Endomorphisms of complex projective spaces

Anna Zdunik

University of Warsaw

Frontiers in Holomorphic Dynamics, 2011

Plan

1. Basic facts about holomorphic endomorphisms of $\mathbb{P}^k$, maximal entropy measure
2. Equilibrium states for Hölder- continuous potentials
3. Construction of the equilibrium measure
4. Stochastic properties of the measure μ_ϕ

Plan

1. Basic facts about holomorphic endomorphisms of $\mathbb{P}^k$, maximal entropy measure
2. Equilibrium states for Hölder- continuous potentials
3. Construction of the equilibrium measure
4. Stochastic properties of the measure μ_ϕ

Plan

1. Basic facts about holomorphic endomorphisms of $\mathbb{P}^k$, maximal entropy measure
2. Equilibrium states for Hölder- continuous potentials
3. Construction of the equilibrium measure
4. Stochastic properties of the measure μ_ϕ

Plan

1. Basic facts about holomorphic endomorphisms of $\mathbb{P}^k$, maximal entropy measure
2. Equilibrium states for Hölder- continuous potentials
3. Construction of the equilibrium measure
4. Stochastic properties of the measure μ_ϕ

Projective space

Space

$$\mathbb{P}^k = \mathbb{C}^{k+1} \setminus \{0\} / \sim$$

$$z \sim w \iff z = \lambda w, \lambda \in \mathbb{C}$$

Endomorphisms

$$F(z_0, z_1, \ldots, z_k) = (f_0(z_0, \ldots, z_k), \ldots, f_k(z_0, \ldots z_k))$$

where f_i are homegenous polynomials of degree d, having no (nontrivial) common roots

Degree

- Algebraic degree– d–the common degree of polynomials $f_0, f_1, \ldots, f_k$
- Topological degree- cardinality of the set $f^{-1}(\{x\})$ for a generic point equals d^k.

Degree

- Algebraic degree– d–the common degree of polynomials $f_0, f_1, \ldots, f_k$
- Topological degree- cardinality of the set $f^{-1}(\{x\})$ for a generic point equals d^k.

Topological entropy

Entropy

$$h_{\text{top}}(f) = \log(deg_{\text{top}}(f)) = k \log d$$

- "$\geq$" follows from a general result of [Misiurewicz, Przytycki]
- "$\leq$" proved by Gromov (uses specific structure of $\mathbb{P}^k$).

Topological entropy

Entropy

$$h_{\text{top}}(f) = \log(deg_{\text{top}}(f)) = k \log d$$

- "$\geq$" follows from a general result of [Misiurewicz, Przytycki]
- "$\leq$" proved by Gromov (uses specific structure of $\mathbb{P}^k$).

Julia set

- "Large Julia set"- defined by normality criterion. Fatou set- complement of the "Large Julia set"
- "Small Julia set"- support of the measure of maximal entropy
- ("intermediate" Julia sets)

Julia set

- "Large Julia set"- defined by normality criterion. Fatou set- complement of the "Large Julia set"
- "Small Julia set"- support of the measure of maximal entropy
- ("intermediate" Julia sets)

Julia set

- "Large Julia set"- defined by normality criterion. Fatou set- complement of the "Large Julia set"
- "Small Julia set"- support of the measure of maximal entropy
- ("intermediate" Julia sets)

Analytic approach

"Analytic" construction of the maximal measure: An invariant (1,1) current T:

- $f^*T = dT$,
- $T = \lim d^{-n}(f^n)^*(\omega)$ where ω is the Fubini-Study form on $\mathbb{P}^k$.
- The measure of maximal entropy is given by T^k.

Dynamical approach

This approach follows one- dimensional construction (Freire, Lopes, Mane; Lyubich).
Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism of degree d. Then there exists an algebraic ("exceptional") set $\mathcal{E}$ such that, for every $x \in \mathbb{P}^k \setminus \mathcal{E}$ the sequence of measures

$$\frac{1}{d^{nk}} \sum_{y \in f^{-n}(x)} \delta_y$$

converges to the common limit μ. The Jacobian of this measure μ equals to d^k, thus $h_\mu(f) = k \log d$.

