# Holomorphic dynamical systems whose orbit spaces give new examples of compact complex manifolds

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Frontiers in Complex Dynamics, Celebrating John Milnor's 80th birthday,
BIRS, Banff, 21-25 February 2011

Consider the linear differential equation in $\mathbb{C}^n$

$$\frac{dz_1}{dT} = \lambda_1 z_1$$

$$\vdots$$

$$\frac{dz_n}{dT} = \lambda_n z_n$$

**We will always consider complex time $T \in \mathbb{C}$.**

This equation can be written in matrix form:

$$\frac{dZ}{dT} = \Lambda Z,$$

where $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ is adiagonal matrix, and $Z = (z_1, \cdots, z_n)^t$ is a column vector.

We will always assume that $\lambda_i \neq 0$ $(i = 1, \cdots, n)$ so that $\Lambda$ is invertible. The equation determines a linear action of $\mathbb{C}$ on $\mathbb{C}^n$, in other words complex flow, $\varphi_T : \mathbb{C}^n \to \mathbb{C}^n$ (Thus, $\varphi_{T_1+T_2} = \varphi_{T_1} \circ \varphi_{T_2}$).

This linear flow is given explicitly by the formula:

$$\varphi_T(Z) = e^{T\Lambda} Z = (e^{T\lambda_1} z_1, \cdots, e^{T\lambda_n} z_n)^t,\ T \in \mathbb{C},\ Z^t \in \mathbb{C}^n.$$

If $Z \neq 0$ the orbit or leaf of $Z$, denoted by $L(Z)$ is parametrized by the map from $\mathbb{C}$ to $\mathbb{C}^n$ given by the function

$$T \mapsto (e^{T\lambda_1} z_1, \cdots, e^{T\lambda_n} z_n)^t, \textit{ where } Z = (z_1, \cdots, z_n)^t.$$

The complement of the origin $\mathbb{C}^n - \{0\}$, is foliated by the orbits of the flow. The orbits are immersed copies of $\mathbb{C}$ or $\mathbb{C}^*$.
There is a dichotomy:

1. The origin of $\mathbb{C}$ is in the convex hull of $\{\lambda_1, \cdots, \lambda_n\}$.
   This happens if and only if there exists a point $Z$ such that its leaf $L(Z)$ does not accumulate at the origin.
2. The origin of $\mathbb{C}$ is *not* in the convex hull of $\{\lambda_1, \cdots, \lambda_n\}$.
   This happens if and only if *every* leaf accumulates at the origin.

In the first case one says that the matrix $\Lambda$ is in the *Poincaré domain*

In the second case one says says that the matrix $\Lambda$ is in the *Siegel domain*

When the origin is in the the convex hull of $\{\lambda_1, \cdots, \lambda_n\}$ and in addition none of the segments $[\lambda_i, \lambda_j]$, $i \neq j$, contains the origin i.e. the origin is not in the diagonals of the polygon with vertices $\lambda_i, \;\; i = 1, \cdots, n$, we say that $\Lambda$ satisfies the *weak hyperbolicity condition*.

This condition is an open condition.

The diagonals divide the interior of the convex hull into a number of open "chambers" and the origin belongs to one of these chambers.

If $\Lambda$ is in the Siegel domain and satisfies the weak hyperbolicity condition we have:

1. There exists an open set $\mathcal{U}$, saturated by the leaves of the flow, such that the space of leaves is a Hausdorff space which is a complex manifold $M$.
2. In fact, there exists a real analytic variety $\widehat{M}$ with a unique isolated singularity at the origin such that $\widehat{M} - \{0\}$ meets transversally every leaf in $U$ in a single point. So that $M$ can be identified with $\widehat{M} - \{0\}$
3. There is a free holomorphic action of $\mathbb{C}^*$ on $M$ such that the orbit space is a compact complex manifold.
4. The set $\mathcal{U}$ is the union of subspaces of $\mathbb{C}^n$ spanned by certain nonempty subsets of the canonical basis of $\mathbb{C}^n$.

Let $\Lambda$ satisfy the weak hyperbolicity condition. Let $\mathcal{S}$ be the union of the Siegel leaves. There is an action of $\mathbb{C}$ on $\mathcal{S}$ given by the flow of the linear system satisfying the weak hyperbolicity condition. There is also an action of $\mathbb{C}^*$ on $\mathcal{S}$ given by scalar multiplication in $\mathbb{C}^n$. These two actions commute, giving a free and holomorphic action of $\mathbb{C} \times \mathbb{C}^*$ on $\mathcal{S}$.

