ON MÜNTZ-JACKSON'S THEOREM

A thesis presented

by

Kitty Jia-ling Fang Lung

to

The Graduate School

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in

Mathematics

State University of New York at Stony Brook

November 1970

STATE UNIVERSITY OF NEW YORK

AT STONY BROOK

THE GRADUATE SCHOOL

ON MÜNTZ - JACKSON'S THEOREM — Thesis by

Kitty Fang Lung

We, the thesis committee for the above candidate for the Ph.D. degree, hereby recommend acceptance of the dissertation.

Raouf Doss (Chairman) (RAOUF DOSS)
P Chun (ADVISOR) (PETER SZUSZ)
Mike Fried (MIKE FRIED)
Václav Dolezal (V. DOLEZAL)
J. T. G. Auchmuty (G. AUCHMUTY)

February 4, 1971

The dissertation is accepted by the Graduate School.

Herbert Weisinger Feb 9, 1971
Herbert Weisinger, Dean

ABSTRACT

Jackson Theorem states that given any function $f(x)$ continuous on $[0,1]$ and any positive integer n, there is a polynomial of degree $\leq n$ such that $|f(x) - p_n(x)| < 12\omega_f(\frac{1}{n})$ where $\omega(\delta) = \max_{x, |t| \leq \delta} |f(x+t) - f(x)|$.

Ch. Müntz showed that for any increasing sequence of nonnegative numbers $\{n_j\}$ the linear combinations of x^{n_j} are dense in $C[0,1]$ if and only if 1) $n_o = 0$ and 2) the series $\sum_{j=1}^{\infty} \frac{1}{n_j}$ diverges.

In 1965, D. J. Newman combined the two theorems. He proved that for any given $f(x) \in L^2[0,1]$ and a sequence of nonnegative integers $\{n_j\}$ such that $n_o = 0, n_{j+1} - n_j \geq 2$ and $\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty$. Then for every natural number s, there is a polynomial $\Pi_s^*(x) = \sum_{j=o}^{s} c_j x^{n_j}$ such that

$$\|f(x) - \sum_{j=o}^{s} c_j x^{n_j}\|_2 \leq 3\omega_f^*(\varepsilon_s)$$

where $\varepsilon_s = \prod_{j=1}^{s} (n_j - \frac{1}{2})/(n_j + \frac{3}{2})$

$$\omega_f^*(\delta) = \max_{|t| \leq \delta} [\int_0^1 |f(x+t) - f(x)|^2 dx]^{1/2}$$

The result we obtained here is to extend Newman's result to continuous functions with the "uniform approximation". We showed the following theorem.

Theorem:

Let $n_0 = 0$, $n_1 = 2 < n_2 < n_3 < \dots$ be a sequence of real numbers satisfying the conditions $\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty$ and $n_{j+1} - n_j \geq 1$.

Now let $f(x)$ be any continuous function in $[0,1]$. Then for every natural number s there is a polynomial

$$\Pi_s^*(x) = \sum_{j=0}^{s} y_j x^{n_j}$$

such that

$$|f(x) - \Pi_s^*(x)| = O(\omega_f(\exp(-\sum_{j=1}^{s} \frac{1}{n_j}))).$$

ACKNOWLEDGMENT

I wish to express my gratitude to Professor Peter Szüsz for suggesting this topic and for his guidance throughout its preparation.

TABLE OF CONTENTS

INTRODUCTION:

A classical theorem of Weierstrass states that to any function $f(x)$ defined and continous in $[0,1]$ and any given $\varepsilon > 0$ there is a polynomial $\Pi_n(x)$ such that

(1) $$|f(x) - \Pi_n(x)| < \varepsilon$$

for $0 \le x \le 1$.

There are different generalizations and improvements of this theorem. The first is the question to make a "quantitative" statement of the above theorem. More precisely if we restrict the degree of the approximating polynomials, how small can ε be chosen. This question has been cleared in 1912 by D. Jackson [1]. His result is as follows.

