Conformally Natural Extensions revisited

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Frontiers in Complex Dynamics
- Celebrating Milnors 80-th birthday. -

Motivation

- What is the three manifold of a Rational Map?

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  How do we extend the action of a rational map on the Riemann sphere to the enclosed hyperbolic ball?
- The object of this talk is to try to convince you that the Douady-Earle extension is a worthwhile answer to the second question.

The Conformal Automorphism Group G of $\mathbb{S}^n$

- Let $G = G_n$ denote the group of Möbius transformations of Möbius space $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ which preserves the n-sphere

$$\mathbb{S}^n = \{(x_1, \ldots x_{n+1}) \in \mathbb{R}^{n+1} |\ x_1^2 + \ldots x_{n+1}^2 = 1\}.$$

as well as the enclosed ball $\mathbb{B}^{n+1}$.

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- Mostow [Mo] proved that any conformal automorphism of $\mathbb{B}^{n+1}$ and/or $\mathbb{S}^n$ is Möbius, i.e.
- G is the common conformal automorphism group of $\mathbb{B}^{n+1}$ and $\mathbb{S}^n$ and is also the hyperbolic isometry group of $\mathbb{B}^{n+1}$.

The G Subgroups G_+, R and R_+.

The automorphism group G is generated by

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- We equip $\mathbb{S}^n$ with the Spherical metric.
- And we denote by $R = R_n$ the subgroup of Spherical/Euclidean isometries,
- we denote by $R_+ := G_+ \cap R$ the subgroup of orientation preserving isometries.

Moving the Origin to $\mathbf{w}$; the Maps $g_{\mathbf{w}}$.

- Motivated by complex Möbius transformations

$$g_w(z) = \frac{z+w}{1+\overline{w}z} = \frac{z(1-|w|^2) + w(1+|z|^2 + w\overline{z} + \overline{w}z)}{1+|w|^2|z|^2 + w\overline{z} + \overline{w}z}$$

- We define $g_{\mathbf{w}} \in G_+$ for $\mathbf{w} \in \mathbb{B}^{n+1}$ by

$$g_{\mathbf{w}}(\mathbf{x}) = \frac{\mathbf{x}(1-|\mathbf{w}|^2) + \mathbf{w}(1+|\mathbf{x}|^2 + 2 < \mathbf{w}, \mathbf{x} >)}{1+|\mathbf{w}|^2|\mathbf{x}|^2 + 2 < \mathbf{w}, \mathbf{x} >},$$

where $< \cdot, \cdot >$ denotes the Euclidean inner product.
- Then $g_{\mathbf{w}}(-\mathbf{w}) = \mathbf{0}$ and $g_{\mathbf{w}}(\mathbf{0}) = \mathbf{w}$,
- $g_{\mathbf{w}}$ stabilizes the hyperbolic geodesic $]-\mathbf{w}/|\mathbf{w}|, \mathbf{w}/|\mathbf{w}|[$
- and $g_{\mathbf{w}}^{-1} = g_{-\mathbf{w}}$.

Generating G II.

- Let $g \in G$ be arbitrary and write $\mathbf{w} = g(\mathbf{0})$. Then g is canonically factorized as

$$g = g_{\mathbf{w}} \circ \rho',$$

where $\rho' = g_{\mathbf{w}}^{-1} \circ g \in R$.

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where $\rho' = g_{\mathbf{w}}^{-1} \circ g \in R$.

- In fact more generally G is generated by the group of Euclidean isometries R and the 1-parameter subgroup $(g_{\mathbf{r}})_{-1<r<1}$, where

$$\mathbf{r} = (r, 0, \ldots, 0) = r\mathbf{e}_1, \qquad -1 < r < 1$$

Natural G Actions

The group G operates

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- $(g \cdot \mathbf{v})(g(\mathbf{z})) = g_*(\mathbf{v})(g(\mathbf{z})) = D_{\mathbf{z}} g(\mathbf{v}(\mathbf{z}))$, for $\mathbf{v} \in \mathcal{F}(\mathbb{B}^{n+1})$ and $\mathbf{z} \in \mathbb{B}^{n+1}$ where $D_{\mathbf{z}} g$ denotes the differential of g at $\mathbf{z}$.