The corresponding quotient will be our compact manifold

$$N := N(\Lambda) = \mathcal{S}/\mathbb{C} \times \mathbb{C}^*.$$

It can be shown that on every Siegel leaf there is a unique point which is closest to the origin. The set of those points coincides with the set

$$M = \{z \in \mathbb{C}^n - \{0\} \mid \sum \lambda_i z_i \bar{z}_i = 0\}$$

and therefore we can identify $M$ with $\mathcal{S}/\mathbb{C}$.

$M$ is a complex cone from the origin so its quotient by the radial action of $\mathbb{R}^+$ can be identified with

$$M_1 = S^{2n-1} \cap M = \{z \in \mathbb{C}^n | \ \sum \lambda_i z_i \bar{z}_i = 0, \ \sum z_i \bar{z}_i = 1\}$$

which is compact. The equations describing $M_1$ are independent, as a result of condition (WH). Therefore $M_1$ is a compact, smooth manifold. To obtain $N$ from $M_1$ it only remains to divide by the scalar action of $S^1$:

$$N = M_1/S^1$$

## BASIC EXAMPLE

In $\mathbb{C}$ consider a non-degenerate triangle with vertices $\lambda_1$, $\lambda_2$ and $\lambda_3$. Suppose that the origin is in the interior of this triangle. Then the open set $\mathcal{U} \subset \mathbb{C}^3$ is the complement of the three coordinate hyperplanes $z_1 = 0$, $z_2 = 0$ and $z_3 = 0$. The set in $\mathbb{C}^3 - \{0\}$ given by the equation $(\star)$

$$\lambda_1|z_1|^2 + \lambda_2|z_2|^2 + \lambda_3|z_3|^2 = 0 \qquad (\star)$$

meets every leaf in $\mathcal{U}$ in exactly one point. So that the space of leaves in $\mathcal{U}$ can be identified with the set, also denoted by $M$, satisfying this equation.
The set $M$ is a complex cone with the origin deleted so that if $Z \in M$ also $cZ \in M$ for al $c \in \mathbb{C}^*$.
Hence one has a free action of $\mathbb{C}^*$ and the quotient $N := M/\mathbb{C}^*$, then a complex, compact manifold of dimension one. In fact $N$ is an elliptic curve.

**Any elliptic curve is obtained this way.**

We see that $N$ is the projectivization of $M$ and therefore $N$ can be identified is the set of points satisfying the following two equations:

$$\lambda_1|z_1|^2 + \lambda_2|z_2|^2 + \lambda_3|z_3|^2 = 0$$

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$$

modulo the natural action of the circle given by

$$(z_1, z_2, z_3) \mapsto (\mu z_1, \mu z_2, \mu z_3),\ |\mu| = 1,\ (z_1, z_2, z_3) \in N.$$

We observe that if we deform the set $\{\lambda_1(t), \cdots, \lambda_n(t)\}$ always satisfying the WH condition, we obtain by transversality manifolds $M_1(t)$ which are diffeomorphic.

We can therefore deform the configuration to a regular polygon with $k \geq 3$ vertices where we may collapse several to a vertex.

Therefore the topology of $M_1(t)$ (and that of $N(t)$ also) is totally described by this final configuration, which can be specified by the multiplicities of those vertices, that is, by the partition $n = n_1 + \cdots + n_k$

It has been shown by López de Medrano that the topology of $M_1$ is given as follows:

Let $d_i = n_i + n_{i+1} + \cdots + n_{i+l-1}$ for $i = 1, \ldots ..., k$ (the subscripts being taken modulo $k = 2l + 1$). Let also $d = min\{d_1, ..., d_k\}$.

These numbers determine the topology of $M_1$:

1. If $k = 3$ then $M_1 = S^{2n_1 -} \times S^{2n_2 - 1} \times S^{2n_3 - 1}$.
2. If $k = 2l + 1 > 3$ then $M_1$ is diffeomorphic to the connected sum of the manifolds $S^{2d_i - 1} \times S^{2n - 2d_i - 2}$, for $i = 1, \ldots, k$, and $M_1$ is $2d - 2$-connected.