Denote $\omega_f(\delta)$ the modulus of continuity of $f(x)$, that is,

$$\omega_f(\delta) = \max_{\substack{x,h \\ |h| \le \delta}} |f(x+h) - f(x)|$$

then there is a polynomial of degree $\le n$ such that

(1') $$|f(x) - \Pi_n(x)| < K\,\omega_f\left(\frac{1}{n}\right)$$

K being a numerical constant.

Weierstrass theorem can be formulated so that the linear-combinations of the powers of x:

(2) $$1, x, x^2, x^3, \ldots$$

are dense in the space of functions continuous in $[0,1]$.

Another question is whether the same is true of a subsequence of (2). S. Bernstein [2] showed that a necessary and sufficient condition for the denseness of the linear combinations of $\{x^{n_j}\}$ in $C[0,1]$ is $n_o = 0$ and

(3) $$\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty.$$

As a generalization of Bernstein's result Ch. Müntz [3] showed that for any sequence $0 = n_o < n_1 < n_2 < \ldots$ of real numbers the condition (3) is necessary and sufficient in order that the linear combinations of x^{n_j} should be dense in $C[0,1]$. (That is he dropped the condition that $n_o, n_1, n_2, \ldots$ should be a subsequence of natural numbers.) There are different proofs of Müntz' beautiful result. Here I mention that of O. Szász [4] based on a determinant formula of Cauchy reproduced also in Natanson's book [5] and that of Paley-Wiener, reproduced in their book [6] based on their result on complex Fourier-transforms.

D. J. Newman [7] combined the two generalizations of Weierstrass theorem, asking about the accuracy of approximation to $f(x)$ by linear combinations of x^{n_j}. He proved the following theorem.

<u>Theorem</u>:

Let $f(x) \in L^2[0,1]$, given any positive integer s and a sequence of nonnegative integers $\{n_j\}$ with the properties 1) $0 = n_o < n_1 < n_2 < \ldots$ 2) the series $\sum_{j=1}^{\infty} \frac{1}{n_j}$ diverges and 3) $n_{j+1} - n_j \geq 2$.

Then there is a polynomial $\sum_{j=0}^{s} c_j x^{n_j}$ such that

$$\|f(x) - \sum_{j=0}^{s} c_j x^{n_j}\|_2 \leq 3\omega_f^*(\varepsilon_s)$$

where $\omega_f^*(\delta)$ denotes the "L^2-modulus of continuity" of $f(x)$. (i.e. $\omega_f^*(\delta) = \max_{|h|\leq\delta} [\int_0^1 |f(x+h) - f(x)|^2]^{1/2}$) and

$\varepsilon_s = \prod_{j=1}^{s} (n_j - \frac{1}{2})/(n_j + \frac{3}{2})$.

Further he showed that his result is essentially the best possible. The question for the accuracy of approximation in "uniform norm" remained open.

The purpose of this dissertation is to give an estimate for the best approximation in the "uniform norm".

We prove the following

<u>Theorem</u>:

Let $n_o = 0$, $n_1 = 2$, $n_1 < n_2 < \ldots$ be a sequence of real numbers such that $n_{j+1} - n_j \geq 1$ and $\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty$. Then for every natural number s there is a polynomial $\Pi_s^*(x)$ satisfying

$$(4) \qquad |f(x) - \Pi_s^*(x)| \leq K\omega_f(\exp(-\sum_{j=1}^{s} \frac{1}{n_j}))$$

for $0 \leq x \leq 1$. Here $\Pi_s^*(x)$ means linear combinations of x^{n_j} with $j \leq s$.

It can be showed that for some more general choice of $\{n_j\}$, we obtain D. Jackson's result which is known to be the best.

The following proof of our theorem is a combination of D. Jackson's theorem and a sharpening of some estimates of O. Szász, which were used in his proof of Müntz' theorem.

