A PROPER PL MAP LOCALLY TRIVIAL IN THE DOMAIN IS A PL FIBRE BUNDLE MAP

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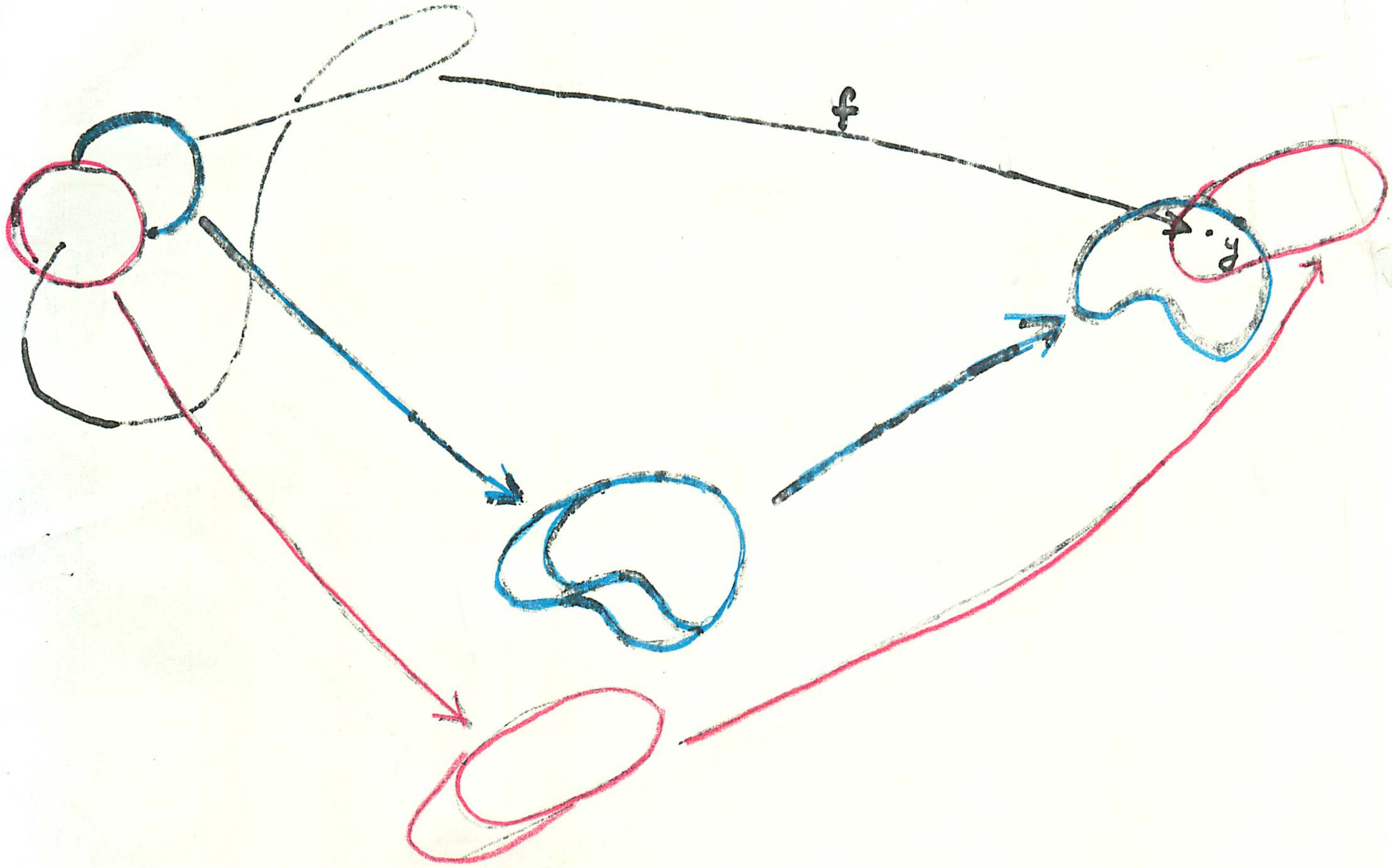
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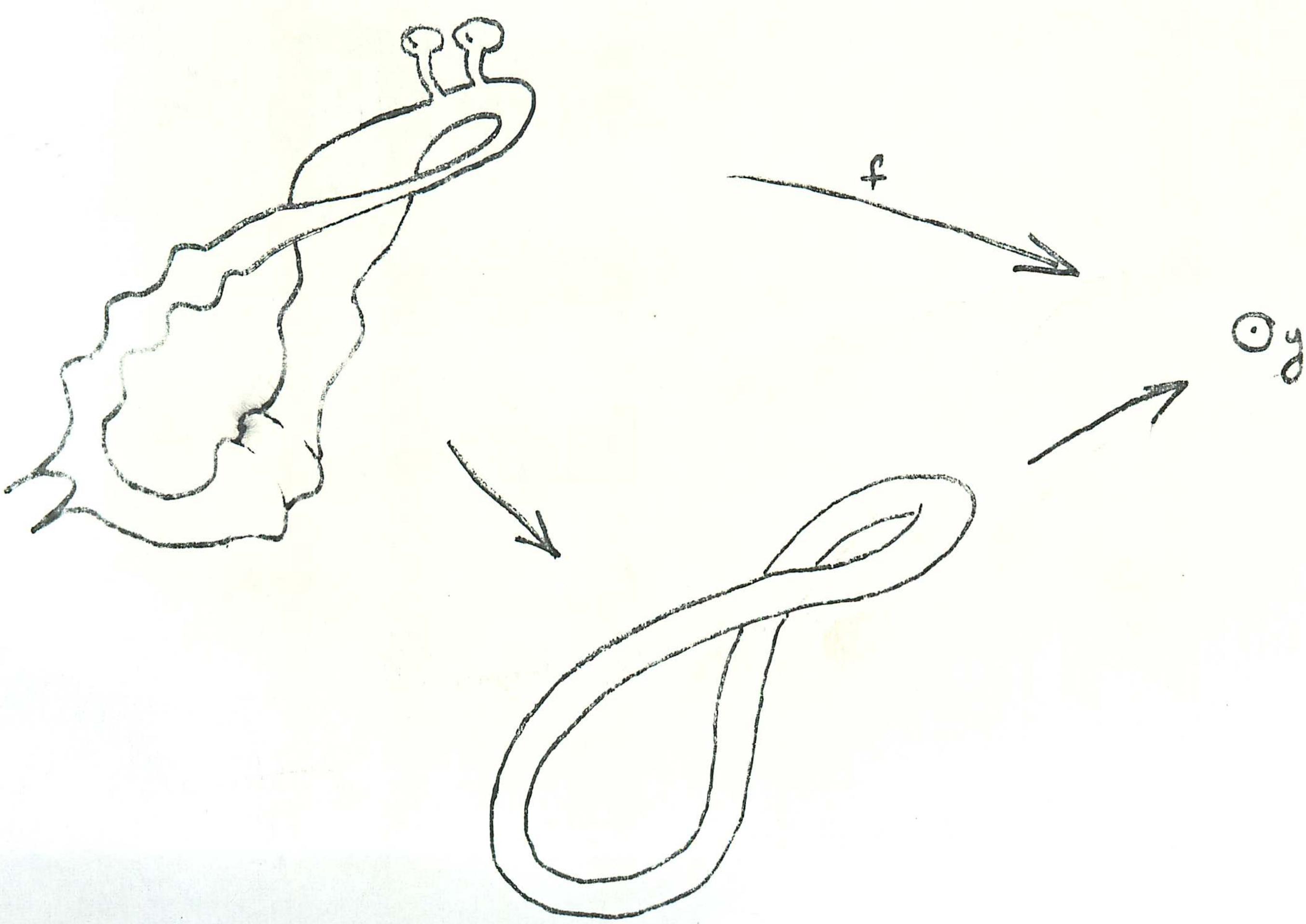
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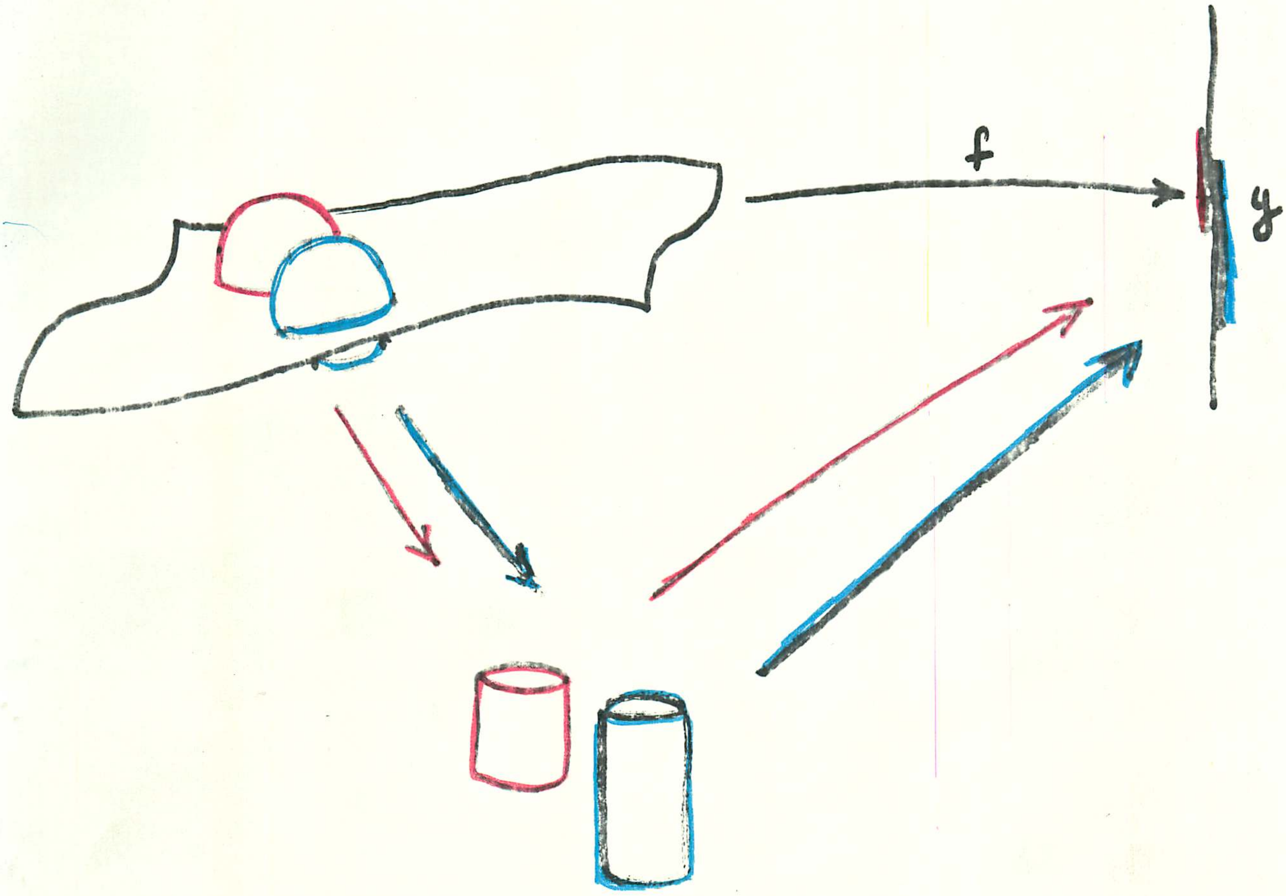
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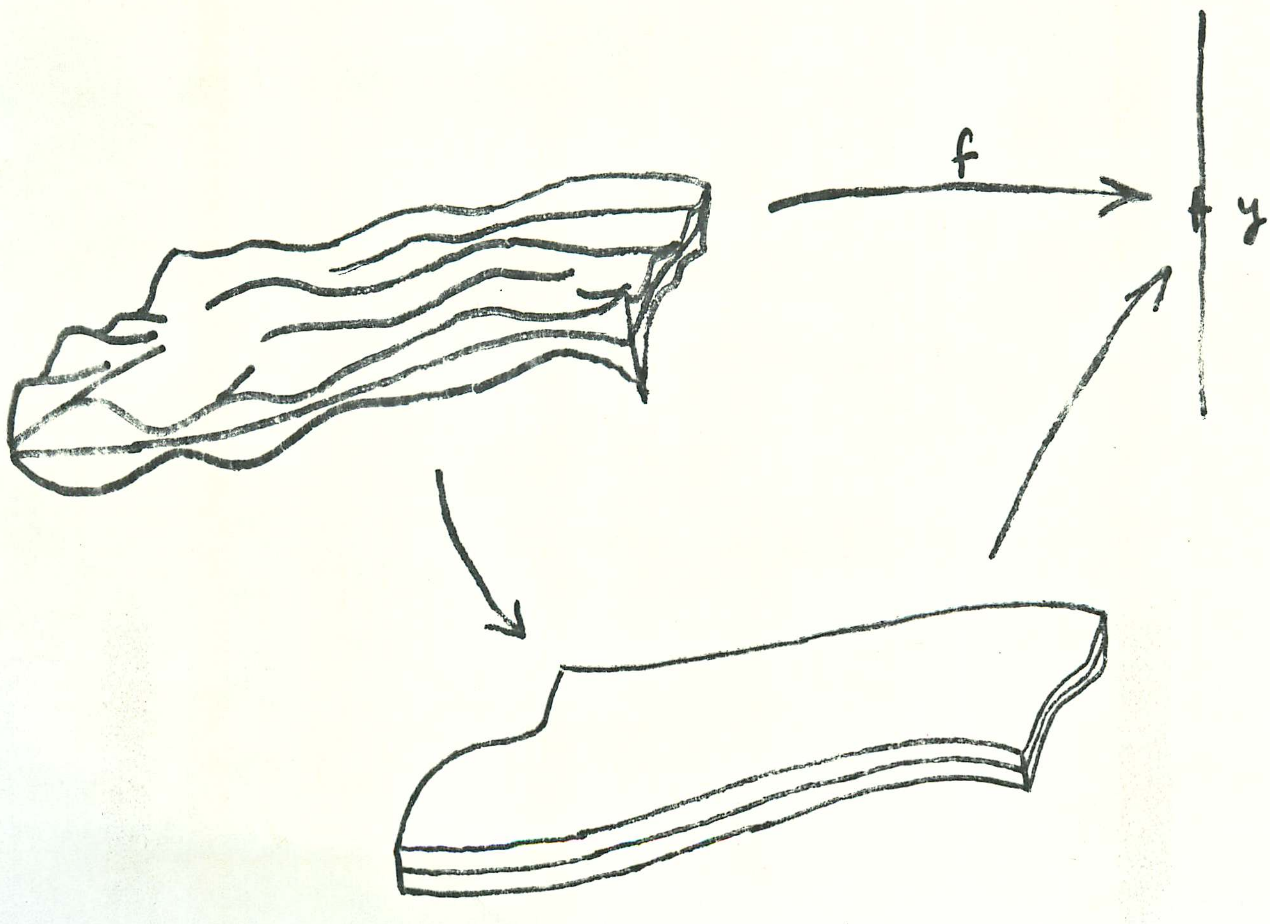
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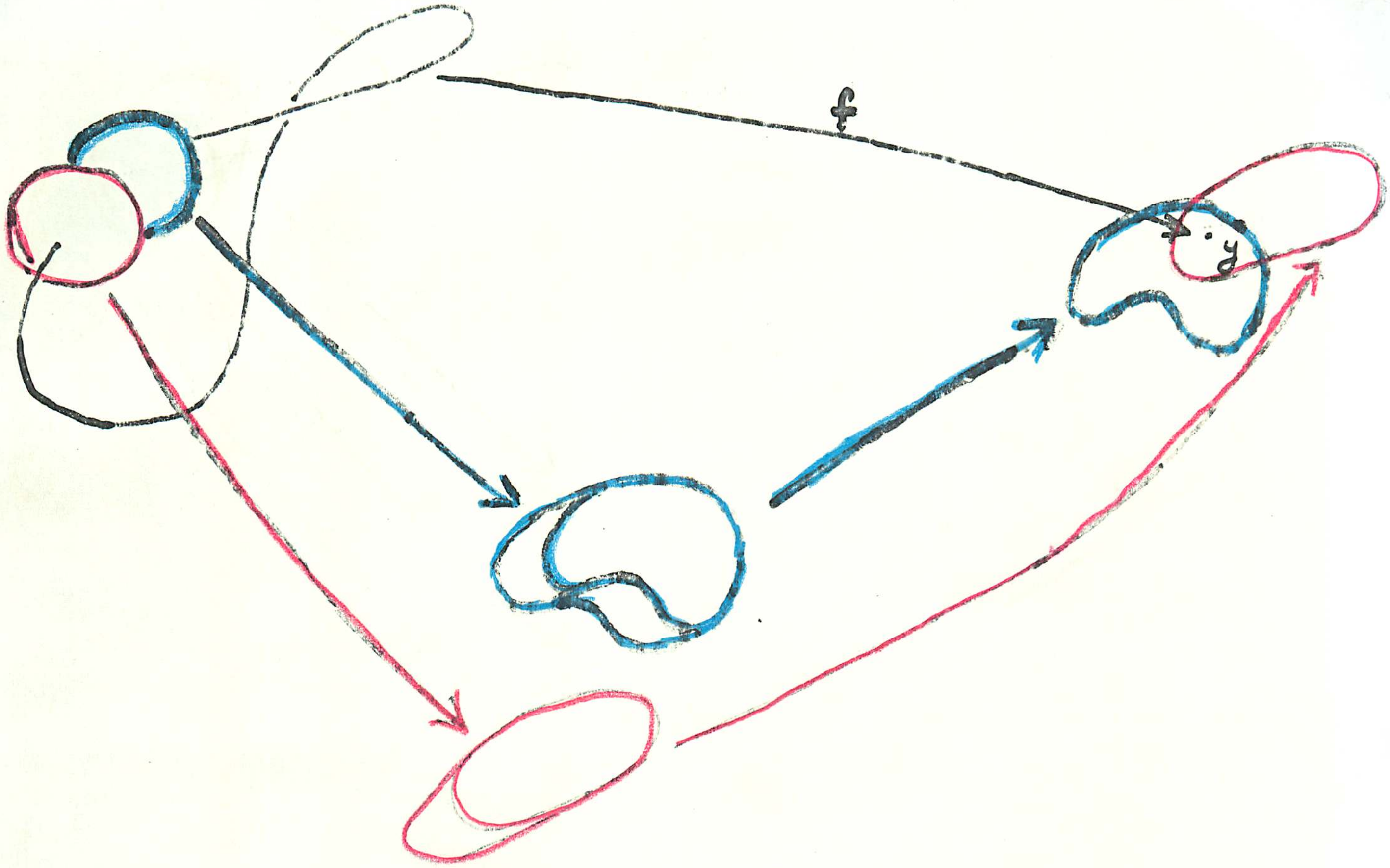
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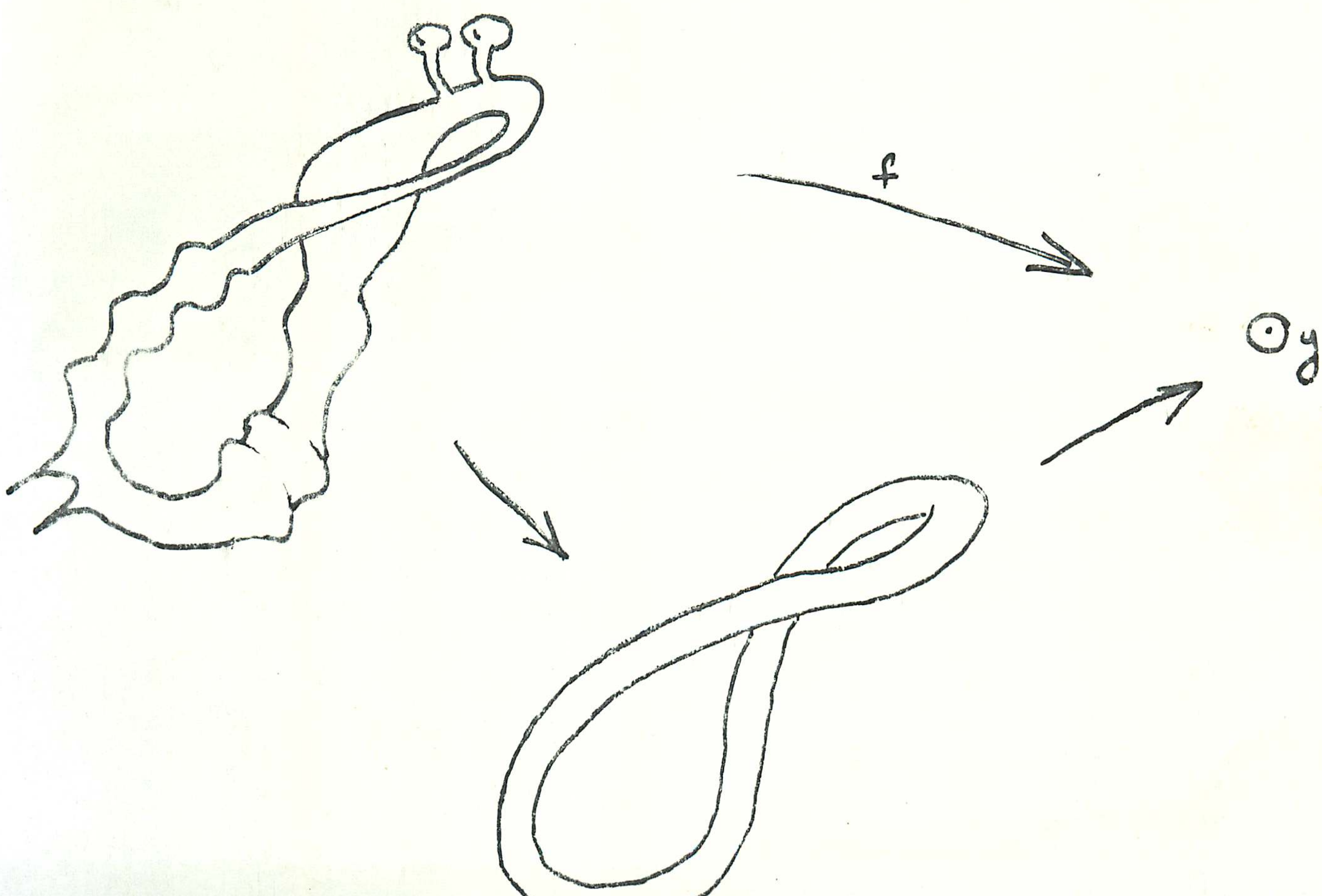
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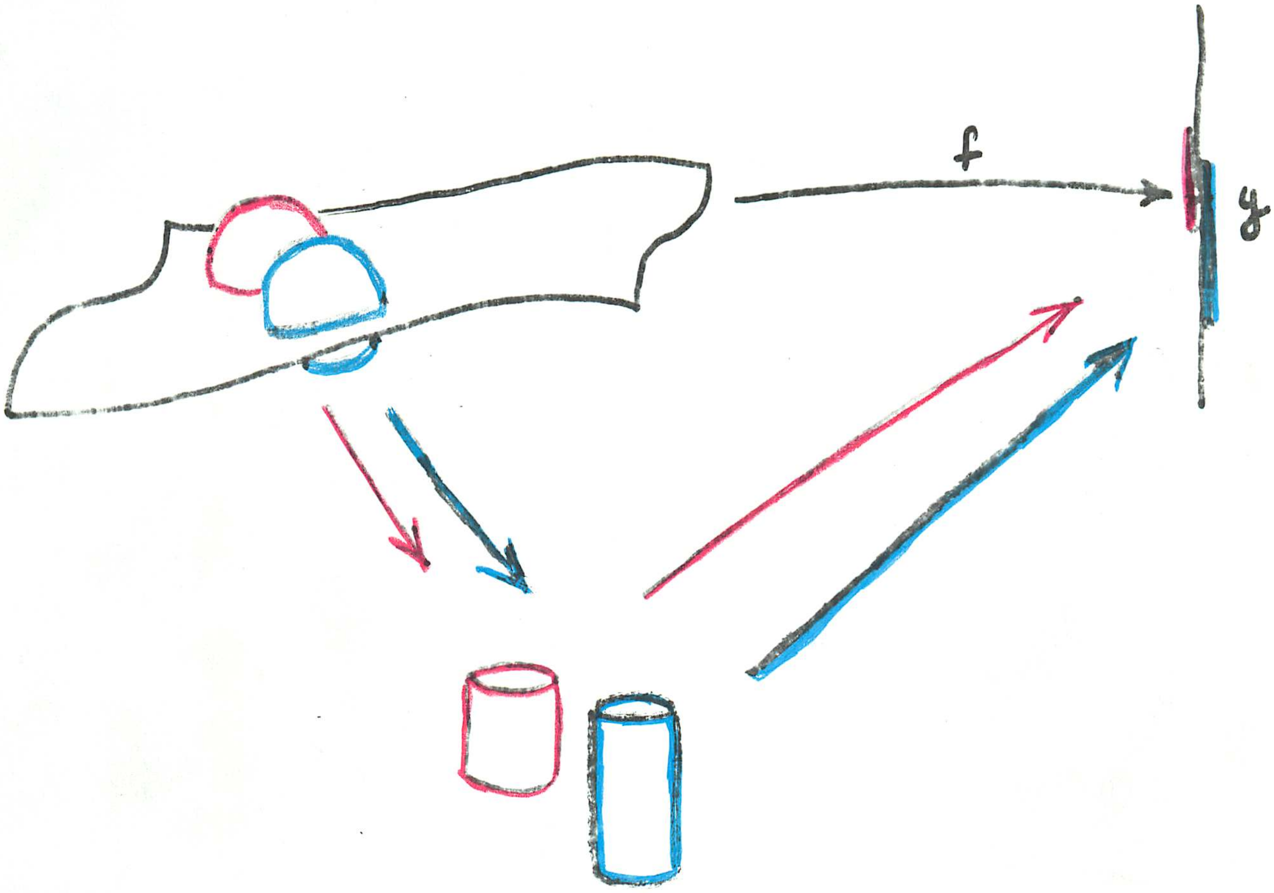
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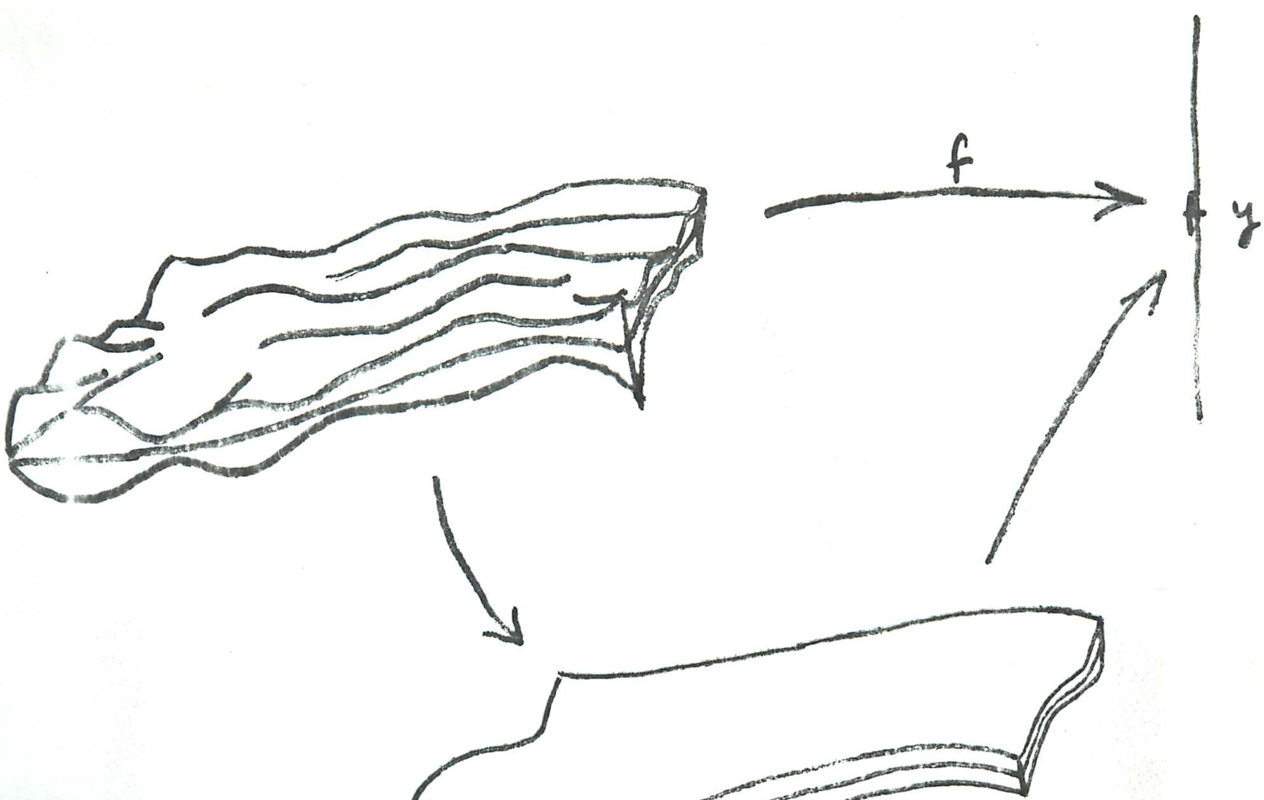
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Abstract

