Moduli Space Maps and Compactifications: A Worked Out Example of Mating

Sarah C. Koch
Harvard University

Moduli Space Maps and Compactifications: A Worked Out Example of Mating

Sarah C. Koch
Harvard University

Joint work with Xavier Buff and Adam Epstein

Moduli Space Maps and Compactifications:
A Worked Out Example of Mating

Sarah C. Koch
Harvard University

Joint work with Xavier Buff and Adam Epstein

Pasting Together Julia Sets: A Worked Out Example of Mating

John Milnor

CONTENTS

2000 AMS Subject Classification: Primary 37F45; Secondary 30D05

Keywords: Julia set, Lattès map, fractal tiling

The operation of "mating" two suitable complex polynomial maps f_1 and f_2 constructs a new dynamical system by carefully pasting together the boundaries of their filled Julia sets so as to obtain a copy of the Riemann sphere, together with a rational map $f_1 \perp\!\!\perp f_2$ from this sphere to itself. This construction is particularly hard to visualize when the filled Julia sets $K(f_i)$ are dendrites, with no interior. This note will work out an explicit example of this type, with effectively computable maps from $K(f_1)$ and $K(f_2)$ onto the Riemann sphere.

1. INTRODUCTION

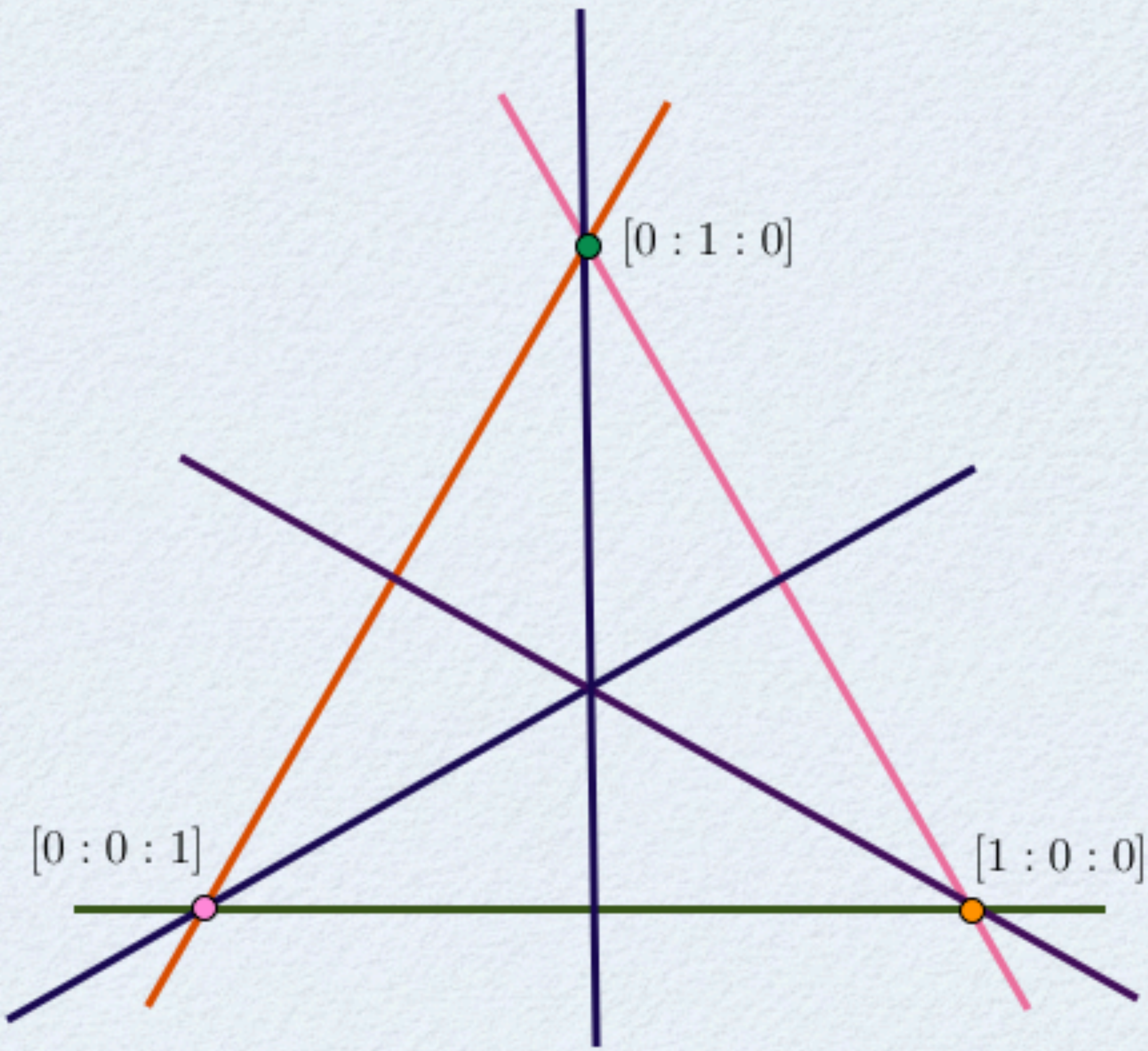
The operation of *mating*, first described by [Douady 83] has been shown to exist for suitable pairs of quadratic polynomial maps by [Tan Lei 90], [Rees 92], and [Shishikura 00]. (See Section 2.) In an attempt to understand this construction, this paper concentrates on one very special example. We consider the (filled) Julia set $K = K(f)$ which is illustrated in Figure 1 and described more precisely in Section 2. The mating $f \perp\!\!\perp f$ exists according to Shishikura. This means that we can form a full Riemann sphere by pasting two copies of $K = \partial K$ together, in such a way that each copy of K covers the full Riemann sphere, while the map f on each copy corresponds to a smooth quadratic rational map from this sphere to itself. We will give a computationally effective description for this particular example, showing just how such a dendrite can map onto a sphere. The construction is closely related to a well known measure-preserving area filling curve, with associated fractal self-similar tiling,[1] which is known as the "Heighway Dragon." The resulting rational map $F \cong f \perp\!\!\perp f$, where $F(z) = (i/2)\,(z + z^{-1})$, can also be described as a *Lattès mapping*, that is as the quotient of a rigid expanding map on a torus. (This is