§1 Lemmata.

Lemma 1.1

Let $f(\theta)$ be a continuous function in $[0,2\Pi]$ with the period 2Π. Let $s_n(\theta) = \sum_{\ell=0}^{n} (a_\ell \cos \ell\theta + b_\ell \sin \ell\theta)$ and if $|s_n(\theta) - f(\theta)| < \varepsilon$ for $0 \leq \theta \leq 2\Pi$ then

$$(1.1) \qquad \begin{cases} |a_\ell| < 2(M+\varepsilon) \\ |b_\ell| < 2(M+\varepsilon) \end{cases}$$

where $M = \max_{\theta\in[0,2\Pi]} |f(\theta)|$.

Proof.

Clear.

Lemma 1.2

Denote by $T_m(x) = \sum_{\ell=0}^{m} C_{m,\ell}\, x^\ell$ the Tchebysheff polynomials of first kind, that is,

$$T_m(x) = \cos(m \text{ arc } \cos x).$$

Then we have

$$(1.2) \qquad |C_{m,\ell}| \leq \frac{2m^{\frac{3}{2}\ell}}{\ell!(\frac{\ell}{2})!}$$

Proof.

As known, we have

$$T_m(x) = \frac{1}{2}[(x+i\sqrt{1-x^2})^m + (x-i\sqrt{1-x^2})^m]$$

$$= \sum_{0\leq k\leq \frac{m}{2}} (-1)^k \binom{m}{2k} x^{m-2k}(1-x^2)^k$$

$$= \sum_{0\le k\le \frac{m}{2}} (-1)^k \binom{m}{2k} x^{m-2k} \Big(\sum_{0\le j\le k} (-1)^j \binom{k}{j} x^{2j} \Big)$$

$$= \sum_{0\le k\le \frac{m}{2}} \sum_{0\le j\le k} (-1)^{k+j} \binom{m}{2k}\binom{k}{j} x^{m-2k+2j}.$$

Now set $m-2k+2j = \ell$, we have

$$T_m(x) = \sum_{0\le \ell\le m} \Big((-1)^{\frac{m-\ell}{2}} \sum_{\frac{m-\ell}{2}\le k\le \frac{m}{2}} \binom{m}{2k}\binom{k}{\frac{m-\ell}{2}} \Big) x^{\ell} \text{ if } 2\,|\,(m-\ell).$$

Therefore, we obtain

$$(1.3)\quad C_{m,\ell} = \begin{cases} (-1)^{\frac{m-\ell}{2}} \sum_{\frac{m-\ell}{2}\le k\le \frac{m}{2}} \binom{m}{2k}\binom{k}{\frac{m-\ell}{2}} & \text{if } 2\,|\,(m-\ell) \\ 0 & \text{otherwise} \end{cases}$$

Hence it suffices to investigate $C_{m,\ell}$ only for the case $2\,|\,(m-\ell)$.

Case (i). $\ell \le \frac{m}{2}$

Since $\binom{m}{2k}$ here takes its maximum for $k = \frac{m}{4}$ and decreases monotonically for $\frac{m}{4} \le k \le \frac{m}{2}$. Hence

$$\binom{m}{2k} = \binom{m}{m-2k} < \binom{m}{\ell} < \frac{m^{\ell}}{\ell!}$$

And since $\binom{k}{\frac{m-\ell}{2}}$ increases monotonically as k increases, we have

$$\binom{k}{\frac{m-\ell}{2}} \le \binom{\frac{m}{2}}{\frac{m-\ell}{2}} < \frac{(\frac{m}{2})^{\frac{\ell}{2}}}{(\frac{\ell}{2})!} \quad .$$

