Monotony of Topological Entropy

Happy Birthday Jack

Sebastian van Strien (Univ of Warwick)

Banff, Feb 2011

Coauthors

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With this in mind, Henk and I are rewriting our 2009 preprint; a completely rewritten version should be in the arxiv in a few weeks.

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On iterated maps of the interval: I,II.

Milnor and Thurston proved

Theorem

The function

$$C^{2,d} \ni g \to h_{top}(g) \in \mathbb{R}^+$$

which associates to each mapping $g \in C^{2,d}$ *its* **topological entropy** $h_{top}(g)$ *is continuous.*

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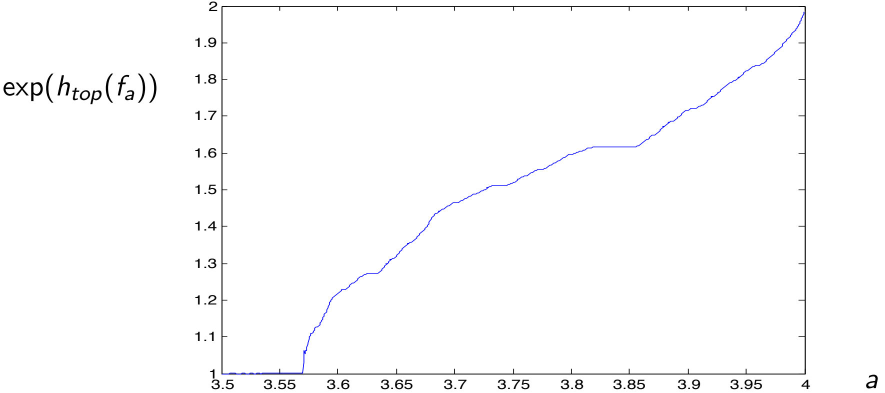
- Misiurewicz & Szlenk: $\exp(h_{top}(f)) =$ growth rate of the number of laps of f^n.
- Milnor & Thurson: it is also equal to the zero of a meromorphic function.

Motivation

About 25 years ago, Sullivan, Milnor, Thurston and Douady & Hubbard all showed that

the *topological entropy* of $x \mapsto ax(1-x)$ increases with $a \in \mathbb{R}$.



In fact, it turns out that periodic orbits disappear when a increases; moreover, as was shown later on (by **Lyubich** and **Graczyk & Swiatek**) hyperbolic maps are dense within this family.

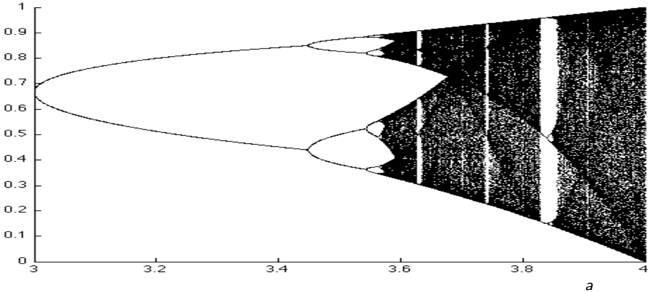


Figure: Bifurcation diagram

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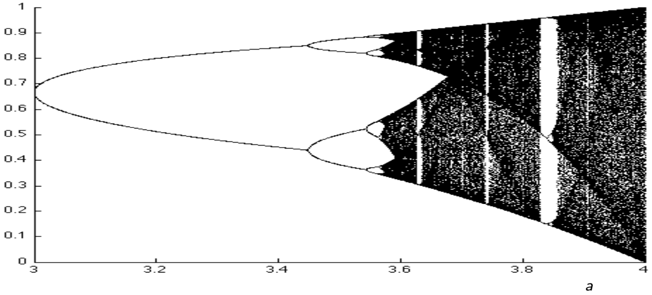

Figure: Bifurcation diagram

In this talk, I want to discuss a generalization of this statement.

Of course monotonicity of

$$a \mapsto h_{top}(f_a)$$

is equivalent to the statement that **isentropes**, i.e. the level sets

$$I(h_0) := \{a; h_{top}(f_a) = h_0\}$$

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This was observation is one reason for Milnor's conjecture on the next slide.

Consider the space P^d_ϵ of real polynomials of degree d with

- all critical points of f are real and contained in $(-1, 1)$;
- $f\{\pm 1\} \subset \{\pm 1\}$;
- with shape ϵ:

$$\epsilon = \begin{cases} +1 & \text{if } f \text{ is increasing at the left endpoint of } [0,1], \\ -1 & \text{otherwise.} \end{cases}$$

Conjecture (Milnor's monotony of entropy conjecture)

Given $\epsilon \in \{-1, 1\}$, isentropes are connected in $f \in P^d_\epsilon$, i.e., the set of $f \in P^d_\epsilon$ with topological entropy equal to h, is connected.

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Theorem (Milnor & Tresser)

The entropy conjecture is true when $d = 3$.

