Invariant Peano curves of expanding Thurston maps.

Daniel Meyer

Jacobs University

February 25, 2011

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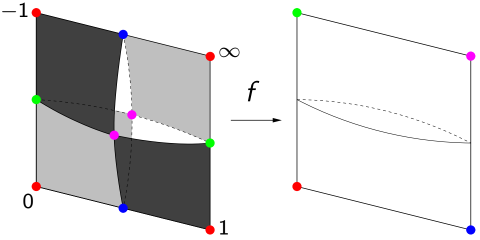

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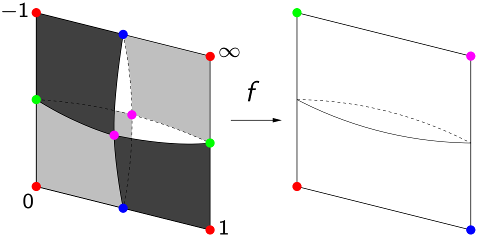

Critical points: ● ● ● $\overset{f}{\mapsto} \{-1, 1, \infty\} \overset{f}{\mapsto} \{0\} \overset{f}{\hookleftarrow}$.

Rational in suitable coordinates.

expanding Thurston Maps

Thurston map $f\colon S^2 \to S^2$:
branched cover of S^2, locally

$$z \mapsto z^d,$$

after *homeomorphic* coordinate changes,
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In general let $\mathcal{C} \supset \operatorname{post}(f)$ be (any) Jordan curve. Then
mesh $S^2 \setminus f^{-n}(\mathcal{C}) \to 0$, as $n \to \infty$.
Independent of the chosen curve $\mathcal{C}$.

Invariant Peano curves

Theorem (M)

f expanding Thurston map. Then for each sufficiently high iterate $F = f^n$ there exists a Peano curve $\gamma\colon S^1 \to S^2$ (onto) such that

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Corresponding to result by Cannon-Thurston.

Lifting Isotopies

Theorem

H_0 isotopy rel. post(G) *lifts to isotopy H_1.*

$$\begin{array}{ccc} S^2 & \xrightarrow{H_1} & S^2 \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} \\ S^2 & \xrightarrow[H_0]{} & S^2 \end{array}$$

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rel. post(G): $H(t,\cdot)$ constant on post(G).

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H_1 isotopy rel. G^{-1} post(G).
H_n lift of H_0 by G^n, then

$$\max_{x \in S^2} \operatorname{diam} H_n(x, \cdot) \leq C\Lambda^{-n},$$

in suitable metric $(\Lambda > 1)$.

Constructing the Peano curve

Explain the construction for f. Show the construction in $\mathbb{C} = \wp^{-1}(\widehat{\mathbb{C}})$ (in the orbifold cover).

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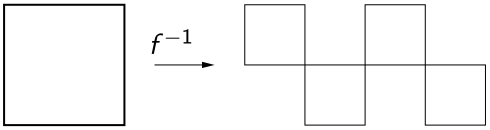

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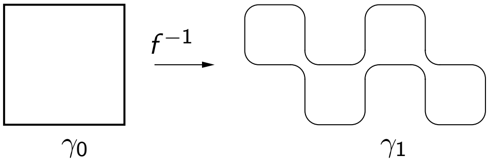

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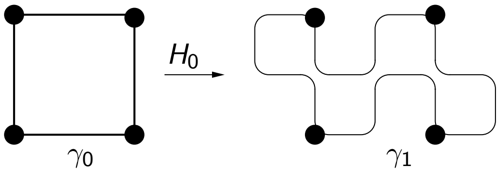

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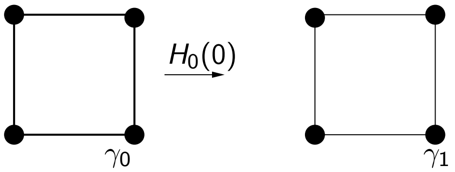

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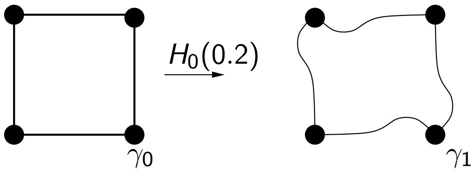

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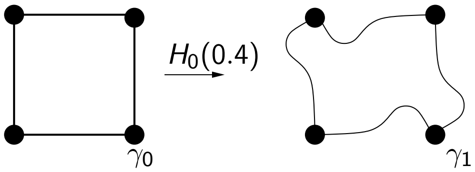