# Higher dimensions

We can generalize all of the previous construction to linear actions of $\mathbb{C}^m$ on $\mathbb{C}^n$ when $n > 2m$.

Let $m$ be a positive integer and $n$ an integer greater than $2m$.

Let $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ be a configuration of $n$ vectors of $\mathbb{C}^m$.
Let $\mathcal{H}(\Lambda_1, \cdots, \Lambda_n) \subset \mathbb{C}^m$ be the convex hull of the $n$-tuple $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$.

One says that $\Lambda$ ist *admissible* if the following holds true:

1) The *Siegel condition*:
The origin 0 belongs to the convex hull $\mathcal{H}(\Lambda_1, \ldots, \Lambda_n)$.

2) The *weak hyperbolicity condition*:
For every $2m$-tuple $(i_1, \cdots, i_{2m})$, $1 \leq i_1 < \cdots < i_{2m} \leq n$ (recall $n > 2m$), one has: $0 \notin \mathcal{H}(\Lambda_{i_1}, \cdots, \Lambda_{i_1})$.

Let $\mathcal{F}$ be the holomorphic foliation of complex projective $(n-1)$-space $\mathbb{P}^{n-1}$ given by the action

$$(T, [z]) \in \mathbb{C}^m \times \mathbb{P}^{n-1} \longmapsto [z_1 \cdot \exp\langle \Lambda_1, T\rangle, \ldots, z_n \cdot \exp\langle \Lambda_n, T\rangle] \in \mathbb{P}^{n-1}$$

$$T = (t_1, \cdots, t_m) \in \mathbb{C}^m,\ \Lambda_i \in \mathbb{C}^m.$$

The brackets denote projective homogeneous coordinates of the corresponding projective space:

$$[z] := [z_1, \cdots, z_n]$$

and $\langle -, - \rangle$ defined by:

$$\langle [z], [w] \rangle := z_1 w_1 + \cdots + z_m w_m$$

is the inner product (*not* the hermitian product).

This action of $\mathbb{C}^m$ is the projectivization of the linear action in $\mathbb{C}^n$ given by the family of $m$ linear commuting vector fields of $\mathbb{C}^n$ given by the diagonal matrices whose eigenvalues are the entries of $\Lambda_i$.
Therefore, consider the lifting of this action and the corresponding foliation $\tilde{\mathcal{F}}$ of $\mathbb{C}^n$ given by the orbits of the action:

$$(T, z) \in \mathbb{C}^m \times \mathbb{C}^n \longmapsto (z_1 \cdot \exp\langle \Lambda_1, T\rangle, \ldots, z_n \cdot \exp\langle \Lambda_n, T\rangle) \in \mathbb{C}^n$$

If $z \in \mathbb{C}^n$ we say that the leaf (or orbit) $L(z)$ of the action of $\mathbb{C}^m$ is a Siegel leaf if 0 *it is not* in the closure of $L(z)$. If 0 is in the closure of $L(z)$ we say that the leaf $L(z)$ is of Poincaré type.

Next we will describe an open set $\mathcal{S}$, of $\mathbb{C}^n$, where the space of leaves is Hausdorff.

For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, let $I(z) = \{j \in \{1, 2 \cdots, n\} | \ z_j \neq 0\}$

$$\mathcal{S} = \{z \in \mathbb{C}^n \quad | \quad 0 \in \mathcal{H}(\{\Lambda_i | \ i \in I(z)\})\}$$

and let $\mathcal{V}$ be the image of $\mathcal{S}$ in $\mathbb{P}^{n-1}$.
Finally, let

$$\mathcal{T} = \{z \in \mathbb{C}^n \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0\}$$

and

$$\mathcal{N} = \{[z] \in \mathbb{P}^{n-1} \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0\}$$

We see from its definition that $\mathcal{S} = \mathbb{C}^n - E$ where $E$ is an analytic set, whose different components correspond to subspaces of $\mathbb{C}^n$ where some coordinates vanish. Therefore $\mathcal{S}$ contains $(\mathbb{C}^*)^n$ and it is invariant under the natural action on $\mathbb{C}^n$ of $(\mathbb{C}^*)^n$ via diagonal and invertible matrices.

