{v_i} = + O_{v_j}$ for all $v_i, v_j \in \delta(D')$. Therefore, it cannot be that $O_{t_o} = -O_{t_1}$, and so $O_{v_i} = + O_{v_j}$ for all $v \in \Gamma$.

To prove the second part of II, we note that each path $P = v_k\overline{w'}$, $v_k \in \Gamma$, $w' \in \{(x_1x_2\ldots x_n)^\gamma\}$ determines a disk D. Therefore, E^2-D has two open connected components, a finite component N, and an infinite component M, one of which contains no vertices or edges of Γ. If D = M, i.e. P determines an infinite disk, then Γ is in the closure of N, the bounded component. Since Γ is locally finite, $o(G) < \infty$, and there can obviously be only one such infinite disk.

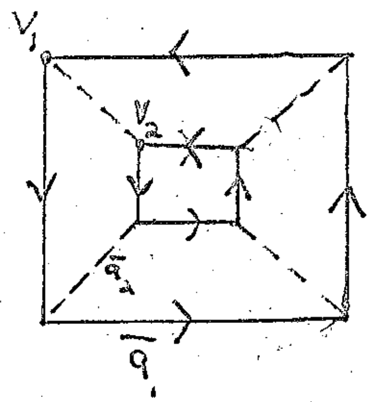
This proves Theorem 3.2.

The theorem states that there exists at most one infinite disk if $o(G) < \infty$. However, it does not predict which cycle determines that disk. For example in the group presented by $G = \langle a_1 a_2; a_1^3, a_2^3, (a_1a_2)^2 \rangle$, the infinite disk can be determined by either $\overline{a_1^3}$ or $\overline{(a_1a_2)^2}$;





The theorem may fail if we allow some generator to be of order 2. For example, let $G = \langle a_1, a_2; a_1^4, a_2^2, a_1 a_2 a_1^{-1} a_2 \rangle$.

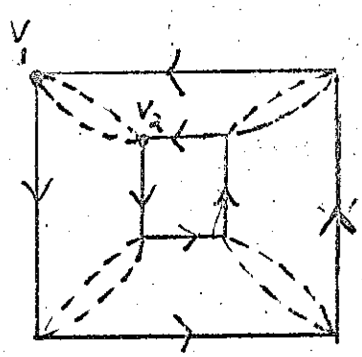


$$Ov_1 = \left\{ \begin{array}{l} a_1 \longrightarrow \\ a_1 \longleftarrow \\ a_2 \quad ? \end{array} \right\}$$

$$Ov_2 = \left\{ \begin{array}{l} a_1 \longleftarrow \\ a_1 \longrightarrow \\ a_2 \quad ? \end{array} \right\}$$

So that $O_{v_1} = -O_{v_2}$

Even if 2-gons are allowed, the situation is not improved;



$$Ov_1 = \left\{ \begin{array}{l} a_1 \longrightarrow \\ a_1 \longleftarrow \\ a_2 \quad ? \\ a_2 \quad ? \end{array} \right\}$$

$$O_{v_2} = \left\{ \begin{array}{l} a_1 \longleftarrow \\ a_1 \longrightarrow \\ a_2 \quad ? \\ a_2 \quad ? \end{array} \right\}$$

And here, even weak point symmetry is impossible.

Corollary 3.2. If G satisfies the hypotheses of Theorem 3.2, then if $R = 1$ in G, $R \in$ $[a_i^{\beta_i}, \{(x_1 x_2 \cdots x_n)^{\gamma}\}]^F$.

Proof. Let $v \in \Gamma$. Then, v is a vertex on the boundaries of the 2n disks, $va_i^{\beta_i}$, $v\overline{w'}$, $w \in \{(x_1 x_2 \cdots x_n)^{\gamma}\}$, at v. The cycle $v_i\overline{R}$, from any $v_i \in \Gamma$ must therefore coincide with edges already in the graph.

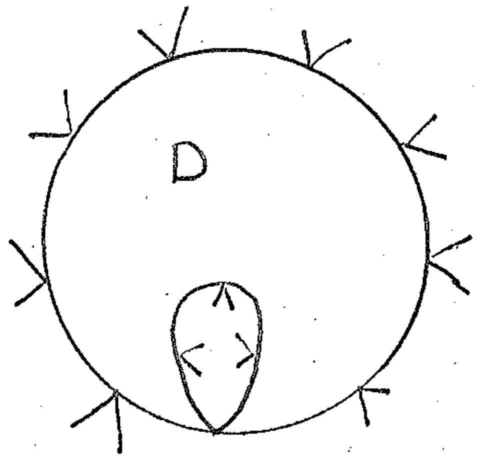
Q.E.D.

So, for example, the group G =

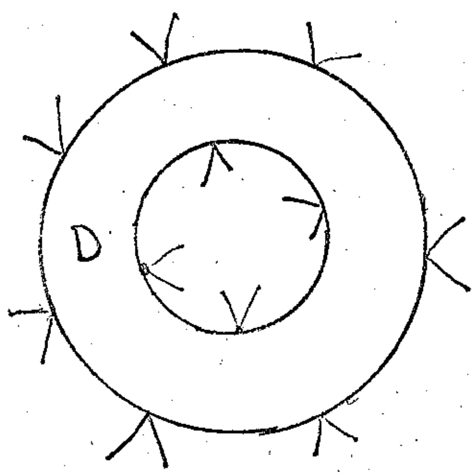
$$\langle a_1, a_2;\ a_1^4,\ a_2^4,\ (a_1 a_2)^4,\ a_1^2 a_2^2 \rangle$$

does not have a locally finite Cayley diagram in E^2 such that $O_{v_i} = \pm O_{v_j}$ for all $v \in \Gamma$, since the relation $(a_1^2 a_2^2) \notin [a_1^4, a_2^4, (a_1 a_2)^4]^F$.

Theorem 3.2 guarantees certain results if given the existence of some disk D such that $\delta(D) = a_i^{\beta_i}$. The same results can be gotten, however, if we merely insist upon the existence of some $R = 1$ in G such that $R \neq a_i^q$, $R \in [a_i^{\beta_i}]^F$. We first need three preliminary results, the first of which states that under certain conditions any cycle C which determines a disk is a Jordan curve. That eliminates the possibility;



A non-simply connected domain D, in the complement of Γ in E^2;

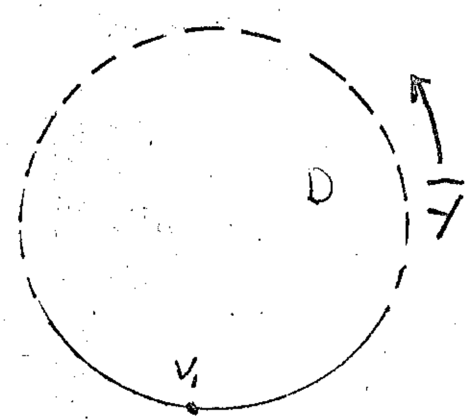


is impossible, since Γ must be connected.

<u>Proposition 3.1.</u> Let G be a group with a locally finite Cayley diagram embedded in E^2. Let $w = 1 \in G$, w of finite length, be such that for some $v_0 \in \Gamma$, $v_0\bar{w}$ is the boundary of some open connected region D which contains no vertices or edges of Γ. Assume that $v_0\bar{w}$ does not meet itself except at vertices. Then, D is a disk, and $\bar{w}$ is a Jordan curve.