Therefore

$$|C_{m,\ell}| \le \sum_{\frac{m-\ell}{2}\le k\le\frac{m}{2}} \binom{m}{2k}\binom{k}{\frac{m-\ell}{2}} < \frac{\ell}{2}\cdot\frac{m^\ell}{\ell!}\cdot\frac{(\frac{m}{2})^{\frac{\ell}{2}}}{(\frac{\ell}{2})!}$$

$$= \frac{m^{\frac{3}{2}\ell}}{\ell!(\frac{\ell}{2})!}\ \frac{\ell}{2^{\frac{\ell}{2}+1}} < \frac{2m^{\frac{3}{2}\ell}}{\ell!(\frac{\ell}{2})!}$$

Case (ii). $\ell > \frac{m}{2}$

Similarly, we have here $\binom{k}{\frac{m-\ell}{2}} < \frac{(\frac{m}{2})^{\frac{\ell}{2}}}{(\frac{\ell}{2})!}$ and

$\binom{m}{2k} \le \binom{m}{\frac{m}{2}} \le \frac{m^{\frac{m}{2}}}{(\frac{m}{2})!} \le \frac{m^\ell}{\ell!}$, for $f(n) = \frac{m^n}{n!}$ is an increasing function of n, if $n \le m$. So we have again $|C_{m,\ell}| < \frac{2m^{\frac{3}{2}\ell}}{\ell!(\frac{\ell}{2})!}$. Hence the proof is complete.

Lemma 1.3

Let $0 < n_1 < n_2 < n_3 < \dots < n_s$ be a given finite set of real numbers. Then for any nonnegative integer m there exists coefficients $y_1, y_2, \dots, y_s$ with

$y_i = \frac{\Delta_s^{(i)}}{\Delta_s}$ such that the linear combination

$$p(x) = \sum_{j=1}^{s} y_j x^{n_j}$$

approximate x^m in $L^2[0,1]$ with the smallest possible mean error $\frac{1}{\sqrt{2m+1}} \prod_{j=1}^{s} \left|\frac{n_j-m}{n_j+m+1}\right|$

i.e. $$\min_{a_1,a_2,\dots,a_s} \int_0^1 (x^m - \sum_{j=1}^{s} a_j x^{n_j})^2 dx = \frac{1}{2m+1} \prod_{j=1}^{s} \left(\frac{n_j-m}{n_j+m+1}\right)^2$$

where Δ_s and $\Delta_s^{(i)}$ are the Gram determinants of the functions $\{x^{n_1}, x^{n_2}, \ldots, x^{n_s}\}$ and $\{x^{n_1}, \ldots, x^{n_{i-1}}, x^m, x^{n_{i+1}}, \ldots, x^{n_s}\}$ respectively.

Proof.

See Natanson [5] vol. II, pp. 36-41.

Lemma 1.4

Under the conditions of Lemma 1.3, for every natural number $m \geq 1$, there is a system $y_1, y_2, \ldots, y_s$ such that

$$(1.5) \qquad |x^m - \sum_{j=1}^{s} y_j x^{n_j}| \leq \sqrt{m} \prod_{j=1}^{s} \left|\frac{n_j - m}{n_j + m - 1}\right|$$

Proof.

We apply lemma 1.3 with x^{m-1} and $n_1-1, n_2-1, \ldots, n_s-1$ which obviously still satisfy the given conditions. So we obtain that for some system $y_1', y_2', \ldots, y_s'$ the inequality

$$\int_0^1 (mx^{m-1} - \sum_{j=1}^{s} y_j' x^{n_j - 1})^2 dx = \frac{m^2}{2m-1} \prod_{j=1}^{s} \left(\frac{n_j - m}{n_j + m - 1}\right)^2$$

Now put $y_j' = n_j y_j$.