A map $f: M \longrightarrow N$ is proper if, for every compact set $C \subset N$, $f^{-1}(C)$ is compact. If f is a piecewise linear (PL) map, then the branch set of f, $B_f \subset M$, can be defined as follows: $x \in M - B_f$ if there exists a neighborhood U_x of x such that $f(U_x) = V_x$ is a closed star neighborhood of f(x), and if there exists a PL homeomorphism $h_x: U_x \longrightarrow B_x \times V_x$, where B_x is a PL ball, such that $f|_{U_x} = \pi_2 \circ h_x: U_x \longrightarrow V_x$, where π_2 is the projection onto the second factor.

Theorem. Let M^m, N^k be PL manifolds, and let $f: M^m \longrightarrow N^k$ be a proper PL map with $B_f = \phi$. Then, for each $y \in N^k$, there is a neighborhood N_y of y, and a PL homeomorphism $g_y: f^{-1}(N_y) \longrightarrow f^{-1}(y) \times N_y$, such that

$$f|_{f^{-1}(N_y)} = \pi_2 \circ g_y: f^{-1}(N_y) \longrightarrow N_y.$$

Thus if N^k is connected, then f is the projection map of a PL fibre bundle.

I understand that C. P. Rourke has proved this result independently and at about the same time.

This theorem is a PL analogue of theorems which have been proved differentiably on manifolds by C. Ehresmann [2] and topologically by J. Cheeger and J. M. Kister [1].

Preface

This thesis was written under the direction of Professor Philip T. Church of Syracuse University, and supported by a teaching assistantship from The State University of New York at Stony Brook and a National Science Foundation graduate fellowship.

I wish to thank the following people and institutions who helped to make this paper possible:

My husband David, for supporting and encouraging my work since I was an undergraduate, and for typing this paper.

The State University of New York at Stony Brook, for providing the setting for the beginning of my interest in mathematical research.

Professor William C. Fox, for encouraging that interest and helping me discover that I was capable of doing mathematical research.

The Saint James Lutheran Church, for providing facilities where I could spend many uninterrupted hours doing the research necessary for this paper.

Syracuse University, for allowing Professor Church to devote time to me.

And especially Professor Church, for his diligence, patience, understanding, and encouragement during the time that I was writing this paper.

Table of Contents

Chapter I

Introduction

Two kinds of maps which are important in the study of any type of manifold (topological, differential, PL) are

1) maps with some properties of embeddings,

and 2) maps with some properties of projections.

In the topological and differential categories there have been many very useful results in the study of maps of both types. In the piecewise linear category, however, most knowledge is about embedding-like maps.

In this paper I obtain a result which gives some information about some projection-like PL maps. My theorem says that, if a PL map is proper, then local triviality in the domain forces the map to be a local projection. The precise statement follows.

<u>Theorem</u>. Let M^m and N^k be PL manifolds and let $f: M^m \longrightarrow N^k$ be a proper PL map with $B_f = \phi$. Then for every y in $f(M^m)$ there is a neighborhood N_y of y and a PL homeomorphism $g_y: f^{-1}(N_y) \longrightarrow f^{-1}(y) \times N_y$ such that $f|_{f^{-1}(N_y)} = \pi_2 \circ g_y: f^{-1}(N_y) \longrightarrow N_y$. Thus, if N^k is connected, f is the projection map of a PL fibre bundle.

Chapters II and III consist of lemmas needed for the proof of the main theorem. Many of the lemmas in these chapters are well known and included only for the sake of completeness. The proofs of these lemmas are included in the appendix for the reader who is not an expert in PL topology.

In Chapter IV I build a neighborhood and a homeomorphism which yield the necessary local trivialization [see the proof sketch in the beginning of the chapter].

Chapter V is a different proof of the theorem in the special case of codimension 1.

In Chapter VI, I state some conjectures and indicate the proof of a theorem which gives insight into a non-manifold generalization of this paper and indicates a possible direction for my future research.

The main theorem in this paper is the PL analogue of theorems which have been proved topologically and differentiably.

The topological result in codimension 0 was presented to the American Mathematical Society by R. S. Palais in 1957 and then published in the following form [7 , Theorem 4.2]:

Theorem. Let X and Y be connected, locally connected, and locally compact Hausdorff spaces. A necessary and sufficient condition for a local homeomorphism $f: X \longrightarrow Y$ to be a finite covering is that the inverse image by f of each compact subset of Y is a compact subset of X.

With a proof that I was able to modify for a PL result [Chapter V], J. G. Timourian obtained the topological result in codimension 1 [11 , Proposition 2.1] and states his proposition as follows:

Proposition. Let M^{p+1} (possibly with boundary) and N^p be connected manifolds. If f: $M^{p+1} \longrightarrow N^p$ is proper and $B_f = \phi$, then f is a fiber bundle map.

In 1969, J. Cheeger and J. M. Kister obtained the topological result in codimension n [1 , page 151]. This result comes from the proof of the main theorem of the paper, which is:

Theorem. There are precisely a countable number of compact topological manifolds (boundary permitted) up to homeomorphism.

In the course of the proof of that theorem, homeomorphisms are obtained which yield the result of interest in this paper, as follows (the notation B(2) means a closed ball in $\mathbb{R}^n$ and the condition involving B(2) compares with $B_f = \phi$):

Remark. Let f: W $\longrightarrow$ Y be a proper monotone map satisfying the following condition. For every x in W there are closed neighborhoods U of f(x) in Y and V of x in W and a homeomorphism h: $B(2) \times U \longrightarrow V$ such that $f \circ h$ is the projection map onto U....Then a local trivialization of f is defined.

In 1947 C. Ehresmann published a proof of the differentiable case with the restriction that the domain be compact [3 , Proposition 1]:

Proposition. If E (a manifold) is compact, every differentiable map p of rank n from E to an n-dimensional manifold gives E the structure of a differentiable fibre space.

The result with the compactness condition removed appeared in a 1950 paper by Ehresmann [2]. J. A. Wolf published a new proof of the same result in 1964 [13 , Corollary 2.4].

In [6, Section 1, 1.6], K. Lamotke generalized Wolf's proof to the case $f: X \longrightarrow Y$ where X is a manifold with boundary.

It is interesting to notice that, in some form, the condition 'proper' appears in every one of these theorems. The condition is actually necessary: a counterexample can be found in each category to the theorem with the 'proper' hypothesis removed.

Let $D \subset \mathbb{R}^2$ be a closed disk and $f: \mathbb{R}^2 - D \longrightarrow \mathbb{R}$ be the restriction of the projection map. Then $B_f = \phi$, but f is not locally trivial. f is clearly continuous and differentiable. I'll show that, with the natural PL structure on $\mathbb{R}^2 - D$ and $\mathbb{R}$, f is a polymap (a generalization of PL map for maps defined on PL spaces which are not necessarily simplicial complexes) and therefore this one situation provides a counterexample to the generalization in all three categories.

<u>Definition</u> [14 , Chapter III, page 5]. A function $f: X \longrightarrow Y$ between PL spaces is called a <u>polymap</u> if the following condition is satisfied: given $g \in F(X)$ then there is $g' \in F(Y)$ and a PL map f' such that $fg = g'f'$.

Applying this definition to the example above, $g: K \longrightarrow \mathbb{R}^2 - D$ is a PL embedding of a simplicial complex. $f \circ g$ is a simplicial complex in $\mathbb{R}$.

Let $L = f \circ g(K)$ and $g'\colon L \longrightarrow \mathbb{R}$ be the inclusion map. Define $f'\colon K \longrightarrow L$ by $f' = (g'^{-1}|_{f \circ g(K)}) \circ f \circ g$, a PL map.

I was told recently that this result has been proved independently by C. P. Rourke but that it is as yet unpublished. A. Haefliger posed this problem to Rourke, who was unable to solve it at the time. It was discussed by several people, including W. Anderson and D. Sullivan, at the 1970 international summer school on manifolds held at the University of Amsterdam, August 17-30, 1970. I understand that Rourke subsequently proved the result. An abstract of my work (with the restriction that $\partial f^{-1}(y) = \phi$) was received by the American Mathematical Society on August 28, 1970. Within a few weeks I was able to modify my proof slightly to remove the restriction on the boundary. Since I have not seen Rourke's work, I do not know whether his proof follows the same line as mine or was done completely differently.

This paper has been written in such a way that it can be read both by a mathematician familiar with PL topology and by a mathematician who has not studied the subject. Many proofs are in the appendix so they do not get in the way of more important work. They are included so the non-PL topologist can read the paper in as much detail as he desires.

Even some of the proofs in the main text are straightforward and yield unexciting results. Therefore, for the convenience of the reader who is interested only in the main results and proofs of this work, *I have starred the results that should be of interest to every reader. The starred results are essentially those I would include in a paper covering this work.*

Chapter II

Simplicial Complexes

In this chapter I state and prove some results about simplicial complexes which are needed for the proof of the main theorem.

The notation and definitions unless otherwise stated are those of Zeeman [14, Chapter 1].

II.1. Notation. A and B are simplexes in E^p, Euclidean p-space.

$A<B$ means A is a face of B.

If K and L are simplicial complexes, then $K\subset L$ means K is a subcomplex of L.

II.2. Definition. The star of a simplex A in K, st(A,K), is defined by $st(A,K) = \{B\in K | A<B\}$.

II.3. Definition [5 , page 7].

$\overline{st}(A,K) = \{B\in K | B$ is a face of an element of $st(A,K)\}$; i.e.,

$\overline{st}(A,K) = \{B\in K | \exists C\in K\colon A<C \text{ and } B<C\}$.

II.4. Convention. If $K\subset L$, $x\in L-K$, then $\overline{st}(x,K)$ is defined to be the empty set ϕ.

II.5. Definition. Simplexes A,B are joinable if their vertices are linearly independent. If A and B are joinable, the join, AB, is the simplex spanned by the vertices of both; otherwise the join is undefined.

II.6. Definition. The link of a simplex A in K, lk(A,K), is defined by lk(A,K) = {B∈K|AB∈K}.

II.7. Definition. Two simplicial complexes K and L are joinable provided:

(i) if A∈K, B∈L, then A,B joinable

(ii) if A,A' ∈ K and B,B' ∈ L then AB∩A'B' is empty or a common face.

If K,L are joinable, we define the join KL = K∪L∪{AB|A∈K,B∈L}.

II.8. Definition. The underlying point set |K| of K is called a Euclidean polyhedron.

II.9. Convention. If A is a simplex, A∉K, then $\overline{st}(A,K) = \phi$ and lk(A,K) = ϕ.

II.10. Lemma. Let K be a simplicial complex with subcomplexes L_1, L_2. Then for any simplex A of K, $\overline{st}(A, L_1 \cup L_2) = \overline{st}(A,L_1) \uplus \overline{st}(A,L_2)$.

II.11. Lemma. Let L_1 and L_2 be subcomplexes of a simplicial complex K. Let A be a simplex of $L_1 \cap L_2$. Then $\overline{st}(A, L_1 \cap L_2)$ $= \overline{st}(A,L_1) \cap \overline{st}(A,L_2)$.

II.12. Lemma. Let K be a simplicial complex, x a vertex of K, and A a simplex of K. If xA can be defined, then

$$lk(A, \overline{st}(x,K)) = x\, lk(A, lk(x,K)).$$

Proof. Let $B \in lk(A, \overline{st}(x,K))$. Then $AB \in \overline{st}(x,K)$. If $x \notin AB$, then $AB \in lk(x,K)$ and $B \in lk(A, lk(x,K)) \subset x\ lk(A, lk(x,K))$.

If $x \in AB$, then $x \in B$ and $B = xC$, $C \in lk(x,K)$. Therefore $AC \in lk(x,K)$. Therefore $C \in lk(A, lk(x,K))$ and $xC \in x\ lk(A, lk(x,K))$.

Therefore $lk(A, \overline{st}(x,K)) \subset x\ lk(A, lk(x,K))$.

Let $B \in x\ lk(A, lk(x,K))$. Then $B = xC$, $C \in lk(A, lk(x,K))$. Therefore $AC \in lk(x,K)$ and $xAC \in \overline{st}(x,K)$. Therefore $B = xC \in lk(A, \overline{st}(x,K))$ and $x\ lk(A, lk(x,K)) \subset lk(A, \overline{st}(x,K))$. □

II.13. Notation. $\dot{A}$ denotes the boundary of a simplex A; $\mathring{A}$ denotes the interior of A.

II.14. Definitions. Let A be a simplex in the simplicial complex K. Choose a point $\hat{A} \in \mathring{A}$. Let $L = (K - st(A,K)) \cup \hat{A}\ \dot{A}\ lk(A,K)$. Then L is a subdivision of K, and we say L is obtained from K by starring A (at $\hat{A}$).

A first derived $K^{(1)}$ of K is obtained by starring all the simplexes of K in some order such that if $A > B$ then A precedes B (for example, in order of decreasing dimension). Another way of defining $K^{(1)}$ is to define the subdivision of each simplex, inductively in order of increasing dimension, by the rule $A' = \hat{A}\ \dot{A}'$.

Therefore a typical simplex of $K^{(1)}$ is $\hat{A}_0 \hat{A}_1 \ldots \hat{A}_p$ where $A_0 < A_1 < \ldots < A_p$ in K.

An r^{th} derived $K^{(r)}$ is defined inductively as the first derived of an $(r-1)^{st}$ derived.

II.15. Notation. Let x_1, x_2 be vertices of a simplicial complex K in E^p with $x_1 \neq x_2$; then $[x_1,x_2]$ denotes the line segment in E^p which connects x_1 and x_2.

If $[x_1,x_2]$ is a simplex of K then $x_{1,2} = [x_1 \hat{,} x_2]$, the point at which $[x_1,x_2]$ is starred in forming $K^{(1)}$ from K.

II.16. Lemma. If $x_1 \neq x_2$ are vertices of K, then either $\overline{st}(x_1,K^{(1)}) \cap \overline{st}(x_2,K^{(1)}) = \phi$, or $[x_1,x_2]$ is a simplex of K and $\overline{st}(x_1,K^{(1)}) \cap \overline{st}(x_2,K^{(1)}) = \overline{st}(x_{1,2}, 1k(x_2,K^{(1)}))$.

I found it necessary to define the following objects to use in the proof of the main theorem.

II.17. Definition. Let x_1, ..., x_m be the vertices of a simplicial complex K.

$$S(x_t,K^{(1)}) = \bigcup_{i=1}^{t} \overline{st}(x_i,K^{(1)})$$

$$T(x_t,K^{(1)}) = \bigcup_{i=t+1}^{m} \overline{st}(x_i,K^{(1)})$$

* II.18. Lemma. Given a simplicial complex K with vertices x_1, ..., x_m. Then $S(x_t,K^{(1)}) \cap \overline{st}(x_{t+1},K^{(1)}) = \bigcup_{i=1}^{t} \overline{st}(x_{i,t+1},1k(x_{t+1},K^{(1)}))$ $= \bigcup_{i=1}^{t} \overline{st}(x_{i,t+1},1k(x_i,K^{(1)}))$, where the union is taken over all i such that $[x_i,x_{t+1}]$ is a simplex of K.

Proof.

$$S(x_t,K^{(1)}) \cap \overline{st}(x_{t+1},K^{(1)}) = (\bigsqcup_{i=1}^{t}\overline{st}(x_i,K^{(1)}))\cap \overline{st}(x_{t+1},K^{(1)})$$

$$= \bigsqcup_{i=1}^{t}(\overline{st}(x_i,K^{(1)}) \cap \overline{st}(x_{t+1},K^{(1)}))$$

$$= \begin{cases} \bigsqcup_{i=1}^{t}\overline{st}(x_{i,t+1},\ lk(x_{t+1},K^{(1)})) \\ \quad\text{or} \\ \bigsqcup_{i=1}^{t}\overline{st}(x_{i,t+1},\ lk(x_i,K^{(1)})) \end{cases}$$

[II.16]. □

II.19. Definition. A map f: K ⟶ L is a continuous map

$$|K| \longrightarrow |L|.$$

Call f simplicial if it maps vertices to vertices and simplexes linearly to simplexes.

Call f piecewise linear (PL) if there exist subdivisions K', L' of K, L with respect to which f is simplicial.

II.20. Lemma [5 , 1.1]. If f: K ⟶ L is a piecewise linear embedding and P is a simplicial complex with $|P|\subset|K|$, then $f|_P$: P ⟶ L is piecewise linear.

II.21. Lemma. Let K_1, K_2, L_1, L_2 be simplicial complexes and $f_1: K_1 \longrightarrow L_1$, $f_2: K_2 \longrightarrow L_2$ be PL embeddings. If $f_1|_{K_1\cap K_2} = f_2|_{K_1\cap K_2}$, then define $f: K_1\cup K_2 \longrightarrow L_1\cup L_2$ by $f|_{K_1} = f_1$, $f|_{K_2} = f_2$. If f is 1-1, then f is PL.

II.22. Lemma. For every vertex x of a simplicial complex K, there is a simplicial homeomorphism $f\colon \mathrm{lk}(x,K^{(1)}) \longrightarrow (\mathrm{lk}(x,K))^{(1)}$.

The proof of this lemma is essentially contained in the proof of [5 , 1.14].

II.23. Lemma. If $f\colon K \longrightarrow L$ is a simplicial homeomorphism and x is a vertex of K, then $f(\overline{\mathrm{st}}(x,K)) = \overline{\mathrm{st}}(f(x),L)$.

The proof is obvious.

Chapter III

PL Manifolds

In this chapter I obtain results involving PL manifolds. The results are limited to those leading to the proof of the main theorem.

I use work of Zeeman [14] and Hudson [5] interchangeably and must be careful about the precise relationships of their definitions.

In this chapter, any definitions without references are my own.

Section 1: Definitions

This section has various definitions of PL manifolds and the fact that they are equivalent. A reader familiar with PL topology can skip this entire section.

III.1.1. Definition [14 , Chapter I, page 4]. A convex linear cell, or cell, A in E^p is a compact nonempty subset given by

$$\begin{cases} \text{linear equations } f_1 = 0, \ldots, f_r = 0 \text{ and} \\ \text{linear inequalities } g_1 \geq 0, \ldots, g_s \geq 0. \end{cases}$$

A face B of A is a cell (i.e., nonempty) obtained by replacing some of the inequalities $g_i \geq 0$ by equations $g_i = 0$.

III.1.2. Definition [14 , Chapter I, page 5]. A convex linear cell complex, or cell complex, K is a finite collection of cells such that

(i) if $A \in K$, then all the faces of A are in K;

(ii) if $A, B \in K$, then $A \cap B$ is empty or a common face.

III.1.3. Definition [5, page 2]. A Euclidean polyhedron (polyhedron) in E^p is any finite union of cells. In particular, a simplicial complex is a Euclidean polyhedron.

III.1.4. Definition [5, page 20]. A piecewise linear n-ball is a polyhedron which is PL homeomorphic to an n-simplex.

A piecewise linear n-sphere is a polyhedron which is PL homeomorphic to the boundary of an (n+1)-simplex.

III.1.5. Definition [14, Chapter II, page 9]. An n-polyball is a polyhedron triangulated by an n-simplex.

An n-polysphere is a polyhedron triangulated by the boundary of an (n+1)-simplex.

NOTE: The two definitions above (III.1.4 and III.1.5) are clearly the same; we shall use the terminology n-ball and n-sphere.

III.1.6. Definition [14, Chapter III, page 2]. We call a complex J a combinatorial n-manifold if the link of each vertex is an (n-1)-sphere or an (n-1)-ball.

III.1.7. Definition [5, page 20]. A PL manifold of dimension n, M^n, is a Euclidean polyhedron in which every point has a closed neighborhood which is a PL n-ball.

III.1.8. Definition [5, page 26]. The complex K is called a combinatorial n-manifold if $\forall A \in K$, lk(A,K) is a sphere or a ball of dimension $n - \dim A - 1$.