Some properties of the maximal measure

- μ is the unique measure of maximal entropy
- For every $x \in \mathrm{supp}\mu$ and every neighbourhood $U \supset x$ $\bigcup_{i=0}^{\infty} f^n(U) \supset \mathbb{P}^k \setminus \mathcal{E}$
- For every probability measure ν such that $\nu(\mathcal{E}) = 0$ we have

$$\frac{f^n * \nu}{d^{kn}} \to \mu$$

- μ is mixing: $\int \phi \cdot \psi \circ f^n d\mu \to \int \phi d\mu \int \psi d\mu$
- CLT holds for μ and Hölder-continuous observables.

Some properties of the maximal measure

- μ is the unique measure of maximal entropy
- For every $x \in \mathrm{supp}\mu$ and every neighbourhood $U \supset x$ $\bigcup_{i=0}^{\infty} f^n(U) \supset \mathbb{P}^k \setminus \mathcal{E}$
- For every probability measure ν such that $\nu(\mathcal{E}) = 0$ we have

$$\frac{f^n * \nu}{d^{kn}} \to \mu$$

- μ is mixing: $\int \phi \cdot \psi \circ f^n d\mu \to \int \phi d\mu \int \psi d\mu$
- CLT holds for μ and Hölder-continuous observables.

Some properties of the maximal measure

- μ is the unique measure of maximal entropy
- For every $x \in \operatorname{supp}\mu$ and every neighbourhood $U \supset x$ $\bigcup_{i=0}^{\infty} f^n(U) \supset \mathbb{P}^k \setminus \mathcal{E}$
- For every probability measure ν such that $\nu(\mathcal{E}) = 0$ we have

$$\frac{f^n * \nu}{d^{kn}} \to \mu$$

- μ is mixing: $\int \phi \cdot \psi \circ f^n d\mu \to \int \phi d\mu \int \psi d\mu$
- CLT holds for μ and Hölder-continuous observables.

Some properties of the maximal measure

- μ is the unique measure of maximal entropy
- For every $x \in \text{supp}\mu$ and every neighbourhood $U \supset x$ $\bigcup_{i=0}^{\infty} f^n(U) \supset \mathbb{P}^k \setminus \mathcal{E}$
- For every probability measure ν such that $\nu(\mathcal{E}) = 0$ we have

$$\frac{f^n * \nu}{d^{kn}} \to \mu$$

- μ is mixing: $\int \phi \cdot \psi \circ f^n d\mu \to \int \phi d\mu \int \psi d\mu$
- CLT holds for μ and Hölder-continuous observables.

Some properties of the maximal measure

- μ is the unique measure of maximal entropy
- For every $x \in \mathrm{supp}\mu$ and every neighbourhood $U \supset x$ $\bigcup_{i=0}^{\infty} f^n(U) \supset \mathbb{P}^k \setminus \mathcal{E}$
- For every probability measure ν such that $\nu(\mathcal{E}) = 0$ we have

$$\frac{f^n * \nu}{d^{kn}} \longrightarrow \mu$$

- μ is mixing: $\int \phi \cdot \psi \circ f^n d\mu \to \int \phi d\mu \int \psi d\mu$
- CLT holds for μ and Hölder-continuous observables.

Equilibrium measures

X- compact metric space, $T : X \to X$ continuous. Given a continuous function $\varphi : X \to X$

$$P(\varphi) = \sup_{\mu} h_\mu(T) + \int \varphi d\mu$$

where $\sup$ is taken over all probability Borel T- invariant measures.

Equilibrium states -One-dimensional case

Denker, Urbański (90's) proved the existence and uniqueness of equilibrium states for Hölder- continuous potentials $\phi : J(f) \to \mathbb{R}$ satisfying

$$\sup \phi < P(\phi) \tag{1}$$

If $\sup \phi - \inf \phi < \log d$ then (1) holds.

