Partial semiconjugacies between rational functions

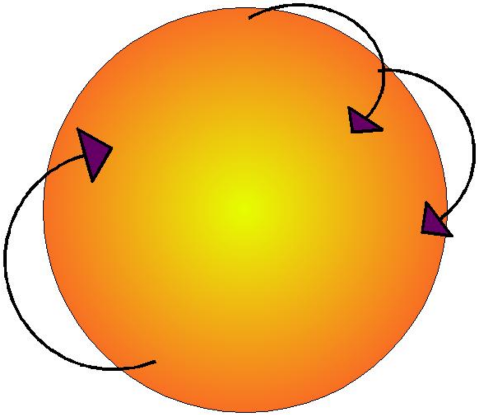
Vladlen Timorin*

*Faculty of Mathematics
National Research University Higher School of Economics,
Moscow

February 24, 2011

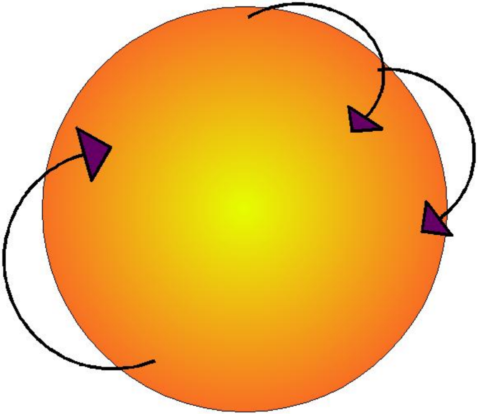
Rational functions

We consider rational functions of one complex variable (i.e. ratios of polynomials) as self-maps of the sphere.



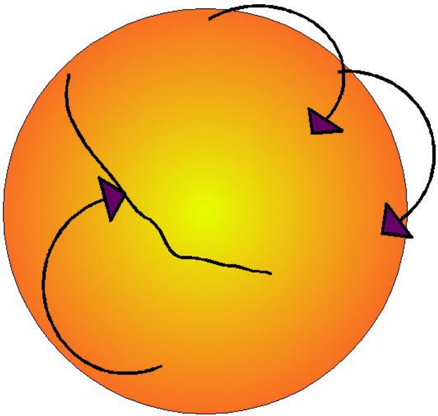
Rational functions

A topological dynamical system $f : S^2 \to S^2$ can be visualized as a sphere with arrows, up to continuous deformations.



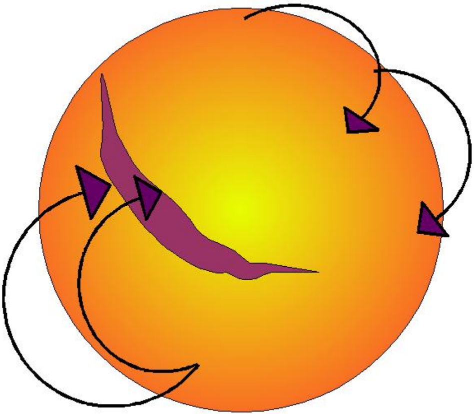
Surgery

To change topological dynamics, we cut the sphere along simple curves.



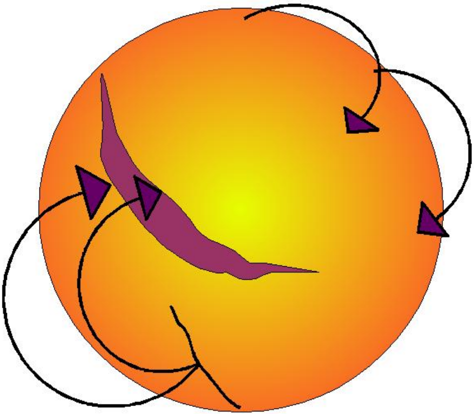
Surgery

Cutting through the tip of an arrow creates a problem...