[1]See Section 4.2 and Figures 7 and 16. This construction was discovered by John Heighway, a physicist at NASA, circa 1966. Compare [Davis and Knuth 65], [Edgar 90], and even [Crichton 90].

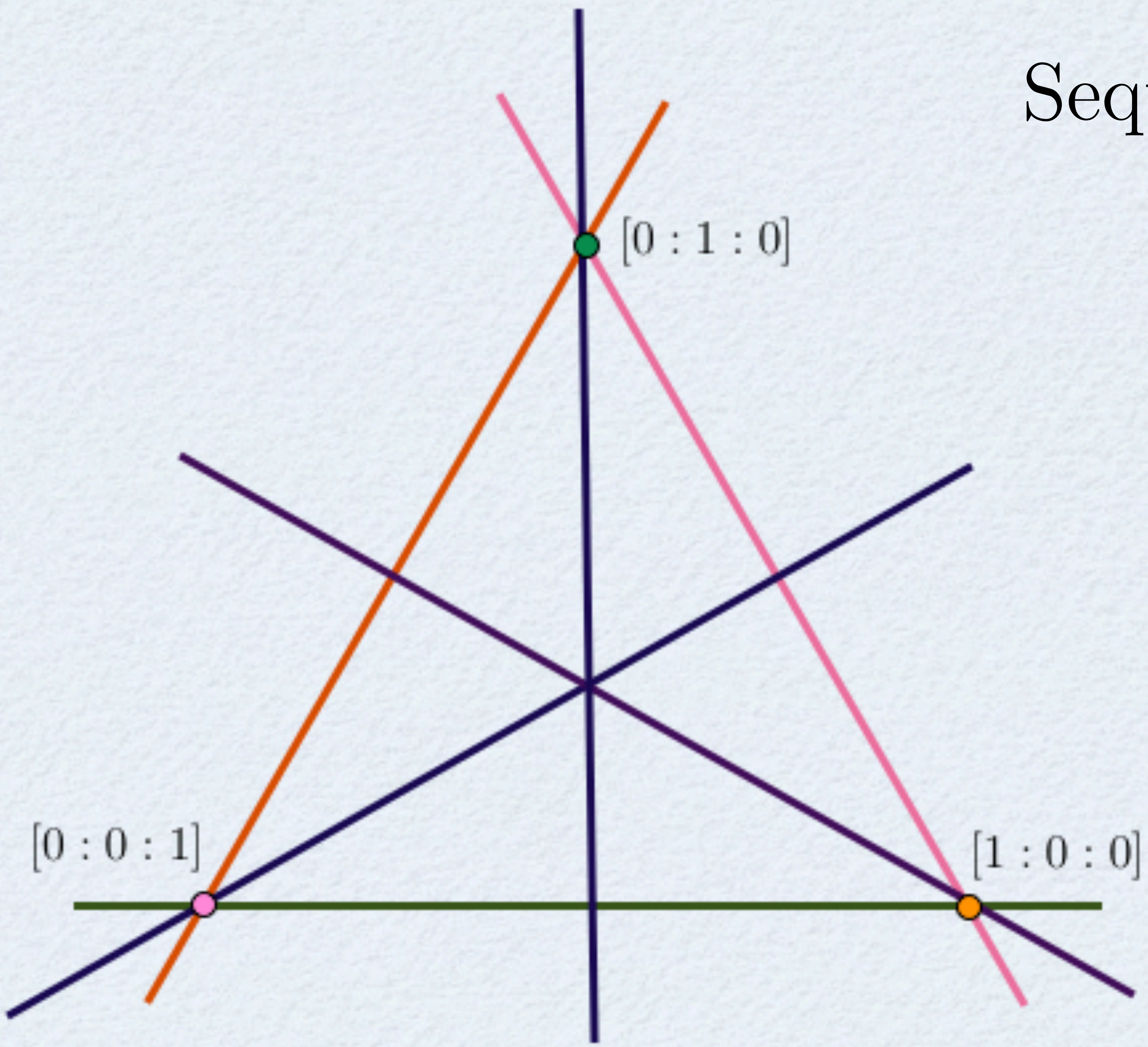
1058-6458/2004$0.50 per page
Experimental Mathematics **13**:1, page 55