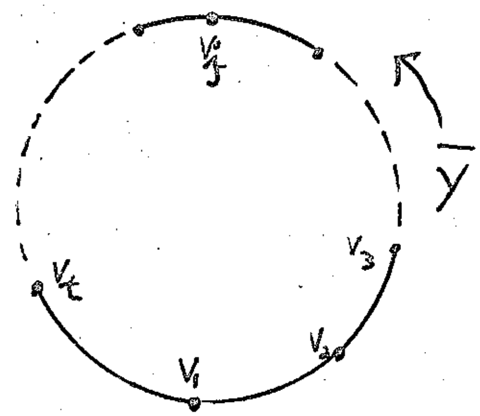
<u>Proof.</u> It will be shown that if $v_0\bar{w}$ is not a Jordan curve then D is not a connected component of the complement of Γ in E^2.

Assume that $v_0\bar{w}$ is not a Jordan curve. Let v_1 be the first vertex on $v_0\bar{w}$ such that $v_1 = v_1\bar{y}$, where $y \in G$ is not empty, and $w = xyz$, x, z G, and $v_0\bar{x} = v_1$. Suppose D is on the left as $v_0\bar{w}$ is traversed counter-clockwise. The proof is identical if we were to assume that D is on the right;



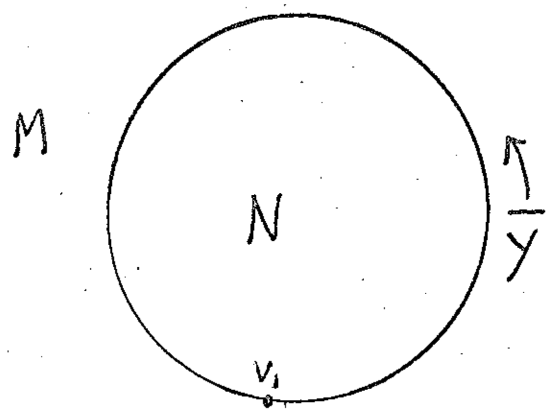
By the choice of v_1, $v_1\bar{y}$ is a Jordan curve. If both x and z are empty, then we are done, since then we would have $w = xyz = y$, and so $v_0\bar{w} = v_1\bar{w}$ is a Jordan curve since $v_1\bar{y}$ is. If either x or z is empty, then $v_1 = v_0$. Without loss of generality, assume that z is not empty.(If z is empty, and x is not, replace $w = xyz = xy$ in the following argument by $w^{-1} = y^{-1}x^{-1}$. Since $v_1\bar{y}$ is a Jordan curve, $v_1\overline{y^{-1}}$ is the same cycle traversed in the opposite direction, and is therefore a Jordan curve as well.)

So the local graph at v_1 contains;



Since $\bar{w}$ is of finite length, and since $\bar{w} = \overline{xyz}$, $\bar{y}$ is of finite length, say t. It will now be shown that $v_1\bar{z} \cap v_1\bar{y} = v_1$, that is, the boundary of D described by $v_0\bar{w}$ has no multiple points at any of $v_2, v_3, \ldots v_t$.

Since $v_1\bar{y}$ is a Jordan curve, the open finite component N, and the open infinite component M, of $E^2 - v_1\bar{y}$ are well defined;



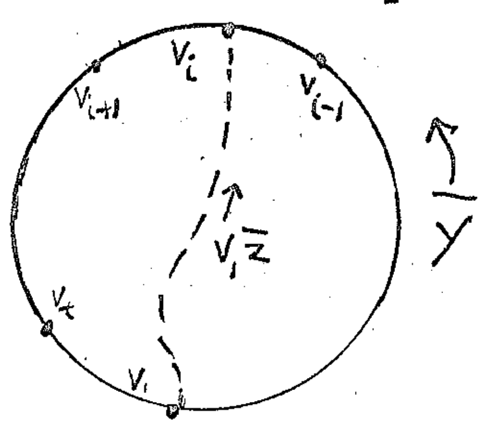
Assume $v_1\bar{z} \cap v_1\bar{y} \neq v_1$. Let v_i be the first vertex on the path $v_1\bar{z}$ such that v_i is also on the path $v_1\bar{y}$ ($|i| > 1$).

Two cases arise;

1. The path $v_1\bar{z}$ enters N.

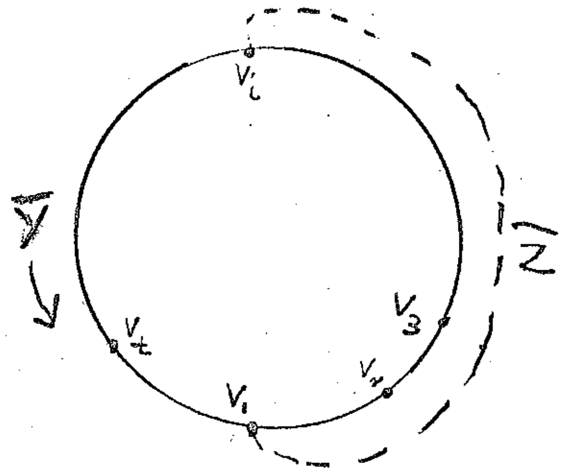
2. The path $v_1\bar{z}$ enters M.

In case 1., where $v_1\bar{z}$ enters the finite component, $v_1\bar{z} \subset N$ until it meets the vertex v_i;



But this creates a disconnection of D, which is impossible, since D was assumed to be connected.

Similarly, in case 2, where $v_1\bar{z}$ enters the infinite component, it must be the case that $v_1\bar{z} \subset M$ until it meets the vertex v_i;



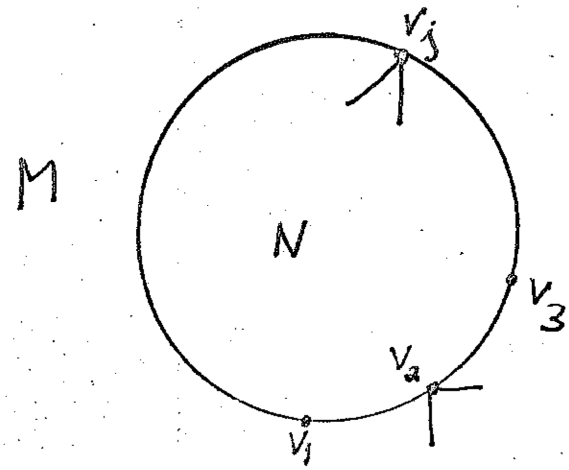
But this too creates an impossible disconnection of D. Therefore, $v_0\bar{w}$ has no multiple points at any of $v_2, v_3, \ldots v_t$.

If there were but 2 edges at each vertex of Γ, then there is nothing to prove, as there could be to multiple points on any cycle in Γ. So assume that there are at least 3 edges at each vertex of Γ. (Each vertex of Γ has the same number of edges.)