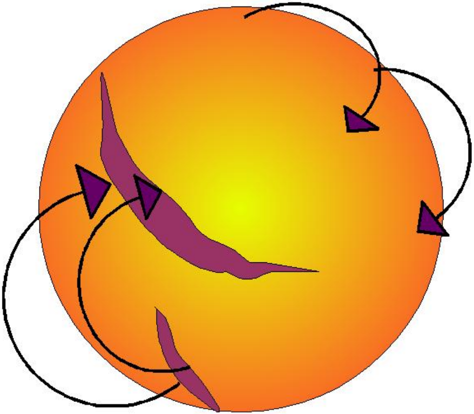
Surgery

and forces us to also cut through the base of the arrow.



Surgery

then we cut through the tips of other arrows, which forces more cuts etc. Once we start cutting, we need to do infinitely many cuts.



The sphere after the cuts

Let $f : S^2 \longrightarrow S^2$ be a branched covering and Z a finite set of simple curves. Cut along Z, $f^{-1}(Z)$, $f^{-2}(Z)$, etc.

We obtain a compact Hausdorff space

$$\hat{S}_{f,Z}$$

as the inverse limit of spheres with finitely many cuts. This space is equipped with arrows.

Hyperbolic rational functions

Definition

A rational function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ is called hyperbolic if it is expanding with respect to some Riemannian metric on a neighborhood of a nonempty closed completely invariant set (the Julia set).

The topological dynamics of hyperbolic rational functions is stable and in many cases easy to understand.

Non-hyperbolic functions

We want to understand at least something about non-hyperbolic rational functions.

Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a non-hyperbolic rational function with at least one super-attracting orbit.

Let $R : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a hyperbolic critically-finite rational function with the same structure of super-attracting orbits.

Non-hyperbolic functions

More precisely,

$$\begin{array}{ccc} \mathbb{C}P^1 & \xrightarrow{R} & \mathbb{C}P^1 \\ \psi \big\downarrow & & \big\downarrow \phi \\ \mathbb{C}P^1 & \xrightarrow{f} & \mathbb{C}P^1 \end{array},$$

where ϕ and ψ are homeomorphisms that coincide on $R(P_R)$ and are isotopic relative to $R(P_R)$.

P_R is the post-critical set of R, consisting of all critical values and their iterated images.
$R(P_R)$ is the post-post-critical set.

Main Theorem

Theorem

If P_R has at least three points, then there exists a finite set Z of simple curves and a continuous semi-conjugacy

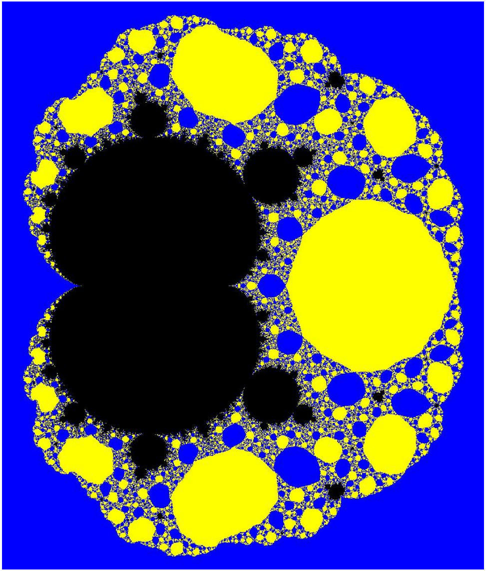
$$\hat{S}_{f,Z} \to (\mathbb{C}P^1, R).$$

In some sense, this semi-conjugacy is holomorphic.