Thursday, February 24, 2011

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^2,\ [x:y:z] \mapsto [yz:xz:xy]$$

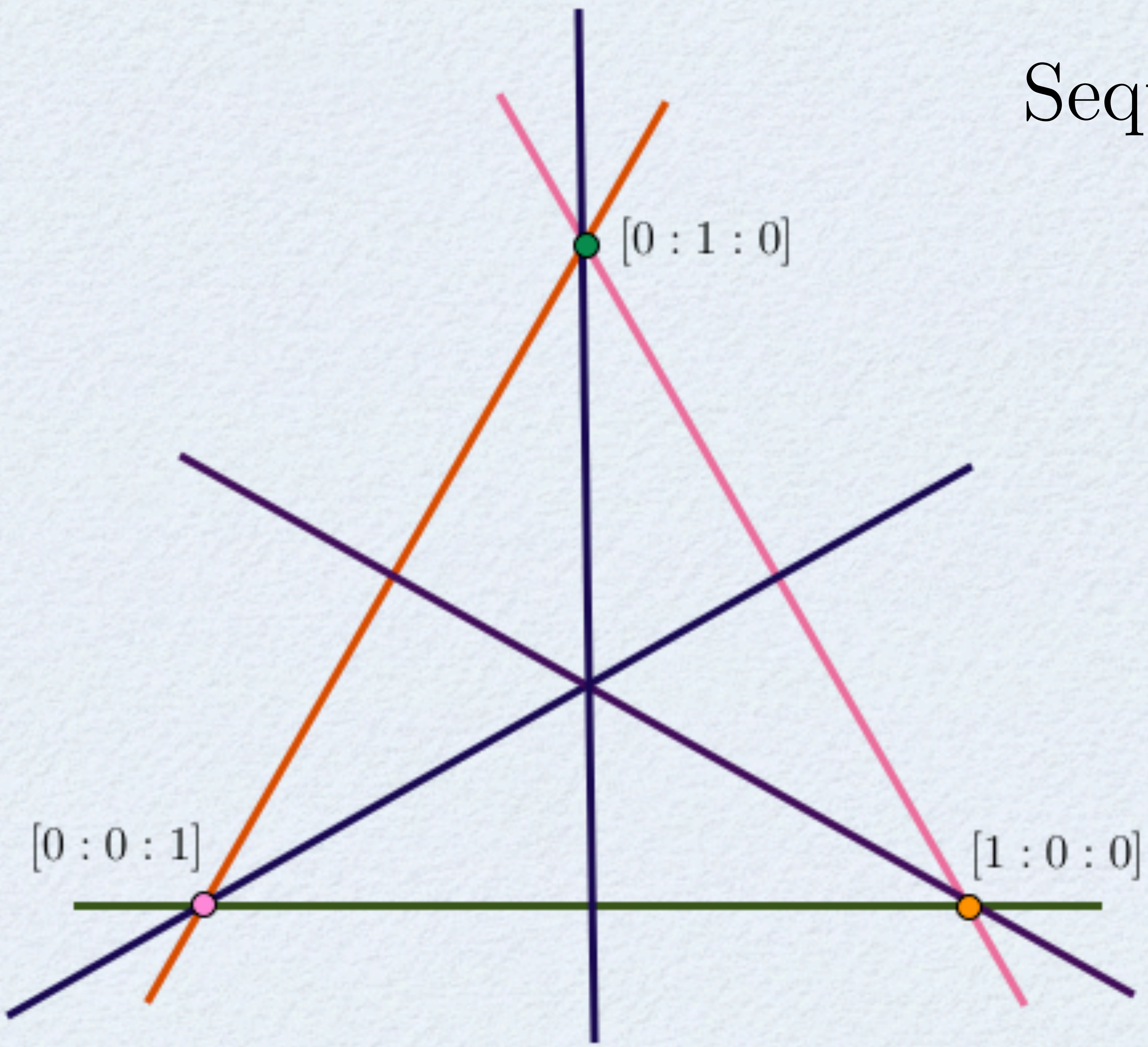


$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^2,\ [x:y:z] \mapsto [yz:xz:xy]$$