III.1.9. Definition. A simplicial complex K is a combinatorial n-manifold if for every vertex $x \in K$, $\overline{st}(x,K)$ is an n-ball.

III.1.10. Lemma. For a simplicial complex K, definitions III.1.6, III.1.7, III.1.8, and III.1.9 are equivalent.

Section 2: $S(x_t,(M^n)^{(1)})$

This section yields some results about $S(x_t,(M^n)^{(1)})$ [II.17] which will be used in the proof of the main theorem.

III.2.1. Definition [14, Chapter III, page 3]. If J is a combinatorial manifold, define the boundary of J, $\text{bdy}(J) = \dot{J}$, to be the subcomplex $\dot{J} = \{A \in J \mid \text{lk}(A,J) \text{ is a ball}\}$ and the interior to be the open subcomplex $\mathring{J} = J - \dot{J}$.

III.2.2. Lemma. Let x be a vertex of M^n, a combinatorial n-manifold; then $\text{lk}(x,M^n) \subset \text{bdy}(\overline{st}(x,M^n))$. $\text{lk}(x,M^n) = \text{bdy}(\overline{st}(x,M^n))$ $\Longleftrightarrow$ $x \notin \text{bdy}(M^n)$.

III.2.3. Lemma. If $x \in \text{bdy}(M^n)$, then $\text{bdy}(\overline{st}(x,M^n)) = \text{lk}(x,M^n) \cup (\text{bdy}(\overline{st}(x,M^n)) \cap \text{bdy}(M^n))$.

III.2.4. Lemma. Let M^n be a PL manifold with vertices $x_1, \ldots, x_m$. Let A be an a-simplex, $A \in S(x_t,(M^n)^{(1)}) \cap \overline{st}(x_{t+1},(M^n)^{(1)})$. Then

$lk(A,\overline{st}(x_i,(M^n)^{(1)}))$ and $lk(A,\overline{st}(x_{t+1},(M^n)^{(1)}))$ are undefined, empty, or (n-a-1)-balls, $i = 1,\ldots,t$. In particular, $A \in bdy(\overline{st}(x_{t+1},(M^n)^{(1)}))$ and if $A \in \overline{st}(x_i,(M^n)^{(1)})$, then $A \in bdy(\overline{st}(x_i,(M^n)^{(1)}))$.

* III.2.5. <u>Lemma</u>. Given a combinatorial n-manifold M^n with vertices $x_1,\ldots,x_m$, then $S(x_t,(M^n)^{(1)}) \cap \overline{st}(x_{t+1},(M^n)^{(1)})$ is a combinatorial (n-1)-manifold (possibly empty) which is contained in $bdy(S(x_t,(M^n)^{(1)})) \cap bdy(\overline{st}(x_{t+1},(M^n)^{(1)}))$, and $S(x_{t+1},(M^n)^{(1)})$ $= S(x_t,(M^n)^{(1)}) \cup \overline{st}(x_{t+1},(M^n)^{(1)})$ is a combinatorial n-manifold. Similarly $T(x_t,(M^n)^{(1)})$ is an n-manifold.

Proof. The lemma is true, trivially, for combinatorial 0-manifolds. Assume it is true for combinatorial (n-1)-manifolds.

By Lemma II.22, there is a simplicial homeomorphism

$$f\colon lk(x_{t+1},(M^n)^{(1)}) \longrightarrow (lk(x_{t+1},M^n))^{(1)}$$

defined on vertices by $A_i \longrightarrow B_i$, where $A_i = x_{t+1}B_i$. $x_{i,t+1}$ [II.15] is a vertex of $lk(x_{t+1},(M^n)^{(1)})$. $x_{i,t+1} = [x_i,\hat{}\,x_{t+1}]$ [II.15]. Therefore $f(x_{i,t+1}) = x_i$, a vertex of $(lk(x_{t+1},M^n))^{(1)}$ and, by Lemma II.23, $f(\overline{st}(x_{i,t+1},\ lk(x_{t+1},(M^n)^{(1)}))) = \overline{st}(x_i,(lk(x_{t+1},M^n))^{(1)})$. $lk(x_{t+1},M^n)$ is an (n-1)-manifold. Let $j(i)$, $i = 1,\ldots,J$, be an enumeration of the vertices of $lk(x_{t+1},M^n)$.

By the inductive hypothesis $S(x_{j(J)},(lk(x_{t+1},M^n))^{(1)})$ is an (n-1)-manifold.

$$S(x_{j(J)},(lk(x_{t+1},M^n))^{(1)}) = \bigcup_{j(i)=1}^{J} \overline{st}(x_{j(i)},(lk(x_{t+1},M^n))^{(1)})$$

$$= \bigcup_{i=1}^{t} \overline{st}(x_i,(lk(x_{t+1},M^n))^{(1)}).$$

$[x_i,x_{t+1}]$ is a simplex of M^n

f is a simplicial homeomorphism.

$$f^{-1}(S(x_{j(J)},(lk(x_{t+1},M^n))^{(1)})) = f^{-1}\Big(\bigcup_{\substack{i=1 \\ [x_i,x_{t+1}] \text{ is a simplex of } M^n}}^{t} \overline{st}(x_i,(lk(x_{t+1},M^n))^{(1)})\Big)$$

$$= \bigcup_{\substack{i=1 \\ [x_i,x_{t+1}] \text{ is a simplex of } M^n}}^{t} f^{-1}(\overline{st}(x_i,(lk(x_{t+1},M^n))^{(1)}))$$

$$= \bigcup_{\substack{i=1 \\ [x_i,x_{t+1}] \text{ is a simplex of } M^n}}^{t} \overline{st}(x_{i,t+1},(lk(x_{t+1},M^n))^{(1)})$$

$$= S(x_t,(M^n)^{(1)}) \cap \overline{st}(x_{t+1},(M^n)^{(1)})$$

[Lemma II.18]

is an (n-1)-manifold.

$S(x_t,(M^n)^{(1)}) \cap \overline{st}(x_{t+1},(M^n)^{(1)}) \subset bdy(S(x_t,(M^n)^{(1)})) \cap bdy(\overline{st}(x_t,(M^n)^{(1)}))$

[Lemma III.2.4].

Therefore $S(x_{t+1},(M^n)^{(1)}) = S(x_t,(M^n)^{(1)}) \cup \overline{st}(x_{t+1},(M^n)^{(1)})$ is a combinatorial n-manifold by [9 , Exercise 6.5.15], which states: The union of two n-manifolds intersecting in an (n-1)-submanifold of their boundaries is an n-manifold. □

III.2.6. Definition [14 , Chapter III, page 13]. If $K \subset J$ are complexes, we say K is full in J if no simplex of J-K has all its vertices in K.

III.2.7. Comment. If J is a simplicial complex and $S = \{x_1,\ldots,x_n\}$ is a set of vertices of J, then

$$K = \{\sigma \in J \mid \text{all the vertices of } \sigma \text{ are in } S\}$$

is clearly a full subcomplex of J, and can be called the full subcomplex of J generated by S.

III.2.8. Definition [14 , Chapter III, page 14]. Let J be a complex and let $X \subset |J|$ [II.8]. The simplicial neighborhood N(X,J) is the smallest subcomplex of J containing a topological neighborhood of X. It consists of all (closed) simplexes of J meeting X, together with their faces.

Now suppose X is a polyhedron in a compact n-manifold M. We construct derived neighborhoods of X in M as follows. Choose a triangulation J,K of M,X. Choose now an r^{th} derived complex $J^{(r)}$ of J. Call $N = N(X,J^{(r)})$ an r^{th}-derived neighborhood of X in M. If r=1 and K is full in J, we call N a derived neighborhood of X in M. A fortiori, if $r \geq 2$, then any r^{th}-derived neighborhood is a derived neighborhood because $K^{(r-1)}$ is full in $J^{(r-1)}$. If J' denotes the first derived, then $N(X,J') = \bigcup \overline{st}(x,J')$, the union taken over all vertices $x \in K$.

III.2.9. Definition [5 , page 57]. Let X be a polyhedron contained in the PL m-manifold M. $N \subset M$ is called a regular neighborhood of X in M if

(i) N is a closed neighborhood of X in M,

(ii) N is an m-manifold, and

(iii) $N \searrow X$.

The fact about regular neighborhoods which we will use is [5 , Theorem 2.11]:

Let $X \subset M$, M an m-manifold, X a polyhedron. Then any derived neighborhood of X is a regular neighborhood.

III.2.10. Lemma. Let M^n be a combinatorial n-manifold with vertices $x_1, \ldots, x_m$. Then $S(x_t, (M^n)^{(1)}) \cap T(x_t, (M^n)^{(1)})$ is an (n-1)-manifold.

The proof of Lemma III.2.10 can be done as a generalization of the proof of III.2.5 but, since the result is known to PL topologists, I'll just include the following proof of M. Cohen because of its elegance and brevity.

Proof. Let $K_1 = \{\sigma \in M^n \mid \{\text{vertices of } \sigma\} \subset \{x_1, \ldots, x_t\}\}$ and

$K_2 = \{\sigma \in M^n \mid \{\text{vertices of } \sigma\} \subset \{x_{t+1}, \ldots, x_m\}\}$.

Then K_1 and K_2 are full subcomplexes of M, and every simplex of M is uniquely the join of a simplex in K_1 and a simplex in K_2.

Hence $(M^n)^{(1)} = N(K_1^{(1)}, (M^n)^{(1)}) \cup N(K_2^{(1)}, (M^n)^{(1)})$ and $\partial N(K_1^{(1)}, (M^n)^{(1)}) = N(K_1^{(1)}, (M^n)^{(1)}) \cap N(K_2^{(1)}, (M^n)^{(1)})$.

Thus the intersection is the boundary of a regular neighborhood in M^n, so that it is a bicollared (n-1)-manifold. Since $N(K_1^{(1)}, (M^n)^{(1)}) = S(x_t, (M^n)^{(1)})$ and $N(K_2^{(1)}, (M^n)^{(1)}) = T(x_t, (M^n)^{(1)})$, we are done. □

Section 3: PL balls

To apply Hudson's isotopy theorem [4, Corollary 3.1] in my proof, I need to show that my map is a locally unknotted isotopy. Results in this section are used for that purpose.

III.3.1. Definition [4 , page 632]. (A,B) is a (q,m)-ball pair if A is a PL q-ball, B is an embedded PL m-ball and either $\dot{A}\cap B=\dot{B}$ (type 1) or $\dot{A}\cap B$ is an (m-1)-ball in $\dot{B}$ (type 2).

We define standard (q,m)-ball pairs as follows:

Type 1. A q-simplex and the join of the barycenter to the boundary of an m-face.

Type 2. A q-simplex together with the join of the barycenter to an (m-1)-face.

A ball pair is unknotted if it is PL homeomorphic to a standard ball pair.

III.3.2. Notation. Let σ be a simplex; then $\Sigma\sigma$ is the suspension of σ and $C(\sigma)$ is the cone over σ.

III.3.3. Definition [14 , Chapter IV, page 1]. The standard (q,m)-ball pair is $\Delta^{q,m} = (\Sigma^{q-m}\Delta^m,\Delta^m)$, where Δ^m is the standard m-simplex, and Σ^{q-m} denotes (q-m)-fold suspension.

This is clearly the same as the standard (q,m)-ball pair, type 1, of Definition III.3.1.

* III.3.4. Lemma. If (A,B) is an unknotted (d,d-1)-ball pair,

type 1 (type 2), and σ is an s-simplex, then $(A\times\sigma, B\times\sigma)$ is an unknotted $(d+s, d+s-1)$-ball pair, type 1 (type 2).

* III.3.5. Lemma. Let K^d, L^d, M^d be d-manifolds such that K^d and L^d are subcomplexes of M^d, $K^d \cup L^d = M^d$, $K^d \cap L^d = (K\cap L)^{d-1}$, a (d-1)-manifold contained in $\partial K^d \cap \partial L^d$, and $\partial M^d \cap (K\cap L)^{d-1}$ is a (d-2)-manifold. Then, for every vertex v in $(K\cap L)^{d-1}$, $(\overline{st}(v,M^d), \overline{st}(v,(K\cap L)^{d-1}))$ is an unknotted ball pair, type 1.

Proof. $\overline{st}(v,M^d) = \overline{st}(v,K^d) \cup \overline{st}(v,L^d)$;

$$\overline{st}(v,(K\cap L)^{d-1}) = \overline{st}(v,K^d) \cap \overline{st}(v,L^d);$$

$$\overline{st}(v,(K\cap L)^{d-1}) \subset (K\cap L)^{d-1} \subset \partial K^d \cap \partial L^d.$$

Therefore $\overline{st}(v,(K\cap L)^{d-1})$ is a face of $\overline{st}(v,K^d)$ and $\overline{st}(v,L^d)$.

There is a homeomorphism $h_1: \overline{st}(v,K^d) \longrightarrow \sigma_d$, a d-simplex, such that $h_1|_{\overline{st}(v,(K\cap L)^{d-1})}$ is a homeomorphism taking $\overline{st}(v,(K\cap L)^{d-1})$ to σ_{d-1}, a face of σ_d.

Let τ_d be a d-simplex such that σ_{d-1} is a face of τ_d and such that $\sigma_d \cap \tau_d = \sigma_{d-1}$.

Extend the homeomorphism $h_1|_{\overline{st}(v,(K\cap L)^{d-1})}$ to a homeomorphism

$$h_2: \overline{st}(v,L^d) \longrightarrow \tau_d.$$

Define $h: \overline{st}(v,M^d) \longrightarrow \sigma_d \cup \tau_d$ by

$$h|_{\overline{st}(v,K^d)} = h_1: \overline{st}(v,K^d) \longrightarrow \sigma_d,$$

$$h|_{\overline{st}(v,L^d)} = h_2: \overline{st}(v,L^d) \longrightarrow \tau_d.$$

The definitions agree on the intersection, which is a closed set, so h is a well-defined map. h is 1-1 since $\sigma_d \cap \tau_d = \sigma_{d-1}$.

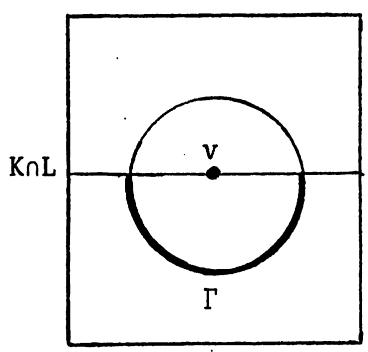
h is PL by Lemma II.21.

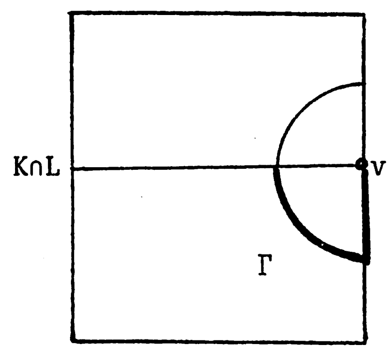
$(h, h|_{\overline{st}(v,(K\cap L)^{d-1})}) : (\overline{st}(v,M^d), \overline{st}(v,(K\cap L)^{d-1}) \longrightarrow (\sigma_d \cup \tau_d, \sigma_{d-1})$ is a map of pairs. $(\sigma_d \cup \tau_d, \sigma_{d-1})$ is homeomorphic to $(\Sigma\sigma_{d-1}, \sigma_{d-1})$ and is therefore an unknotted ball pair, type 1. Therefore $(\overline{st}(v,M^d), \overline{st}(v,(K\cap L)^{d-1})$ is an unknotted ball pair, type 1. □

* III.3.6. <u>Lemma</u>. Let K, L, M, K∩L be as in Lemma III.3.5, and let Γ be a (d-1)-manifold contained in K∩L such that $\Gamma \cap \partial(K\cap L)$ is a (d-2)-manifold. If $v \in \partial\Gamma$, then $(\overline{st}(v,K\cap L), \overline{st}(v,\Gamma))$ is PL homeomorphic to $(\Sigma\sigma, C(\sigma))$, where the cone point is one of the suspension points.

Part of the following proof is due to M. Cohen and replaces a longer proof of mine.

Proof. $\overline{st}(v,K\cap L) = D^{d-1}$, a (d-1)-ball;
$\overline{st}(v,\Gamma) = A^{d-1}$, a (d-1)-ball.





$A^{d-2} = \partial A^{d-1} \cap \partial D^{d-1} = lk(v,\Gamma) \cup \overline{st}(v, \Gamma\cap\partial(K\cap L))$. If $v \in int(K\cap L)$, then $\overline{st}(v, \Gamma\cap\partial(K\cap L)) = \phi$ and $A^{d-2} = lk(v,\Gamma)$, a (d-2)-ball. If $v \in \partial(K\cap L)$, then $\overline{st}(v, \Gamma\cap\partial(K\cap L))$ is a (d-2)-ball.

$\mathrm{lk}(v,\Gamma) \cap \overline{\mathrm{st}}(v, \Gamma\cap\partial(K\cap L)) = \mathrm{lk}(v, \Gamma\cap\partial(K\cap L))$, a (d-3)-ball. Therefore A^{d-2} is a (d-2)-ball.

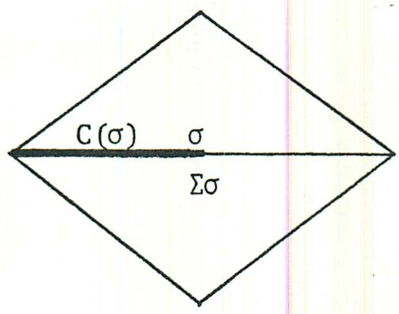
Choose a cone point $p \notin D^{d-1}$. $D^{d-1} \cup C(\partial D^{d-1}) = S^{d-1}$, a (d-1)-sphere. $D^{d-1} \cap C(\partial D^{d-1}) = \partial D^{d-1}$; $C(\partial D^{d-1})$ is a (d-1)-ball. $A^{d-1} \cap C(\partial D^{d-1}) = A^{d-1} \cap \partial D^{d-1} = \partial A^{d-1} \cap \partial D^{d-1} = A^{d-2}$. Therefore $A^{d-1} \cup C(\partial D^{d-1})$ is a (d-1)-ball. $\overline{D^{d-1}-A^{d-1}} = \overline{S^{d-1}-(A^{d-1} \cup C(\partial D^{d-1}))}$, a (d-1)-ball [14 , Chapter III, Theorem 3]. Also, ∂A^{d-1} is a (d-2)-sphere and $A^{d-2} \subset \partial A^{d-1}$, so $\overline{\partial A^{d-2}-A^{d-2}}$ is a (d-2)-ball [14 , Chapter III, Theorem 3]. Therefore $A^{d-1} \cap (\overline{D^{d-1}-A^{d-1}}) = \partial A^{d-1}-A^{d-2}$ is a (d-2)-ball.

Using methods from the proof of Lemma III.3.5, we find $h\colon (D^{d-1}, A^{d-1}, A^{d-1}\cap(\overline{D^{d-1}-A^{d-1}})) \longrightarrow (\sigma^{d-1} \cup \tau^{d-1}, \sigma^{d-1}, \sigma^{d-2})$, where $\sigma^{d-2} = \sigma^{d-1} \cap \tau^{d-1}$. Since $\sigma^{d-1} \cup \tau^{d-1}$ is PL homeomorphic to $\Sigma\sigma^{d-2}$ and σ^{d-1} is PL homeomorphic to $C(\sigma^{d-1})$, we are done. □

* III.3.7. Lemma. If K, L, M, $K\cap L$, and Γ are as in Lemma III.3.6, then, for every vertex v in Γ, $(\overline{\mathrm{st}}(v,M), \overline{\mathrm{st}}(v,\Gamma))$ is an unknotted ball pair.

Proof. If $v \in \mathrm{int}\,\Gamma$ then $\overline{\mathrm{st}}(v,\Gamma) = \overline{\mathrm{st}}(v,K\cap L)$ and Lemma III.3.5 yields the desired result.

If $v \in \partial\Gamma$, $(\overline{\mathrm{st}}(v,K\cap L), \overline{\mathrm{st}}(v,\Gamma))$ is PL homeomorphic to $(\Sigma\sigma, C(\sigma))$ [Lemma III.3.6]. By the proof of Lemma III.3.5, $\overline{\mathrm{st}}(v,M)$ is PL homeomorphic to $\Sigma(\overline{\mathrm{st}}(v,K\cap L))$. Therefore $(\overline{\mathrm{st}}(v,M), \overline{\mathrm{st}}(v,K\cap L), \overline{\mathrm{st}}(v,\Gamma))$ is PL homeomorphic to $(\Sigma^2\sigma, \Sigma\sigma, C(\sigma))$.



$\Sigma^2\sigma$

By techniques of the proof of Lemma A.2.8, $(\Sigma^2\sigma, C(\sigma))$ is an unknotted ball pair, type 2. □

III.3.8. Definition. If M, N are PL manifolds and $N \subset M$, then $\text{int}_M N$, the interior of N relative to M, is the largest open set *in M* contained in N.

$\text{bdy}_M N$, the boundary of N relative to M, is equal to $\partial N - \partial M$.

$\text{cl}_M N$, the closure of N relative to M, is the smallest closed set *in M* containing N.

* III.3.9. Lemma. If B^{n-1} is a face of an n-ball B^n contained in a PL n-manifold M^n and $B^{n-1} \cap \partial M^n \subset \partial B^{n-1}$, then there is a PL n-ball $B'^n \subset M^n$ such that $B'^n \cap B^n = B^{n-1}$ and such that if $x \in \text{int}_{M^n} B^{n-1}$, then $B'^n \cup B^n$ is a neighborhood of x in M^n.

Proof. Let $a \in \text{int}\, B^n$. Then B^n is a regular neighborhood of a. Let N be a regular neighborhood of B^n in M^n. Apply [9 , Exercise 6.4.5, page 129] with X={a} to conclude that $N - B^n$ is PL homeomorphic to $\text{bdy}_{M^n} B^n \times I$ by ψ.

Define $B'^n = \psi^{-1}(B^{n-1} \times I)$. □

Section 4. PL n-isotopies and PL structures

None of the work in this section is original; it is a selection of definitions and results from J. F. P. Hudson, all of which are needed in my proof.

III.4.1. Notation [4 , page 631]. Let I^n denote the n-cube; that is, the subset of Euclidean n-space with coordinates t_i satisfying $0 \le t_i \le 1$ for $i=1,\dots,n$.

Let M be a compact PL m-manifold in a PL q-manifold Q, not necessarily compact.

III.4.2. Definition [4 , page 631]. A PL n-isotopy of M in Q is a PL embedding $i\colon M \times I^n \longrightarrow Q \times I^n$ which commutes with projections onto the second factor. So, for each $t \in I^n$, there is an embedding $i_t\colon M \longrightarrow Q$ given by the relation $i(x,t) = (i_t x, t)$ for all x in M.

III.4.3. Definition [4 , page 631]. An ambient PL n-isotopy of Q is a PL homeomorphism $h\colon Q \times I^n \longrightarrow Q \times I^n$ which commutes with the projections onto the second factor and is such that $h_0\colon Q \longrightarrow Q$ is the identity (0 being the origin).

III.4.4. Definition [4 , page 632]. A PL n-isotopy i of M in Q is allowable if, for some PL (m-1)-submanifold N of ∂M, $i_t^{-1}(\partial Q)=N$ for all t in I^n. N may be empty and it may be the whole of ∂M. A PL embedding $i\colon M \longrightarrow Q$ is allowable if $i^{-1}(Q)$ is a PL (m-1)-submanifold of ∂M.

III.4.5. Definition [4 , page 632]. If $i: M \longrightarrow Q$ is an allowable embedding, i is locally unknotted at a point x of M if there are closed neighborhoods A, B of $i(x)$ in Q, $i(M)$, respectively, such that (A,B) is an unknotted ball pair.

If $i: M \longrightarrow Q$ is an allowable embedding, i is locally unknotted if it is locally unknotted at every point of M.

An allowable n-isotopy $i: M\times I^n \longrightarrow Q\times I^n$ is locally unknotted if, for every simplex σ linearly embedded in I^n, $0 \le \dim \sigma \le n$, $i|_{M\times\sigma}: M\times\sigma \longrightarrow Q\times\sigma$ is a locally unknotted embedding.

III.4.6. Lemma [4 , Corollary 3.1]. If $i: M\times I^n \longrightarrow Q\times I^n$ is an allowable locally unknotted n-isotopy, then i may be extended to an ambient n-isotopy of Q.