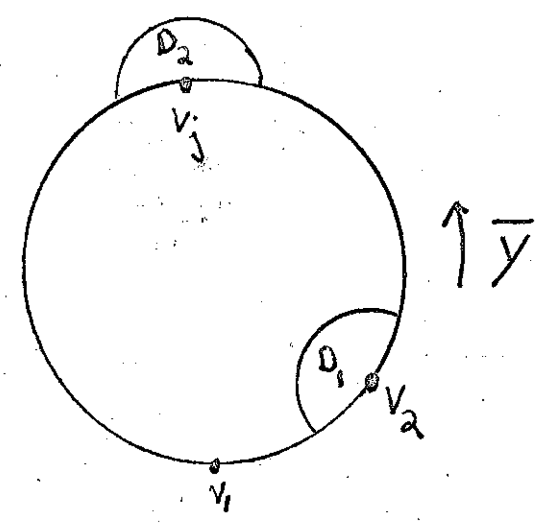
Assume that some edge at v_2 (not on $\delta(D)$) leads into M. (If it leads into N, the argument is the same with M and N switched.) Since the number of edges at v_2 is at least 3, and by the above argument at most 2 of these can be boundary edges, there must be at least one edge at v_2 not on δ (D).

Since the arc $\overline{v_1 v_2 v_3}$ is on $\delta(D)$, all edges at v_2 must lead into M.

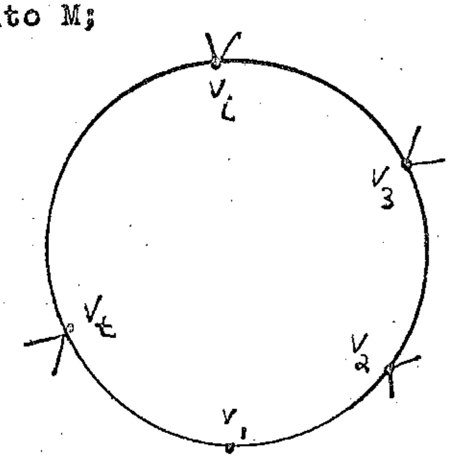
If, in addition, some edge not on δ (D) leads into N from some vertex $v_j \in v_1\bar{y}$, $j = 3,4,\ldots t$, then all edges at v_j not on δ (D) lead into N, and the local graph contains;



Therefore, for some sufficiently small neighborhood S about v_2, we have $D_1 \equiv N \cap S \subset D$, since $\overline{v_1 v_2 v_3} \subset \delta(D)$. Similarly, for some neighborhood S' about v_j, we have $D_2 \equiv S' \cap M \subset D$. Therefore we have;

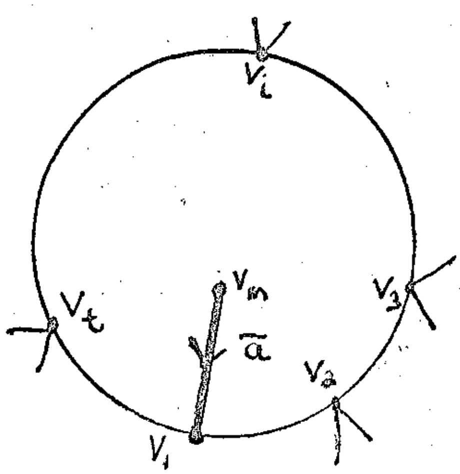


and therefore, D is disconnected, a contradiction. Thus, all edges to $v_1\overline{y}$ not on $\delta(D)$ lead from $v_2, v_3, \ldots v_t$ into M;



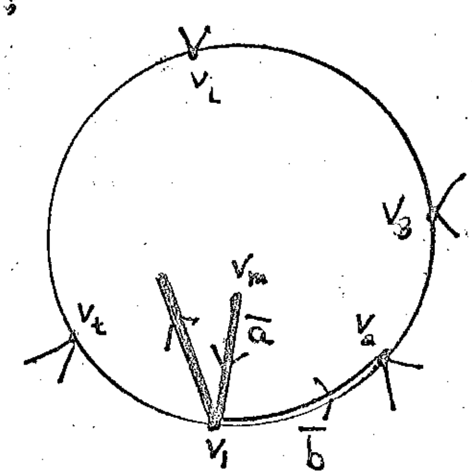
Therefore, the only edges of Γ leading into N must do so from v_1. If there are no edges from v_1 to N, then $v_1\overline{y}$ determines a disk which must be all of D since D is connected, and we are done.

So assume that there is an edge from v_1 into N. Call it $\overline{a^{-1}}$. Since $a = a_i$, some $i \in (1,2,\ldots n)$, $a \neq 1$, and so $v_1\overline{a_1^{-1}} \neq v_1$. Also, $v_1\overline{a^{-1}} \neq v_i \in v_1\overline{y}$, since all connections to the v_i must lead to M. Therefore, $v_1\overline{a^{-1}} \in N$;



Consider the path $v_1\overline{(a^{-1})^k}$, $k = 1,2,\ldots$. Since Γ is locally finite, and all connections to $v_1 \in v_1\overline{y}$ must be made from M, this path must meet itself. But this implies that $a^q = 1$, some q, $2 \leq q < \infty$. If $q = 2$, it follows immediately that the arc $\overline{v_1 v_2} = \overline{b} \neq \overline{a}$. If $q \geq 3$, and $\overline{v_1 v_2} = \overline{a}$, it is impossible to complete the polygon $v_m\overline{a^q}$ since no $\overline{a}$ edge can lead from $v_2, v_3, \ldots v_t$ to N, and the polygon $v_m\overline{a^q}$ must be a disk by Theorem 3.1.

Therefore $\overline{v_1 v_2} = \overline{b} = \overline{a}_j \neq \overline{a}$, some $j \in (1,2,\ldots n)$. A similar argument shows that $v_1 v_t \neq \overline{a}$, and that $v_1\overline{a} \notin M$, and so, $v_1\overline{a} \in N$;



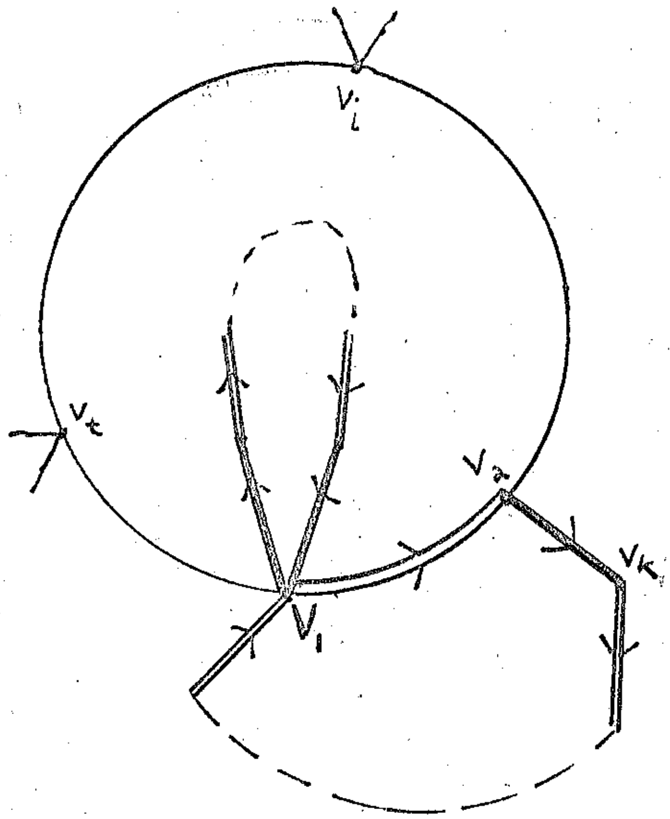
It will now be shown that (ab) has finite order. If (ab) has infinite order, consider the path $v_m\overline{b^i}$, $i = 1,2,\ldots$. If it meets itself, b has finite order. If it does not meet itself, the path must leave N, and must therefore meet $v_1\overline{y}$ at v_1, since v_1 is the only vertex with possible connections to N. But this implies that $v_m\overline{a} = v_m\overline{b^p}$ for some p, which is impossible since a and b are independent. Therefore, b has finite order.