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- $(g \cdot \mathbf{v})(g(\mathbf{z})) = g_*(\mathbf{v})(g(\mathbf{z})) = D_{\mathbf{z}} g(\mathbf{v}(\mathbf{z}))$, for $\mathbf{v} \in \mathcal{F}(\mathbb{B}^{n+1})$ and $\mathbf{z} \in \mathbb{B}^{n+1}$ where $D_{\mathbf{z}} g$ denotes the differential of g at $\mathbf{z}$.
- $G \times G$ operates on the spaces $\mathrm{End}(\mathbb{B}^{n+1}), \mathcal{C}(\mathbb{B}^{n+1})$ and $\mathrm{End}(\mathbb{S}^n), \mathcal{C}(\mathbb{S}^n)$ of endomorphisms and continuous endomorphisms of $\mathbb{B}^{n+1}$ and $\mathbb{S}^n$ respectively by

$$(g, h)\phi := g \circ \phi \circ h^{-1}.$$

The Notion of Conformal Naturality

- If G operates on the spaces $\mathbb{X}$ and $\mathbb{Y}$ then a map $T : \mathbb{X} \longrightarrow \mathbb{Y}$ is called G equivariant or conformally natural if

$$\forall\, g \in G, \quad \forall\, x \in \mathbb{X} \quad : T(g \cdot x) = g \cdot T(x).$$

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- And if $G \times G$ operates on both $\mathbb{X}$ and $\mathbb{Y}$ then conformal naturality of T is taken to mean $G \times G$-equivariance, i.e.

$$\forall\, g, h \in G, \quad \forall\, x \in \mathbb{X} \quad : T(g\ x\ h^{-1}) = g\ T(x)\ h^{-1}.$$

Normalized Euclidean Lebesgue Measure

Denote by η_0 the normalized Euclidean Lebesgue measure on $\mathbb{S}^n$,

$$\eta_0(A) = \frac{1}{\mathrm{Vol}(\mathbb{S}^n)} \int \dots \int_A \mathrm{d}\,L, \qquad \mathrm{Vol}(\mathbb{S}^n) = \int \dots \int_A \mathrm{d}\,L,$$

where L denotes Lebesgue measure. We shall henceforth also write

$$\eta_0(A) = \int_A \mathrm{d}\eta_0\,.$$

Then η_0 is the unique R invariant probability measure, i.e. $g_*(\eta_0) = \eta_0$ for every element $g \in R$.

The Douady-Earle Extension Theorem

Let $\mathcal{E}(\mathbb{S}^n)$ denote the space of Borel measurable endomorphisms $\phi : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ such that $\phi_* \eta_{\mathbf{0}}$ has no atoms, i.e. such that $\eta_{\mathbf{0}}(\phi^{-1}(\underline{\zeta})) = 0$ for any point $\underline{\zeta} \in \mathbb{S}^n$.

Theorem

There is a conformally natural extension operator

$$E : \mathcal{E}(\mathbb{S}^n) \longrightarrow \mathrm{End}(\overline{\mathbb{B}}^{n+1})$$

More precisely $\forall\ f \in \mathcal{E}(\mathbb{S}^n)$ *the map $E(f)$ is real analytic in $\mathbb{B}^{n+1}$ continuous at $\mathbb{S}^n$ whenever f is continuous and for all $g, h \in G$:*

$$g \circ E(f) \circ h^{-1} = E(g \circ f \circ h^{-1}).$$

Conformal Barycenter of Probability Measures

- Define a probability measure $\mu \in \mathcal{P}(\mathbb{S}^n)$ to be admissible, if $\mu(\{\mathbf{z}\}) < 1/2$ for all $\mathbf{z} \in \mathbb{S}^n$.
- Let $\mathcal{P}'(\mathbb{S}^n)$ denote the space of admissible probability measures.

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- Let $\mathcal{P}'(\mathbb{S}^n)$ denote the space of admissible probability measures.
- To each $\mu \in \mathcal{P}'(\mathbb{S}^n)$ we shall assign a point $B(\mu) \in \mathbb{B}^{n+1}$ the conformal Barycenter so that the map

$$\mu \mapsto B(\mu) : \mathcal{P}'(\mathbb{S}^n) \to \mathbb{B}^{n+1}$$

is conformally natural and normalized by

$$B(\mu) = \mathbf{0} \quad \Leftrightarrow \quad \int_{\mathbb{S}^n} \underline{\zeta}\, \mathrm{d}\mu(\underline{\zeta}) = \mathbf{0}$$