III.4.7. Definition [5 , page 76]. Let X be a topological space. A coordinate map (f,P) is a topological embedding $f: P \longrightarrow X$ of a Euclidean polyhedron P. Two such maps (f,P) and (g,Q) are compatible provided that if $f(P) \cap g(Q) \neq \phi$ there exists a coordinate map (h,R) such that $h(R) = g(Q) \cap f(P)$ and $f^{-1}h$ and $g^{-1}h$ are PL maps. Equivalently, we say that (f,P) and (g,Q) are compatible if $f^{-1}(g(Q))$ is a sub-polyhedron of P and $g^{-1}f: f^{-1}(g(Q)) \longrightarrow Q$ is a PL map.

III.4.8. Definition [5 , page 77]. A PL structure F on X is a family of coordinate maps such that

(i) Any two elements of F are compatible.

(ii) For all $x \in X$, there exists $(f,P) \in F$ such that $f(P)$ is a topological neighborhood of x in X.

(iii) $\mathcal{F}$ is maximal; i.e., if (f,P) is compatible with every map of $\mathcal{F}$, then $(f,P) \in \mathcal{F}$.

If X is a second countable Hausdorff space, the pair $(X,\mathcal{F})$ is called a PL space. $(X,\mathcal{F})$ will sometimes be referred to as X.

III.4.9. Definition [5 , page 79]. The PL space $(X,\mathcal{F})$ is called a PL m-manifold if for all $x \in X$ there exists $h: \Delta^m \longrightarrow X$ with $(h,\Delta^m) \in \mathcal{F}$ and $x \in \text{int}_X\, h(\Delta^m)$.

III.4.10. Lemma [5 , Lemma 3.4]. If $(X,\mathcal{F})$ is a PL m-manifold and $C \subset X$ is compact, then there exists $(h,R) \in \mathcal{F}$ such that

(i) R is a combinatorial m-manifold, and

(ii) $C \subset \text{int}_X\, h(R)$.

Chapter IV

PL Fibre Bundles

This chapter contains the proof of the main theorem of this paper.

IV.1. Definition [8 , page 319]. A function f from a space X to a space Y is said to be proper if it is continuous and if for every bounded set A of Y, $f^{-1}(A)$ is a bounded set of X.

The property of proper maps which I need is that for every compact set C of Y, $f^{-1}(C)$ is a compact subset of X.

IV.2. Definition [8 , page 90]. A fibre bundle $\xi = (E,B,F,p)$ consists of a total space E, a base space B, a fibre F and a bundle projection $p: E \longrightarrow B$ such that there exists an open covering $\{U\}$ of B and, for each $U \in \{U\}$, a homeomorphism $\psi_U: U \times F \longrightarrow p^{-1}(U)$ such that the composite

$$U \times F \xrightarrow{\psi_U} p^{-1}(U) \xrightarrow{p} U$$

is the projection onto the first factor.

IV.3. Definition. A PL fibre bundle is defined as the fibre bundle of IV.2 with spaces and maps replaced by PL spaces and PL maps.

IV.4. Definition [11 , 1.1]. Let f be a map of M^n (possibly with boundary) into N^p. The branch set $B_f \subset M^n$ is defined by: x is an element of $M^n - B_f$ if and only if f at x is locally topologically equivalent to the natural projection map of E^n or E^n_+ onto E^p, where E^n_+ is (closed) Euclidean half-space.

I modified this definition to define the branch set for a PL map.

IV.5. Definition. If $f: M \longrightarrow N$ is PL then the branch set of f, $B_f \subset M$, can be defined as follows: $x \in M-B_f$ if there exists a neighborhood U_x of x such that $f(U_x) = V_x$ is a closed star neighborhood of f(x), and if there exists a PL homeomorphism $h_x: U_x \longrightarrow B_x \times V_x$, where B_x is a PL ball, such that

$$f|_{U_x} = \pi_2 \circ h_x: U_x \longrightarrow V_x.$$

U_x is called a trivializing neighborhood of x and h_x a trivializing homeomorphism.

* IV.6. Theorem. Let M^m, N^k be PL manifolds and let $f: M^m \longrightarrow N^k$ be a proper PL map with $B_f = \phi$. Then for every y in $f(M^m)$ there is a neighborhood N_y of y and a PL homeomorphism $g_y: f^{-1}(N_y) \longrightarrow f^{-1}(y) \times N_y$ such that

$$f|_{f^{-1}(N_y)} = \pi_2 \circ g_y: f^{-1}(N_y) \longrightarrow N_y.$$

Thus, if N^k is connected, f is the projection map of a PL fibre bundle.

Proof sketch. The proof is done by building the neighborhood N_y and the PL homeomorphism g_y as follows.

In Section 1, I subdivide the fibre so that the star of any

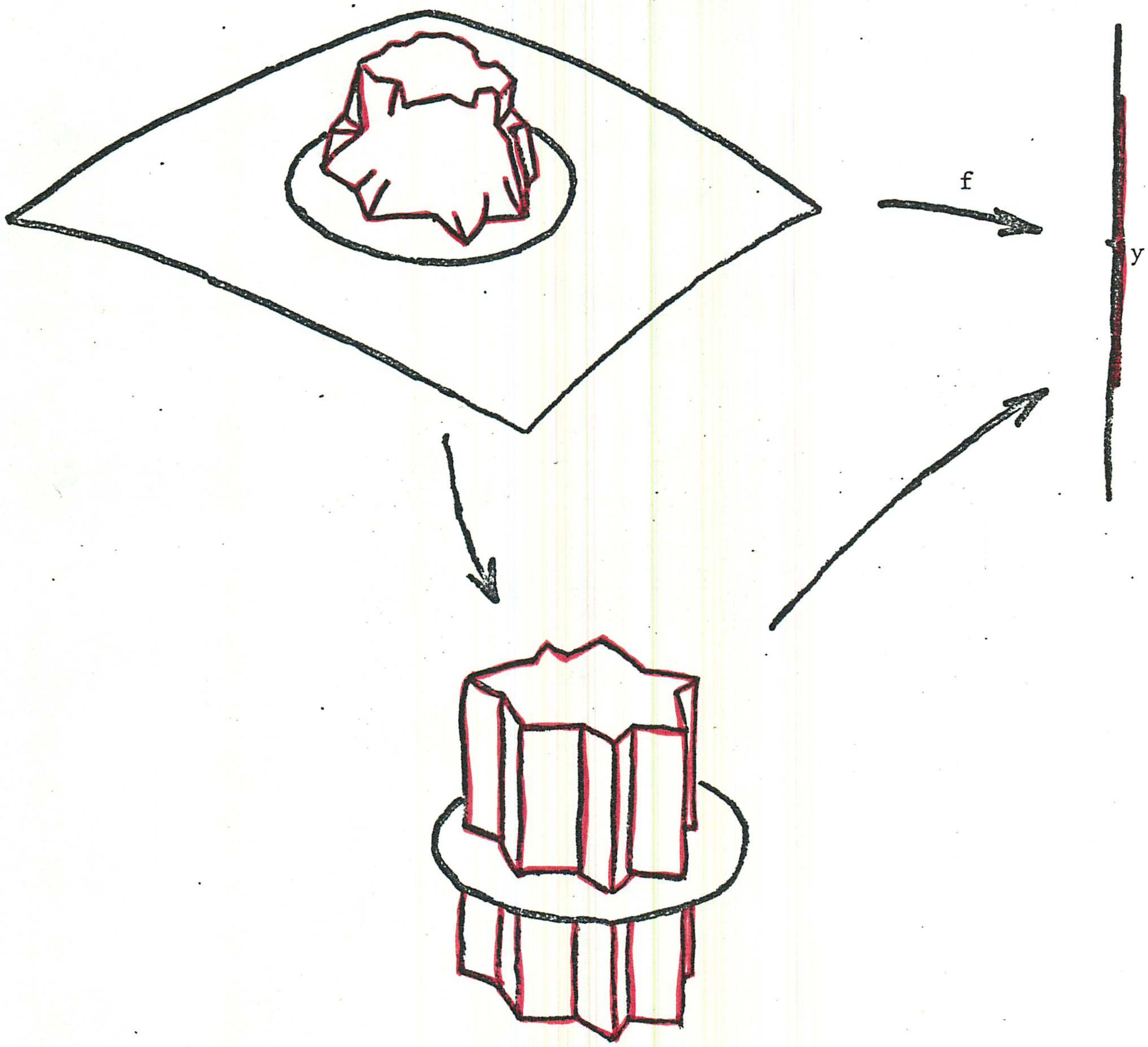


Figure 1

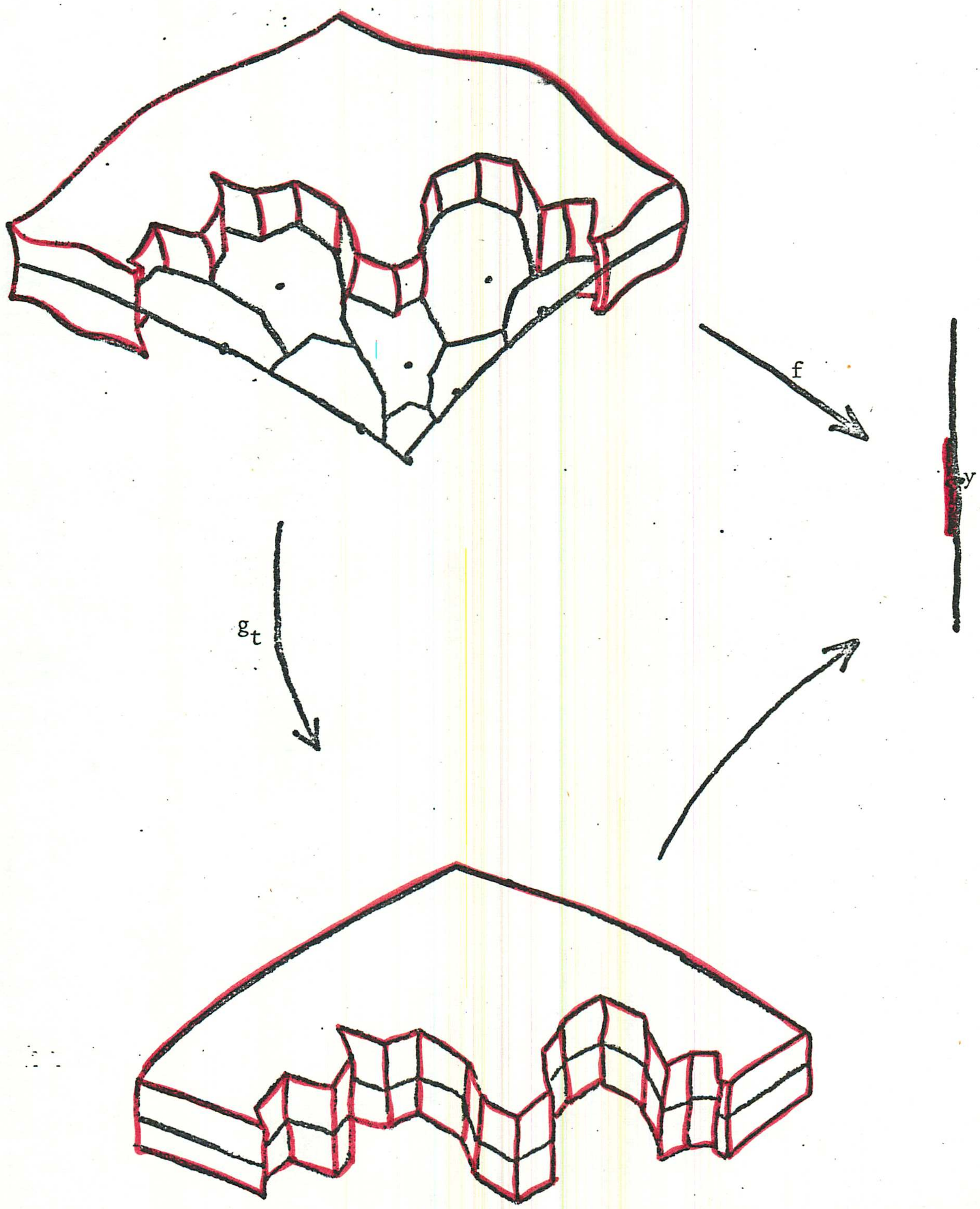


Figure 2

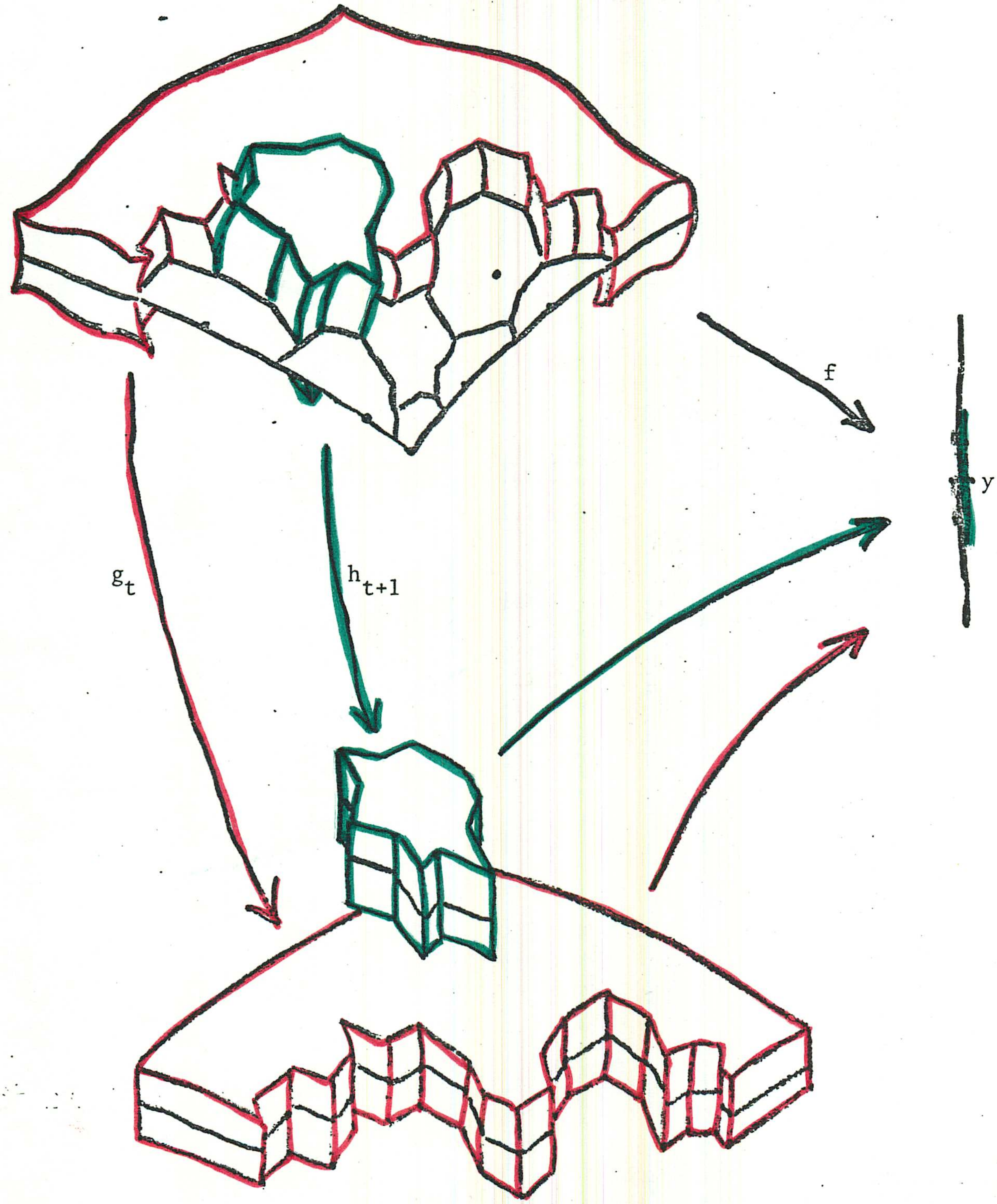


Figure 3

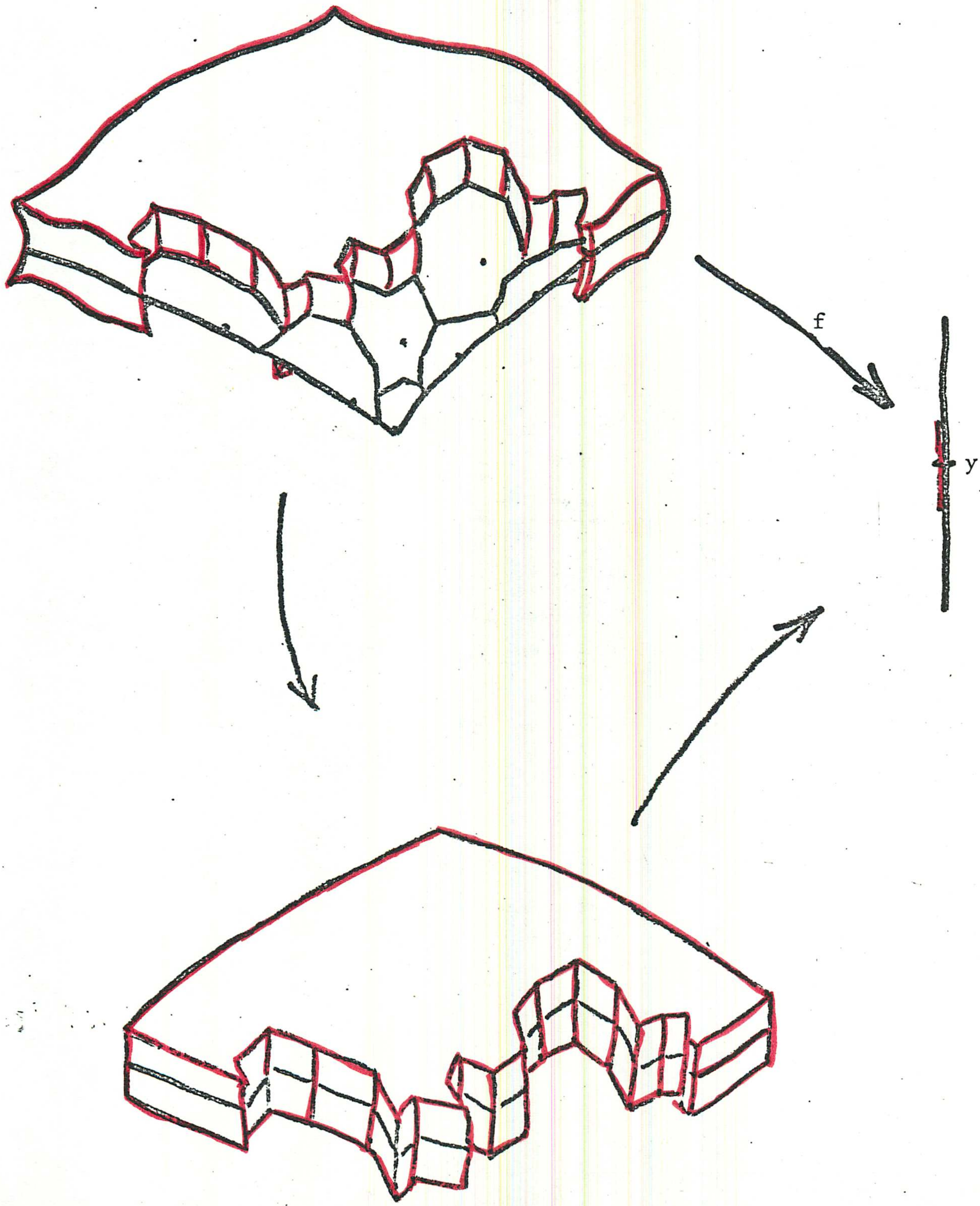


Figure 4

vertex is contained in a trivializing neighborhood [IV.5]. Now I take the stars of these vertices in a derived subdivision. These stars then have the property that they cover the fibre and fit together nicely.

In Section 2, I order the stars and, using a restriction of a trivializing homeomorphism [IV.5.], define an appropriate PL homeomorphism on a "block" containing the first star. (See Figure 1.)

In Section 3, assuming a map g_t defined on a "block" containing the first t stars (Figure 2), and a map h_{t+1} on a "block" over the $(t+1)^{st}$ star (Figure 3), I use an isotopy extension theorem of J. F. P. Hudson [4 , Corollary 3.1] to "extend" g_t to a PL homeomorphism on a "block" containing the first t+1 stars (Figure 4).

In Section 4, I show that, when the function has been extended to a "block" containing the fibre, the "block" can be shrunk so that the function does satisfy the conclusion of the theorem.

Proof.

<u>Section 1</u>. Let $y \in f(M^m)$. Since f is a proper PL map, $f^{-1}(y)$ is a compact subset of M^m and [III.4.10] there exists $(h,R) \in \mathcal{F}$ [III.4.8] with (i) R a combinatorial m-manifold, and

(ii) $f^{-1}(y) \subset \operatorname{int}_{M^m} h(R)$.

We shall identify R with h(R) and call h(R) a combinatorial manifold. Then $f^{-1}(y)$ is a combinatorial manifold.

Let $\operatorname{int}_{M^m} U_1 \cup \ldots \cup \operatorname{int}_{M^m} U_n$ be a cover of $f^{-1}(y)$. Suppose $f(U_i) = V_i$, a neighborhood of y, and suppose that there are PL balls

(1.1) $$B_i = f^{-1}(y) \cap U_i$$

and PL homeomorphisms $h_i\colon U_i \longrightarrow B_i \times V_i$ satisfying the following commutative diagrams:

(A) $$\begin{array}{ccc} U_i & \xrightarrow{f} & V_i \\ {\scriptstyle h_i}\downarrow & \nearrow {\scriptstyle \pi_2} & \\ B_i \times V_i & & \end{array}$$

h_i is chosen so that for each u in $f^{-1}(y) \cap U_i$,

(1.2) $$h_i(u) = (u,y).$$

This can be done, since $B_f = \phi$ and $f^{-1}(y)$ is compact.

Let $(f^{-1}(y))_1$ be a subdivision of $f^{-1}(y)$ which is so fine that for all vertices $x \in (f^{-1}(y))_1$, $\overline{st}(x,(f^{-1}(y))_1) \subset \text{int}_{M^m} U_i$ for some i.

Let $(f^{-1}(y))_2$ be a first derived subdivision of $(f^{-1}(y))_1$.

Let $v_1, \ldots, v_\nu$ be the vertices of $(f^{-1}(y))$. Define a function $\theta\colon \{1,\ldots,\nu\} \longrightarrow \{1,\ldots,n\}$ by $\theta(j) = \min\{i \mid \overline{st}(v_j,(f^{-1}(y))_2) \subset \text{int}_{M^m} U_i\}$. Then $\overline{st}(v_j,(f^{-1}(y))_2) \subset \text{int}_{M^m} U_{\theta(j)} \cap (f^{-1}(y))_2$. Therefore $h_{\theta(j)}$ is defined on $\overline{st}(v_j,(f^{-1}(y))_2)$ and

$$h_{\theta(j)}(\overline{st}(v_j,(f^{-1}(y))_2)) = \overline{st}(v_j,(f^{-1}(y))_2) \times \{y\} \subset \text{int}_{f^{-1}(y)} B_{\theta(j)} \times \{y\}.$$

Identify $\overline{st}(v_j,(f^{-1}(y))_2)$ with

$h_{\theta(j)}(\overline{st}(v_j,(f^{-1}(y))_2) = \overline{st}(v_j,(f^{-1}(y))_2) \times \{y\}$ and, under this identification, say

(1.3) $$\overline{st}(v_j,(f^{-1}(y))_2) \subset \text{int}_{f^{-1}(y)} B_{\theta(j)}.$$

Let

(1.4) $$K_t = S(v_t,(f^{-1}(y))_2) \text{ and } L_t = T(v_t,(f^{-1}(y))_2) \qquad [\text{II}.17].$$

Section 2. Let $W_1 = V_{\theta(1)}$ and recall $K_1 = \overline{st}(v_1,(f^{-1}(y))_2)$. Let $N_1 = h^{-1}_{\theta(1)}(\overline{st}(v_1,(f^{-1}(y))_2) \times W\ )$ with a triangulation that subdivides its cell structure.

Let $g_1 \colon N_1 \longrightarrow K_1 \times W_1$ be defined by

$$g_1 = h_{\theta(1)}|_{N_1} \colon N_1 \longrightarrow K_1 \times W_1 .$$

Then g_1 is a PL [II.20] homeomorphism which satisfies the following commutative diagram:

(B)

$$\begin{array}{ccc} N_1 & \xrightarrow{f} & W_1 \\ {\scriptstyle g_1}\downarrow & \nearrow {\scriptstyle \pi_2} & \\ K_1 \times W_1 & & \end{array}$$

and for each $u \in f^{-1}(y) \cap N_1$, $g_1(u) = (u,y)$. (See Figure 1.)