**This is the direct connection of toric varieties and the complex manifolds we will obtain**

Another characterization of $\mathcal{S}$ is the following:

$$\mathcal{S} = \{z \in \mathbb{C}^n \mid 0 \text{ is not in the closure of the leaf of } \tilde{\mathcal{F}} \text{ through } z\}$$

in other words $\mathcal{S}$ is the union of the Siegel Leaves. It is open and invariant under the action of $\mathbb{C}^m$

The weak hyperbolicity condition implies that the system of quadratic equations which define $\mathcal{T}$ et $\mathcal{N}$, given before, are of maximal rank in in every point.

The Siegel condition implies that both $\mathcal{T}$ and $\mathcal{N}$ nonempty. One also shows that $\tilde{\mathcal{F}}$ is a non singular foliation when restricted to $\mathcal{S}$ and that $\mathcal{T}$ is a smooth manifold which meets every leaf of $\tilde{\mathcal{F}}$ contained in $\mathcal{S}$ and it is transverse to the leaves. In other words: the quotient space $\tilde{\mathcal{F}}$ restricted to $\mathcal{S}$ can be canonically identified to $\mathcal{T}$ and therefore it is a Hausdorff space.

An important fact is that $\mathcal{T}$ can be given the structure of a complex manifold which we denote by $M$.

In the same way, $\mathcal{N}$ can be identified with $\mathcal{F}$ restricted to $V$ and therefore becomes a compact complex manifold. Let us denote by $N$ this complex manifold. The complex dimension of $M$ is $n-m$ and the complex dimension of $N$ is $n-m-1$.

The natural projection $M \to N$, induced by the projection $\mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$, is a principal $\mathbb{C}^*$-bundle.

Let $M_1$ denote the total space of the associated circle bundle. It has the same homotopy type as $M$ but has the advantage of being compact. Let us note that $M_1$ can be identified with the transverse intersection of the cone $\mathcal{T}$ with the unit sphere $\mathbb{S}^{2n-1}$ of $\mathbb{C}^n$.

Therefore we define:

$$M_1 = \{z \in \mathbb{C}^n \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0, \ \sum_{i=1}^{n} |z_i|^2 = 1\}$$

The space $\mathcal{S}$ has the same homotopy of $M$ and therefore the same homotopy type of $M_1$.

(i) If $n = 2m + 1$, then the convex hull of the $\Lambda_i$'s is combinatorially equal to the $2m + 1$-simplex of $\mathbb{C}^m \simeq \mathbb{R}^{2m}$. If we take out one of the $\Lambda_i$'s, 0 does not belong to the convex closure of the others. In other words, $\mathcal{S}$ is equal to $(\mathbb{C}^*)^n$ **One can show that $N$ is a complex torus and that every complex (in particular any abelian variety) is obtained this way**.

(ii) If $m = 1$ Let us define for, $n \geq 4$:

$$\Lambda_1 = 1 \qquad \Lambda_2 = i \qquad \Lambda_3 = \ldots = \Lambda_n = -1 - i \ .$$

One can prove in this case that $\mathcal{S}$ is equal to $(\mathbb{C}^*)^2 \times \mathbb{C}^{n-2} \setminus \{0\}$.
Let us consider the two equations which define $\mathcal{T}$:

$$|z_1|^2 = |z_3|^2 + \ldots + |z_n|^2$$

$$|z_2|^2 = |z_3|^2 + \ldots + |z_n|^2.$$

If we intersect $\mathcal{T}$ with the unit sphere $\mathbb{C}^n$ we see that this intersection is diffeomorphic to $\mathbb{S}^{2n-5} \times \mathbb{S}^1 \times \mathbb{S}^1$ and one shows that $N$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{2n-5}$. In particular, for $n = 4$, one obtains all linear Hopf surfaces.

(iii) If $m = 1$ let:

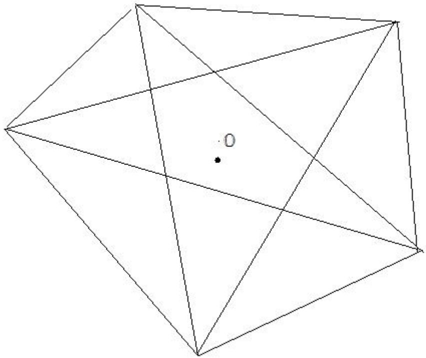
$$\Lambda_1 = 1 \qquad \Lambda_2 = \Lambda_3 = i \qquad \Lambda_4 = \Lambda_5 = -1 - i\ .$$

The same reasoning as before shows that $N$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3$.