We have by Schwarz' inequality

$$|x^m - \sum_{j=1}^{s} y_j x^{n_j}| = |\int_0^x (mt^{m-1} - \sum_{j=1}^{s} n_j y_j t^{n_j - 1}) dt|$$

$$\leq x^{\frac{1}{2}} \cdot [\int_0^x (mt^{m-1} - \sum_{j=1}^{s} n_j y_j t^{n_j - 1})^2 dt]^{1/2}$$

$$\leq [\int_0^1 (mt^{m-1} - \sum_{j=1}^{s} n_j y_j t^{n_j - 1})^2 dt]^{1/2}$$

$$\leq \sqrt{m} \prod_{j=1}^{s} \left|\frac{n_j - m}{n_j + m - 1}\right|$$

and our lemma is proved.

Lemma 1.5

Let $0 < n_1 < n_2 < n_3 < \dots < n_s$ and $s \geq 1$ be given as of lemma 1.3. We have for any given $\delta \geq 1+\varepsilon$

$$(1.6) \qquad \prod_{j=1}^{s} \left|\frac{n_j-m}{n_j+m-1}\right| < K \exp\Big(-(2m-1) \sum_{\delta m<n_j\leq n_s} \frac{1}{n_j}\Big)$$

K being a constant depending only on δ.

Proof.

We have

$$\prod_{j=1}^{s} \left|\frac{n_j-m}{n_j+m-1}\right| \leq \prod_{\delta m<n_j\leq n_s} \left|\frac{n_j-m}{n_j+m-1}\right|$$

$$= \exp\{\sum_{\delta m<n_j\leq n_s} [\log(n_j-m) - \log(n_j+m-1)]\}$$

$$= \exp\{\sum_{\delta m<n_j\leq n_s} [\log(1-\frac{m}{n_j}) - \log(1+\frac{m-1}{n_j})]\}$$

$$= \exp\{\sum_{\delta m<n_j\leq n_s} [(-\frac{m}{n_j} - \frac{1}{2}(\frac{m}{n_j})^2 - \frac{1}{3}(\frac{m}{n_j})^3 - \dots)$$

$$- (\frac{m-1}{n_j} - \frac{1}{2}(\frac{m-1}{n_j})^2 + \frac{1}{3}(\frac{m-1}{n_j})^3 - \dots)]\}$$

$$= \exp\{\sum_{\delta m<n_j\leq n_s} (-\frac{m}{n_j} + O(1)) - (\frac{m-1}{n_j} + O(1))\}$$

$$\leq K \exp\{-(2m-1) \sum_{\delta m<n_j\leq n_s} \frac{1}{n_j}\}.$$

§2 Completion of the proof of the theorem.

Now we are in a position to finish our proof. We formulate our theorem again:

Theorem.

Let $0 = n_o < n_1 = 2 < n_3 < \ldots$ be a sequence of real numbers satisfying

$$(2.1) \qquad \sum_{j=1}^{\infty} \frac{1}{n_j} = \infty$$

and

$$(2.2) \qquad n_{j+1} - n_j \geq 1.$$

Further, let $f(x)$ be any function continuous in $[0,1]$. Then for every natural number $s \geq 1$ there is a linear combination of the x^{n_j}'s

$$\Pi_s^*(x) = \sum_{j=o}^{s} y_j x^{n_j}$$

such that

$$(2.3) \qquad |f(x) - \Pi_s^*(x)| = O(\omega_f(\exp(-\sum_{j=1}^{s} \frac{1}{n_j})))$$

where $\omega_f(\delta)$ is the modulus of continuity of $f(x)$.

Proof.

Without loss of generality we may suppose that our function is even; namely for any given continuous function in $[0,1]$, we may define it this way by setting $f(-x) = f(x)$ for $0 \leq x \leq 1$. We then continue $f(x)$ on the real line to have the period 2.

Now let $x = \cos\theta$ for $-1 \le x \le 1$. Thus $f(x) = f(\cos\theta) = \varphi(\theta)$ is a continuous function with the period Π.