Section 3. Assume a PL homeomorphism $g_t \colon N_t \longrightarrow K_t \times W_t$ ($N_t \subset M^m$, $N_t \cap f^{-1}(y) = K_t$, W_t a neighborhood of y) such that the following diagram is commutative:

(C)

$$\begin{array}{ccc} N_t & \xrightarrow{f} & W_t \\ {\scriptstyle g_t}\downarrow & \nearrow {\scriptstyle \pi_2} & \\ K_t \times W_t & & \end{array}$$

and such that for each u in $f^{-1}(y) \cap N_t$,

(3.1) $$g_t(u) = (u,y).$$ (See Figure 2.)

Construct N_{t+1} and a PL homeomorphism $g_{t+1} \colon N_{t+1} \longrightarrow K_{t+1} \times W_{t+1}$, where W_{t+1} is a neighborhood of y, $N_{t+1} \cap f^{-1}(y) = K_{t+1}$, g_{t+1} satisfies the following commutative diagram:

(D)
$$\begin{array}{ccc} N_{t+1} & \xrightarrow{f} & W_{t+1} \\ g_{t+1} \downarrow & \nearrow \pi_2 & \\ K_{t+1} \times W_{t+1} & & \end{array}$$

and, for each $u \in f^{-1}(y) \cap N_{t+1}$, $g_{t+1}(u) = (u,y)$.

In doing this, I use the PL homeomorphism $h_{\theta(t+1)}: U_{\theta(t+1)} \longrightarrow B_{\theta(t+1)} \times V_{\theta(t+1)}$ to build g_{t+1}.

First I create the situation pictured in Figure 3; then I use Hudson's theorem [4, Corollary 3.1] to build the desired map (as pictured in Figure 4). In the next few paragraphs, I adjust the situation and find a map G_{t+1} to which I can apply Hudson's theorem.

The homeomorphism $h_{\theta(t+1)}$ satisfies the following commutative diagram:

(E)
$$\begin{array}{ccc} U_{\theta(t+1)} & \xrightarrow{f} & V_{\theta(t+1)} \\ h_{\theta(t+1)} \downarrow & \nearrow \pi_2 & \\ B_{\theta(t+1)} \times V_{\theta(t+1)} & & \end{array}$$

(3.2) $\quad g_t^{-1}((K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)) \times \{y\}) \subset \operatorname{int}_{M^m} U_{\theta(t+1)}$.

Therefore $\exists V_{t+1}$, a neighborhood of y, $V_{t+1} \subset V_{\theta(t+1)}$, such that $g_t^{-1}((K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)) \times V_{t+1}) \subset U_{\theta(t+1)} = h_{\theta(t+1)}^{-1}(B_{\theta(t+1)})$.

Let M_{t+1} be a triangulation of the cell complex $h_{\theta(t+1)}^{-1}((\overline{st}(v_{t+1},(f^{-1}(y))_2) \times V_{t+1}))$, and

(3.3) $\quad h_{t+1} = h_{\theta(t+1)}|_{M_{t+1}} : M_{t+1} \longrightarrow \overline{st}(v_{t+1},(f^{-1}(y))_2) \times V_{t+1}$.

Then $M_{t+1} \cap f^{-1}(y) = \overline{st}(v_{t+1},(f^{-1}(y))_2)$.

Restrict diagram (E) to:

(F)
$$\begin{array}{ccc} M_{t+1} & \xrightarrow{f} & V_{t+1} \\ h_{t+1} \downarrow & \nearrow \pi_2 & \\ \overline{st}(v_{t+1},(f^{-1}(y))_2) \times V_{t+1} & & \end{array}$$

This restriction of $h_{\theta(t+1)}$ is PL [II.20].

Let $(S(v_t,(f^{-1}(y))_2))_2$ be a second derived subdivision of $S(v_t,(f^{-1}(y))_2)$ [II.14].

Let $(K_t \cap L_t)_2$ be a second derived subdivision of $K_t \cap L_t$ and

(3.4) $\Gamma_t = N(S(v_t,(f^{-1}(y))_2) \cap \overline{st}(v_{t+1},(f^{-1}(y))_2), \quad (K_t \cap L_t)_2)$,

a derived neighborhood [III.2.8] in the (d-1)-manifold $K_t \cap L_t$ [III.2.10].

We shall temporarily drop the subscript t from Γ_t, K_t, L_t for ease in notation.

Since Γ is a regular neighborhood [14 , Chapter III, Theorem 8] in a (d-1)-manifold, Γ is a (d-1)-manifold [14 , Chapter III, page 22].

Since Γ is a derived neighborhood, Γ meets the boundary regularly [5, page 65], and $\Gamma \cap \partial(K \cap L)$ is a regular neighborhood of $S(v_t,(f^{-1}(y))_2 \cap \overline{st}(v_{t+1},(f^{-1}(y))_2 \cap \partial(K \cap L)$ in $\partial(K \cap L)$, a (d-2)-manifold [III.2.10].

Therefore $\Gamma \cap \partial(K \cap L)$ is a (d-2)-manifold [14 , Chapter III, page 22].

<u>Claim</u>: $\Gamma \cap \partial(K \cap L) = \Gamma \cap \partial(f^{-1}(y))$.

True because $\Gamma \cap \partial(K \cap L) \subset \Gamma \cap \partial(f^{-1}(y))$ and

(3.5) $\Gamma \cap \partial(f^{-1}(y)) \subset K \cap L \cap \partial(f^{-1}(y)) \subset \partial(K \cap L)$.

Therefore

(3.6) $\Gamma \cap (\partial(f^{-1}(y)))_4$ is a (d-2)-manifold.

By (3.5), $\Gamma \cap (\partial(f^{-1}(y)))_4 \subset \partial(K \cap L) \cap \Gamma \subset \partial\Gamma$ since Γ and $K \cap L$ are both (d-1)-manifolds. In fact, $(\partial(f^{-1}(y)))_4$ is a subcomplex of $\partial\Gamma$; so

(3.7) $$\Gamma \cap (\partial(f^{-1}(y)))_4 \text{ is a submanifold of } \partial\Gamma.$$

$$|\Gamma| \subset |K_t| = |S(v_t, (f^{-1}(y))_2)|.$$

Therefore $|\Gamma \times W'_{t+1}| \subset |S(v_t, (f^{-1}(y))_2) \times W_t| = |g_t(N_t)|$ and g_t^{-1} is defined on $\Gamma \times W'_{t+1}$.

Also, $$\begin{aligned} |\Gamma| &\subset |N(S(v_t, (f^{-1}(y))_2) \cap \overline{st}(v_t, (f^{-1}(y))_2), K_t)| \\ &\subset |\overline{st}(v_{t+1}, (f^{-1}(y))_1)| \\ &\subset |\operatorname{int}_{f^{-1}(y)} B_{\theta(t+1)}| \qquad [(1.3)]. \end{aligned}$$

Since $g_t(u) = (u,y)$ when $u \in f^{-1}(y)$,

$$g_t^{-1}(\Gamma \times \{y\}) = \Gamma \subset \operatorname{int}_{f^{-1}(y)} B_{\theta(t+1)} \subset \operatorname{int}_{M^m} U_{\theta(t+1)}.$$

Therefore there is a neighborhood

(3.8) $$W''_{t+1} \subset W_t \cap V_{t+1}$$

such that

(3.9) $$g_t^{-1}(\Gamma \times W''_{t+1}) \subset \operatorname{int}_{M^m} U_{\theta(t+1)}$$

and such that

(3.10) $$W''_{t+1} \text{ is PL homeomorphic to } I^k.$$

Define $G_{t+1}\colon \Gamma \times W''_{t+1} \longrightarrow B_{\theta(t+1)} \times V_{\theta(t+1)}$ by $G_{t+1} = h_{\theta(t+1)} \circ g_t^{-1}$. Since g_t^{-1} and $h_{\theta(t+1)}$ satisfy diagrams (C) and (E) respectively, $G_{t+1}(\Gamma \times W''_{t+1}) \subset B_{\theta(t+1)} \times W''_{t+1}$.

(G) $$\begin{array}{ccccc} & & U_{\theta(t+1)} & & \\ & {\scriptstyle g_t^{-1}} \nearrow & & \searrow {\scriptstyle h_{\theta(t+1)}} & \\ \Gamma \times W''_{t+1} & & \xrightarrow[G_{t+1}]{} & & B_{\theta(t+1)} \times W''_{t+1} \end{array}$$

We shall now show that G_{t+1} satisfies the hypotheses of the corollary to Hudson's isotopy theorem [4, Corollary 3.1].

These are:

1) Γ is a compact PL (d-1)-manifold.

2) $B_{\theta(t+1)}$ is a PL manifold.

3) G_{t+1} is a PL k-isotopy [III.4.2].

4) G_{t+1} is allowable [III.4.4].

5) G_{t+1} is locally unknotted [III.4.5].

1) Γ is compact because it is defined as a closed neighborhood in a compact manifold [(3.4)]. Γ is a PL (d-1)-manifold [(3.6)].

2) $B_{\theta(t+1)}$ is a PL d-ball [(1.1)].

3) Since g_t^{-1} and $h_{\theta(t+1)}$ satisfy diagrams (C) and (E), respectively, and W''_{t+1} is PL homeomorphic to I^k [(3.10)], $G_{t+1}: \Gamma \times W''_{t+1} \longrightarrow B_{\theta(t+1)} \times W''_{t+1}$ is a PL k-isotopy.

4)
$$G_{t+1}(\Gamma \times W''_{t+1}) = h_{\theta(t+1)} \circ g_t^{-1}(\Gamma \times W''_{t+1})$$
$$\subset h_{\theta(t+1)}(\mathrm{int}_{M^m} U_{\theta(t+1)}) \qquad [(3.9)]$$
$$\subset \mathrm{int}_{f^{-1}(y)} B_{\theta(t+1)} \times V_{\theta(t+1)}.$$

Therefore, since $G_{t+1}: \Gamma \times W''_{t+1} \longrightarrow B_{\theta(t+1)} \times W''_{t+1}$ is a PL k-isotopy, in particular G_{t+1} is an embedding, and must preserve boundary points: $(G_{t+1})_w^{-1}(\partial B_{\theta(t+1)}) = \Gamma \cap \partial f^{-1}(y)$, which is a (d-2)-submanifold of $\partial\Gamma$ [(3.7)]. Therefore G_{t+1} is an allowable isotopy [III.4.4].

5) To show that $G_{t+1}: \Gamma \times W''_{t+1} \longrightarrow B_{\theta(t+1)} \times W''_{t+1}$ is locally

unknotted [III.4.5], I must show that, for any s-simplex Σ linearly embedded in W''_{t+1} and any point $(\gamma,\sigma) \in \Gamma\times\Sigma$, there are closed neighborhoods A, B of $G_{t+1}(\gamma,\sigma)$ in $B_{\theta(t+1)} \times \Sigma$, $G_{t+1}(\Gamma\times\Sigma)$, respectively, such that (A,B) is an unknotted ball pair.

Suppose (γ,σ) is a point of $\Gamma\times\Sigma$ where Σ is linearly embedded in W''_{t+1}. Then there is a vertex v of Γ' such that $\overline{st}(v,\Gamma') \times \Sigma$ is a closed neighborhood of (γ,σ) in $\Gamma\times\Sigma$. Also, $\overline{st}(v,(f^{-1}(y))_5) \times \Sigma$ is a closed neighborhood of (γ,σ) in $f^{-1}(y) \times \Sigma$.

By Lemma III.3.7, $(\overline{st}(v,(f^{-1}(y))_5), \overline{st}(v,\Gamma'))$ is an unknotted (d,d-1)-ball pair.

By Lemma III.3.4, $(\overline{st}(v,(f^{-1}(y))_5) \times \Sigma, \overline{st}(v,\Gamma') \times \Sigma)$ is an unknotted (d+s,d+s-1)-ball pair.

We shall now build an appropriate PL ball pair (A,B) and a PL homeomorphism h_v: $(\overline{st}(v,(f^{-1}(y))_5) \times \Sigma, \overline{st}(v,\Gamma') \times \Sigma) \longrightarrow (A,B)$, showing thereby that (A,B) is unknotted.

$$\begin{aligned}\overline{st}(v,(f^{-1}(y))_5) &= \overline{st}(v,(K_t)_3 \cup (L_t)_3)\\ &= \overline{st}(v,(K_t)_3) \cup \overline{st}(v,(L_t)_3).\end{aligned}$$

$\overline{st}(v,(K_t)_3) \times \Sigma$ is a PL (d+s)-ball. $\overline{st}(v,(K_t)_3) \times \Sigma \subset K_t \times W''_{t+1}$

$$\subset K_t \times W_t \quad [(3.8)],$$

the domain of g_t^{-1} [diagram (C)].

$$\begin{aligned}g_t^{-1}(\overline{st}(v,(K_t)_3) \times \Sigma &\subset g_t^{-1}(\overline{st}(v,(K_t)_3) \times W''_{t+1})\\ &\subset g_t^{-1}(\overline{st}(v,(K_t)_3) \times V_{t+1}) && [(3.8)]\\ &\subset g_t^{-1}(\overline{st}(v_{t+1}, K_t) \times V_{t+1})\\ &\subset U_{\theta(t+1)} && [(3.2)]\\ &= h^{-1}_{\theta(t+1)}(B_{\theta(t+1)} \times V_{\theta(t+1)}).\end{aligned}$$

Therefore $h_{\theta(t+1)} \circ g_t^{-1}$ can be defined on $\overline{st}(v,(K_t)_3) \times \Sigma$.

Define $f_v = h_{\theta(t+1)} \circ g_t^{-1} \colon \overline{st}(v,(K_t)_3) \times \Sigma \longrightarrow B_{\theta(t+1)} \times \Sigma$.
Then f_v is an embedding, $f_v(\overline{st}(v,(K_t)_3) \times \Sigma)$ is a PL (d+s)-ball, and $f_v\colon \overline{st}(v,(K_t)_3) \times \Sigma \rightarrowtail\!\!\twoheadrightarrow f_v(\overline{st}(v,(K_t)_3) \times \Sigma)$ is a PL homeomorphism.

$\overline{st}(v,(K\cap L)_3) \times \Sigma$ is a face of $\overline{st}(v,(K_t)_3) \times \Sigma$ because $(K\cap L)_3$ is a (d-1)-manifold [III.2.10] contained in $\partial(K_t)_3$. Therefore $f_v(\overline{st}(v,(K\cap L)_3) \times \Sigma)$ is a face of $f_v(\overline{st}(v,(K_t)_3) \times \Sigma)$.

By Lemma III.3.9, there is a PL (d+s)-ball $B^{d+s} \subset B_{\theta(t+1)} \times \Sigma$ such that $B^{d+s} \cap f_v(\overline{st}(v,(K_t)_3) \times \Sigma) = f_v(\overline{st}(v,(K\cap L)_3) \times \Sigma)$.

Extend

$$f_v\Big|_{\overline{st}(v,(K\cap L)_3) \times \Sigma} \colon \overline{st}(v,(K\cap L)_3) \times \Sigma \longrightarrow f_v(\overline{st}(v,(K\cap L)_3) \times \Sigma)$$

to a PL homeomorphism $g_v\colon \overline{st}(v,(L_t)_3) \times \Sigma \longrightarrow B^{d+s}$ [14, Chapter III, Theorem 2].
Let $h_v\colon \overline{st}(v,(f^{-1}(y))_5) \times \Sigma \longrightarrow B_{\theta(t+1)} \times \Sigma$ be defined by

$$h_v\Big|_{\overline{st}(v,(K_t)_3) \times \Sigma} = f_v\colon \overline{st}(v,(K_t)_3) \times \Sigma \longrightarrow f_v(\overline{st}(v,(K_t)_3) \times \Sigma),$$

$$h_v\Big|_{\overline{st}(v,(L_t)_3) \times \Sigma} = g_v\colon \overline{st}(v,(L_t)_3) \times \Sigma \longrightarrow B^{d+s}.$$

Then h_v is a PL homeomorphism.

$$\left(h_v,\ h_v\Big|_{\overline{st}(v,\Gamma')\times\Sigma}\right)\colon (\overline{st}(v,(f^{-1}(y))_5)\times\Sigma,\ \overline{st}(v,\Gamma')\times\Sigma)$$

$$\longrightarrow (B^{d+1}\cup f_v(\overline{st}(v,(K_t)_3)\times\Sigma),\ f_v(\overline{st}(v,\Gamma')\times\Sigma))$$

is a map of pairs.

$$f_v|_{\Gamma\times\Sigma} = h_{\theta(t+1)}\circ g_t^{-1}|_{\Gamma\times\Sigma} = G_{t+1}|_{\Gamma\times\Sigma}.$$

By Lemma III.3.9, $B^{d+s} \cup f_v(\overline{st}(v,(K_t)_3) \times \Sigma)$ is a neighborhood of $f_v(\gamma,\sigma) = G_{t+1}(\gamma,\sigma)$ in $B_{\theta(t+1)} \times \Sigma$.

Since $f_v|_{\overline{st}(v,\Gamma')\times\Sigma}\colon \overline{st}(v,\Gamma')\times\Sigma \longrightarrow f_v(\overline{st}(v,\Gamma')\times\Sigma)$ is a PL homeomorphism, $f_v(\overline{st}(v,\Gamma') \times \Sigma) = G_{t+1}(\overline{st}(v,\Gamma') \times \Sigma)$ is a PL neighborhood of $f_v(\gamma,\sigma) = G_{t+1}(\gamma,\sigma)$ in $f_v(\Gamma'\times\Sigma) = G_{t+1}(\Gamma'\times\Sigma)$.

Since $(B^{d+s} \cup f_v(\overline{st}(v,(K_t)_3) \times \Sigma), f_v(\overline{st}(v,(K_t)_3) \times \Sigma))$ is PL homeomorphic to an unknotted ball pair, G_{t+1} is locally unknotted.

We have now proved that G_{t+1} satisfies 1)-5), the hypotheses of Hudson's isotopy extension theorem [4, Theorem 3].

We may now apply the corollary of the theorem to find an ambient k-isotopy

$$H'_{t+1}: B_{\theta(t+1)} \times W''_{t+1} \longrightarrow B_{\theta(t+1)} \times W''_{t+1}$$

extending G_{t+1}.

Since $H'_{t+1}: B_{\theta(t+1)} \times W''_{t+1} \longrightarrow B_{\theta(t+1)} \times W''_{t+1}$ is an ambient isotopy,

$H'_{t+1}|_{B_{\theta(t+1)}\times\{w'\}}: B_{\theta(t+1)} \times \{w'\} \longrightarrow B_{\theta(t+1)} \times \{w'\}$ is the identity,

where w' is the point in the boundary of W'_{t+1} corresponding to the point $(0,0,0,\ldots,0)$ of I^k.

$(H'_{t+1})_y: B_{\theta(t+1)} \longrightarrow B_{\theta(t+1)}$, defined by $(H'_{t+1})_y(b) = \pi_1 \circ H'_{t+1}(b,y)$, is a PL homeomorphism.

Define a layer preserving PL homeomorphism

$$H_{t+1}: B_{\theta(t+1)} \times W''_{t+1} \longrightarrow B_{\theta(t+1)} \times W''_{t+1}$$

by $H_{t+1} = ((H'_{t+1})_y^{-1} \times \mathrm{id}) \circ (H'_{t+1})$. Then

$$\begin{aligned} H_{t+1}(b,y) &= ((H'_{t+1})_y^{-1} \times \mathrm{id}) \circ (H'_{t+1})(b,y) \\ &= ((H'_{t+1})_y^{-1} \times \mathrm{id}) \circ ((H'_{t+1})_y \times \mathrm{id})(b,y) \\ &= (b,y). \end{aligned} \tag{3.11}$$

Let $\overline{U}_{t+1} = h^{-1}_{\theta(t+1)}(B_{\theta(t+1)} \times W''_{t+1})$. Recalling that $G_{t+1} = h_{\theta(t+1)} \circ g_t^{-1}|_{\Gamma\times W''_{t+1}}$, we now have the following commutative diagram:

(H)

$$\begin{array}{ccccc} \Gamma\times W''_{t+1} & \xrightarrow{g_t^{-1}|\Gamma\times W''_{t+1}} & \overline{U}_{t+1} & \xrightarrow{h_{\theta(t+1)}} & B_{\theta(t+1)}\times W''_{t+1} \\ & \searrow^{\text{inclusion}} & & & \downarrow H_{t+1} \\ & & & & B_{\theta(t+1)}\times W''_{t+1} \end{array}$$

Therefore $g_t^{-1}\big|_{\Gamma\times W''_{t+1}} = h_{\theta(t+1)}^{-1}\circ H_{t+1}^{-1}\circ\text{inclusion}\colon \Gamma\times W''_{t+1} \longrightarrow \overline{U}_{t+1}$,

or

(3.12) $$g_t^{-1}\big|_{\Gamma\times W''_{t+1}} = h_{\theta(t+1)}^{-1}\circ H_{t+1}^{-1}\big|_{\Gamma\times W''_{t+1}}.$$

Let

(3.13) $$\psi_{t+1} = h_{\theta(t+1)}^{-1}\circ H_{t+1}^{-1}.$$

(This is the map we need to "extend" g_t to a "block" over the $(t+1)^{st}$ star; see the proof sketch.)

$g_t^{-1}\colon K_t\times W''_{t+1} \longrightarrow M^m$ is a PL embedding.

$\psi_{t+1}\colon \overline{st}(v_{t+1},(f^{-1}(y))_2)\times W''_{t+1} \longrightarrow M^m$ is a PL embedding.

$$(K_t\times W''_{t+1})\cap(\overline{st}(v_{t+1},(f^{-1}(y))_2)\times W''_{t+1}) = (K_t\cap\overline{st}(v_{t+1},(f^{-1}(y))_2))\times W''_{t+1}$$

$$\subset \Gamma\times W''_{t+1} \qquad [(3.4)].$$

$$g_t^{-1}\big|_{\Gamma\times W''_{t+1}} = h_{\theta(t+1)}^{-1}\circ H_{t+1}^{-1}\big|_{\Gamma\times W''_{t+1}} \qquad [(3.12)]$$

$$= \psi_{t+1}\big|_{\Gamma\times W''_{t+1}} \qquad [(3.13)].$$

Therefore g_t^{-1} and ψ_{t+1} agree on the intersection of their domains, a closed set, so they give a well-defined map

$$\gamma_{t+1}\colon (K_t\cup\overline{st}(v_{t+1},(f^{-1}(y))_2))\times W''_{t+1} \longrightarrow M^m.$$

Since $K_{t+1} = K_t\cup\overline{st}(v_{t+1},(f^{-1}(y))_2)$ [(1.4)],

$\gamma_{t+1}\colon K_{t+1}\times W''_{t+1} \longrightarrow M^m$.

γ_{t+1} satisfies all the conclusions of the theorem, except that it may not be 1-1. We now find a neighborhood W_{t+1} of y, $W_{t+1}\subset W''_{t+1}$, such that $\gamma_{t+1}|_{K_{t+1}\times W_{t+1}}$ is 1-1.

1) Let $\gamma = \gamma_{t+1}$. $\gamma|_{K_{t+1}\times\{y\}}$ is 1-1 because

$$(3.14)\qquad \gamma(k,y) = \begin{cases} g_t^-(k,y) = k & [(3.1)],\text{ or} \\ \psi_{t+1}(k,y) = h^{-1}_{(t+1)}\circ H^{-1}_{t+1}(k,y) & \\ \qquad = h^{-1}_{(t+1)}(k,y) & [(3.11)] \\ \qquad = k & [(1.2)]. \end{cases}$$

Now we shall show that there is a neighborhood $W_{t+1} \subset W''_{t+1}$ of y so that $\gamma|_{K_{t+1}\times W_{t+1}}$ is 1-1.

If $\gamma(k_1,s_1) = \gamma(k_2,s_2)$, then $s_1 = s_2$ because γ is a layer map.

2) Suppose that in every neighborhood of y there are distinct points (k_1,s) and (k_2,s) such that $\gamma(k_1,s) = \gamma(k_2,s)$. Then there is a sequence $\{s_n\}$, $s_n \to y$, and points $(p_n,s_n) \neq (q_n,s_n)$ such that $\gamma(p_n,s_n) = \gamma(q_n,s_n)$. We may suppose $p_n \to p$ and $q_n \to q$. Then $\gamma(p,y) = \gamma(q,y)$ and $p=q$ is in $K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)$.

$H_{t+1}\circ h_{\theta(t+1)}$ is defined on a neighborhood of $K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)$ in M^m.