On obtains this in this way examples of Calabi-Eckmann.

A Calabi-Eckmann manifold is a complex manifold whose underlying smooth manifold is a product of odd-dimensional spheres $\mathbb{S}^{2n-1} \times \mathbb{S}^{2m-1}$.
One can show that every linear Calabi-Eckmann manifold is obtained this way.

(iv) In the case that one has the configuration given by the vertices of the pentagon




Santiago López de Medrano has shown that $M_1$ is diffeomorphic to the connected sum of five copies of $\mathbb{S}^3 \times \mathbb{S}^4$. The manifold $N$, is the quotient of $M_1$ by the orbits of a free action of $\mathbb{S}^1$.

In the examples (ii) and (iv), one obtains *non-symplectic* manifolds, since their second de Rham cohomology group vanishes.
This is a general fact in the manifolds we have obtained:
In general the manifold $N$ is a compact,complex manifold which is not symplectic.

### THEOREM

*The following properties are two-by-two equivalent: (i) $N$ est symplectic.*

*(ii) $N$ is a Kähler manifold.*
*(iii) $N$ is a complex torus*
*(iv) One has $n = 2m + 1$.*

We recall that a *toric variety* $X$ is a normal projective variety containing an algebraic torus $(\mathbb{C}^*)^n$ as a dense subset, such that the action of the torus on itself extends to the whole variety.

In other words: there exists an algebraic action of $(\mathbb{C}^*)^n$ on $X$ such that there is a dense principal orbit and therefore $X$ is an equivariant compactification of the principal orbit.

A toric variety $X$ is *quasi-regular* if its singularities are quotient singularities (in other words if $X$ is a complex orbifold)

Let $(\Lambda_1, \cdots, \Lambda_n)$ be a configuration admissible i.e. it satisfies both the Siegel and weak hyperbolicity conditions as before.
Consider the system of equations:

$$\sum_{i=1}^{n} s_i \Lambda_i = 0$$

$$\sum_{i=1}^{n} s_i = 0$$

We say that the configuration satisfies condition (K) if the dimension over $\mathbb{Q}$ of the vector space of rational solutions of the system above is maximal, in other words is of dimension $n - 2m - 1$.

THEOREM

*Let $N$ be one of our manifolds corresponding to a configuration which satisfies condition (K).*
*Then $N$ is a Seifert fibration in complex torii of dimension $m$ over a quasi-regular, projective, toric variety of dimension $n-2m-1$.*

This theorem has the following:

COROLLARY

*Let $N$ satisfy the conditions of the above theorem. Then the algebraic reduction of $N$ is a quasi-regular, projective, toric variety of dimension $n-2m-1$.*

As a particular case of the previous theorem one recovers the elliptic fibrations used by E. Calabi et B. Eckmann to provide the product of spheres $\mathbb{S}^{2p-1} \times \mathbb{S}^{2q-1}$ (pour $p > 1$ et $q > 1$) with a complex structure. This generalization is given by the following

## Definition

A generalized Calabi-Eckmann fibration is the fibration obtained by the previous theorem.

Since we know, fixing $m$ and $n$, that the set of configurations satisfying condition (K) is dense in the space of admissible configurations on obtains:

## COROLLARY

*Every manifold N corresponding to an admissible configuration is a small deformation of a generalized Calabi-Eckmann fibration*

THEOREM

*Let $X$ be a projective, quasi-regular, toric variety. Then there exists $m > 0$ and a manifold $N$ corresponding to an admissible configuration which admits a generalized Calabi-Eckmann over $X$ and whose fibres are compact complex tori of complex dimension $m$.*
*Furthermore, if $X$ is nonsingular (smooth), one can choose $m$ and $N$ such that the fibration is a holomorphic principal fibration.*

Let $N$ be as before. Let, as before,

$$M_1 = \{z \in \mathbb{C}^n \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0, \ \sum_{i=1}^{n} |z_i|^2 = 1\}$$