Let k be a natural number to be determined later. From Jackson's theorem, for every given $\varphi(\theta) \in C_{2\Pi}$ there is a trigonometric polynomial $g_k(\theta)$ of degree $\le k$ such that

$$|\varphi(\theta) - g_k(\theta)| \le 12\omega_\varphi(\tfrac{1}{k}) \qquad 0 \le \theta \le 2\Pi.$$

Since f(x) is even, so is the trigonometric polynomial which best approximates it. That is, $g_k(\theta)$ is a trigonometric polynomial of cosines alone. Therefore, $g_k(\theta)$ can be written in the form

$$g_k(\theta) = \sum_{\ell=0}^{k} a_\ell \cos \ell\theta = \sum_{\ell=0}^{k} a_\ell T_\ell(x) = p_k(x).$$

Then a simple computation will show that

$$\omega_\varphi(\delta) \le \omega_f(\delta).$$

Hence we obtain

$$|f(x) - p_k(x)| \le 12\omega_f(\tfrac{1}{k}) \qquad -1 \le x \le 1.$$

Moreover, since f(x) is even; there exists always an even polynomial $\Pi_k(x)$ which satisfies the condition

(2.4) $$|f(x) - \Pi_k(x)| \le 12\omega_f(\tfrac{1}{k}).$$

For we can set

$$\Pi_k(x) = \frac{p_k(x) + p_k(-x)}{2} = \sum_{0\le\ell\le[\frac{k}{2}]} d_{2\ell}T_{2\ell}(x).$$

From lemma 1.1 we obtain

$$(2.5) \qquad |d_{2\ell}| \le C_1 \qquad \ell = 0,1,2,\ldots,[\tfrac{k}{2}]$$

C_1 being a constant independent of k. Now we write

$$(2.6) \qquad \Pi_k(x) = \sum_{0\le\ell\le[\frac{k}{2}]} y_{2\ell}x^{2\ell}$$

where

$$(2.7) \qquad y_{2\ell} = \sum_{\ell\le m\le[\frac{k}{2}]} d_{2m}C_{2m,2\ell}$$

and $C_{2m,2\ell}$ having the same meaning as in lemma 1.2. From lemma 1.1 and lemma 1.2 it follows that

$$(2.8) \qquad |y_{2\ell}| \le \sum_{\ell\le m\le[\frac{k}{2}]} |d_{2m}C_{2m,2\ell}| \le C_1 \frac{k^{3\ell+1}}{(2\ell)!\ell!}$$

Let a natural number s be given. According to lemma 1.4 and lemma 1.5, for every given positive integer ℓ there is a linear combination of x^{n_j} with $j \le s$; say $\Pi^*_{2\ell,s}(x)$ such that

$$(2.9) \qquad |x^{2\ell} - \Pi^*_{2\ell,s}(x)| \le C_2\exp(-(4\ell-1)\sum_{2\ell\delta<n_j\le n_s} \frac{1}{n_j})$$

Now we define

$$\Pi^*_s(x) = \sum_{0\le\ell\le[\frac{k}{2}]} y_{2\ell}\Pi^*_{2\ell,s}(x).$$

Clearly

$$(2.10) \qquad \begin{aligned} |f(x) - \Pi_s^*(x)| &\le |f(x) - \Pi_k(x)| + |\Pi_k(x) - \Pi_s^*(x)| \\ |f(x) - \Pi_s^*(x)| &\le |f(x) - \Pi_k(x)| \\ &+ \sum_{0\le \ell \le [\frac{k}{2}]} |y_{2\ell}(x^{2\ell} - \Pi_{2\ell,s}^*(x))| \end{aligned}$$

The first term of the right-hand side of (2.10) is, because of (2.4), $O(\omega_f(\frac{1}{k}))$. Furthermore, since we have $n_o = 0$ and $n_1 = 2$, it thus follows that $|1 - \Pi_{o,s}^*(x)| = 0$ and $|x^2 - \Pi_{2,s}^*(x)| = 0$, and therefore the second term on the right-hand side of (2.10) can be replaced by

$$(2.11) \qquad \sum_{2\le \ell \le [\frac{k}{2}]} |y_{2\ell}| \; |x^{2\ell} - \Pi_{2\ell,s}^*(x)|$$