Denker, Przytycki, Urbański - studied spectral properties of Perron-Frobenius operator related to the potential ϕ, in particular they proved CLT for Hölder continuous 'observables'.

Equilibrium states in multidimensional case

Joint work with Mariusz Urbański

Theorem.
Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a ("regular") holomorphic endomorphism. Then there exists a positive number $\kappa(f)$ such that for every Hölder - continuous potential ϕ with $\sup \phi - \inf \phi < \kappa(f)$ there exists a unique equilibrium state μ_ϕ.

This equilibrium state is equivalent to a conformal measure- the eigenmeasure of the conjugate Perron -Frobenius operator. The dynamical system (f, μ_ϕ) is metrically exact. The corresponding normalized Perron Frobenius operator, acting in $C(J(f))$ is almost -periodic.
$\kappa(f) = \log d$ in many cases (always?).

Local degree

A_p-critical periodic set is the union of all irreducible varieties that are contained in the critical set C and which are periodic under some f^l, $l \leq p$.
E_n^p is the set of points $z \in \mathbb{P}^k$ such that $f^i \in A_p$ for some $i \leq n$.

Proposition: For every $\beta > 0$ there exists $p = p(\beta)$ such that for every $n > N$ and for every $x \notin E_n^p$

$$\mathrm{Card}\{j \leq n : f^j(x) \in C\} \leq n\beta$$

It follows from a much more general result [Ch. Favre] that for all $x \in \mathbb{P}^k$ the limit $d(x) = \lim_{n\to\infty}(deg_x f^n)^{\frac{1}{n}}$ exists. Moreover, if $d(x) > 1$ then there exists an irreducible periodic variety $V \subset C$ such that $f^i(x) \in V$ for some $i \geq 0$.

Perron-Frobenius operator:

$$\mathcal{L}_\phi(g)(x) = \sum_{y \in f^{-1}(x)} \exp \phi(y) g(y)$$

Normalized $\mathcal{L}$: $\hat{\mathcal{L}} = \frac{1}{\lambda}\mathcal{L}$ where λ is an eigenvalue of the conjugate Perron- Frobenius operator.
Aim:

- Prove that
$$c < \hat{\mathcal{L}}^n(1) < C \tag{2}$$
- Prove that the normalized Perron- Frobenius operator is almost periodic: For every $g \in C(J)$ the family of iterates $\hat{\mathcal{L}}^n(g)$ is equicontinuous.
- Prove that
$$P(\phi) = \log \lambda \tag{3}$$

Perron-Frobenius operator:

$$\mathcal{L}_\phi(g)(x) = \sum_{y \in f^{-1}(x)} \exp \phi(y) g(y)$$

Normalized $\mathcal{L}$: $\hat{\mathcal{L}} = \frac{1}{\lambda}\mathcal{L}$ where λ is an eigenvalue of the conjugate Perron- Frobenius operator.
Aim:

- Prove that
$$c < \hat{\mathcal{L}}^n(1) < C \tag{2}$$
- Prove that the normalized Perron- Frobenius operator is almost periodic: For every $g \in C(J)$ the family of iterates $\hat{\mathcal{L}}^n(g)$ is equicontinuous.
- Prove that
$$P(\phi) = \log \lambda \tag{3}$$

Perron-Frobenius operator:

$$\mathcal{L}_\phi(g)(x) = \sum_{y \in f^{-1}(x)} \exp \phi(y) g(y)$$

Normalized $\mathcal{L}$: $\hat{\mathcal{L}} = \frac{1}{\lambda}\mathcal{L}$ where λ is an eigenvalue of the conjugate Perron- Frobenius operator.
Aim:

- Prove that
$$c < \hat{\mathcal{L}}^n(1) < C \tag{2}$$
- Prove that the normalized Perron- Frobenius operator is almost periodic: For every $g \in C(J)$ the family of iterates $\hat{\mathcal{L}}^n(g)$ is equicontinuous.
- Prove that
$$P(\phi) = \log \lambda \tag{3}$$

We extend the function ϕ twice: $\tilde{\phi}$ is Hölder continuous in a neighbourhood U of J with the same exponent α, $\sup \tilde{\phi} = \sup \phi$. This extension is used to prove (2).