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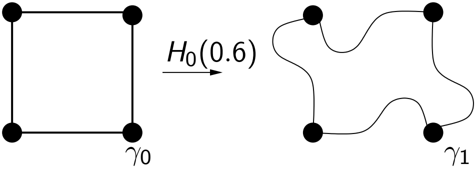

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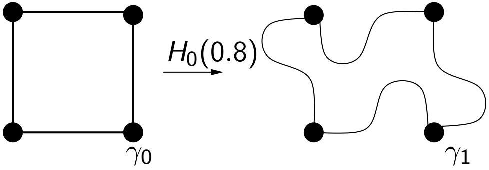

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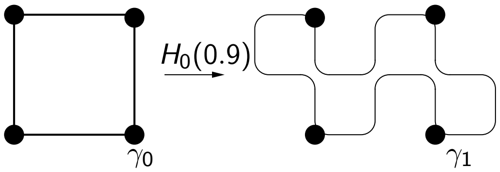

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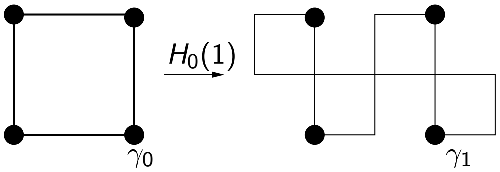

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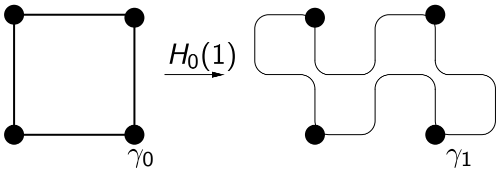

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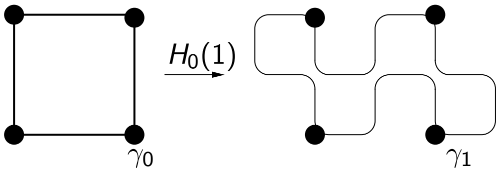

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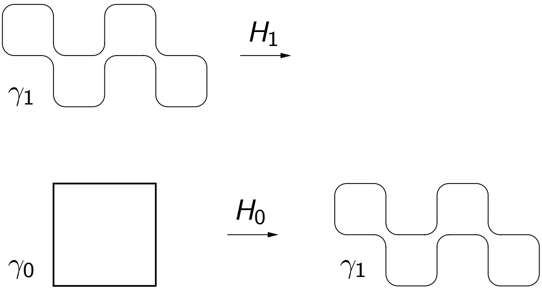

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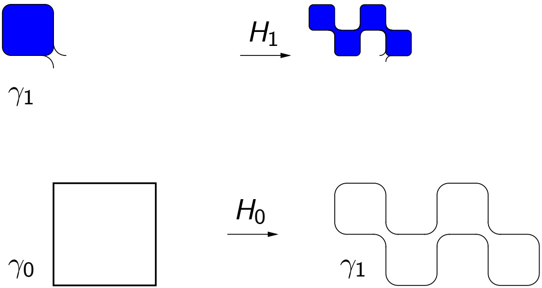

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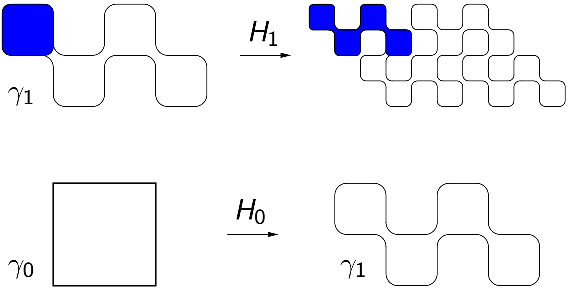

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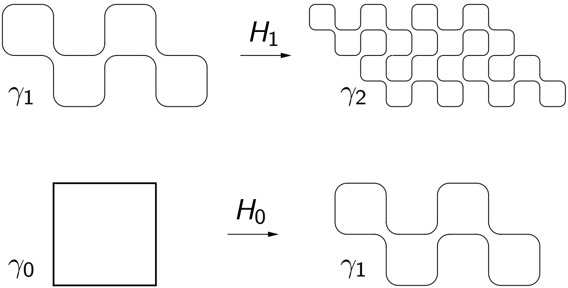