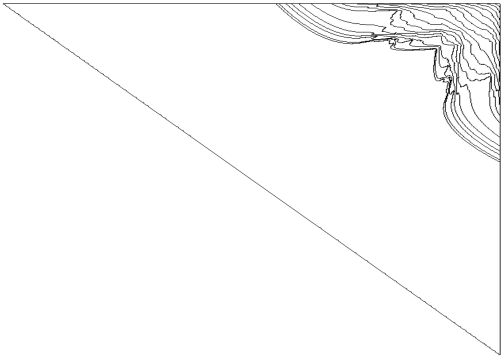


Figure: Isotropies in entropy for cubic maps. The horizontal and vertical axis determine the position of the first resp. second critical value.

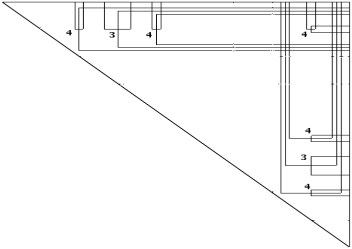
Milnor and Tresser

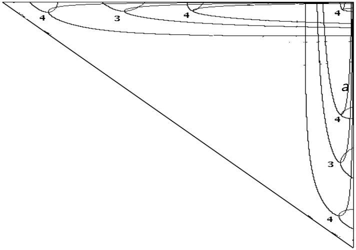
analyse bifurcation curves,

see figures on the right.

They use planar topology

to show 'bones' are connected.





The aim of this talk is to discuss a theorem proving Milnor's conjecture:

Theorem (Monotonicity of entropy, Bruin and SvS, 2009)

Fix $\epsilon \in \{-1, 1\}$. Isentropes in P^d_ϵ are connected.

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Theorem (Monotonicity of entropy, Bruin and SvS, 2009)

Fix $\epsilon \in \{-1, 1\}$. Isentropes in P_ϵ^d are connected.

We have not yet proved

Conjecture (Milnor)

Fix $\epsilon \in \{-1, 1\}$. Isentropes in P_ϵ^d are contractible.

but are very hopeful, for reasons I will explain.

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Question (Thurston)

Does there exist a dense set of level sets $H \subset [0, \log(d)]$ so that for any $h_0 \in H$, the isentrope $I(h_0)$ in P^d_ϵ contains a dense set of hyperbolic maps?

As usual, by definition **hyperbolic** maps are maps so that each critical point is in the basin of a periodic attractor.

How to analyse higher dimensional parameter space?

Higher dimensional parameter spaces are very complicated.

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- Following Milnor & Tresser we use **stunted sawtooth maps**, which form a model for d-modal interval maps.
- We restrict to **admissable sawtooth maps** (i.e. 'absence of Levy cycles') and prove that isentropes within this set are **contractible**.

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- We restrict to **admissable sawtooth maps** (i.e. 'absence of Levy cycles') and prove that isentropes within this set are **contractible**.
- A discussion on how to relate the spaces of polynomials with stunted sawtooth maps in a suitable manner.

One crucial ingredient for our proof is a result used by Kozlovski, Shen and SvS to prove hyperbolic maps are dense in P^d, for any d:

Theorem (Rigidity)

Let $f, g \in P^d$. Assume that f and g are partially conjugate and that f, g are conformally conjugate restricted to their immediate basins of periodic attractors. Then $f = g$.

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In fact, we need a version of this theorem which gives a description of the **partial hyperbolic deformation space**. (Generalising the notion of hyperbolic component.)

Let $B(f)$ consists of all points x so that $f^n(x)$ tends to a (possibly one-sided) periodic attractor.

- We say that two d-modal maps $f, g\colon [-1,1] \to [-1,1]$ are **partially conjugate** if there is a homeomorphism $h\colon [-1,1] \to [-1,1]$ such that
  - h maps $B(f)$ onto $B(g)$;
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- Let $\mathcal{PH}(f)$ be the set of maps which are partially conjugate to f.
- $\mathcal{PH}^o(f)$ consists of maps $g \in \mathcal{PH}(f)$ with
  - only hyperbolic periodic points and
  - no critical point of g maps to the boundary of a component of $B(g)$.

Theorem (Description of partial conjugacy class)

Let $f \in P_\varepsilon^d$. Then

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Analogously to the **Douady-Hubbard** result for quadratic maps if each periodic attractor has precisely one critical point in its basin. Then $\mathcal{PH}^o(f)$ is parametrized by multipliers at the periodic attractors.

More generally, if there are **several critical points in the basin** of one periodic attractor then $\mathcal{PH}^o(f)$ is parametrized by Boetcher functions (and for example critical relations unfold transversally).