A Vector Field for a Probability Measures

Proposition

The map $V : \mathcal{P}(\mathbb{S}^n) \longrightarrow \mathcal{F}(\mathbb{B}^{n+1})$, *which to a probability measure* $\mu \in \mathcal{P}(\mathbb{S}^n)$ *assigns the vector field*

$$\begin{aligned} V_\mu(\mathbf{w}) &= \frac{1-|\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} \underline{\zeta}\, \mathrm{d}(g_{-\mathbf{w}})_* \mu(\underline{\zeta}), \\ &= \frac{1-|\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\underline{\zeta})\, \mathrm{d}\mu(\underline{\zeta}), \qquad \mathbf{w} \in \mathbb{B}^{n+1}, \end{aligned}$$

is the unique conformally natural map from $\mathcal{P}(\mathbb{S}^n)$ *to* $\mathcal{F}(\mathbb{B}^{n+1})$ *satisfying the normalizing condition*

$$V_\mu(\mathbf{0}) = \frac{1}{2} \int_{\mathbb{S}^n} \underline{\zeta}\, \mathrm{d}\mu(\underline{\zeta}).$$

Unique Zero of V_μ, for μ Admissible

- Proposition

  For each admissible probability measure $\mu \in \mathcal{P}'(\mathbb{S}^n)$ the vector field V_μ has a unique zero in $\mathbb{B}^{n+1}$.

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  For each admissible probability measure $\mu \in \mathcal{P}'(\mathbb{S}^n)$ the vector field V_μ has a unique zero in $\mathbb{B}^{n+1}$.

  - The proof relies on two elementary lemmas: Go to proof outline

The Conformal Barycenter map.

Definition

Define a conformally natural map $B : \mathcal{P}'(\mathbb{S}^n) \longrightarrow \mathbb{B}^{n+1}$ by setting $B(\mu)$ equal to the unique zero $\mathbf{w} \in \mathbb{B}^{n+1}$ of the vector field

$$V_\mu(\mathbf{w}) = \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\underline{\zeta}) \, \mathrm{d}\mu(\underline{\zeta}).$$

Then B satisfies:

$$B(\mu) = \mathbf{0} \quad \Leftrightarrow \quad \int_{\mathbb{S}^n} \underline{\zeta} \, \mathrm{d}\mu(\underline{\zeta}) = \mathbf{0}$$

and $B(\mu)$ is called the Conformal Barycenter of μ.

Harmonic Measure on $\mathbb{S}^n$ with center $\mathbf{w} \in \mathbb{B}^{n+1}$.

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- For $\mathbf{w} \in \mathbb{B}^{n+1}$ the harmonic measure with center $\mathbf{w}$ is the measure $\eta_{\mathbf{w}} = (g_{\mathbf{w}})_* \eta_0 = g_* \eta_{\mathbf{0}}$, for $g \in G$ with $g(0) = \mathbf{w}$.

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- The normalized Lebegue measure $\eta_{\mathbf{0}}$ is the unique R-invariant probability measure on $\mathbb{S}^n$.
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- A simple computation shows that

$$\int_{\mathbb{S}^n} f(\underline{\zeta})\, \mathrm{d}\eta_{\mathbf{w}}(\underline{\zeta}) = \int_{\mathbb{S}^n} f(\underline{\zeta}) \left(\frac{1 - |\mathbf{w}|^2}{|\underline{\zeta} - \mathbf{w}|^2} \right)^n \mathrm{d}\eta_{\mathbf{0}}(\underline{\zeta}).$$

The Right space for Extensions

- Recall that $\mathcal{E}(\mathbb{S}^n)$ denotes the space of Borel measureable endomorphisms $\phi : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ such that $\phi_* \eta_{\mathbf{0}}$ has no atoms.
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- For such mappings the measures $\phi_* \eta_{\mathbf{z}}$ has no atoms neither for any $\mathbf{z} \in \mathbb{B}^{n+1}$.
- And let $\mathrm{End}(\overline{\mathbb{B}}^{n+1})$ denote the space of measurable endomorphisms of $\overline{\mathbb{B}}^{n+1}$, whose restrictions to $\mathbb{B}^{n+1}$ are also endomorphisms of $\mathbb{B}^{n+1}$.