Therefore $\delta = H_{t+1}\circ h_{\theta(t+1)}\circ\gamma$ is defined on an open neighborhood N of $(K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)) \times \{y\}$ in $K_{t+1} \times W''_{t+1}$.

Let V be a derived neighborhood of $K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)$ in $K_t \cup \overline{st}(v_{t+1},(f^{-1}(y))_2)$ and W a neighborhood of y such that $V\times W \subset N$. Then δ is defined on $V\times W$.

$$\delta\Big|_{(V \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)) \times W}$$

$$= H_{t+1} \circ h_{\theta(t+1)} \circ \gamma\Big|_{(V \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)) \times W}$$

$$= H_{t+1} \circ h_{\theta(t+1)} \circ h^{-1}_{\theta(t+1)} \circ H^{-1}_{t+1}$$

(3.15) $= \text{identity}.$

(3.16) $\delta\Big|_{\Gamma\times W} = \text{identity}$, or $\delta\Big|_{\Gamma\times W} = H_{t+1} \circ h_{\theta(t+1)} \circ g_t^{-1}\Big|_{\Gamma\times W} = \text{identity}$ [(3.12)].

γ is a layer map and $H_{t+1} \circ h_{\theta(t+1)}$ preserves layers, so δ is a layer map.

(3.17) Let $X = V \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)$, a finite polyhedron, and $Y = cl(bdy_{f^{-1}(y)}(V \cap K_t) - \Gamma)$ [III.3.8].

Claim 1: $Y \cap (K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)) = \phi$.

Let x be a point of $K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)$. Then $x \in int_{K\cap L}\, \Gamma$. Let $A = bdy_{f^{-1}(y)}(V\cap K)$. Then $x \in int_{K\cap L}\, A$, because $V\cap K\cap L \subset A$ and V is a derived neighborhood of $K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)$, hence of x. Since $(int_{K\cap L}\, A)\cap(int_{K\cap L}\, \Gamma) \subset int_A\, \Gamma \cap int_\Gamma\, A$, $x \in int_A\, \Gamma$. Therefore x cannot be in the closure of $A-\Gamma$.

Claim 2: $Y \cap \overline{st}(v_{t+1},(f^{-1}(y))_2) \subset (K_t-\Gamma) \cap \overline{st}(v_{t+1},(f^{-1}(y))_2) = \phi$.

Therefore $Y\cap X = Y \cap V \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)$

$$= Y \cap (K_t \cup \overline{st}(v_{t+1},(f^{-1}(y))_2)) \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)$$

$$= Y \cap ((K_t \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)) \cup \overline{st}(v_{t+1},(f^{-1}(y))_2))$$

$$= (Y\cap(K_t\cap \overline{st}(v_{t+1},(f^{-1}(y))_2)))\cup(Y\cap \overline{st}(v_{t+1},(f^{-1}(y))_2))$$

$$= \phi \cup (Y \cap \overline{st}(v_{t+1},(f^{-1}(y))_2)) \quad \text{[Claim 1]}$$

$$= \phi \quad \text{[Claim 2]}.$$

For $u \in W$, $\delta_u : V \longrightarrow B_{\theta(t+1)}$ can be defined by $\delta_u(x) = \pi_1 \circ \delta(x,u)$. $\delta_y(V \cap K_t) = V \cap K_t \not\supset X$. Therefore, by continuity, there is a neighborhood of y (which we assume is W) on which $\delta_u(V \cap K_t) \not\supset X$ for all u in the neighborhood.

$$\begin{aligned} &\delta_u(V \cap K_t) \cap \delta_u(V \cap \overline{st}(x_{t+1},(f^{-1}(y))_2)) \\ &\quad = \delta_u(V \cap K_t) \cap (V \cap \overline{st}(x_{t+1},(f^{-1}(y))_2)) && [(3.15)] \\ &\quad = \delta_u(V \cap K_t) \cap X && [(3.17)]. \end{aligned}$$

Claim 3: If $\delta_u(V \cap K_t) \cap X \neq \phi$, then $\delta_u(Y) \cap X \neq \phi$.

Proof. Let $x \in (\delta_u(V \cap K_t) \cap X) - Y$. Then

(i) $x \in \Gamma$,

(ii) $x \in \partial f^{-1}(y)$, or

(iii) $x \in \text{int}(V \cap K)$.

(i) is impossible because $\delta_u|_\Gamma$ = inclusion [(3.16)].

(ii) $\delta_u|_{V \cap K_t}$ is a homeomorphism. Therefore $\delta_u(x) \in \partial f^{-1}(y) \cap X$ and $\exists\, x' \in \text{int}(V \cap K)$ such that $\delta_u(x') \in X - \partial f^{-1}(y)$, and we may limit our attention to (iii).

(iii) If $\delta_u(x) \in \partial X$, then, since $\delta_u|_Y$ is a homeomorphism, $\exists\, x' \in \text{int}(V \cap K)$ such that $\delta_u(x') \in \text{int } X$. Therefore we may suppose $\delta_u(x) \in \text{int } X$. X is a regular neighborhood of the ball $K_t \cap \overline{st}(x_{t+1},(f^{-1}(y))_2)$ in $\overline{st}(x_{t+1},(f^{-1}(y))_2)$. Therefore X is a ball [5 , Theorem 2.11]. Since $X \not\subset \delta_u(V \cap K_t)$, $\exists$ a point z in $\text{int } X - \delta_u(V \cap K)$.

Join $\delta_u(x)$ and z by an arc α in int X, an open ball. $\exists\, r \in (\text{image } \alpha \cap \partial(\delta_u(V \cap K)))$ since $\delta_u(V \cap K)$ is a ball. $r \in \text{int } X$. Therefore $r \notin \delta_u(V \cap \partial f^{-1}(y))$ because the homeomorphism δ_u takes boundary to boundary and $r \notin \delta_u(\Gamma)$ since $\delta_u|_\Gamma$ = identity.

Therefore $r \in \delta_u(Y)$, $\delta_u(Y) \cap X \neq \phi$, and we have proved Claim 3.

We have now shown that if δ_u is not 1-1, then $\delta_u(Y) \cap X = \phi$. Since $\delta_y(Y) = Y$, $\delta_y(Y) \cap X = \phi$. Therefore, since X and Y are compact, $\exists$ a neighborhood W of y which is sufficiently small so that $u \in W$ implies $\delta_u(Y) \cap X = \phi$; i.e., δ_u is 1-1. Therefore $\delta|_{V\times W}$ is 1-1, and $\gamma|_{K_{t+1}\times W}$ is 1-1.

Let $W_{t+1} = W$. We have now found a neighborhood, W_{t+1}, so that $\gamma_{t+1}|_{K_{t+1}\times W_{t+1}}$ is 1-1.

By Lemma II.2.1, $\gamma_{t+1}|_{K_{t+1}\times W_{t+1}}$ is PL.

Let $N_{t+1} = \gamma_{t+1}(K_{t+1} \times W_{t+1}) \subset M^m$.

(3.18) $$g_{t+1} = \gamma_{t+1}^{-1} \cdot g_{t+1} \text{ is PL.}$$

Then $g_{t+1}: N_{t+1} \longrightarrow K_{t+1} \times W_{t+1}$.

$N_{t+1} \cap f^{-1}(y) = \gamma_{t+1}(K_{t+1} \times W_{t+1}) \cap f^{-1}(y) = K_{t+1}$, and g_{t+1} satisfies the following commutative diagram:

(I) $$\begin{array}{ccc} N_{t+1} & \xrightarrow{f} & W_{t+1} \\ {\scriptstyle g_{t+1}}\downarrow & \nearrow {\scriptstyle \pi_2} & \\ K_{t+1}\times W_{t+1} & & \end{array}$$

Also, $g_{t+1}(u) = \gamma_{t+1}^{-1}(u)$ [(3.18)]

$= (u,y)$ [(3.14)].

<u>Section 4</u>. Letting $t = \nu$, we now have $N_\nu \subset M^m$, $N_\nu \cap f^{-1}(y) = K_\nu = f^{-1}(y)$, W_ν a neighborhood of y, and a PL homeomorphism $g_\nu\colon N_\nu \longrightarrow K_\nu \times W_\nu$ with $g_\nu(u) = (u,y)$ satisfying the following commutative diagram:

(J)
$$\begin{array}{ccc} N_\nu & \xrightarrow{\ f\ } & W_\nu \\ {\scriptstyle g_\nu}\downarrow & \nearrow{\scriptstyle \pi_2} & \\ K_\nu \times W_\nu & & \end{array}$$

$\gamma_\nu(K_\nu \times \mathring{W}_\nu)$ is open in $h(R)$ and therefore in M^m. Therefore there is a neighborhood of y, W'_ν, such that $f^{-1}(W'_\nu) \subset K_\nu \times \mathring{W}_\nu$ and, in fact, by continuity, $f^{-1}(W'_\nu) = \gamma_\nu(K_\nu \times W'_\nu)$.

Let $N'_\nu = f^{-1}(W'_\nu)$. $g'_\nu = g_\nu|_{f^{-1}(W'_\nu)}$. g'_ν is PL [II.20]. $g'_\nu\colon N'_\nu \longrightarrow K_\nu \times W'_\nu$ is a PL homeomorphism satisfying the commutative diagram:

(K)
$$\begin{array}{ccc} N'_\nu & \xrightarrow{\ f\ } & W'_\nu \\ {\scriptstyle g'_\nu}\downarrow & \nearrow{\scriptstyle \pi_2} & \\ K_\nu \times W'_\nu & & \end{array}$$

Therefore f is locally trivial and, if N^k is connected, f is the projection map of a PL fibre bundle. □

Chapter V

Codimension 1

J. G. Timourian proved a result similar to Theorem IV.6 for a map between topological manifolds when n-k = 1 [11 , 2.1]. His proof can be modified as follows [V.3] to yield Theorem IV.6 in the special case n-k = 1.

V.1. Definition [12 , page 70]. A map $f: X \longrightarrow Y$ is monotone if $f^{-1}(y)$ is connected for every $y \in Y$.

V.2. Lemma. Let M^{k+1}, N^k be PL manifolds and let $f: M^{k+1} \longrightarrow N^k$ be PL, monotone, and proper, with $B_f = \phi$; then, for every y in N^k, $f^{-1}(y) = \phi$ or $f^{-1}(y)$ is PL homeomorphic to $\dot{\Delta}^2$, the boundary of a 2-simplex, or to I = [0,1].

Proof. $f^{-1}(y) \neq \phi \Longrightarrow f^{-1}(y)$ is a cell complex of dimension 1 which is a manifold and therefore homeomorphic to S^1 or I. By the Hauptvermütung, which is true in dimension 1, $f^{-1}(y)$ is PL homeomorphic to $\dot{\Delta}^2$ or I. □

* V.3. Theorem. Let M^{k+1}, N^k be PL manifolds, and let $g: M^{k+1} \longrightarrow N^k$ be a proper PL map with $B_g = \phi$. Then for every y in $g(M^{k+1})$, there is a neighborhood N_y of y and a PL homeomorphism $g_y: g^{-1}(N_y) \longrightarrow g^{-1}(y) \times N_y$ such that $g|_{g^{-1}(N_y)} = \pi_2 \circ g_y: f^{-1}(N_y) \longrightarrow N_y$. Thus, if N^k is connected, then g is the projection map of the PL fibre bundle $(M^{k+1}, N^k, g^{-1}(y), g)$.

Proof: By a simple modification of [10 , (2.3)], there is a factorization of g into $h \circ f$, where f is a monotone PL map and h is a PL covering map. Since h is a local homeomorphism, $B_f = \phi$ and it suffices to show that $f: M^{k+1} \longrightarrow N^k$ satisfies the condition: for every y in $f(M^{k+1})$ there is a neighborhood N_y of y and a PL homeomorphism $g_y: f^{-1}(N_y) \longrightarrow f^{-1}(y) \times N_y$ such that $f|_{f^{-1}(N_y)} = \pi_2 \circ g_y: f^{-1}(N_y) \longrightarrow N_y$.

Let $y \in f(M^{k+1})$. By Lemma V.2, $f^{-1}(y)$ is PL homeomorphic to $\dot{\Delta}^2$ or I. Let $\{\mathring{U}_x\}$ be a covering of $f^{-1}(y)$, where U_x is a neighborhood of x. Since $B_f = \phi$, U_x can be chosen so that there are homeomorphisms h_x which satisfy the following commutative diagrams:

(L)
$$\begin{array}{ccc} U_x & \xrightarrow{\ f\ } & f(U_x) = \overline{V}_x \\ {\scriptstyle h_x}\downarrow & \nearrow {\scriptstyle \pi_1} & \\ \overline{V}_x \times J & & \end{array}$$

Let $\mathring{U}_{x_1} \cup \ldots \cup \mathring{U}_{x_n}$ be a finite subcovering such that $\mathring{U}_{x_i} \cap \mathring{U}_{x_j} \cap f^{-1}(y)$ is empty if $j \not\equiv (i-1, i, i+1) \bmod n$, and $\mathring{U}_{x_i} \cap f^{-1}(y) \neq \phi$. Rename the PL homeomorphisms $h_i: U_{x_i} \longrightarrow J \times \overline{V}_{x_i}$.

If H is a PL homeomorphism, $H: f^{-1}(y) \longrightarrow I$, then reindex so that $H^{-1}(0) \subset \mathring{U}_{x_1}$, $H^{-1}(1) \subset \mathring{U}_{x_n}$.

Let W be a neighborhood of y such that $f^{-1}(W) \subset \bigsqcup_{i=1}^{n} \mathring{U}_i$. Let $V_{x_i} = f^{-1}(W) \cap U_{x_i}$. Then the following diagram commutes:

(M)
$$\begin{array}{ccc} V_{x_i} & \xrightarrow{\ f\ } & W \\ {\scriptstyle h_i|V_{x_i}}\downarrow & \nearrow {\scriptstyle \pi_1} & \\ W \times J & & \end{array}$$

Let $(N)_1$ be a subdivision of N^k which includes y as a vertex.

Let $N^{(1)}$ be the barycentric subdivision of $(N)_1$. Let $N^{(b)}$ be the barycentric subdivision of $N^{(b-1)}$. Let $\overline{st}^{(b)}(y) = \overline{st}(y,N^{(b)})$.

$$\text{Let } \bar{i} = \begin{cases} i \text{ if } f^{-1}(y) \text{ is homeomorphic to } I, \\ i \bmod n \text{ if } f^{-1}(y) \text{ is homeomorphic to } \mathring{\Delta}^2. \end{cases}$$

Then there is an integer N such that for every $i = 1,\ldots,n$ with $V_{x_i} \cap V_{x_{\overline{i-1}}} \neq \phi$, there is a t_i in J such that

$|h_i^{-1}(\overline{st}^{(N)}(y)\times\{t_i\})| \subset |V_{x_i} \cap V_{x_{\overline{i-1}}}|$ and $|\overline{st}^{(N)}(y)| \subset |W|$.

This is true because:

$f^{-1}(y) \cap \mathring{V}_{x_i} \cap \mathring{V}_{x_{\overline{i-1}}} \neq \phi$.

Since $V_{x_i} = h_i^{-1}(W \times J)$ there is a t_i such that

$f^{-1}(y) \cap \mathring{V}_{x_i} \cap \mathring{V}_{x_{\overline{i-1}}} \cap h_i^{-1}(W \times \{t_i\}) \neq \phi$.

$\mathring{V}_{x_i} \cap \mathring{V}_{x_{\overline{i-1}}}$ is an open set containing

$f^{-1}(y) \cap \mathring{V}_{x_i} \cap \mathring{V}_{x_{\overline{i-1}}} \cap h^{-1}(W \times \{t_i\})$.

As $\overline{st}^{(b)}(y) \to y$, $h_i^{-1}(\overline{st}^{(b)}(y) \times \{t_i\}) \to h_i^{-1}(y,t_i)$.

Since $\mathring{V}_{x_i} \cap \mathring{V}_{x_{\overline{i-1}}}$ is open, there is an N_i such that

$b \geq N_i$ implies $h_i^{-1}(\overline{st}^{(b)}(y) \times \{t_i\}) \subset \mathring{V}_{x_i} \cap \mathring{V}_{x_{\overline{i-1}}}$.

Choose $\overline{N}$ such that $b \geq \overline{N}$ implies $\overline{st}^{(b)}(y) \subset W$.

Let $N = \max(N_i,\overline{N})$.

Let $T_i\colon U_{x_i} \times [-1,1] \longrightarrow U_{x_i} \times [-1+t_i,1+t_i]$ be defined by

$$(z,t) \mapsto (z,t+t_i).$$

Let $\overline{h}_i = T_i \circ h_i$.

Without loss of generality, we may assume $t_i = 0$ and

$\pi_1 \circ \overline{h}_i \circ \overline{h}_{i+1}^{-1}(\overline{st}^{(N)}(y) \times \{0\}) > 0$.

Look at the following diagram of PL maps:

(N) $$\begin{array}{lcl} \overline{st}^{(N)}(y)\times J & \xrightarrow{h_1^{-1}} & \\ \overline{st}^{(N)}(y)\times J & \xrightarrow{h_2^{-1}} & M^{k+1} \xrightarrow{f} N^{(N)} \\ \vdots & & \\ \overline{st}^{(N)}(y)\times J & \xrightarrow{h_n^{-1}} & \end{array}$$

This is a one-way tree in C [14 , Chapter I, page 7 and 14 , Chapter I, page 10]. Therefore there is a simplicial subdivision [14 , Chapter I, Theorem I]:

(O) $$\begin{array}{lcl} (\overline{st}^{(N)}(y)\times J)_1 & \xrightarrow{h_1^{-1}} & \\ (\overline{st}^{(N)}(y)\times J)_2 & \xrightarrow{h_2^{-1}} & (M^{k+1})_1 \xrightarrow{f} (N^{(N)})_1 \\ \vdots & & \\ (\overline{st}^{(N)}(y)\times J)_n & \xrightarrow{h_n^{-1}} & \end{array}$$

Recalling diagram (L) and restricting $\overline{h}_i^{-1}$ to $(\overline{st}^{(N)}(y) \times J)_i$, we have the following commutative diagrams:

(P) $$\begin{array}{ccc} (M^{k+1})_1 & \xrightarrow{f} & (N^{(N)})_1 \\ \downarrow h_i^{-1} & \nearrow \pi_1 & \\ (\overline{st}^{(N)}(y)\times J)_i & & \end{array}$$

Since f and h_i^{-1} are simplicial, π_1 is simplicial.

Let $(\overline{st}^{(N)}(y))_1 = \overline{st}(y,(N^{(N)})_1)$ and restrict diagram (P) to:

(Q) $$\begin{array}{ccc} f^{-1}((\overline{st}^{(N)}(y))_1) & \xrightarrow{f} & (\overline{st}^{(N)}(y))_1 \\ \downarrow h_i^{-1} & \nearrow \pi_1 & \\ ((\overline{st}^{(N)}(y))_1\times J)_i & & \end{array}$$

If $f^{-1}(y)$ is homeomorphic to $\dot{\Delta}^2$, let $W_i = h_i^{-1}((\overline{st}^{(N)}(y))_1\times\{0\})$.

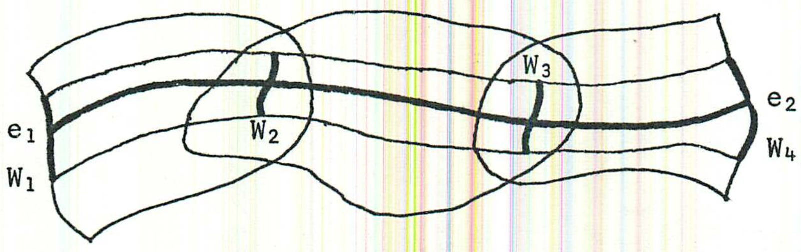
If $f^{-1}(y)$ is homeomorphic to I, let

$$W_i = h_i^{-1}((\overline{st}^{(N)}(y))_1 \times \{0\}),\ i = 2, \ldots, n;$$

$$W_1 = h_1^{-1}((\overline{st}^{(N)}(y))_1 \times h_1(e_1)), \text{ where } H(e_1) = 0;$$

$$W_{n+1} = h_n^{-1}((\overline{st}^{(N)}(y))_1 \times h_n(e_2)), \text{ where } H(e_2) = 1.$$

Look at the following picture for the case of the line:

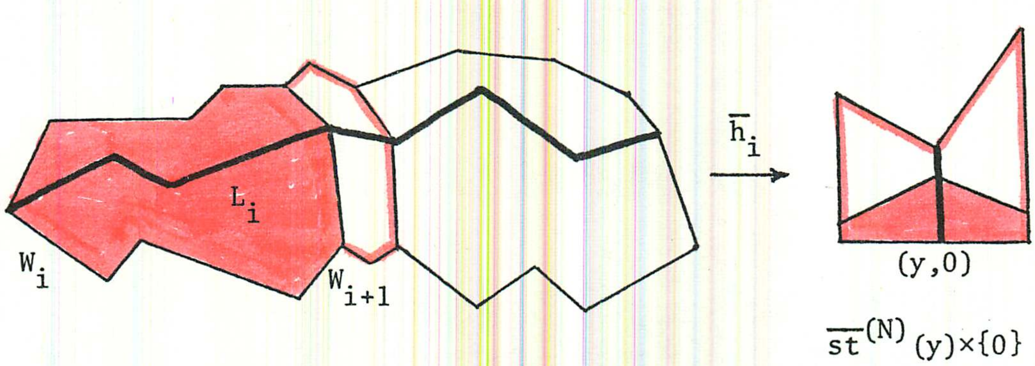


This picture looks as it does at the ends because the e_i are not branch points.

$W_i \cup W_{\overline{i+1}}$ separates $f^{-1}((\overline{st}^{(N)}(y))_1)$.

Let $L_i = W_i \cup W_{\overline{i+1}} \cup$ the component of $(f^{-1}((\overline{st}^{(N)}(y))_1) - (W_i \cup W_{\overline{i+1}}))$ which lies in V_{x_i}.

$h_i : h_i^{-1}(((\overline{st}^{(N)}(y))_1 \times J)_i) \longrightarrow ((\overline{st}^{(N)}(y))_1 \times J)_i$ is a simplicial homeomorphism.



$h_i(W_{i+1}) = h_i \circ h_{i+1}^{-1}((\overline{st}^{(N)}(y))_1 \times \{0\})$. $h_i \circ h_{i+1}^{-1}(x,0) = (x, t_i(x))$ by diagram (L). Since h_i, h_{i+1} are simplicial, $t_i : (\overline{st}^{(N)}(y))_1 \longrightarrow (I)_1$ is simplicial. $h_i(W_{i+1})$ is the graph of t_i. Therefore every p-simplex of $h_i(W_{i+1})$ is the join of a unique point $q = (y, t_i(y))$ with a unique $(p-1)$-simplex τ of $h_i(f^{-1}((\dot{\overline{st}}^{(N)}(y))_1) \cap h_i(W_{i+1})$. Let $B_i = (h_i(L_i)) \cap ((\dot{\overline{st}}^{(N)}(y))_1 \times I)$. Define $\overline{g}_i : h_i(L_i) \longrightarrow (\overline{st}^{(N)}(y))_1 \times [\frac{i-1}{n}, \frac{i}{n}]$ as follows:

$$\overline{g}_i(x,0) = (x, \tfrac{i-1}{n}) \text{ for all } x \in (\overline{st}^{(N)}(y))_1.$$

$\overline{g}_i\big|_{\{y\}\times[0,t_i(y)]} : \{y\} \times [0, t_i(y)] \longrightarrow \{y\} \times [\frac{i-1}{n}, \frac{i}{n}]$ is given by $(y,0) \mapsto (y, \frac{i-1}{n})$, $(y, t_i(y)) \mapsto (y, \frac{i}{n})$, and extend linearly.