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In fact, bifurcations near $f \in \mathcal{PH}(f) \setminus \mathcal{PH}^o(f)$ are also generic:

- saddle-node (creation of one-sided attractor, which then becomes becomes an attracting + repelling pair)
- pitchfork (a two-sided attractor, which becomes repelling and spins off a pair of attracting orbits)
- period-doubling (multiplier -1)
- homoclinic bifurcation (with a critical point hitting the boundary of the basin)

Second Ingredient: model for the parameter space, relating polynomials with their combinatorics

- Given a piecewise monotone d-modal map f with turning points $c_1, \ldots, c_d$, associate to $x \in [-1, 1]$ its **itinerary** $i_f(x)$ consisting of symbols from the alphabet

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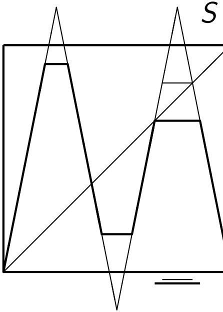
- The **kneading invariant** $\nu(f)$ of f is defined as

$$\nu(f) := (\nu_1, \dots, \nu_d).$$

Any kneading sequence which is realized by some piecewise monotone d-modal map is called *admissible*.

- The space of kneadings with the natural topology is not connected.
- So it is easier to work in a better space, the space of stunted sawtooth maps which are stunted versions of some fixed map S with slope $\pm\lambda$.



The sawtooth map S

Two stunted sawtooth maps,

with different third plateaus.

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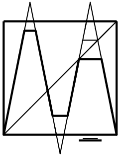
$$\lim_{y \downarrow s_i} i_S(y) = \nu_i := \lim_{x \downarrow c_i} i_f(x)$$



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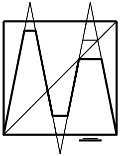

- Such a point s_i **exists**, because all kneading sequences are realized by S. It is **unique** since S is expanding and so distinct points have different different kneading sequences.

Assigning a stunted sawtooth map to a polynomial

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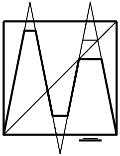

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Associate to each polynomial the stunted seesaw map $\Psi(f)$

- which is constant on a plateau Z_i with right endpoint s_i
- which agrees with S outside $\cup Z_i$.

What is good and bad about the space $\mathcal{S}^d$?

The map

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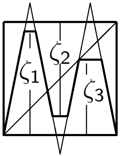

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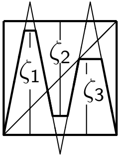

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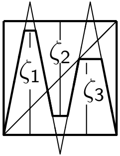

- $T \mapsto h_{top}(T)$ is monotone increasing in each parameter ζ_i (describing the **height** of the i-th plateau.
- Using this, it is easy to show **isentropes are connected (and even contractible)**

- Polynomial maps have **no wandering intervals**. Hence if the endpoints of an interval containing two distinct critical points have the same itineraries, then the interval is contained in the basin of a periodic attractor.

Addressing non-surjectivity of Ψ: non-degerate sawtooth maps

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- Analogously, $\mathcal{S}_*^d \subset \mathcal{S}^d$ consists of maps T so that if
  - an interval J contains two plateaus **and**
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Theorem

The space of maps in $\mathcal{S}^d_*$ with constant entropy is connected and even contractible.

Equivalence classes of sawtooth maps

Define the *plateau-basin* $\mathcal{PB}(T)$:

$$\mathcal{PB}(T) = \{y;\, T^k(y) \in \text{interior}(\cup_{i=1}^{d} Z_{i,T})$$

$$\text{for some } k \geq 0\}.$$

In order to ignore what happens within basin of periodic attractors, define

$$\langle T \rangle = \{\tilde{T} \in \mathcal{S}^d; \mathcal{PB}(\tilde{T}) = \mathcal{PB}(T)\}$$

and

$$[T] = \text{closure}(\langle T \rangle).$$

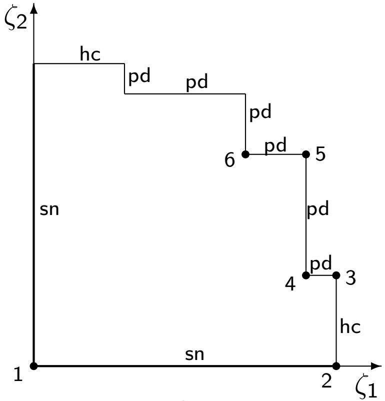

Figure: The case of a periodic component W of $W(T)$ of period $s_1 + s_2$ so that W and the component W' of $\mathcal{PB}(T)$ containing $T^{s_1}(W)$ both contain a plateau.

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Proposition (Injectivity)

If $f_1, f_2 \in P^d$ *and* $[\Psi(f_1)] \cap [\Psi(f_2)] \neq \emptyset$ *then* $\overline{\mathcal{PH}(f_1)} \cap \overline{\mathcal{PH}(f_2)} \neq \emptyset$.

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For each $T \in \mathcal{S}^d_*$ there exists $f \in P^d$ so that $T \in [\Psi(f)]$.

Proposition (Injectivity)

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The upshot

Theorem

If K is closed and connected then $\Psi^{-1}(K) = \{f; [\Psi(f)] \cap K \neq \emptyset\}$ is connected.

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Corollary

Isotropy sets in P^d, i.e. level sets of constant topological entropy, are connected.

Question

Are isentropes contractible?

Probably yes, but this is work in progress.