The Douady-Earle extension operator E

The Douady-Earle extension operator E is the map $E : \mathcal{E}(\mathbb{S}^n) \longrightarrow \mathrm{End}(\overline{\mathbb{B}}^{n+1})$ defined as follows:

For $\varphi \in \mathcal{E}(\mathbb{S}^n)$ the map $E(\varphi) = \Phi : \overline{\mathbb{B}^{n+1}} \longrightarrow \overline{\mathbb{B}^{n+1}}$ is given by the formulas

$$\Phi(\mathbf{z}) = \begin{cases} \varphi(\mathbf{z}), & \mathbf{z} \in \mathbb{S}^n, \\ B((\varphi \circ g_{\mathbf{z}})_*(\eta_0)) = B(\varphi_*(\eta_{\mathbf{z}})), & \mathbf{z} \in \mathbb{B}^{n+1} \end{cases}$$

The D-E Extension is Conformally Natural

- The map $\varphi \mapsto E(\varphi) = \Phi$ is conformally natural, i.e. for all $g, h \in G$:

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  by conformal naturallity of E and because $B(\eta_{\mathbf{0}}) = \mathbf{0}$.

- And hence

$$\forall\, g \in G : \qquad E(g_{|\mathbb{S}^n}) = g$$

  by conformal naturality.

Smoothness of the D-E Extensions

- Proposition

  Let $\varphi \in \mathcal{E}(\mathbb{S}^n)$ *and let* $\Phi = E(\varphi)$. *If* ϕ *is continuous at some point* $\underline{\zeta}_0 \in \mathbb{S}^n$ *then so is* Φ. *In particular if* φ *is continuous then* Φ *is continuous on* $\mathbb{S}^n$.

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- Proposition

  Let $\varphi \in \mathcal{E}(\mathbb{S}^n)$ and $E(\varphi) = \Phi$ be as above. Then Φ is real-analytic in $\mathbb{B}^{n+1}$.

Go to proof outline

Questions and Observations I

Questions that naturally arises are:
For $\widehat{f}$ a rational map on the Riemann sphere.

- BQ 1: How many of the properties of $\widehat{f}$ are inherited by $E(\widehat{f})$?
- BQ 2: What are the geometric and dynamical properties of the D-E extension $E(\widehat{f})$?

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For $\widehat{f}$ a rational map on the Riemann sphere.

- BQ 1: How many of the properties of $\widehat{f}$ are inherited by $E(\widehat{f})$?
- BQ 2: What are the geometric and dynamical properties of the D-E extension $E(\widehat{f})$?
- OBS 1: By elementary topology $E(\widehat{f})$ is a proper map, that is the preimage of any compact set is compact.
- OBS 2: And moreover for any point $\mathbf{w} \in \overline{\mathbb{B}^{n+1}}$ the preimage $E(\widehat{f})^{-1}(\mathbf{w})$ is a real analytic set.

Natural Questions II

Question 1: Is $E(f)$ a discrete map?

Question 2: Is $E(f)$ an open map?

Question 3: Is $E(f)$ a map of the same degree as f?

Question 4: Is the Julia set of $E(f)$ equal to the convex hull of the Julia set for f?

Finite Blaschke Products Setup

- Identify $\mathbb{C}$ with, $\{\mathbf{x} = (x_1, x_2, x_3) | \ x_3 = 0\} \subset \mathbb{R}^3$ and write $z = x + iy$ for the point $(x, y, 0)$. Then

$$\mathbb{D} = \{\mathbf{x} \in \mathbb{R}^3 | \ |z|^2 = x_1^2 + x_2^2 < 1, x_3 = 0\}$$
$$\mathbb{S}^1 = \{\mathbf{x} \in \mathbb{R}^3 | \ |z|^2 = 1, x_3 = 0\},$$

- Then stereographic projection S of $\overline{\mathbb{C}}$ onto $\mathbb{S}^2$ from the north pole $N = \mathbf{e}_3 \in \mathbb{R}^3$ is the map

$$S(z) = \left(\frac{2z}{1 + |z|^2}, \frac{|z|^2 - 1}{1 + |z|^2} \right) = \frac{1}{1 + |z|^2}(2x, 2y, |z|^2 - 1).$$

- For $f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ a rational map we shall write $\widehat{f}$ for its conjugate by S, i.e.:

$$\widehat{f}(S(z)) = S(f(z)).$$

Finite Blaschke Products

Consider finite Blaschke products

$$f(z) = \sigma \prod_{j=1}^{d} \frac{z + a_j}{1 + \overline{a}_j z}, \qquad |\sigma| = 1, \quad a_j \in \mathbb{D}$$