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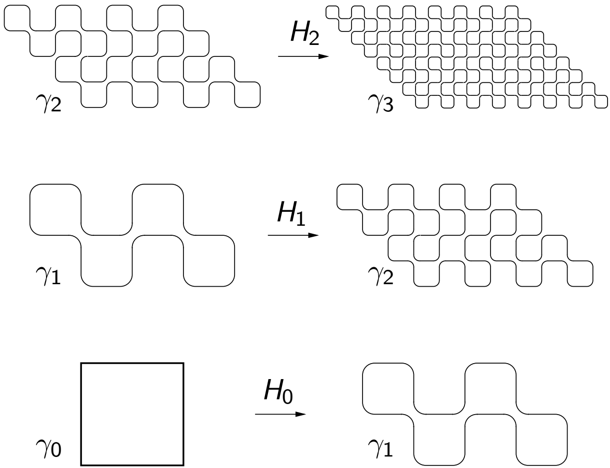

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Theorem (M)

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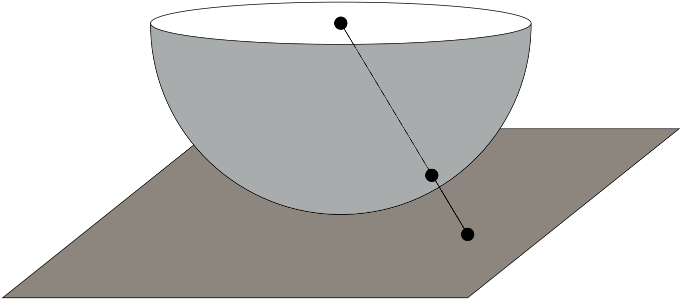
The gnomonic projection maps $\mathbb{C}$ to a hemisphere.



Action of polynomial on $\mathbb{C}$, becomes action on hemisphere.

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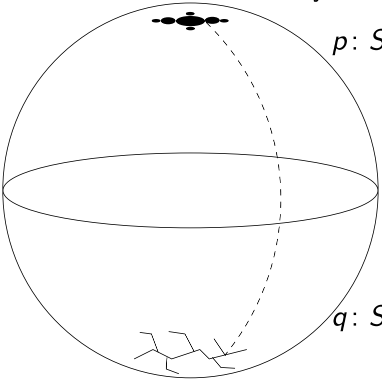

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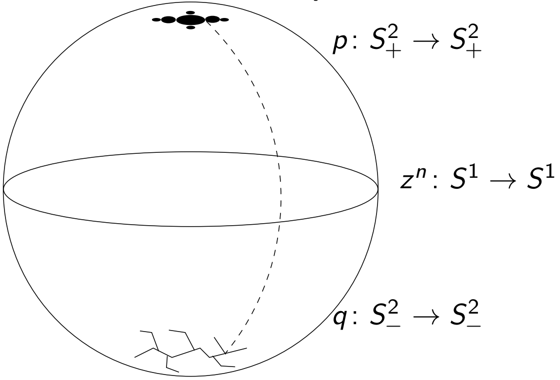

Collapse each external ray, map extends.

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$$\sigma_{p,q} \colon S^1 = \partial\overline{\mathbb{D}} \to \partial\mathcal{K}_{p,q}.$$

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Map on $\mathcal{K}_p \amalg \mathcal{K}_q = \mathcal{K}_p \sqcup \mathcal{K}_q / \sim$ given by

$$p \amalg q|_{\mathcal{K}_p} = p \quad p \amalg q|_{\mathcal{K}_q} = q,$$

well defined. This is the mating of p, q.

Quotients

$\gamma\colon S^1 \to S^2$ Peano curve.
Let $x \sim y$ iff $\gamma(x) = \gamma(y)$, $x, y \in S^1$.
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It is possible to "generate" $\sim$ from finite data.

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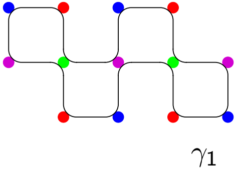

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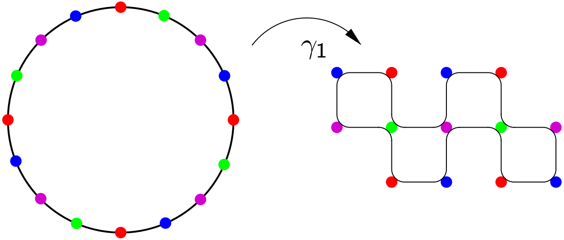