Next, we extend $\tilde{\phi}$ to a continuous function $\hat{\phi}$ defined on the whole $\mathbb{P}^k$ so that

$$\hat{\mathcal{L}}^n_{\hat{\phi}}(1) < C, \quad \text{thus} \;\; \mathcal{L}^n_{\hat{\phi}}(1)(x) < C\lambda^n$$

This extension is used to prove (3).

Theorem. Let $\psi : \mathbb{P}^k \to \mathbb{R}$ be a continuous function. Assume that there exist $\lambda > 0$ and $Q > 0$ such that $\sup \psi + (k-1)\log d < \log \lambda$ and

$$\mathcal{L}_\psi^n(1)(x) \leq Q\lambda^n$$

for every $x \in \mathbb{P}^k$. Then $P(\psi) \leq \log \lambda$.
Proof... leads to the estimate the integral

$$\int_{\mathbb{P}^k} \exp S_n\psi(\omega + f^*\omega + \cdots + (f^{(n-1)})^*\omega)^k$$

where $S_n\psi(x) = \psi(x) + \psi(f(x)) + \cdots + \psi(f^{n-1}(x))$
How to use the estimate on iterates $\mathcal{L}_\psi^n(1)$:
Observation:

$$\int_{\mathbb{P}^k} \exp S_n\psi(f^{(n)})^*\omega^k = \int_{\mathbb{P}^k} \mathcal{L}_\psi^n(1)d\omega^k$$

Uniqueness of the equilibrium state

Proposition: Suppose that $\phi : J \to \mathbb{R}$ is an "admissible" potential and let $g : J \to \mathbb{R}$ is a Hölder continuous function. Then the function

$$t \mapsto P(\phi + tg)$$

is differentiable in a neighbourhood of zero and

$$\frac{d}{dt}_{|t=t_0} P(\phi + tg) = \int g d\mu_{\phi + t_0 g}$$

Fine Inducing in one and several dimensions

Joint work with M. Szostakiewicz and M. Urbański

The map f is replaced by an infinite Iterated Function System

$$F : \bigcup U_i \to U$$

U is (holomorphically equivalent to) a ball. The map F restricted to each U_i is a holomorphic isomorphism onto U given by some iterate $f^{n(i)}$ of f. Moreover,$\mathrm{cl}(U_i) \cap \mathrm{cl}(U_j) = \emptyset$ for $i \neq j$ and $\mu_\phi(\bigcup U_i) = \mu_\phi(U)$.

Let

$$V_N = \bigcup_{i:n(i)>N} U_i$$

Main estimate: $\mu_\phi(V_N) < \exp(-N\delta)$ for some positive δ.

Exponential decay of correlation and CLT

Theorem. For every $\alpha \le 1$, every α- Hölder continuous function $\psi : J(f) \to \mathbb{R}$, every $\eta \in L^\infty(\mu_\phi)$

$$|\int \psi \cdot \eta \circ f^n - \int \psi d\mu_\phi \int \eta d\mu_\phi| = O(\theta^n)$$

with some constant $\theta < 1$, depending on α.

Remark: In dimension one, this gives an alternative proof of N. Haydn's result.

Theorem (Corollary). CLT holds for Hölder continuous functions $\psi : J(f) \to \mathbb{R}$) such that ϕ is not cohomologous to a constant in $L^2(\mu_\phi)$.

Some other conclusions for one- dimensional case

Theorem: Hausdorff dimension of the equilibrium measure μ_ϕ is (typically) smaller than the Hausdorff dimension h of the Julia set. Exceptions:

- f is expanding on J and $\phi + h\log|f'|$ is cohomologous to a constant.
- $\#(P(f) \cap J(f)) \leq 4$

This generalizes (and gives an alternative proof) of the result about dimension of maximal measure

Theorem. The function

$$t \mapsto P(t\phi)$$

is real- analytic (in some neighbourhood of 1)