Proposition

For f a finite Blaschke product the D-E extension $E(\widehat{f})$

- *maps $\mathbb{D}$ onto $\mathbb{D}$,*
- *preserves the upper and lower hemispheres, ${\mathbb{S}^2}_+, {\mathbb{S}^2}_-$ and*
- *on $\mathbb{D}$ the partial derivative $\partial E(\widehat{f})/\partial x_3(z) = g(z)\mathbf{e}_3$ for some positive real analytical function $g : \mathbb{D} \longrightarrow \mathbb{R}_+$.*

The special case $f(z) = z^d$

- Write $M_t(z) = tz$ for $0 < t$ and $\mathbb{D}_t = \widehat{M}_t(\mathbb{D})$ the hyperbolic geodesic disk with boundary the circle $\widehat{M}_t(\mathbb{S}^1)$.

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- Corollary

  For $f(z) = z^d$ (i.e. $a_j = 0$ for all j):
  - $E(\widehat{f})(z) = z^d \cdot h(|z|^2)$ *for some real analytical function h with $r^d h(r)$ increasing and $h(r) \to 1$ as $r \nearrow 1$.*
  - $E(\widehat{f})$ *maps $\mathbb{D}_t$ onto $\mathbb{D}_{t^d}$ by a degree d covering and*
  - $E(\widehat{f})$ *maps the interval $[\mathbf{0}, \mathbf{e}_3[$ onto itself by an increasing diffeomorphism.*

- Conjecture
  For all finite Blaschke products f we have $f = E(\widehat{f})$ *on* $\mathbb{D}$.
- Conjecture
  For all finite Blaschke products f with $f(0) = 0$ *we have* $E(\widehat{f})(\mathbf{0}) = \mathbf{0}$.

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  - Motivation: Fuchsian groups plus ...

Inner Functions

- For inner functions, that is for holomorphic selfmaps $f : \mathbb{D} \longrightarrow \mathbb{D}$ of the unit disc $\mathbb{D} \subset \mathbb{C}$ with boundary values in $\mathbb{S}^1$ a.e., the D-E extension simply recovers f from its boundary values.

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- More precisely for an inner function the radial limit

$$f^{\#}(\zeta) = \lim_{r \nearrow 1} f(r\zeta)$$

exists for a.e. $\zeta \in \mathbb{S}^1$ and the measure $f^{\#}_{*}\eta_0$ is absolutely continuous with respect to η_0 so that $f^{\#} \in \mathcal{E}(\mathbb{S}^1)$.

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- **Proposition**

If $f : \mathbb{D} \longrightarrow \mathbb{D}$ is an inner function then

$$E(f^{\#})(z) = f(z), \qquad \forall z \in \mathbb{D}.$$

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Appendix with Further Details

- Proof of Unique zero of V_μ for μ admissible.
- Proof of Real Analyticity of the D-E Extensions

Zeros of V_μ are Isolated stable Equilibria

- Lemma

  For any admissible probability measure $\mu \in \mathcal{P}'(\mathbb{S}^n)$ any zero $\mathbf{v} \in \mathbb{B}^{n+1}$ of the vector field $V = V_\mu$ is an isolated stable equilibrium.

Zeros of V_μ are Isolated stable Equilibria

- Lemma
  For any admissible probability measure $\mu \in \mathcal{P}'(\mathbb{S}^n)$ any zero $\mathbf{v} \in \mathbb{B}^{n+1}$ of the vector field $V = V_\mu$ is an isolated stable equilibrium.
- Proof.
  By conformal naturallity it suffices to consider the case $\mathbf{v} = \mathbf{0}$. By an elementary computation we find

$$\mathrm{Jac}_V(\mathbf{0})(\underline{\epsilon}) = -\int_{\mathbb{S}^n} (\underline{\epsilon} - \underline{\zeta} < \underline{\epsilon}, \underline{\zeta} >) \, \mathrm{d}\mu(\underline{\zeta})$$

and thus $\mathrm{Jac}_V(\mathbf{0})$ is non singular. In fact $\mathbf{v} = \mathbf{0}$ is a sink since

$$< \underline{\epsilon}, \mathrm{Jac}_V(\mathbf{0})(\underline{\epsilon}) >= -\int_{\mathbb{S}^n} (< \underline{\epsilon}, \underline{\epsilon} > -(< \underline{\epsilon}, \underline{\zeta} >)^2) \, \mathrm{d}\mu(\underline{\zeta}) \; < 0.$$

V_μ Points Inwards near the Boundary $\mathbb{S}^n$.