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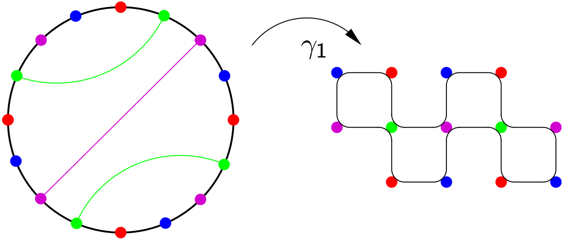

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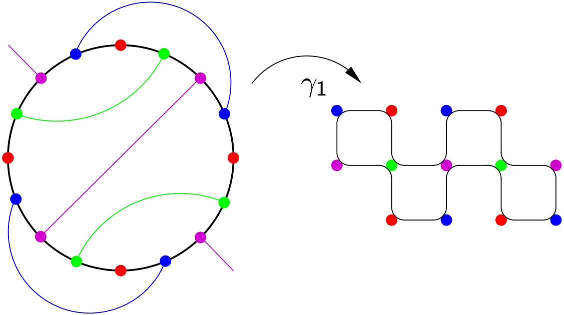

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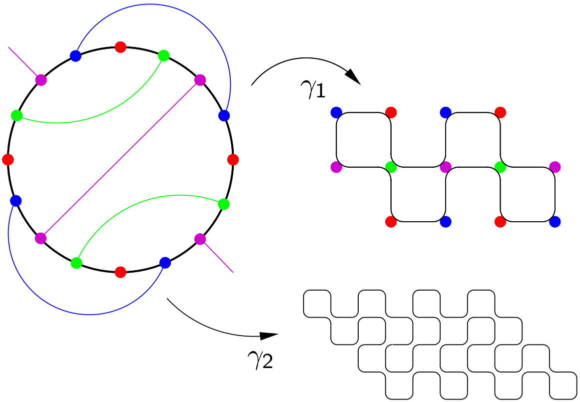

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The *closure* of the union of $\stackrel{n}{\sim}$ is $\sim$.

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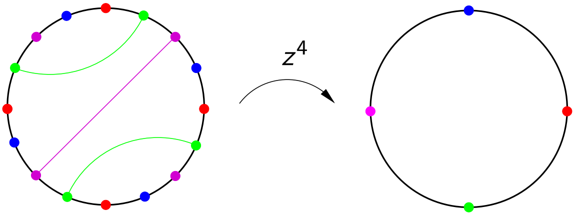

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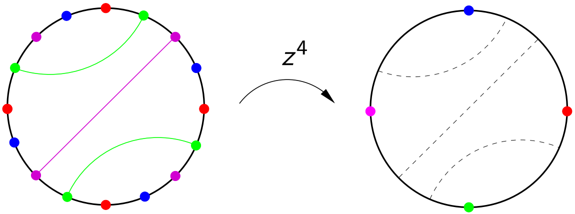

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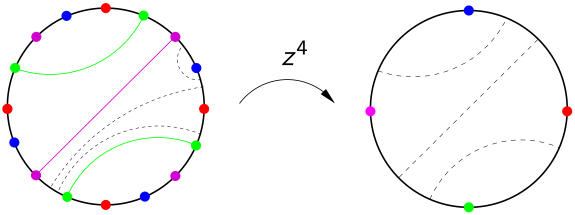

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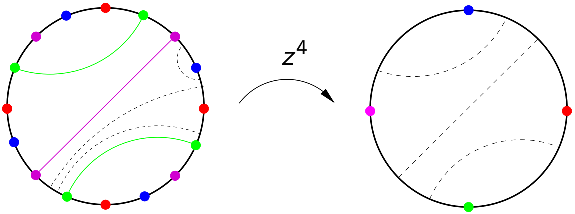

From this one obtains $\overset{2}{\sim}$, in the limit $\sim$.

Classification

Thus $\overset{1}{\sim}$ (the picture before) contains all the necessary information to recover the map G. For our example The following points are identified (in $S^1 = \mathbb{R}/\mathbb{Z}$) by γ_1.

in the "interior" $\left(\frac{2}{16}, \frac{10}{16}\right), \left(\frac{3}{16}, \frac{7}{16}\right), \left(\frac{11}{16}, \frac{11}{16}\right);$

in the "exterior" $\left(\frac{1}{16}, \frac{5}{16}\right), \left(\frac{6}{16}, \frac{14}{16}\right), \left(\frac{9}{13}, \frac{13}{16}\right).$

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These numbers form critical portraits of two polynomials (Poirier). These are the polynomials that "mate" to G.