Let $x \in \dot{\overline{st}}^{(N)}(y)$. We have already defined $\overline{g}_i(x,0) = (x, \frac{i-1}{n})$, $\overline{g}_i(x, t_i(x)) = (x, \frac{i}{n})$. Extend this linearly to a map

$$\overline{g}_i\big|_{(\{x\}\times\mathbb{R})\cap B_i} : (\{x\} \times \mathbb{R}) \cap B_i \longrightarrow \{x\} \times [\tfrac{i-1}{n}, \tfrac{i}{n}].$$

Now $\overline{g}_i$ has been defined on all the vertices of $h_i(L_i)$ and can be extended linearly. Notice also that the following diagram commutes:

$$\text{(R)} \qquad \begin{array}{ccc} h_i(L_i) & \xrightarrow{\text{proj}} & (\overline{st}^{(N)}(y))_1 \\ {\scriptstyle \overline{g}_i}\big\downarrow & \nearrow {\scriptstyle \pi_1} & \\ (\overline{st}^{(N)}(y))_1 \times [\frac{i-1}{n}, \frac{i}{n}] & & \end{array}$$

Define $u: (\overline{st}^{(N)}(y))_1 \times I \longrightarrow (\overline{st}^{(N)}(y))_1 \times \dot{\Delta}^2$ by

$u\big|_{(\overline{st}^{(N)}(y))_1\times[0,\frac{1}{3}]} = u_1 : (\overline{st}^{(N)}(y))_1 \times [0, \frac{1}{3}] \longrightarrow (\overline{st}^{(N)}(y))_1 \times [(0,0),(1,0)]$,

$u\big|_{(\overline{st}^{(N)}(y))_1\times[\frac{1}{3},\frac{2}{3}]} = u_2 : (\overline{st}^{(N)}(y))_1 \times [\frac{1}{3}, \frac{2}{3}] \longrightarrow (\overline{st}^{(N)}(y))_1 \times [(1,0),(0,1)]$,

$u\big|_{(\overline{st}^{(N)}(y))_1\times[\frac{2}{3},1]} = u_3 : (\overline{st}^{(N)}(y))_1 \times [\frac{2}{3}, 1] \longrightarrow (\overline{st}^{(N)}(y))_1 \times [(0,1),(0,0)]$,

where u_i is the identity on the first component and linear on the second.

If $f^{-1}(y) \approx I$, define $\overline{g}: f^{-1}((\overline{st}^{(N)}(y))_1) \longrightarrow (\overline{st}^{(N)}(y))_1 \times [0,1]$ by $\overline{g}\big|_{L_i} = \overline{g}_i \circ h_i\big|_{L_i}$.

If $f^{-1}(y) \approx \dot{\Delta}^2$, define $\overline{g}: f^{-1}((\overline{st}^{(N)}(y))_1) \longrightarrow (\overline{st}^{(N)}(y))_1 \times \dot{\Delta}^2$

by $\overline{g}|_{L_i} = u \circ \overline{g}_i \circ \overline{h}_i|_{L_i}$.

g is a PL homeomorphism.

Since $\overline{h}_i$ satisfies diagram (L) and since $\pi_1 \circ \overline{g}_i = \pi_1$ and $\pi_1 \circ u = \pi_1$, the following diagram commutes:

$$\text{(S)} \qquad \begin{array}{ccc} f^{-1}((\overline{st}^{(N)}(y))_1) & \xrightarrow{f} & (\overline{st}^{(N)}(y))_1 \\ \big\downarrow \overline{g} & \nearrow \pi_1 & \\ (\overline{st}^{(N)}(y)) \times f^{-1}(y) & & \end{array}$$

Therefore f is locally trivial at y and, if N^k is connected, f is the projection map of a PL fibre bundle. □

Chapter VI

Non-manifold Possibilities

While the proof of Theorem IV.6 relies heavily on the fact that the fibre, $f^{-1}(y)$, is a manifold, the fact that the range and domain of the function are manifolds is needed only to satisfy the hypotheses of Hudson's n-isotopy extension theorem [4, Theorem 3]. If this theorem were replaced by one with the n-simplex replaced by a simplicial complex, then the proof in chapter IV would yield the following generalization of Theorem IV.6.

VI.1. Conjecture. Let M, N be PL spaces and let $f: M \longrightarrow N$ be a proper PL map with $B_f = \phi$ and $f^{-1}(y)$ a PL manifold. Then there is a neighborhood N_y of y and a PL homeomorphism $g_y: f^{-1}(N_y) \longrightarrow f^{-1}(y) \times N_y$ such that $f|_{f^{-1}(N_y)} = \pi_2 \circ g_y: f^{-1}(N_y) \longrightarrow N_y$.

It is natural to ask whether more information can be obtained by looking at a much more general version of Theorem IV.6, where M, N, and $f^{-1}(y)$ are PL spaces and the definition of B_f is modified so that it doesn't force any nice manifold-like structure in either the domain or the range.

VI.2. Definition. If $f: M \longrightarrow N$ is PL then the branch set of f, $B_f \subset M$, can be defined as follows: $x \in M - B_f$ if there exists a neighborhood U_x of x such that $f(U_x) = V_x$, a closed star neighborhood of f(x), and if there exists a PL homeomorphism $h_x: U_x \longrightarrow B_x \times V_x$, where B_x is a simplicial complex, such that $f|_{U_x} = \pi_2 \circ h_x: U_x \longrightarrow V_x$.

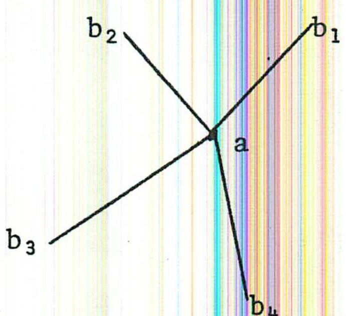
With this new definition of B_f, the following conjecture is very general.

VI.3. Conjecture. Let M, N be PL spaces. Suppose $f: M \longrightarrow N$ is a proper PL map with $B_f = \phi$. Then for every $y \in f(M)$, there is a neighborhood N_y of y and a PL homeomorphism $g_y: f^{-1}(N_y) \longrightarrow f^{-1}(y) \times N_y$ such that $f|_{f^{-1}(N_y)} = \pi_2 \circ g_y: f^{-1}(N_y) \longrightarrow N_y$. Thus, if N is connected, f is the projection map of a PL fibre bundle.

The proof of Theorem IV.6, with some minor modifications, is also a proof that Conjecture VI.4 implies Conjecture VI.3.

VI.4. Conjecture. Let K_1, K_2, L be simplicial complexes. Let $f: K_1 \times \overline{st}(y,L) \longrightarrow K_2 \times \overline{st}(y,L)$ be a level-preserving PL embedding. Then there is a neighborhood N_y of y, $|N_y| \subset |\overline{st}(y,L)|$, and a PL homeomorphism $H: \overline{st}(y,K_2) \times N_y \longrightarrow \overline{st}(y,K_2) \times N_y$ which satisfies $H_t \circ f_t(x) = f_y(x)$, i.e., $H \circ f(x,t) = (\pi_1 \circ f(x,y),t)$, $\forall x \in \overline{st}(y,K_2)$, $t \in N_y$.

VI.5. Counterexample to VI.4. Take $K_1 = p$, a point; $K_2 = \bigcup_{i=1}^{4} [a,b_i]$, four intervals meeting at the point a:



$y = 0$, $\overline{st}(y,L) = [-1,1]$. Define $f: p \times [-1,1] \longrightarrow K_2 \times [-1,1]$ by $f(p,t) = ((1-t)a+tb_1,t)$ if $t \in [0,1]$, $((1-|t|)a+|t|b_2,t)$ if $t \in [-1,0]$.

Although Conjecture VI.4 is untrue, it and the fact that Conjecture VI.4 implies Conjecture VI.3 are of interest because they shed light on what kind of information is needed to prove a more general case of Theorem IV.6. It is likely that with some added hypotheses, Conjecture VI.4 will be true and will imply a version of Conjecture IV.3.

Appendix

The appendix consists of proofs of some of the lemmas in the main body of the paper and lemmas needed for the proofs.

I felt that putting these proofs in the main text would have added unnecessary complication but include them here for the sake of completeness.

Section 1: Proofs of some lemmas in Chapter II.

A.1.1 (Lemma II.10). If $A \notin L_1 \cup L_2$, then $\overline{st}(A, L_1 \cup L_2) = \phi$, $\overline{st}(A, L_1) = \phi$, and $\overline{st}(A, L_2) = \phi$.

i) Let $A \in L_1 \cap L_2$.

Let $B \in \overline{st}(A, L_i)$. Then there is a simplex C of L_i such that $A < C$, $B < C$. Since $C \in L_1 \cup L_2$, $B \in \overline{st}(A, L_1 \cup L_2)$. Therefore $\overline{st}(A, L_1) \cup \overline{st}(A, L_2) \subset \overline{st}(A, L_1 \cup L_2)$.

Let $B \in \overline{st}(A, L_1 \cup L_2)$. Then there is a simplex C of $L_1 \cup L_2$ such that $A < C$ and $B < C$. Since $C \in L_1 \cup L_2$, $C \in L_i$, $i = 1$ or 2. $A \in L_1 \cap L_2 \subset L_i$. Therefore $B \in \overline{st}(A, L_i) \subset \overline{st}(A, L_1) \cup \overline{st}(A, L_2)$. Therefore $\overline{st}(A, L_1 \cup L_2) \subset \overline{st}(A, L_1) \cup \overline{st}(A, L_2)$.

ii) Let $A \in L_i - L_j$.

$\overline{st}(A, L_1) \cup \overline{st}(A, L_2) = \overline{st}(A, L_i) \cup \phi \subset \overline{st}(A, L_1 \cup L_2)$.

Let $B \in \overline{st}(A, L_1 \cup L_2)$. Then there is a simplex C of $L_1 \cup L_2$ such that $A < C$ and $B < C$. Since A is not in L_j and L_j is a simplicial complex, A is not a face of any simplex of L_j. Therefore $C \notin L_j$, $C \in L_i$, $B \in \overline{st}(A, L_i)$ and $\overline{st}(A, L_1 \cup L_2) \subset \overline{st}(A, L_1) \cup \overline{st}(A, L_2)$. □

A.1.2 (Lemma II.11). Let $B \in \overline{st}(A, L_1 \cap L_2)$. Then there is a simplex C in $L_1 \cap L_2$ such that $A < C$, $B < C$. Since $C \in L_i$, $B \in \overline{st}(A, L_i)$ and $\overline{st}(A, L_1 \cap L_2) \subset \overline{st}(A, L_1) \cap \overline{st}(A, L_2)$.

Let $B \in \overline{st}(A, L_1) \cap \overline{st}(A, L_2)$. Then there are simplexes C_1 in L_1 and C_2 in L_2 such that $A < C_1$, $B < C_1$, $A < C_2$, $B < C_2$. $C_1 \cap C_2$ is a face of C_1 and C_2. $A < C_1 \cap C_2$ and $B < C_1 \cap C_2$. $C_1 \cap C_2$ is in $L_1 \cap L_2$. Therefore $B \in \overline{st}(A, L_1 \cap L_2)$ and $\overline{st}(A, L_1) \cap \overline{st}(A, L_2) \subset \overline{st}(A, L_1 \cap L_2)$. □

A.1.3 (Lemma II.16). Since $K^{(1)}$ is a first derived subdivision of K, every simplex A of $K^{(1)}$ is of the form $\hat{A}_0 \hat{A}_1 \ldots \hat{A}_p$ where $A_0 < A_1 < \ldots < A_p$ and A_j is a simplex of K, $j=0,\ldots,p$.

$\overline{st}(x_1, K^{(1)}) = \{A \in K^{(1)} \mid \exists B \in K^{(1)}: x_1 < B \text{ and } A < B\}$

$= \{\hat{A}_0 \ldots \hat{A}_p \in K^{(1)} \mid A_0 = x_1 \text{ or } x_1 \hat{A}_0 \ldots \hat{A}_p \text{ is a simplex of } K^{(1)}\}$.

$\overline{st}(x_2, K^{(1)}) = \{\hat{B}_0 \ldots \hat{B}_q \in K^{(1)} \mid B_0 = x_2 \text{ or } x_2 \hat{B}_0 \ldots \hat{B}_q \text{ is a simplex of } K^{(1)}\}$.

Since $x_1 \neq x_2$ and since the second simplex in the expression for a simplex of $K^{(1)}$ must always have dimension at least one, $x_1 \hat{A}_0 \ldots \hat{A}_p \notin \overline{st}(x_2, K^{(1)})$ and $x_2 \hat{B}_0 \ldots \hat{B}_q \notin \overline{st}(x_1, K^{(1)})$. Therefore

$\overline{st}(x_1, K^{(1)}) \cap \overline{st}(x_2, K^{(1)}) = \{C = \hat{C}_0 \ldots \hat{C}_r \in K^{(1)} \mid x_1 C \in K^{(1)} \text{ and } x_2 C \in K^{(1)}\}$.

If $\overline{st}(x_1, K^{(1)}) \cap \overline{st}(x_2, K^{(1)}) \neq \phi$, there is a simplex C_0 of K satisfying $x_1 < C_0$ and $x_2 < C_0$. Therefore $[x_1, x_2] < C_0$ and $[x_1, x_2]$ is a simplex of K. $\overline{st}(x_1, K^{(1)}) \cap \overline{st}(x_2, K^{(1)})$

$= \{\hat{C}_0 \ldots \hat{C}_r \in K^{(1)} \mid x_1 \text{ and } x_2 \text{ are both vertices of } C_0\}$

$= \{\hat{C}_0 \ldots \hat{C}_r \in K^{(1)} \mid [x_1, x_2] < C_0\}$.

$lk(x_2, K^{(1)}) = \{\hat{D}_0 \ldots \hat{D}_\ell \in K^{(1)} \mid x_2 \hat{D}_0 \ldots \hat{D}_\ell \in K^{(1)}\}$.

$x_2 x_{1,2} \in K^{(1)}$, so $x_{1,2} \in lk(x_2, K^{(1)})$. $\overline{st}(x_{1,2}, lk(x_2, K^{(1)}))$

$= \{D = \hat{D}_0 \ldots \hat{D}_\ell \in lk(x_2, K^{(1)}) \mid \exists E = \hat{E}_0 \ldots \hat{E}_k \in lk(x_2, K^{(1)}): x_{1,2} < E, D < E\}$.

Since $D \in lk(x_2, K^{(1)})$, $x_2 D \in K^{(1)}$. Since $E \in lk(x_2, K^{(1)})$, $x_2 E \in K^{(1)}$.

(i) Let $C \in \overline{st}(x_1, K^{(1)}) \cap \overline{st}(x_2, K^{(1)})$. Then $[x_1, x_2] < C_0$. Therefore $x_2 \hat{C}_0 \ldots \hat{C}_r \in K^{(1)}$ and $C \in lk(x_2, K^{(1)})$.

If $[x_1, x_2] = C_0$, then $x_{1,2} = \hat{C}_0$, $x_{1,2} < C$, and $C \in \overline{st}(x_{1,2},\ lk(x_2, K^{(1)}))$.

If $[x_1, x_2] \neq C_0$, then $x_{1,2}C \in K^{(1)}$, $x_{1,2} < x_{1,2}C$, $C < x_{1,2}C$, and $C \in \overline{st}(x_{1,2},\ lk(x_2, K^{(1)}))$.

Therefore $\overline{st}(x_1, K^{(1)}) \cap \overline{st}(x_2, K^{(1)}) \subset \overline{st}(x_{1,2},\ lk(x_2, K^{(1)}))$.

(ii) Let $D \in \overline{st}(x_{1,2},\ lk(x_2, K^{(1)}))$. Then $D = \hat{D}_0 \ldots \hat{D}_\ell \in lk(x_2, K^{(1)}) \subset \overline{st}(x_2, K^{(1)})$. Since $D \in \overline{st}(x_{1,2},\ lk(x_2, K^{(1)}))$, there is a simplex E in $lk(x_2, K^{(1)})$ which satisfies $x_{1,2} < E$, $D < E$, $E = \hat{E}_0 \ldots \hat{E}_m$, and $x_2 E \in K^{(1)}$. Since $x_{1,2} < E$ and $x_2 E \in K^{(1)}$, $[x_1, x_2] = E_0$. Therefore $x_1 E$ is a simplex of $K^{(1)}$ and $E \in \overline{st}(x_1, K^{(1)})$.

Therefore $\overline{st}(x_{1,2},\ lk(x_2, K^{(1)})) \subset \overline{st}(x_1, K^{(1)}) \cap \overline{st}(x_2, K^{(1)})$. □

A.1.4 (Lemma II.21). Subdivide K_1, L_1, K_2, L_2 to $(K_1)_1$, $(L_1)_1$, $(K_2)_1$, $(L_2)_1$ so that $f_1: (K_1)_1 \longrightarrow (L_1)_1$ and $f_2: (K_2)_1 \longrightarrow (L_2)_1$ are simplicial. Subdivide $(K_1)_1$, $(K_2)_1$, $K_1 \cup K_2$ to $(K_1)_2$, $(K_2)_2$, $(K_1 \cup K_2)_2$ so that $(K_1)_2$ and $(K_2)_2$ are subcomplexes of $(K_1 \cup K_2)_2$ [14 , Chapter I, Lemma 4, Corollary 2]. Then $(K_1)_2 \cap (K_2)_2$ is a subcomplex of $(K_1)_2$, $(K_2)_2$, and $(K_1 \cup K_2)_2$.

$f_1((K_1)_2)$ is a subdivision of $f_1((K_1)_1) \subset (L_1)_1$. Extend this to a subdivision $(L_1)_2$ of $(L_1)_1$ [14, Chapter I, Lemma 3]. Since $f_1((K_1)_2 \cap (K_2)_2) = f_2((K_1)_2 \cap (K_2)_2)$ and f is 1-1, $(L_1)_2 \cup f_2((K_2)_2)$ is a subdivision of $L_1 \cup f_2((K_2)_1)$. Extend this to a subdivision $(L_1 \cup L_2)_2$ of $L_1 \cup L_2$.

$f: (K_1 \cup K_2)_2 \longrightarrow (L_1 \cup L_2)_2$ is simplicial, so f is PL. □

Section 2: Proofs of some lemmas in Chapter III.

A.2.1 (Lemma III.1.10).

(i) Definition III.1.6 ==> Definition III.1.9.

If K is a combinatorial n-manifold, then lk(x,K) is an (n-1)-sphere or (n-1)-ball for any vertex x of K. $\overline{st}(x,K)$ = x lk(x,K) [5 , page 7]. Therefore $\overline{st}(x,K)$ is an n-ball [5 , Lemma 1.13].

(ii) Definition III.1.9 ==> Definition III.1.7.

If K is a combinatorial n-manifold, then $\overline{st}(x,K)$ is a cover of K by n-balls. Therefore K is a PL n-manifold.

(iii) Definition III.1.7 ==> Definition III.1.8.

[5 , Corollary 1.16].

(iv) Definition III.1.8 ==> Definition III.1.6.

[14 , Lemma 9, Corollary 1]. □

A.2.2. Lemma. If x is a vertex of a simplicial complex K, then lk(x, $\overline{st}(x,K)$) = lk(x,K).

Proof. Let $A \in$ lk(x, $\overline{st}(x,K)$). Then $xA \in \overline{st}(x,K) \subset K$. Therefore $xA \in K$, $A \in$ lk(x,K), and lk(x, $\overline{st}(x,K)$) $\subset$ lk(x,K).

Let $A \in$ lk(x,K). Then $xA \in K$. Therefore $A \in \overline{st}(x,K)$ and $x \notin A$. Therefore $xA \in \overline{st}(x,K)$, $A \in$ lk(x, $\overline{st}(x,K)$), and lk(x,K) $\subset$ lk(x, $\overline{st}(x,K)$). □

A.2.3 (Lemma III.2.2). Let $A \in \text{lk}(x,M^n)$. $\text{lk}(A, \overline{\text{st}}(x,M^n)) = x\ \text{lk}(A, \text{lk}(x,M^n))$ [II.12]. $\text{lk}(A, \text{lk}(x,M^n))$ is a sphere or a ball. Therefore $x\ \text{lk}(A, \text{lk}(x,M^n))$ is a ball, $A \in \text{bdy}(\overline{\text{st}}(x,M^n))$, and $\text{lk}(x,M^n) \subset \text{bdy}(\overline{\text{st}}(x,M^n))$.

If $\text{lk}(x,M^n) \neq \text{bdy}(\overline{\text{st}}(x,M^n))$, then $\text{lk}(x,M^n) \subsetneqq \text{bdy}(\overline{\text{st}}(x,M^n))$ and there is a simplex A in $\text{bdy}(\overline{\text{st}}(x,M^n)) - \text{lk}(x,M^n)$. Since $A \notin \text{lk}(x,M^n)$, $xA \notin M^n$. Since $A \in \overline{\text{st}}(x,M^n)$, there is a simplex B in M^n which satisfies $A < B$, $x < B$. If $x \nless A$, then $xA < B$ and $xA \in M^n$. Contradiction. Therefore $x < A$.

$A \in \text{bdy}(\overline{\text{st}}(x,M^n))$, a simplicial complex. Therefore $x \in \text{bdy}(\overline{\text{st}}(x,M^n))$ and $\text{lk}(x, \overline{\text{st}}(x,M^n))$ is a ball. $\text{lk}(x,M^n) = \text{lk}(x, \overline{\text{st}}(x,M^n))$ [A.2.2]. Therefore $\text{lk}(x,M^n)$ is a ball and $x \in \text{bdy}(M^n)$.

Conversely, if $x \in \text{bdy}(M^n)$, then $\text{lk}(x,M^n)$ is a ball. Therefore $\text{lk}(x, \overline{\text{st}}(x,M^n))$ is a ball [A.2.2] and $x \in \text{bdy}(\overline{\text{st}}(x,M^n))$. Since $x \notin \text{lk}(x,M^n)$, $x \in \text{bdy}(\overline{\text{st}}(x,M^n)) - \text{lk}(x,M^n)$ and $\text{lk}(x,M^n) \neq \text{bdy}(\overline{\text{st}}(x,M^n))$. □

A.2.4 (Lemma III.2.3). $\text{lk}(x,M^n) \subset \text{bdy}(\overline{\text{st}}(x,M^n))$ [III.2.2]. $\text{bdy}(\overline{\text{st}}(x,M^n)) \cap \text{bdy}(M^n) \subset \text{bdy}(\overline{\text{st}}(x,M^n))$. Therefore $\text{lk}(x,M^n) \cup (\text{bdy}(\overline{\text{st}}(x,M^n)) \cap \text{bdy}(M^n) \subset \text{bdy}(\overline{\text{st}}(x,M^n))$.

Let $A \in \text{bdy}(\overline{\text{st}}(x,M^n))$. If $A \notin \text{lk}(x,M^n)$, then $A \in \overline{\text{st}}(x,M^n)$ and $x < A$. Therefore $\text{lk}(A,M^n) \subset \overline{\text{st}}(A,M^n) \subset \overline{\text{st}}(x,M^n)$. Since $A \in \text{bdy}(\overline{\text{st}}(x,M^n))$, $\text{lk}(A, \overline{\text{st}}(x,M^n))$ is a ball. $\text{lk}(A, \overline{\text{st}}(x,M^n)) = \overline{\text{st}}(x,M^n) \cap \text{lk}(A,M^n) = \text{lk}(A,M^n)$. Therefore $\text{lk}(A,M^n)$ is a ball and $A \in \text{bdy}(M^n)$. Therefore $A \in \text{bdy}(\overline{\text{st}}(x,M^n)) \cap \text{bdy}(M^n)$. □

A.2.5 (Lemma III.2.4). $A \in S(x_t, (M^n)^{(1)}) \cap \overline{st}(x_{t+1}, (M^n)^{(1)})$ implies that there is an i such that $A \in \overline{st}(x_i, (M^n)^{(1)}) \cap \overline{st}(x_{t+1}, (M^n)^{(1)})$. So $A \in \text{bdy}(\overline{st}(x_i, (M^n)^{(1)}) \cap \text{bdy}(\overline{st}(x_{t+1}, (M^n)^{(1)})$.

If $A \notin \overline{st}(x_i, (M^n)^{(1)})$, then $lk(A, \overline{st}(x_i, (M^n)^{(1)})) = \phi$, otherwise $x_i A$ can be defined, so we may apply Lemma II.12, and

(2.1) $lk(A, \overline{st}(x_i, (M^n)^{(1)})) = x_i \; lk(A, lk(x_i, (M^n)^{(1)}))$.

$lk(x_i, (M^n)^{(1)})$ is an (n-1)-ball or sphere and either

i) $A \notin lk(x_i, (M^n)^{(1)})$ so $lk(A, lk(x_i, (M^n)^{(1)})$, and therefore also $lk(A, \overline{st}(x_i, (M^n)^{(1)})$ [(2.1)], is undefined or empty, or

ii) $lk(A, lk(x_i, (M^n)^{(1)}))$ is an (n - a - 2)-ball or sphere and $x_i \; lk(A, lk(x_i, (M^n)^{(1)}))$ is an (n - a - 1)-ball. Therefore $lk(A, \overline{st}(x_i, (M^n)^{(1)}))$ is an (n - a - 1)-ball [(2.1)]. □

A.2.6. Notation. If σ^d is a d-simplex, $b(\sigma^d)$ is the barycenter [5, page 9] of σ^d. $I = [0,1]$. σ^{d-1} is a (d-1)-face of σ^d. $o(\sigma^{d-1})$ is the vertex of σ^d opposite σ^{d-1}; i.e., $\sigma^d = o(\sigma^{d-1})\sigma^{d-1}$.