Let us remark that the standard action of the torus $(\mathbb{S}^1)^n$ on $\mathbb{C}^n$

$$(\exp i\theta, z) \in (\mathbb{S}^1)^n \times \mathbb{C}^n \longmapsto (\exp i\theta_1 \cdot z_1, \dots, \exp i\theta_n \cdot z_n) \in \mathbb{C}^n \quad (\star\star)$$

leaves $M_1$ invariant. The quotient of $M_1$ by this action can be identified, via the difféomorphism $r \in \mathbb{R}^+ \to r^2 \in \mathbb{R}^+$, to

$$K = \{r \in (\mathbb{R}^+)^n \quad | \quad \sum_{i=1}^{n} r_i \Lambda_i = 0, \ \sum_{i=1}^{n} r_i = 1\}$$

LEMMA

*The quotient $K$ is a convex polytope of dimension $n - 2m - 1$ with $n - k$ hyperfaces.*

**Proof.** By definition $K$ is the intersection of the space $A$ of solutions of an affine system with the closed sets $r_i \geq 0$. Each one of these closed sets defines an affine half-space $A \cap \{r_i \geq 0\}$ in the affine space $A$. In other words, $K$ is the intersection of a finite number of affine half-spaces. Since this intersection is bounded (since $M_1$ is compact), one obtains indeed a convex polytope. The weak hyperbolicity condition implies that the affine system that defines $K$ is of maximal rank. Hence, $K$ is of dimension $n - 2m - 1$.

Let us consider in more detail the definition of $K$. The points $r \in K$ verifying $r_i > 0$ for all $i$ are the points which belong to the interior of the convex polytope. They correspond to the points $z$ de $M_1$ which also belong to $(\mathbb{C}^*)^n$, i.e. to the points of $M_1$ such that the orbit under the action $(\star\star)$ is isomorphic to $(\mathbb{S}^1)^n$. The points which belong to a hyperface are exactly the points $r$ of $K$ having all of its coordinates *except one* equal to zero. They correspond to the points $z$ de $M_1$ which have a unique coordinate equal to zero, i.e. such that its orbit under the action $(\star\star)$ is isomorphic to $(\mathbb{S}^1)^{n-1}$. One obtains from the definition of $K$ that there exist points of $K$ having all coordinates different from zero except the i$^{th}$ coordinate if and only 0 belongs to the convex envelope of the configuration formed by the $\Lambda_j$ with $j$ different from $i$; hence if and only if $\Lambda_i$ is a point which can be eliminated keeping the conditions of Siegel and weak hyperbolicity. therefore one has $n - k$ hyperfaces. $\square$

One calls the convex polytope $K$ *the associated polytope* . One central idea is that the topology of the manifolds $M_1$, and therefore of the manifolds $N$, is codified by the combinatorial type of the polytope $K$. To make this idea more precise, it is interesting to push to the end the reasoning involved in the proof of the preceding lemma. One had seen that

$$K_i = K \cap \{r_i = 0,\ r_j > 0 \text{ for } j \neq i\}$$

is nonempty, and therefore is a hyperface de $K$, if and only if

$$0 \in \mathcal{H}((\Lambda_j)_{j \neq i}) \ .$$

Analogously, given $I$ a subset of $\{1, \ldots, n\}$, the set

$$K_I = K \cap \{r_i = 0 \text{ for } i \in I,\ r_j > 0 \text{ for } j \notin I\}$$

is nonempty, and therefore it is a facet of $K$ of codimension equal the cardinality of $I$, if and only if

$$0 \in \mathcal{H}((\Lambda_j)_{j \notin I})$$

One has therefore stablished a very important correspondence between two convex polytopes: the polytope $K$ on one hand and the convex hull of the $\Lambda_i$'s on the other hand.
This correspondence allows us to to prove the following result:

THEOREM

*(i) The polytope $K$ is simple, in other words, it is the dual of a simplicial polytope.*
*(ii) Let $P$ be a simple convex polytope. Then there exists manifolds $N$, as described before, whose associated polytope is combinatorially equivalent to $P$.*

# Sketch of the proof.

The first part is a direct consequence of the existence of the correspondence. One translates the weak hyperbolicity condition in the combinatorics of $K$ to deduce that each vertex of $K$ is a vertex of exactly $n-2m-1$ edges, and this number is precisely the dimension of $K$. This property characterizes simple polytopes. To prove (ii), one needs to reconstruct the convex hull of the $\Lambda_i$'s from the polytope $P$. The correspondence described before can be expressed in the following way: The convex hull of the $\Lambda_i$'s must be a Gale diagram of the polytope which is dual to $P$ . There are classical methods in combinatorics and convex geometry to construct such diagrams and this permits to finish the proof. $\square$

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