Now consider the path $v_1\overline{(ab)^\gamma}$, $\gamma = 1,2,\ldots$. Since Γ is locally finite, this path must either meet itself, or must leave N. If the former case, Lemma 3.1 immediately implies that (ab) has finite order. If $v_1\overline{(ab)^\gamma}$ leaves N, it must first meet the path $v_1\overline{y}$ and can only do so at the vertex v_1. Therefore, in this case too, the path meets itself (at v_1), and again, (ab) has finite order by Lemma 3.1. So, $(ab)^\alpha = 1$, $\alpha < \infty$, and $|\alpha| > 1$ since a and b are independent.

It will now be shown that the existence of the edge $\overline{a}$ from v_m to v_1 leads to a contradiction, and so there can be no edges at all from v_1 into N, and so the Jordan curve $v_1\overline{y}$ determines D, and D is a disk.

Consider the path $v_m\overline{(ab)^\alpha}$, which ends at v_m since $(ab)^\alpha = 1$. Since $v_m\overline{ab}$ ends at v_2, and since $|\alpha| > 1$, $v_2\overline{(ab)^{\alpha-1}}$ ends at v_m.

Therefore, the path $v_2\overline{(ab)^{\alpha-1}}$ must go from M into N. This can only happen via v_1. Therefore, we have;



and so for some β, $v_1\overline{(ba)^{\beta}b}$ ends at v_1, and so $(ba)^{\beta}b = 1$, $\beta > 1$ since a and b are independent.

Let v_k be the end of the path $v_1\overline{ba}$. Therefore, since $(ba)^{\beta}b = 1$, the path $v_k\overline{(ba)^{\beta}b}$ ends at v_k. But, then, $v_k = v_k\overline{(ba)^{\beta}b} = v_1\overline{ba}\cdot\overline{(ba)^{\beta}b} = v_1\overline{(ba)^{\beta}b\cdot ab} = v_1(\overline{ab})$.

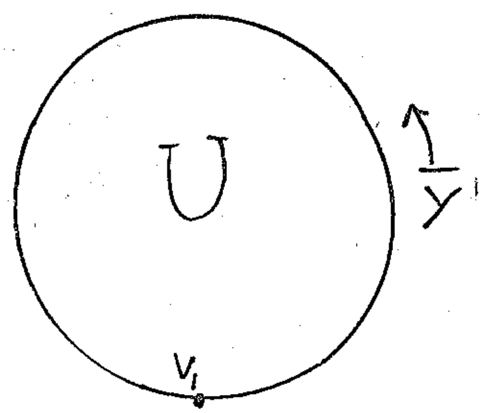
Therefore, $v_1(\overline{ab}) = v_k = v_1(\overline{ba})$. But this is impossible since $v_1(\overline{ab})$ ends in N, and $v_1(\overline{ba})$ ends in M.

Q.E.D.

Preparatory to Theorem 3.3 we have:

<u>Lemma 3.5</u>. Let $G = \langle a_1, a_2, \ldots a_n; R_1, R_2, \ldots R_m \rangle$, not a free group, $a_i^2 \neq 1$. Assume G has a Cayley diagram $\Gamma \subset E^2$. If $v \in \Gamma$, the path $v\overline{R_j}$ determines at least one finite open region U whose boundary is a Jordan curve.

<u>Proof</u>. Since $R_j = 1$ in G, the path $v\overline{R_j}$ is a cycle beginning and ending at v. If $v\overline{R_j}$ is not a Jordan curve itself, let v_1 be the first vertex on $v\overline{R_j}$ which is reached twice, i.e. $R_j \simeq xyz$, $y = 1$ in G, y not empty, x and z not both empty. Therefore the local graph at v_1 contains;



and U is the desired region. To show that U is open (in fact that $U \neq \emptyset$) we need only show that y has length ≥ 3. But this follows immediately from the facts that the $\langle a_i \rangle$ are independent and $a_i^2 \neq 1$.

Q.E.D.

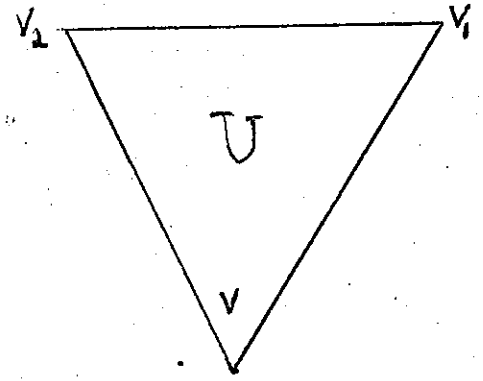
Lemma 3.6. Let $G = \langle a_1, a_2, \ldots a_n : R_1, R_2 \ldots R_m \rangle$, not a free group, $a_i^2 \neq 1$. Assume G has a locally finite Cayley diagram Γ embedded in E^2. If $v_0 \in \Gamma$, then there exists some $v \in \Gamma$ on the path $v_0\overline{R_j}$ such that v is on the boundary of some disk D. That is, every relation R_j contains some subrelation R_j' such that $v\overline{R_j}$ determines a disk.

Proof. By Lemma 3.5, $v_0\overline{R_j}$ determines some finite connected open region U whose boundary is a Jordan curve. Let $v \in \delta(U)$. By Proposition 3.1, it suffices to show that v is on the boundary of some open connected region D which contains no edges or vertices of Γ, and that $\delta(D) = v\overline{w}$, some $w = 1$ in G, does not meet itself except perhaps at some vertices on $v\overline{w}$.

The proof will be by induction on N, the number of vertices in U. Since Γ is locally finite, $N < \infty$.

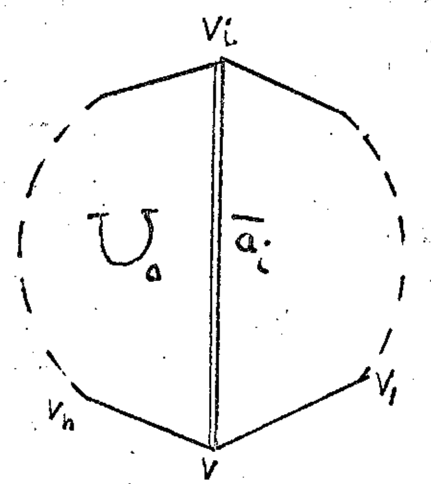
Say $N = 0$. This case will be proven by induction on E, the number of edges in $\delta(U)$.