We now decompose (2.11) into two parts according to $\ell = 2$ and $\ell > 2$, we have

$$(2.12) \qquad \begin{aligned} &\sum_{2\le \ell \le [\frac{k}{2}]} |y_{2\ell}| \; |x^{2\ell} - \Pi_{2\ell,s}^*(x)| \\ &= |y_4| \; |x^4 - \Pi_{4,s}^*(x)| + \sum_{3\le \ell \le [\frac{k}{2}]} |y_{2\ell}| \; |x^{2\ell} - \Pi_{2\ell,s}^*(x)| \\ &= \Sigma_1 + \Sigma_2 . \end{aligned}$$

Now from formula (1.3) we can easily see that

$$C_{2m,4} = O(m^4).$$

Hence according to (2.5) and (2.7) we obtain

$$|y_4| = \Big|\sum_{2\le m \le [\frac{k}{2}]} d_{2m} C_{2m,4}\Big| = O(k^5).$$

But by (1.5) and (1.6) we get

$$|x^4 - \Pi^*_{4,s}(x)| \le C_3 \exp(-(4\cdot 2-1) \sum_{4\delta<n_j\le n_s} \frac{1}{n_j})$$

and so

$$(2.13) \qquad \Sigma_1 \le C_4 k^5 \exp(-(4\cdot 2-1) \sum_{4\delta<n_j\le n_s} \frac{1}{n_j})$$

Similarly, combining (2.5), (2.7) with (1.5) and (1.6) give the upper estimation of Σ_2.

$$(2.14) \qquad \Sigma_2 \le C_5 \sum_{3\le \ell\le [\frac{k}{2}]} \frac{\sqrt{2\ell}}{(2\ell)!\ell!} k^{3\ell+1} \exp(-(4\ell-1) \sum_{2\ell\delta<n_j\le n_s} \frac{1}{n_j})$$

Now for every given natural number s, let a positive integer k be determined in the following way.

$$(2.15) \qquad k = [\exp(\sum_{j=1}^{s} \frac{1}{n_j} - 1)]$$

Hence

$$(2.16) \qquad \sum_{j=1}^{s} \frac{1}{n_j} \ge \log k + 1$$

and therefore

$$(2.17) \qquad \exp(-(4\ell-1) \sum_{j=1}^{s} \frac{1}{n_j}) \le \frac{1}{k^{4\ell-1}} \exp(-(4\ell-1))$$

But

$$(2.18) \qquad \exp(-(4\ell-1) \sum_{2\ell\delta<n_j\le n_s} \frac{1}{n_j})$$

$$= \exp(-(4\ell-1)(\sum_{j=1}^{s} \frac{1}{n_j} - \sum_{2\le n_j\le 2\ell\delta} \frac{1}{n_j})).$$

We now make use of our assumption $n_{j+1} - n_j \ge 1$, it follows at once that

$$(2.19)\qquad \sum_{2\le n_j\le 2\ell\delta} \frac{1}{n_j} \le \log 2\ell\delta = \log \ell + C.$$

If δ is chosen so that $1 < \delta < \frac{e}{2}$, we then have $C = \log 2\delta < 1$.

Combining (2.19) with (2.16) and (2.18), we shall have

$$(2.20)\qquad \exp(-(4\ell-1)\sum_{2\ell\delta<n_j\le n_s}\frac{1}{n_j})$$

$$\le \exp\{-(4\ell-1)(\log k+1) + (4\ell-1)(\log \ell+C)\}$$

$$= \frac{\ell^{4\ell-1}}{k^{4\ell-1}} \exp\{-(4\ell-1)(1-C)\}.$$

We now estimate Σ_1 of (2.12), from (2.13) and (2.20) we get

$$\Sigma_1 \le C_4 k^5 \cdot \frac{2^7}{k^7} \exp(-7(1-C)) = O(\frac{1}{k^2}).$$

Similarly, from (2.14) and (2.20) we have

$$\Sigma_2 \le C_5 \sum_{3\le\ell\le[\frac{k}{2}]} \frac{\sqrt{2\ell}\ k^{3\ell+1}}{(2\ell)!\ell!} \frac{\ell^{4\ell-1}}{k^{4\ell-1}} \exp(-(4\ell-1)(1-C))$$