A.2.7. Lemma. $(\sigma^{d-1} \times I) \cup ((b(\sigma^d)\sigma^{d-1}) \times \dot{I})$ is a d-ball.

Proof.

$(\sigma^{d-1} \times I) \cup ((b(\sigma^d)\sigma^{d-1}) \times \dot{I}) = (\sigma^{d-1} \times I) \cup ((b(\sigma^d)\sigma^{d-1}) \times \{0\}) \cup ((b(\sigma^d)\sigma^{d-1}) \times \{1\})$.

$\sigma^{d-1} \times I$ and $(b(\sigma^d)\sigma^{d-1}) \times \{0\}$ are both d-balls.

$(\sigma^{d-1}\times I) \cap ((b(\sigma^d)\sigma^{d-1})\times\{0\}) = \sigma^{d-1}\times\{0\}$, a (d-1)-ball.

$\sigma^{d-1}\times\{0\} \subset \sigma^{d-1}\times\mathring{I} \subset \mathrm{bdy}(\sigma^{d-1}\times I)$. $\sigma^{d-1}\times\{0\} \subset \mathrm{bdy}((b(\sigma^d)\sigma^{d-1})\times\{0\})$.

Therefore $(\sigma^{d-1}\times I) \cup ((b(\sigma^d)\sigma^{d-1})\times\{0\})$ is a d-ball.

$(b(\sigma^d)\sigma^{d-1})\times\{1\}$ is a d-ball.

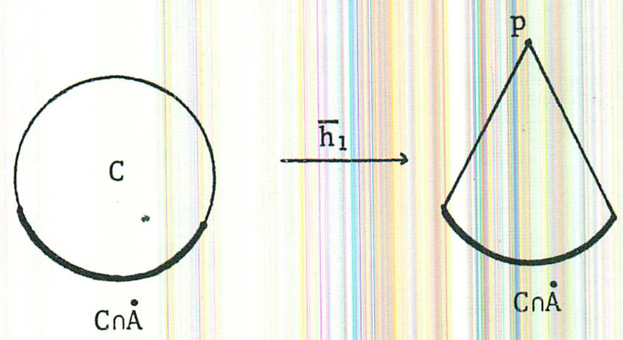
$((\sigma^{d-1}\times I) \cup ((b(\sigma^d)\sigma^{d-1})\times\{0\})) \cap (b(\sigma^d)\sigma^{d-1}\times\{1\}) = \sigma^{d-1}\times\{1\}$, a (d-1)-ball.

$\sigma^{d-1}\times\{1\} \subset \mathrm{bdy}((\sigma^{d-1}\times I)\cup((b(\sigma^d)\sigma^{d-1})\times\{0\})) \cap \mathrm{bdy}((b(\sigma^d)\sigma^{d-1})\times\{1\})$.

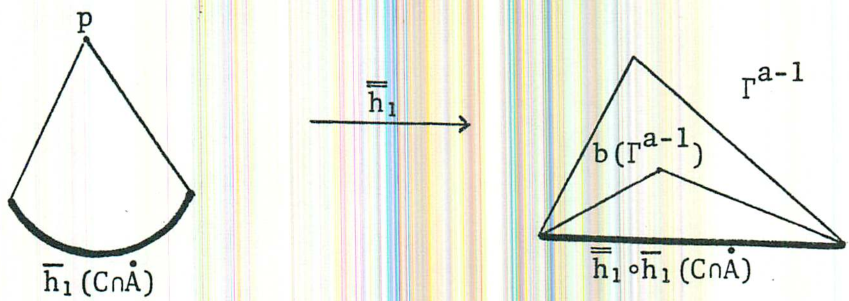
Therefore $(\sigma^{d-1}\times I) \cup ((b(\sigma^d)\sigma^{d-1})\times\{0\}) \cup ((b(\sigma^d)\sigma^{d-1})\times\{1\})$ is a d-ball. □

A.2.8. Lemma. Suppose (A,B) is an unknotted (a,a-1)-ball pair, type 1, and $(A,C) \subset (A,B)$ is an (a,a-1)-ball pair, type 2. Then (A,C) is an unknotted (a,a-1)-ball pair, type 2.

Proof: $C \cap \mathring{A}$ is an (a-2)-face of C. Therefore there is a homeomorphism $\overline{h}_1 : C \longrightarrow p(C \cap \mathring{A})$, where p is a point and $\overline{h}_1\big|_{C\cap\mathring{A}}(x) = x$, $x \in C \cap \mathring{A}$.

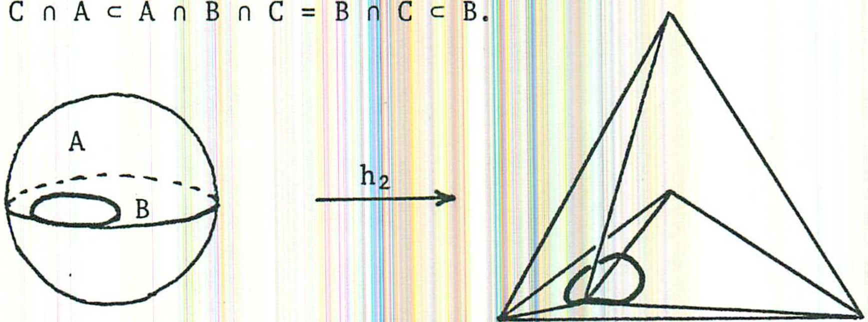


Let Γ^{a-1} be an (a-1)-simplex with barycenter $b(\Gamma^{a-1})$. Let Γ^{a-2} be an (a-2)-face of Γ^{a-1}. There is a PL embedding $\overline{\overline{h}}_1 : \overline{h}_1(C) \longrightarrow \Gamma^{a-1}$ such that $\overline{\overline{h}}_1 \circ \overline{h}_1(C) = b(\Gamma^{a-1})\Gamma^{a-2}$, $\overline{\overline{h}}_1(p) = b(\Gamma^{a-1})$, and $\overline{\overline{h}}_1 \circ \overline{h}_1(C \cap \mathring{A}) = \Gamma^{a-2}$.

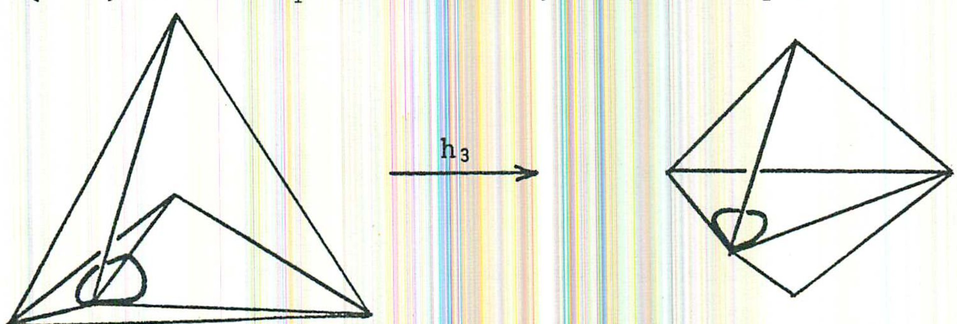


Let $h_1 = \bar{\bar{h}}_1 \circ \bar{h}_1 : C \longrightarrow \Gamma^{a-1}$.

Since (A,B) is an unknotted (a,a-1)-ball pair, type 1, there is a homeomorphism $h_2\colon (A,B) \longrightarrow (\sigma^a, b(\sigma^a)\dot{\sigma}^{a-1})$, where $b(\sigma^a)$ is the barycenter of σ^a, an a-simplex, and σ^{a-1} is an (a-1)-face of σ^a [III.3.1]. $\dot{B} = \dot{A} \cap B$, $C \cap \dot{A} \subset \dot{C}$, $C \subset B$. Therefore $C \cap \dot{A} \subset C \subset B$. Therefore $C \cap \dot{A} \subset \dot{A} \cap B \cap \dot{C} = \dot{B} \cap \dot{C} \subset B$.

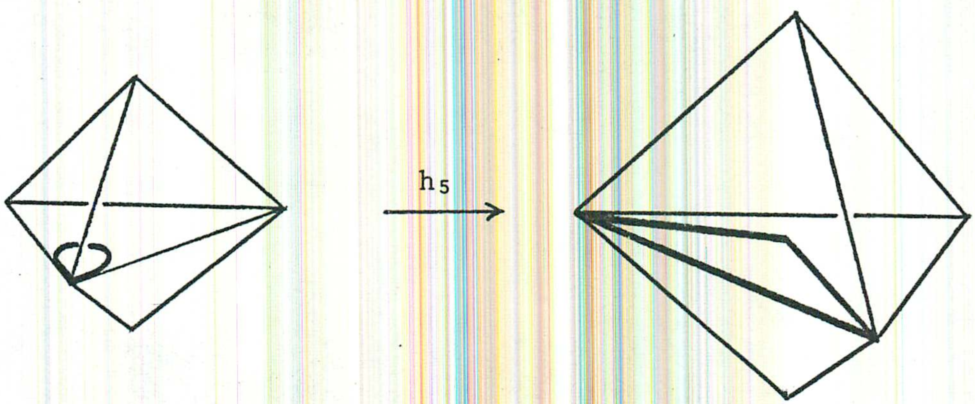


There is a homeomorphism $h_3\colon (\sigma^a, b(\sigma^a)\dot{\sigma}^{a-1}) \longrightarrow (\Sigma(\tau^{a-1}), \tau^{a-1})$, where $\Sigma(\tau^{a-1})$ is the suspension of τ^{a-1}, an (a-1)-simplex.



$h_1 \circ h_2^{-1} \circ h_3^{-1}|_{h_3 \circ h_2(C)} : h_3 \circ h_2(C) \longrightarrow \Gamma^{a-1}$ is a PL embedding. $h_1 \circ h_2^{-1} \circ h_3^{-1}(h_3 \circ h_2(C)) = b(\Gamma^{a-1})\Gamma^{a-2}$. Extend this to a PL homeomorphism $h_4\colon \tau^{a-1} \longrightarrow \Gamma^{a-1}$. Extend h_4 to a PL homeomorphism

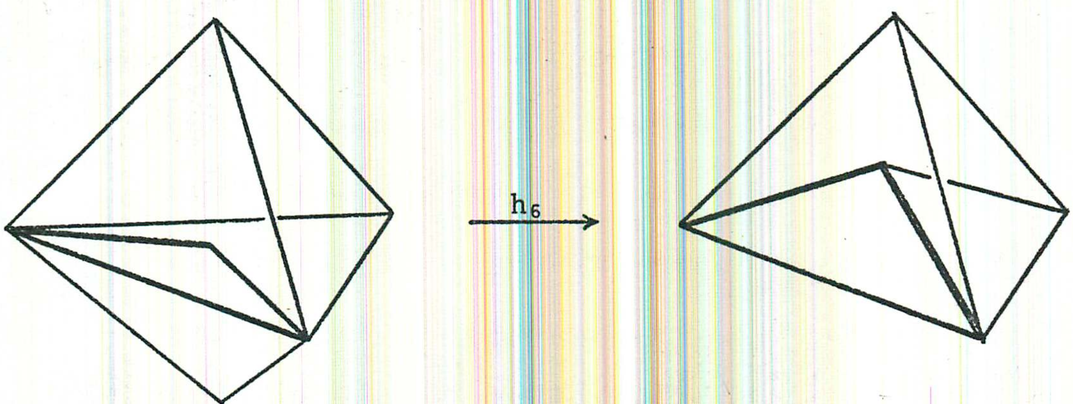
$$h_5\colon (\Sigma(\tau^{a-1}),\tau^{a-1}) \longrightarrow (\Sigma(\Gamma^{a-1}),\Gamma^{a-1}).$$



There is a homeomorphism

$$h_6\colon (\Sigma(\Gamma^{a-1}),\Gamma^{a-1}) \longrightarrow (\Delta^a, b(\Delta^a)\dot{\Delta}^{a-1}),$$

where Δ^a is an a-simplex, such that $h_6(b(\Gamma^{a-1})) = b(\Delta^a)$, $h_6(\Gamma^{a-2}) = \Delta^{a-2}$, an (a-2)-simplex of $\dot{\Delta}^{a-1}$. Therefore $h_6(b(\Gamma^{a-1})\Gamma^{a-2}) = b(\Delta^a)\Delta^{a-2}$.



Let $h = h_6 \circ h_5 \circ h_3 \circ h_2\colon (A,C) \longrightarrow (\Delta^a, b(\Delta^a)\Delta^{a-2})$. Therefore (A,C) is an unknotted ball pair. □

A.2.9. Lemma. Let I = [0,1]. If (A,B) is an (a,b)-ball pair, type 1 (type 2), then (A×I,B×I) is an (a+1,b+1)-ball pair, type 1 (type 2).

Proof.

$\text{bdy}(A\times I) \cap (B\times I) = ((\dot{A}\times I) \cup (A\times \dot{I}))\cap(B\times I)$

$= ((\dot{A}\times I)\cap(B\times I)) \cup ((A\times\dot{I})\cap(B\times I))$

$= ((\dot{A}\cap B)\times(I\cap I)) \cup ((A\cap B)\times(\dot{I}\cap\dot{I}))$

$= ((\dot{A}\cap B)\times I) \cup (B\times\dot{I})$.

If (A,B) is an (a,b)-ball pair, type 1, then $\dot{A}\cap B = \dot{B}$. Therefore

$\text{bdy}(A\times I) \cap (B\times I) = (\dot{B}\times I) \cup (B\times\dot{I}) = \text{bdy}(B\times I)$.

If (A,B) is an (a,b)-ball pair, type 2, then $\dot{A}\cap B = C$, a $(b-1)$-ball in $\dot{B}$.

Therefore

$\text{bdy}(A\times I) \cap (B\times I) = (C\times I) \cup (B\times\dot{I}) = (C\times I) \cup (B\times\{0\}) \cup (B\times\{1\})$.

$C\times I$ is a b-ball contained in $\dot{B}\times I \subset \text{bdy}(B\times I)$. $B\times\{0\}$, $B\times\{1\}$ are b-balls contained in $B\times\dot{I} \subset \text{bdy}(B\times I)$. $(C\times I) \cap (B\times\{0\}) = (B\cap C)\times\{0\} = C\times\{0\}$, a $(b-1)$-ball. $C\times\{0\} \subset C\times\dot{I} \subset \text{bdy}(C\times I)$. $C\times\{0\} \subset \dot{B}\times\{0\} \subset \text{bdy}(B\times\{0\})$.

Therefore $(C\times I) \cup (B\times\{0\})$ is a b-ball.

$((C\times I) \cup (B\times\{0\})) \cap (B\times\{1\}) = C\times\{1\}$, a $(b-1)$-ball.

$C\times\{1\} \subset \text{bdy}((C\times I) \cup (B\times\{0\})) \cap \text{bdy}(B\times\{1\})$.

Therefore $(C\times I) \cup (B\times\{0\}) \cup (B\times\{1\})$ is a b-ball contained in $\text{bdy}(B\times I)$. □

A.2.10 (Lemma III.3.4). Since σ^s is PL homeomorphic to I^s, we may prove the lemma for I, and then apply this result s times for the general case.

By Lemma A.2.9, $(A\times I, B\times I)$ is a ball pair of the same type as (A,B). Suppose first that (A,B) is an unknotted (a,b)-ball pair, type 1. Then there is a homeomorphism h_1: $(A,B) \longrightarrow (\sigma^d, b(\sigma^d)\dot{\sigma}^{d-1})$.

$h_1\times\text{id}$: $(A\times I, B\times I) \longrightarrow (\sigma^d\times I, b(\sigma^d)\dot{\sigma}^{d-1}\times I)$, where id is the identity map, is a PL homeomorphism. By Lemma A.2.6,

$(\sigma^{d-1}\times I) \cup (b(\sigma^d)\sigma^{d-1}\times \dot{I})$ is a d-ball with boundary $(\dot{\sigma}^{d-1}\times I) \cup (b(\sigma^d)\dot{\sigma}^{d-1}\times \dot{I})$.

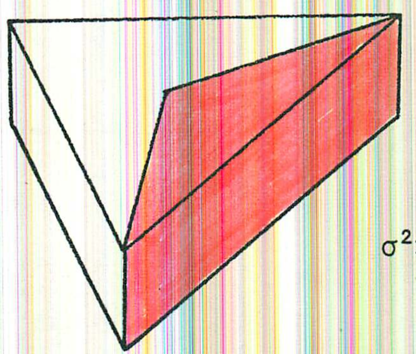
Let τ^d be a face of the (d+1)-simplex τ^{d+1}.

$$f_1 : (\sigma^{d-1}\times I) \cup (b(\sigma^d)\sigma^{d-1}\times \dot{I}) \longrightarrow \tau^d.$$

$b(\sigma^d)\dot{\sigma}^{d-1}\times I$ is a d-ball with boundary $(\dot{\sigma}^{d-1}\times I) \cup (b(\sigma^d)\dot{\sigma}^{d-1}\times \dot{I})$.

Let $f_2 : b(\sigma^d)\dot{\sigma}^{d-1}\times I \longrightarrow b(\tau^{d+1})\dot{\tau}^d$ be the extension of

$$f_1\Big|_{(\dot{\sigma}^{d-1}\times I)\cup(b(\sigma^d)\dot{\sigma}^{d-1}\times\dot{I})} : \text{Def } f_1 \longrightarrow \dot{\tau}^d.$$



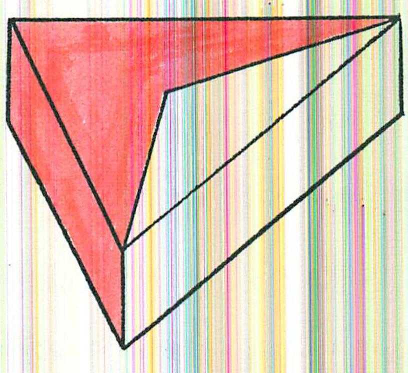
$\sigma^2\times I$

Since f_1 and f_2 agree on the intersection of their domains, a closed set, we may define a homeomorphism of d-spheres

$$f_3 : (\sigma^{d-1}\times I)\cup(b(\sigma^d)\sigma^{d-1}\times I)\cup(b(\sigma^d)\dot{\sigma}^{d-1})\times I \longrightarrow \tau^d \cup b(\tau^{d+1})\dot{\tau}^d.$$

Extend f_3 to the interior, to the homeomorphism of balls

$$f : b(\sigma^d)\sigma^{d-1}\times I \longrightarrow b(\tau^{d+1})\tau^d.$$



Let $g: o(\sigma^{d-1})b(\sigma^d)\dot{\sigma}^{d-1}\times I \longrightarrow o(\tau^d)b(\tau^{d+1})\dot{\tau}^d$ be the extension of $f|_{b(\sigma^d)\dot{\sigma}^{d-1}\times I} : b(\sigma^d)\dot{\sigma}^{d-1}\times I \longrightarrow b(\tau^{d+1})\dot{\tau}^d$.

$\sigma^d\times I = (b(\sigma^d)\sigma^{d-1}\times I) \cup (o(\sigma^{d-1})b(\sigma^d)\dot{\sigma}^{d-1}\times I)$.

Define $H: \sigma^d\times I \longrightarrow \tau^{d+1}$ by $H|_{b(\sigma^d)\sigma^{d-1}\times I} = f$, $H|_{o(\sigma^{d-1})b(\sigma^d)\dot{\sigma}^{d-1}\times I} = g$.

Then H is a PL homeomorphism and $H(b(\sigma^d)\dot{\sigma}^{d-1}\times I) = b(\tau^{d+1})\dot{\tau}^d$.

Therefore $(A\times I, B\times I)$ is an unknotted ball pair, type 1.

If (A,B) is an unknotted ball pair, type 2, then there are homeomorphisms $h_2: (A,B) \longrightarrow (\sigma^d, b(\sigma^d)\sigma^{d-2})$, $h_2\times id: (A\times I, B\times I) \longrightarrow (\sigma^d\times I, b(\sigma^d)\sigma^{d-2}\times I)$. We may apply Lemma A.2.8 to $(\sigma^d\times I, b(\sigma^d)\sigma^{d-2}\times I) \subset (\sigma^d\times I, b(\sigma^d)\dot{\sigma}^{d-1}\times I)$ to find that $(\sigma^d\times I, b(\sigma^d)\sigma^{d-2}\times I)$, and therefore $(A\times I, B\times I)$, is an unknotted ball pair, type 2. □

Bibliography

1. J. Cheeger and J. M. Kister, *Counting topological manifolds*, Topology, Vol. 9, pp. 149-151.

2. C. Ehresmann, *Les connexions infinitesimales dans un espace fibré differentiable*, Colloque de Topologie, Bruxelles (1950), pp. 29-55.

3. ______, *Sur les espaces fibrés differentiables*, Comp. Rend. Acad. Sci. Paris 224 (1947), pp. 1611-1612.

4. J. F. P. Hudson, *Extending piecewise linear isotopies*, Proc. Lond. Math. Soc. 3, 16 (1966), pp. 651-668.

5. ______, *Piecewise Linear Topology*, W. A. Benjamin, Inc., New York, 1969.

6. K. Lamotke, *Die Störung einer isolierten Hyperflächen-Singularität*, (to appear).

7. R. S. Palais, *Natural operations on differential forms*, Trans. Amer. Math. Soc. 92 (1959), pp. 125-141.

8. E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Company, New York, 1966.

9. J. R. Stallings, *Lectures on Polyhedral Topology*, Tata Institute of Fundamental Research, Bombay, 1968.

10. J. G. Timourian, *Fiber bundles with discrete singular set*, J. Math. Mech. 18 (1968), pp. 61-70.

11. ______, *Maps with discrete branch sets between manifolds of codimension one*, Canadian J. Math. 21 (1969), pp. 660-668.

12. G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ. Vol. 28, Amer. Math. Soc., New York, 1942.

13. J. A. Wolf, *Differentiable fibre spaces and mappings compatible with Riemannian metrics*, Mich. J. Math. 11 (1964), pp. 65-70.

14. E. C. Zeeman, *Seminar on Combinatorial Topology*, Institute des Hautes Etudes Scientifiques, 1963.

Index of Notation

$\mathrm{bdy}(J)$	boundary of J	III.2.1
$\mathrm{bdy}_M N$	boundary in M of N	III.3.8
B_f	branch set of f	IV.4
$b(\sigma^d)$	barycenter of σ^d	A.2.6
$\mathrm{cl}_M N$	closure in M of N	III.3.8
$C(\sigma)$	cone over σ	III.3.2
E^p	Euclidean p-space	II.1
F	a PL structure	III.4.8
I^n	n-cube	III.4.1
$\mathrm{int}_M N$	interior in M of N	III.3.8
$K = K_t = S(v_t,(f^{-1}(y))_2)$		IV (1.4)
$\mathrm{lk}(A,K)$	link of A in K	II.6
$L = L_t = T(v_t,(f^{-1}(y))_2)$		IV (1.4)
$N(X,J)$	simplicial neighborhood	III.2.8
$N(X,J^{(r)})$	r^{th} derived neighborhood	III.2.8
$o(\sigma^{d-1})$	the vertex opposite σ^{d-1}	A.2.6

PL	piecewise linear	II.19
$st(A,K)$	star of A in K	II.2
$\overline{st}(A,K)$	closed star of A in K	II.3
$S(x_t,K^{(1)}) = \bigcup_{i=1}^{t} \overline{st}(x_i,K^{(1)})$		II.17
$T(x_t,K^{(1)}) = \bigcup_{i=t+1}^{m} \overline{st}(x_i,K^{(1)})$		II.17
$\Gamma = \Gamma_t = N(S(v_t,(f^{-1}(y))_2) \cap \overline{st}(v_{t+1},(f^{-1}(y))_2), (K \cap L)_2)$		IV (3.4)
$\Sigma\sigma$	suspension of σ	III.3.2
$\dot{A}$	boundary of A	II.13
$\mathring{A}$	interior of A	II.13
$\hat{A}$	a point in $\mathring{A}$	II.14
$\dot{J}$	boundary of J	III.2.1
$\mathring{J}$	interior of J	III.2.1
$A < B$	A is a face of B	II.1
$K \subset L$	K is a subcomplex of L	II.1
$\lvert K \rvert$	the underlying point set	II.8
$K^{(1)}$	first derived	II.14
$K_r = K^{(r)}$	r^{th} derived	II.14
$[x_1,x_2]$	line segment	II.15
$x_{1,2} = \widehat{[x_1,x_2]}$	point in $\mathring{[x_1,x_2]}$	II.15

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