Since $a_i^2 \neq 1$, and the $\langle a_i \rangle$ are independent, $E \geqslant 3$. If $E = 3$, $N = 0$, the subgraph must be;



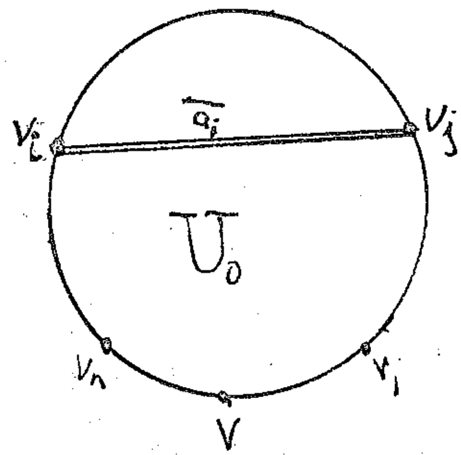
Since U contains no vertices of Γ, any edges in U must be of length 1 and must join 2 vertices of $\delta(U)$. But this is clearly impossible since $a_i^2 \neq 1$, and the $\langle a_i \rangle$ are independent. Therefore, U cannot contain any edges, and so $v \in \delta(U)$, U a disk.

Assume that the lemma is true for $N = 0$, $E < n$, and suppose that $\delta(U)$ has length n. As above, any edges in U must join 2 vertices $v_i, v_j \in \delta(U)$, $i \neq j$. Pick any such edge, say $\overline{a_i}$. If $v = v_i$ (or v_j), the local graph at v is;



Since $\delta(U) = \overline{vv_1 \ldots v_i \ldots v_n v}$ is a Jordan curve, so is $\delta(U_o)$ $\overline{vv_i \ldots v_n v}$, and so $\delta(U_o)$ does not meet itself. But, $\delta(U_o)$ must contain less than n edges, and by the induction assumption, v is on the boundary of some disk $D \subseteq U_o \subset U$.

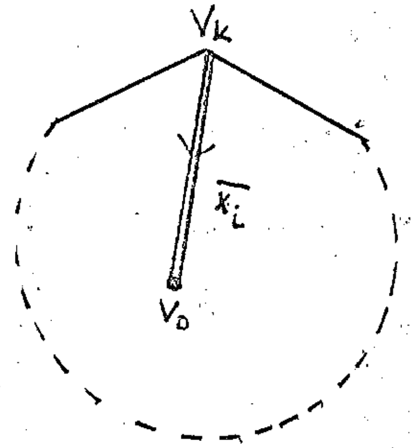
If $v \neq v_i$, $v \neq v_j$, the local graph at v is;



and the same argument shows that v is on the boundary of some disk $D \subseteq U_0 \subset U$.

Therefore, the proposition is true for $N = 0$.

Assume that the proposition is true for $N < q$, and that the region U contains q vertices. Pick one, say v_t, $t \in (1,2,\ldots q)$. Since Γ is connected, v_t must be connected to $\delta(U)$ by some path, and so there must be some edge, say $\overline{x_i} = \overline{a_i^{\pm 1}}$, from some $v_k \in \delta(U)$ to some $v_0 \in U$;

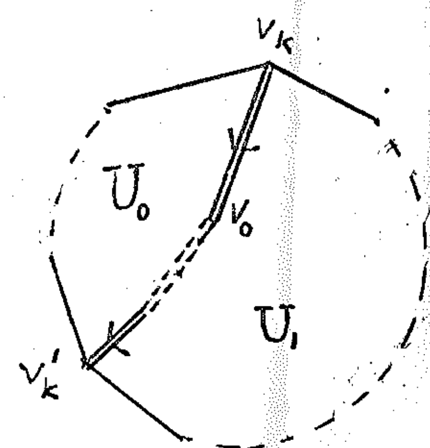


Consider the path $v_k\overline{x_i{}^{\gamma}}$, γ sufficiently large. Since $a_i{}^2 \neq 1 \neq x_i{}^2$, there are three cases;

1. The path $P = v_k\overline{x_i{}^{\gamma}}$ meets $\delta(U)$ at some vertex other than v_k.
2. P meets $\delta(U)$ at v_k.
3. P does not meet $\delta(U)$.

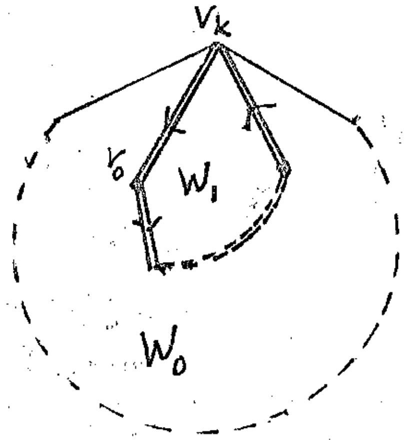
Since Γ is locally finite, P must either meet itself or leave the finite region U. If P meets itself, $x_i{}^m = 1 = a_i{}^m$, some m such that $3 \leq m < \infty$, so that $v_k\overline{x_i{}^m}$ ends at v_k. If P leaves U, it must first meet $\delta(U)$. So in either case, 3. is impossible.

Case 1 implies the following local graph;



where $U = U_0 \cup U_1$, and the number of vertices in U_0 is less than q, as is the number of vertices in U_1. But $v \in \delta(U_1)$ or $v \in \delta(U_0)$ (or both if $v = v_k$ or $v = v_k'$). Therefore, by the induction assumption, since v is on the boundary of some cycle containing less than q vertices, v is on the boundary of some disk $D \subseteq U_0$ or $D \subseteq U_1$, so that $D < U_0 \cup U_1 = U$.

Case 2. gives the following subgraph;



where $U = W_0 \cup W_1$, and each of W_0, W_1 contains less than q vertices. In addition, each of $\delta(W_0)$, $\delta(W_1)$ does not meet itself except perhaps at some vertices(v_k for $\delta(W_0)$). As above, since v is on the boundary of some cycle containing less than q vertices, v is on the boundary of some disk $D \subseteq U$.

Q.E.D.

We are now ready to obtain the results of Theorem 3.2 by assuming the weaker condition $G = G'/R \neq G$, G' a free product of finite cyclic groups of order ≥ 3, whereas in Theorem 3.2, we needed the existence of some disk D whose boundary(by definition a Jordan curve) was not of one color.

Theorem 3.3. Let $G = \langle a_1, a_2, \ldots a_n;\ a_1^{\beta_1}, a_2^{\beta_2}, \ldots a_n^{\beta_n}, R \rangle$ $\not\cong \langle a_1, a_2 \ldots a_n;\ a_1^{\beta_1},\ a_2^{\beta_2}, \ldots a_n^{\beta_n} \rangle$, $n \geq 2$, $3 \leq \beta_i < \infty$, β_i the true order of a_i. Assume that G has a locally finite Cayley diagram $\Gamma \subset E^2$ such that $O_{v_i} = \pm O_{v_j}$, all $v \in \Gamma$. Then;

I. $w = (x_1 x_2 \ldots x_n)^{\gamma} = 1 \in G$ where $x_j = a_i^{\epsilon_i}$, $\epsilon_i = \pm 1$, $1 < \gamma < \infty$, $x_q \neq x_k^{\pm 1}$,

II. For any $v \in \Gamma$, and any $w' \in \{(x_1 x_2 \ldots x_n)^{\gamma}\}$, $v\overline{w'}$ is a disk, and every disk in Γ is either $v_i a_i^{\beta_i}$ or $v_j \overline{w'}$; v_i, $v_j \in \Gamma$, $w' \in \{(x_1 x_2 \ldots x_n)^{\gamma}\}$. All but at most one of the disks in Γ are finite disks, and if $o(G) < \infty$, all such disks are finite.