or

$$(2.21)\qquad \Sigma_2 = \frac{C_5}{k} \sum_{3\le\ell\le[\frac{k}{2}]} \frac{\sqrt{2\ell}}{(2\ell)!\ell!} \frac{\ell^{4\ell-1}}{k^{\ell-3}} \exp(-(4\ell-1)(1-C))$$

Consider the ratio:

$$\gamma = \lim_{\ell\to\infty} \frac{\sqrt{2\ell+2}}{(2\ell+2)!(\ell+1)!} \frac{\ell^{4\ell+3}}{k^{\ell-2}} \exp(-(4\ell+3)(1-C))$$

$$\cdot \frac{(2\ell)!\ell!}{\sqrt{2\ell}} \frac{k^{\ell-3}}{\ell^{4\ell-1}} \exp((4\ell-1)(1-C))$$

$$= \lim_{\ell\to\infty} \frac{\ell^4}{(2\ell+2)(2\ell+1)(\ell+1)k} \exp(-4(1-C)).$$

Since $\ell \leq [\frac{k}{2}]$ and $C < 1$, it is clear that $\gamma < 1$. Thus we may write

$$(2.22) \qquad \Sigma_2 = O(\tfrac{1}{k})$$

and therefore

$$|\Pi_k(x) - \Pi_s^*(x)| = \Sigma_1 + \Sigma_2 = O(\tfrac{1}{k}).$$

Finally, we have

$$|f(x) - \Pi_s^*(x)| = O(\omega_f(\tfrac{1}{k})) + O(\tfrac{1}{k}) = O(\omega_f(\tfrac{1}{k}))$$

unless $f(x) \equiv$ constant. In which case, the Müntz-Jackson theorem is obviously true.

Hence we have in general

$$(2.23) \qquad |f(x) - \Pi_s^*(x)| = O(\omega_f(\tfrac{1}{k})).$$

From (2.15) it follows that

$$(2.24) \qquad \exp(-(\sum_{j=1}^{s} \frac{1}{n_j} - 1)) \leq \frac{1}{k} \leq \exp(-(\sum_{j=1}^{s} \frac{1}{n_j} - 1)) + 1$$

Because of (2.24) we may rewrite (2.23) as

$$(2.25) \qquad |f(x) - \Pi_s^*(x)| = O(\omega_f(\exp(-\sum_{j=1}^{s} \frac{1}{n_j})))$$

for every natural number s.

The theorem is herewith completely proved.

References

[1] D. Jackson: On approximation by trigonometric sums and polynomials. Trans. Amer. Math. Soc. 11 (1912) pp. 491-515.

[2] S. Bernstein: Sur les recherches recentes relatives a la meilleure approximation des fonctions continues par les polynomes. Proc. of the Fifth International Congress of Mathematicians. Cambridge, 1913, Vol. I, pp. 256-266.

[3] Ch. Müntz: Über den Approximationssatz von Weierstrass. Schwarz-Festschift (1914) pp. 303-312.

[4] O. Szász: Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen. Math. Ann. 77 (1916) pp. 482-496.

[5] Natanson: Constructive theory of Functions I-II, New York, 1964.

[6] Paley-Wiener: Fourier Transforms in the complex domain. Trans. Amer. Math. Soc. 19 (1934).

[7] D. J. Newman: A Müntz-Jackson theorem. Amer. Journ. of Math. 87 (1965) pp. 940-944.