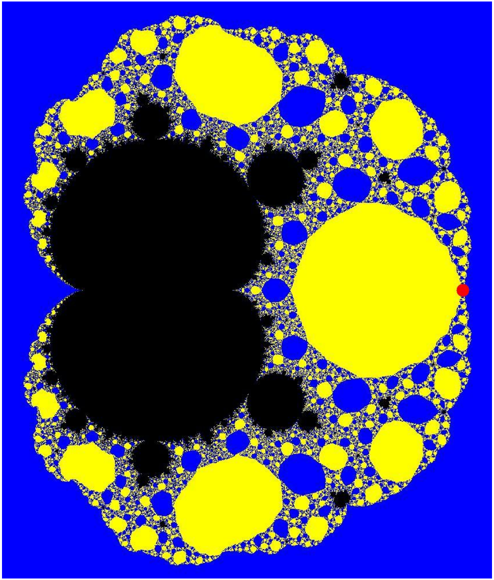
Remark

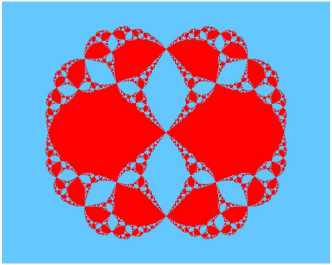
The set Z can be defined explicitly. Moreover, there is a lot of freedom in the choice of R and Z.

The parameter plane of maps $c/(z^2 + 2z)$

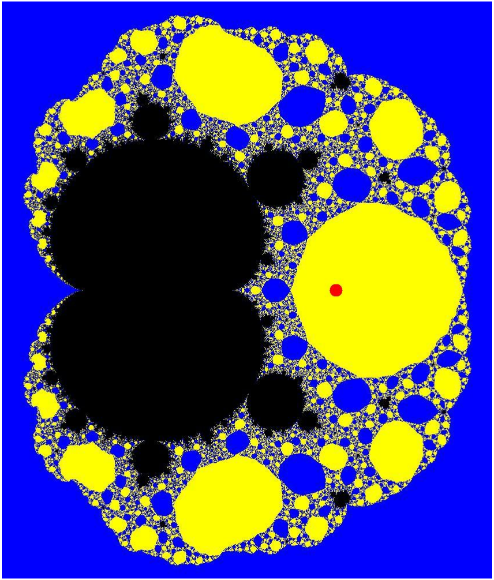


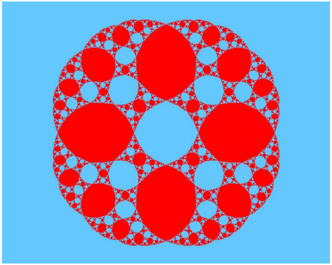
The parameter plane of maps $c/(z^2+2z)$



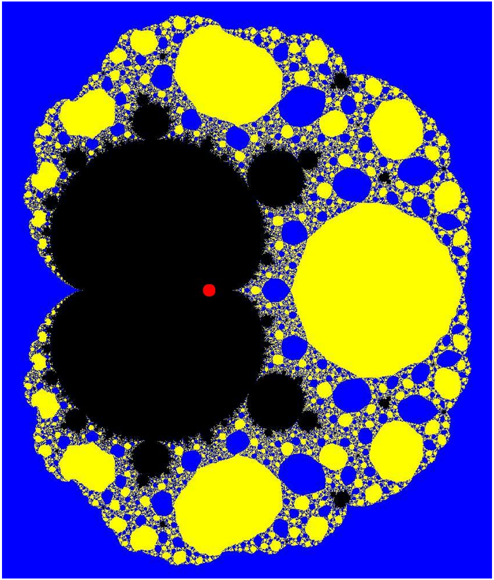


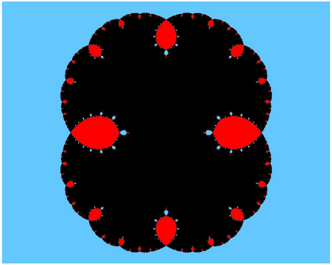
The parameter plane of maps $c/(z^2 + 2z)$