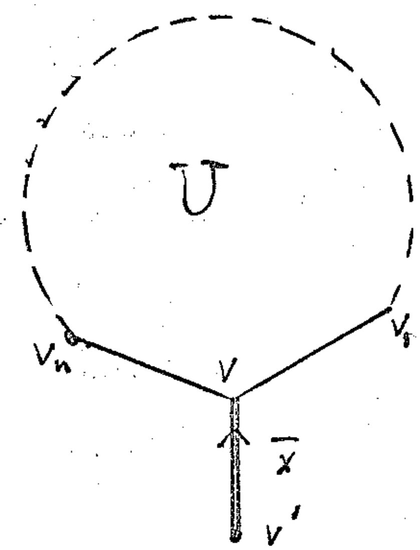
III. $O_{v_i} = \pm O_{v_j}$, all $v \in \Gamma$.

Proof. We will show that the hypotheses imply that there is some disk in Γ not of one color, and then the theorem will follow from Theorem 3.2.

If any expressions of the form $a_i^{\beta_i}$ appear in the relation R, delete them and repeat until none remain. Since this is merely a Tietze Transformation, we still have G. We obtain a new relation R' which is not the empty word.

Pick any $v_o \in \Gamma$. By Lemma 3.5, $v_o\overline{R'}$ determines at least one finite connected open region whose boundary is a Jordan curve. If either the finite or infinite region determined by the boundary of this region, U, contains no vertices or edges of Γ, the theorem follows immediately from Theorem 3.2 by the definition of R'.

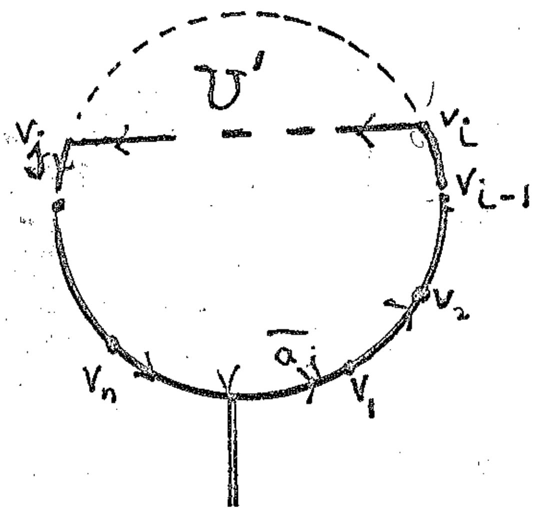
So assume that $\delta(U)$ does not determine a disk. Thus there must be some edge, call it $\overline{x}$, from the infinite region determined by $\delta(U)$ to some vertex $v \in \delta(U)$, for if not, $\delta(U)$ determines an infinite disk. Call $\overline{vx^{-1}}$ v', and so the local graph at v is;



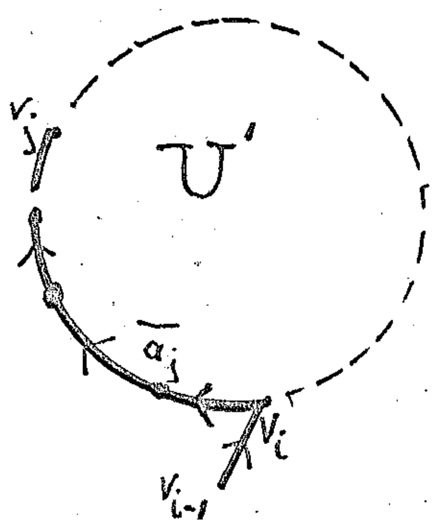
Without loss of generality, assume that $\overline{x^{-1}} = \overline{vv'}$ is the first edge in the clockwise ordering of the edges about v after the edge $\overline{vv_1}$.

By Lemma 3.6, v is on the boundary of some disk $D \subseteq U$. If $\delta(D) \neq \overline{va_j^{\beta_j}}$, then the result follows from Theorem 3.2, so assume that $\delta(D) = \overline{va_j^{\beta_j}}$, some j.

Without loss of generality, assume that the generator a_j corresponding to the edge $\overline{vv_1}$ is not the same as the generator corresponding to the edge $\overline{v_n v}$. For if they were the same, the local graph at v would contain;

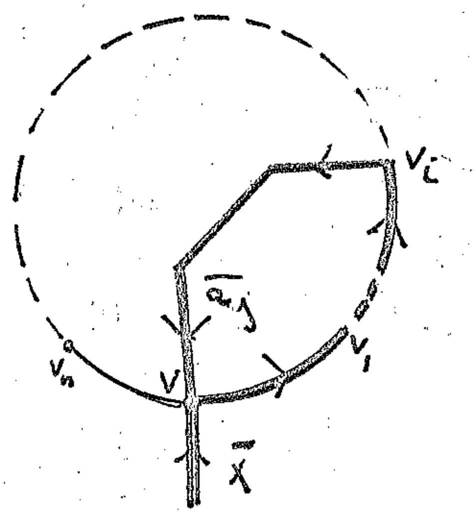


since $\delta(U) = va_1^{\overline{a_1}}$, and we could then consider in the argument the region U', which looks like;

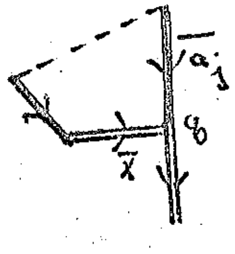


and now the 2 edges of $\delta(U')$ at v_i correspond to different generators.

So $\overline{vv_1} = \overline{a_j}$, $\overline{v_n v} \neq \overline{a_j^{\pm 1}}$, and there is a disk D, $v \in D \subset U$ whose boundary is $\overline{va_j^{\beta_j}}$. The local graph at v must therefore contain;



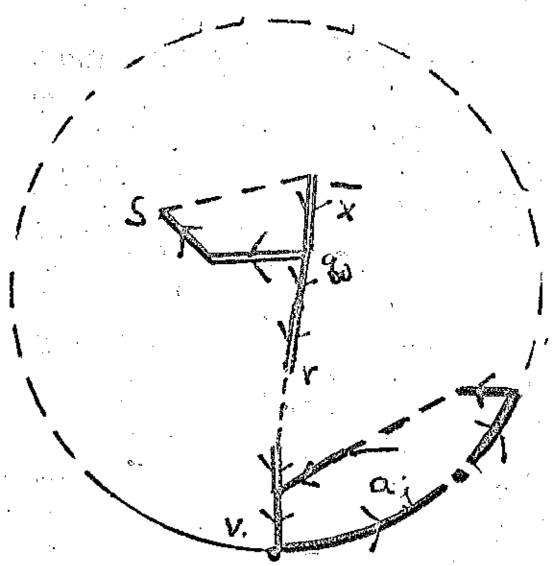
Consider the path $P = v\overline{(a_j^{-1} x^{-1})^q}$, q = 1,2,... .
It will be shown that P meets $\delta(U)$ at some vertex other than v. Assume not. Since Γ is locally finite, U contains but a finite number of vertices of P. Since P does not meet $\delta(U)$, P cannot leave $U \cup \delta(U)$, so that P must meet itself at some vertex q. If the local graph at q contains either