- Lemma
  For any admissible probability measure $\mu \in \mathcal{P}'(\mathbb{S}^n)$ *there exists* $r \in \,]0,1[$ *such that* $V_\mu(\mathbf{w})$ *points inwards at any point* $\mathbf{w} \in \mathbb{B}^{n+1}$ *with* $r \leq |\mathbf{w}| < 1$*, i.e.* $< V_\mu(\mathbf{w}), \mathbf{w} > < 0$.

Return to Proposition

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- Proof of Uniqueness of zero of V_μ:

  The Poincaré-Hoppf (Index) theorem [Mi, see also Lemma 3, p 36].

Return to Proposition

Proof of Real Analyticity of the D-E Extensions

Proposition

Let $\varphi \in \mathcal{E}(\mathbb{S}^n)$ and $E(\varphi) = \Phi$ be as above. Then Φ is real-analytic in $\mathbb{B}^{n+1}$.

Proof of real analyticity

$\Phi(\mathbf{z})$ is the unique zero of the vector field

$$V_{\varphi_*(\eta_{\mathbf{z}})}(\mathbf{w}) = \frac{1 - |\mathbf{w}|^2}{2} \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\varphi(\underline{\zeta})) \left(\frac{1 - |\mathbf{z}|^2}{|\mathbf{z} - \underline{\zeta}|^2} \right)^n \mathrm{d}\eta_{\mathbf{0}}(\underline{\zeta}).$$

Proof of Differentiability Cont.

- Thus $\forall\, \mathbf{z} \in \mathbb{B}^{n+1}$ the value $\mathbf{w} = \Phi(\mathbf{z}) \in \mathbb{B}^{n+1}$ is the unique point such that:

$$F(\mathbf{z}, \mathbf{w}) = \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\varphi(\underline{\zeta})) \left(\frac{1 - |\mathbf{z}|^2}{|\mathbf{z} - \underline{\zeta}|^2} \right)^n \mathrm{d}\eta_{\mathbf{0}}(\underline{\zeta}) = \mathbf{0}.$$

Proof of Differentiability Cont.

- Thus $\forall$ $\mathbf{z} \in \mathbb{B}^{n+1}$ the value $\mathbf{w} = \Phi(\mathbf{z}) \in \mathbb{B}^{n+1}$ is the unique point such that:

$$F(\mathbf{z}, \mathbf{w}) = \int_{\mathbb{S}^n} g_{-\mathbf{w}}(\varphi(\underline{\zeta})) \left(\frac{1 - |\mathbf{z}|^2}{|\mathbf{z} - \underline{\zeta}|^2} \right)^n \mathrm{d}\eta_{\mathbf{0}}(\underline{\zeta}) = \mathbf{0}.$$

- Since F is a real-analytical function of $(\mathbf{z}, \mathbf{w}) \in \mathbb{B}^{n+1} \times \mathbb{B}^{n+1}$, the implicit function theorem implies Φ is real analytic provided $\mathrm{J}_{\mathbf{w}} F = \frac{\partial F}{\partial \mathbf{w}}(\mathbf{z}, \mathbf{w})$ is non-singular whenever $F(\mathbf{z}, \mathbf{w}) = \mathbf{0}$.

Proof of Differentiability Cont.

- Thus $\forall\ \mathbf{z} \in \mathbb{B}^{n+1}$ the value $\mathbf{w} = \Phi(\mathbf{z}) \in \mathbb{B}^{n+1}$ is the unique point such that:

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- Since F is a real-analytical function of $(\mathbf{z}, \mathbf{w}) \in \mathbb{B}^{n+1} \times \mathbb{B}^{n+1}$, the implicit function theorem implies Φ is real analytic provided $\mathrm{J}_{\mathbf{w}} F = \frac{\partial F}{\partial \mathbf{w}}(\mathbf{z}, \mathbf{w})$ is non-singular whenever $F(\mathbf{z}, \mathbf{w}) = \mathbf{0}$.
- By conformal naturality we can suppose $\mathbf{z} = \mathbf{w} = \mathbf{0}$. A straight forward computation shows that

$$\mathrm{J}_{\mathbf{w}} F(\underline{\epsilon}) = -2 \int_{\mathbb{S}^n} (\underline{\epsilon} - < \underline{\epsilon}, \phi(\zeta) > \phi(\zeta)) \, d\eta_{\mathbf{0}}(\zeta).$$