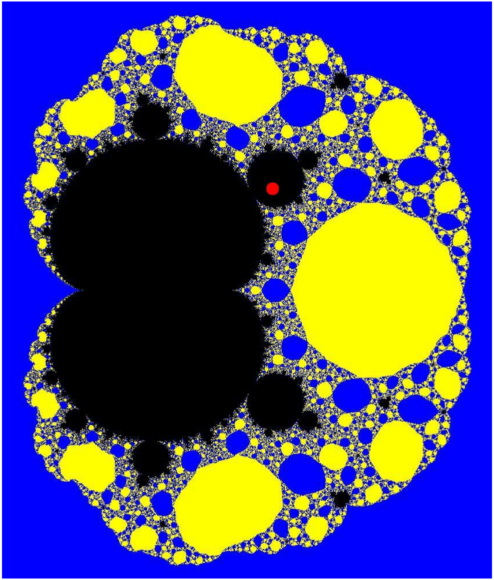


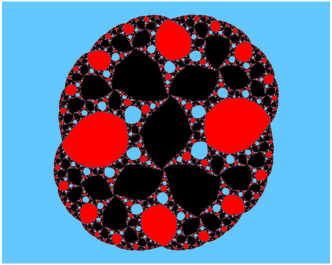
The parameter plane of maps $c/(z^2 + 2z)$



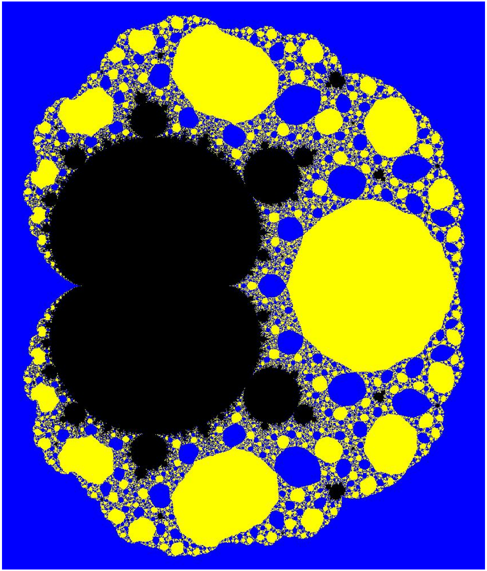


The parameter plane of maps $c/(z^2 + 2z)$





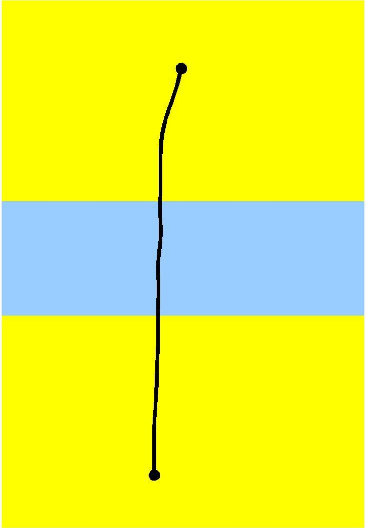
The parameter plane of maps $c/(z^2 + 2z)$



In this example:

- $f = \frac{c}{z^2+2z}$ is any map, which is not critically finite,
- R is the center of any blue or yellow component,
- Z is a single simple curve.

Path homeomorphisms

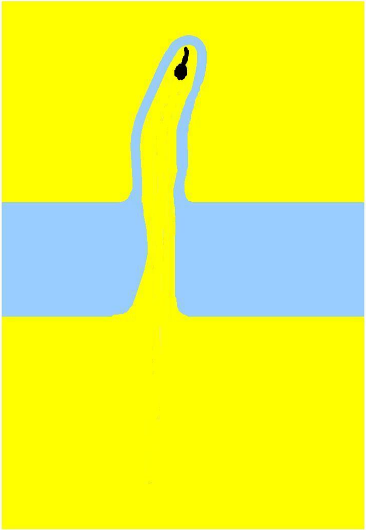


Definition

Let $\beta : [0, 1] \longrightarrow S^2$ be a simple path. Define a path homeomorphism $\sigma_\beta : S^2 \rightarrow S^2$ as a homeomorphism such that

- $\sigma_\beta(\beta(0)) = \beta(1)$,
- $\sigma_\beta(x) = x$ except in a narrow tube around $\beta[0, 1]$.