or

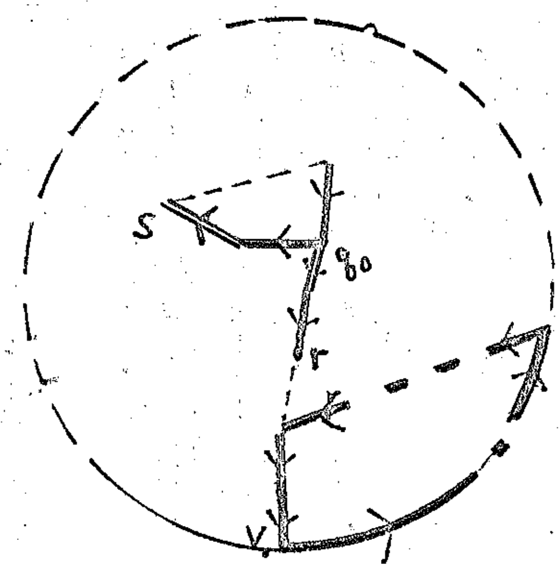
then $(a_j^{-1}x^{-1})$ has finite order by inspection. Therefore, P must meet itself at v (with an $\overline{x^{-1}}$ edge), and it must do so without first meeting any $v_j \in \mathcal{S}(U)$ by assumption. But, if this $\overline{x^{-1}}$ edge comes into v from U, it is impossible to complete the $\bar{x}$ polygon at v to be a disk, contrary to Theorem 3.1. And, by assumption, the $\overline{x^{-1}}$ edge cannot come from $v_n \in \mathcal{S}(U)$. Therefore, P must meet itself in one of the following two ways;

1.



or

2.



where q_o is defined to be the first vertex $\in \Gamma$ where P meets itself. It will be shown that case 2. is impossible. The proof for case 1. is identical.

By inspection case 2. yields;

$q_o(a_j^{-1}x^{-1})^k a_j^{-1} = q_o$, some $k \geq 1$ (since x and a_j are independent) and $r(\overline{a_j^{-1}x^{-1}}) = q_o$.

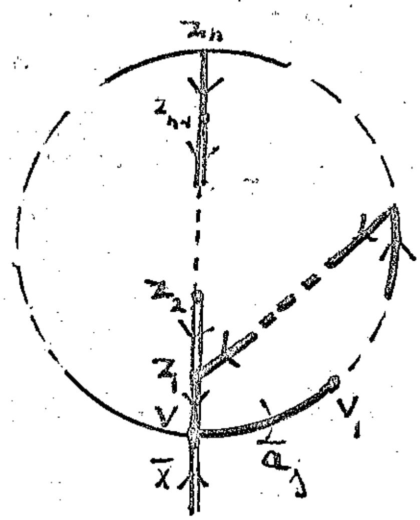
Therefore, $r(\overline{(a_j^{-1}x^{-1})(a_j^{-1}x^{-1})^k a_j^{-1}} = q_o \Longrightarrow$

$$r(\overline{(a_j^{-1}x^{-1})^k a_j^{-1}(x^{-1}a_j^{-1})} = q_o .$$

Therefore, $r(\overline{(a_j^{-1}x^{-1})^k a_j^{-1}} = q_o\overline{a_j x} \equiv s$.

But, $(a_j^{-1}x^{-1})^k a_j^{-1} = 1$, so that $r = s$ which contradicts the definition of q_o.

Therefore, P must meet $\delta(U)$ at some vertex other than v. Say P meets $\delta(U)$ with an $\overline{a_j^{-1}}$ edge at some vertex z_n. The proof is identical if P meets $\delta(U)$ with an x^{-1} edge. We thus have the subgraph;



By the definition of $\bar{x}$, O_v $\begin{cases} a_j \longleftarrow \\ a_j \longrightarrow \\ \bar{x} \longleftarrow \\ \cdot \\ \cdot \end{cases}$

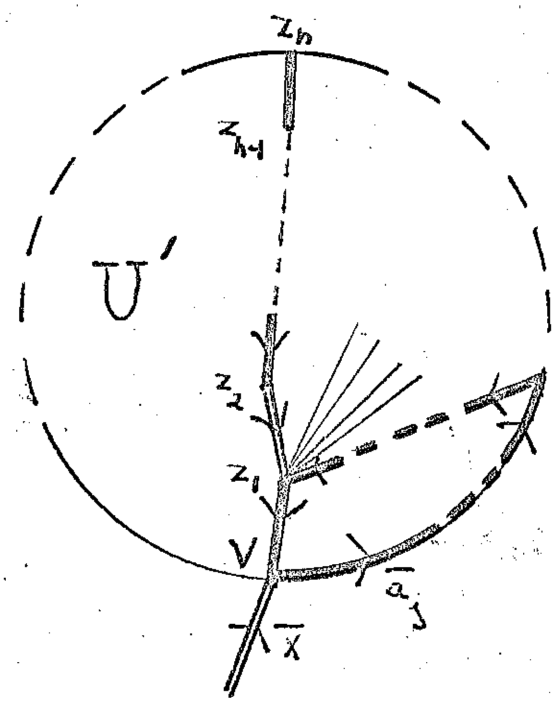
and since $O_{v_i} = \pm O_{v_j}$, all $v \in \Gamma$, it must be that

$O_{z_1} = \pm O_v$, and since the clockwise ordering of the edges

at z_1 begins $\begin{Bmatrix} a_j \longleftarrow \\ a_j \longrightarrow \end{Bmatrix}$

the clockwise ordering of the edges at z_1 must be $\begin{cases} a_j \longleftarrow \\ a_j \longrightarrow \\ x \longleftarrow \\ \cdot \\ \cdot \end{cases}$

and so $z_1 x^{-1} = \overline{z_1 z_2}$ is the first edge after $a_j \longrightarrow$ in the clockwise ordering about z_1;



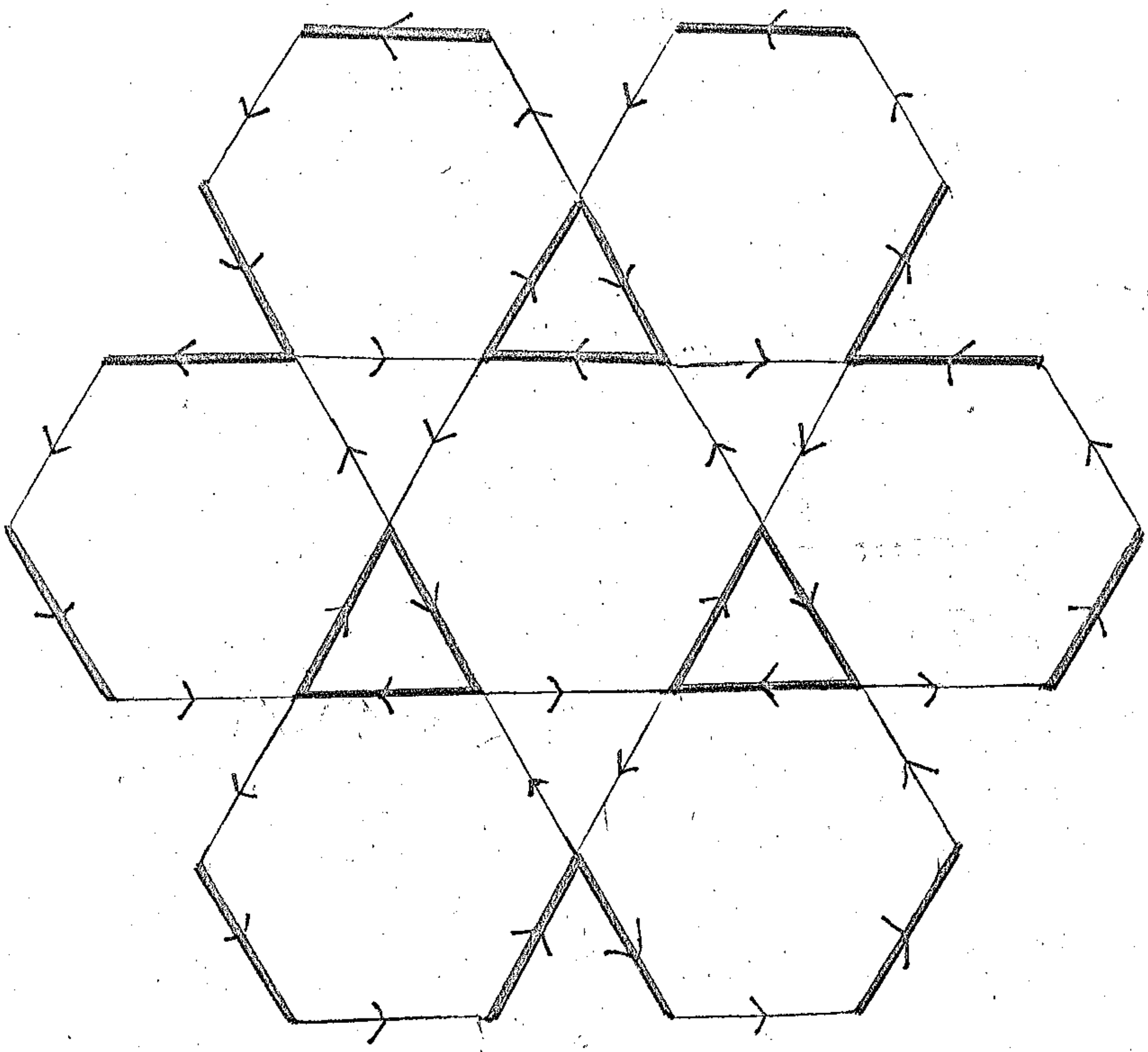
Define the region U' as in the above diagram. Therefore, $z_1 \in \delta(U')$, and by Lemma 3.6, z_1 is on the boundary of some $D' \subseteq U'$, D' a disk. Since there are no edges from z_1 into U' by the above argument, v, z_1 and z_2 are three consecutive vertices on $\delta(D')$. Therefore, $\delta(D')$ has two consecutive edges which are not the same color. Therefore, $\delta(D') \neq z_1\overline{a_i^{\beta_i}}$, any i. The result now follows from Theorem 3.2.

Q.E.D.

Groups of the type described in Theorem 3.3 may or may not admit Cayley diagrams $\subset E^2$ where the disks are regular polygons(all edges of each polygon has the same length - a "paving" of E^2). If not, the Cayley diagram $\Gamma \subset E^2$ of G is obtained by first drawing the Cayley diagram $\Gamma' \subset E^2$ of G', the free product of the finite cyclic groups, and then making the proper identifications corresponding to the $\{(x_1x_2,\ldots x_n)^{\gamma}\}$.

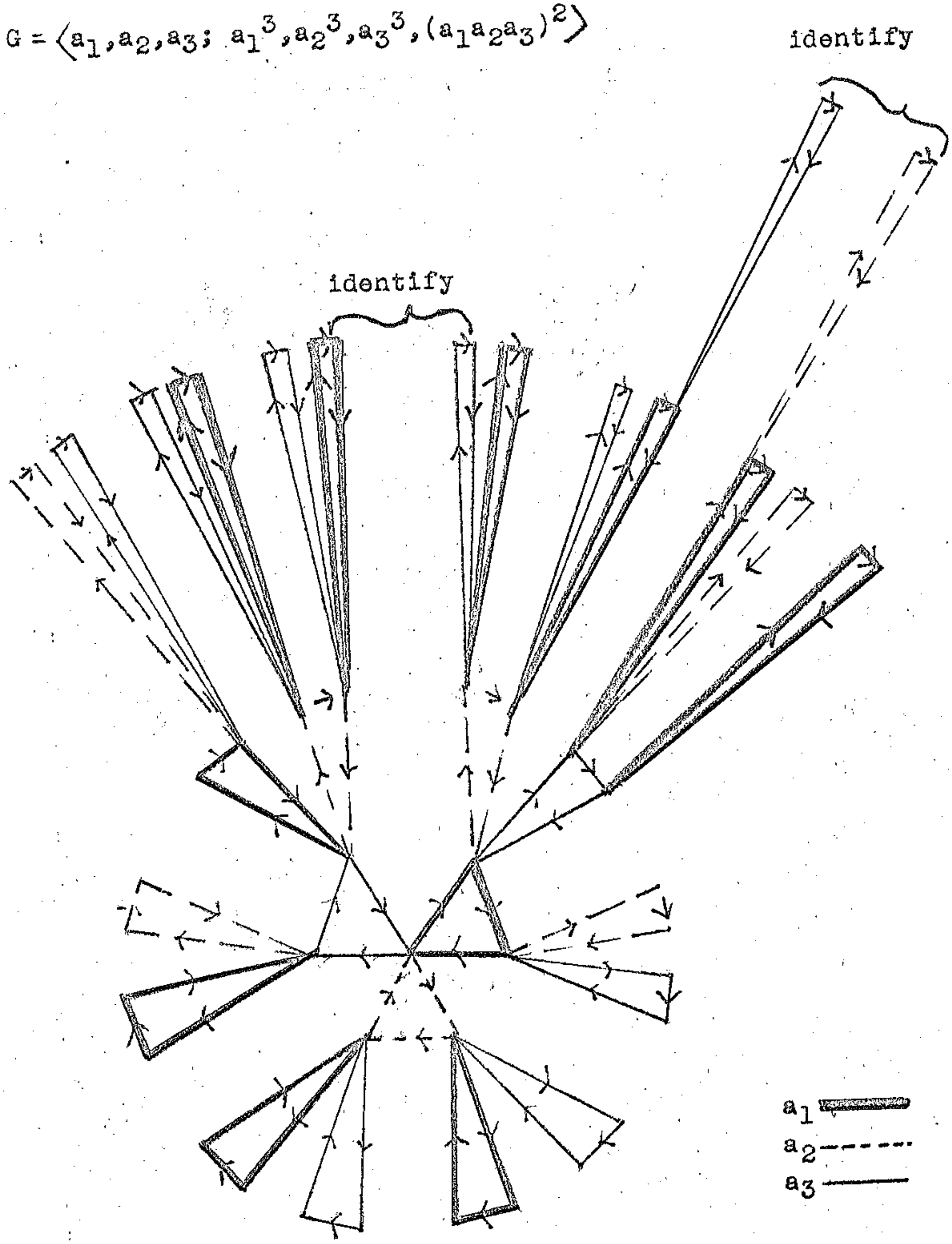
An example of each of these two types is shown on the following pages.

$$G = \langle a_1, a_2; a_1^3, a_2^3, (a_1a_2)^3 \rangle$$



Etc.

$G = \langle a_1, a_2, a_3;\ a_1^3, a_2^3, a_3^3, (a_1a_2a_3)^2 \rangle$



Etc.

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