Path homeomorphisms



Definition

Let $\beta : [0,1] \to S^2$ be a simple path. Define a path homeomorphism $\sigma_\beta : S^2 \to S^2$ as a homeomorphism such that

- $\sigma_\beta(\beta(0)) = \beta(1)$,
- $\sigma_\beta(x) = x$ except in a narrow tube around $\beta[0,1]$.

Path homeomorphisms

Theorem (M. Rees)

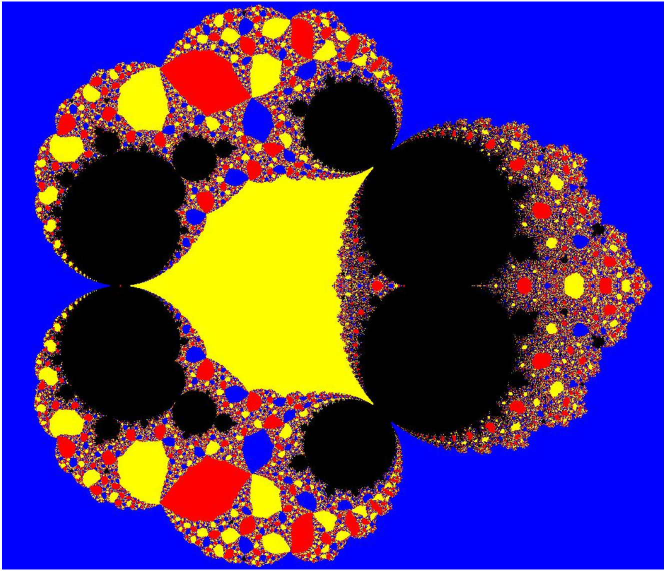
There exists a simple path $\beta : [0,1] \to \mathbb{C}P^1$ such that $\sigma_\beta \circ f$ is a critically finite branched covering Thurston equivalent to R.

$\beta(0)$ is a critical value of f;
$\beta(1)$ is a preperiodic point of f that gets eventually mapped to the super-attracting cycle $\{0, \infty\}$.
β is only defined up to homotopy (fixing endpoints, relative to the forward f-orbit of $\beta(1)$)

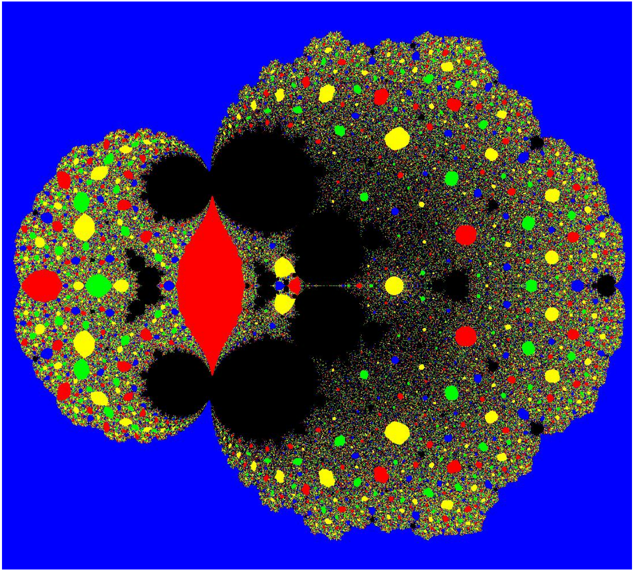
The curve Z

$$Z = f^{-1}(\beta[0,1]).$$

Parameter plane $Per_3(0)$



Parameter plane $Per_4(0)$



Congratulations!

Happy Birthday, Jack!