Sequence of algebraic degrees:

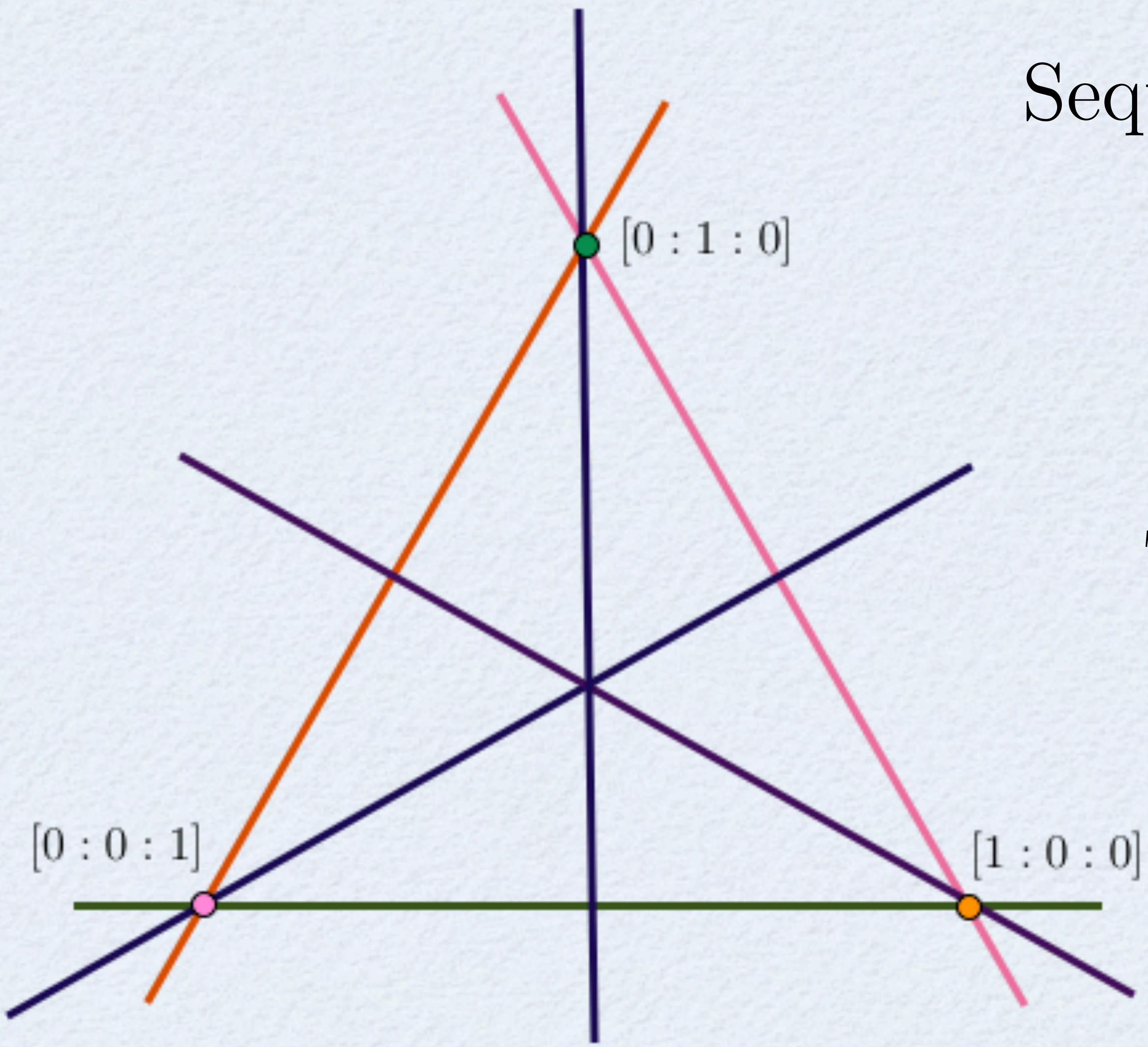
$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^2,\ [x:y:z] \mapsto [yz:xz:xy]$$



Sequence of algebraic degrees:

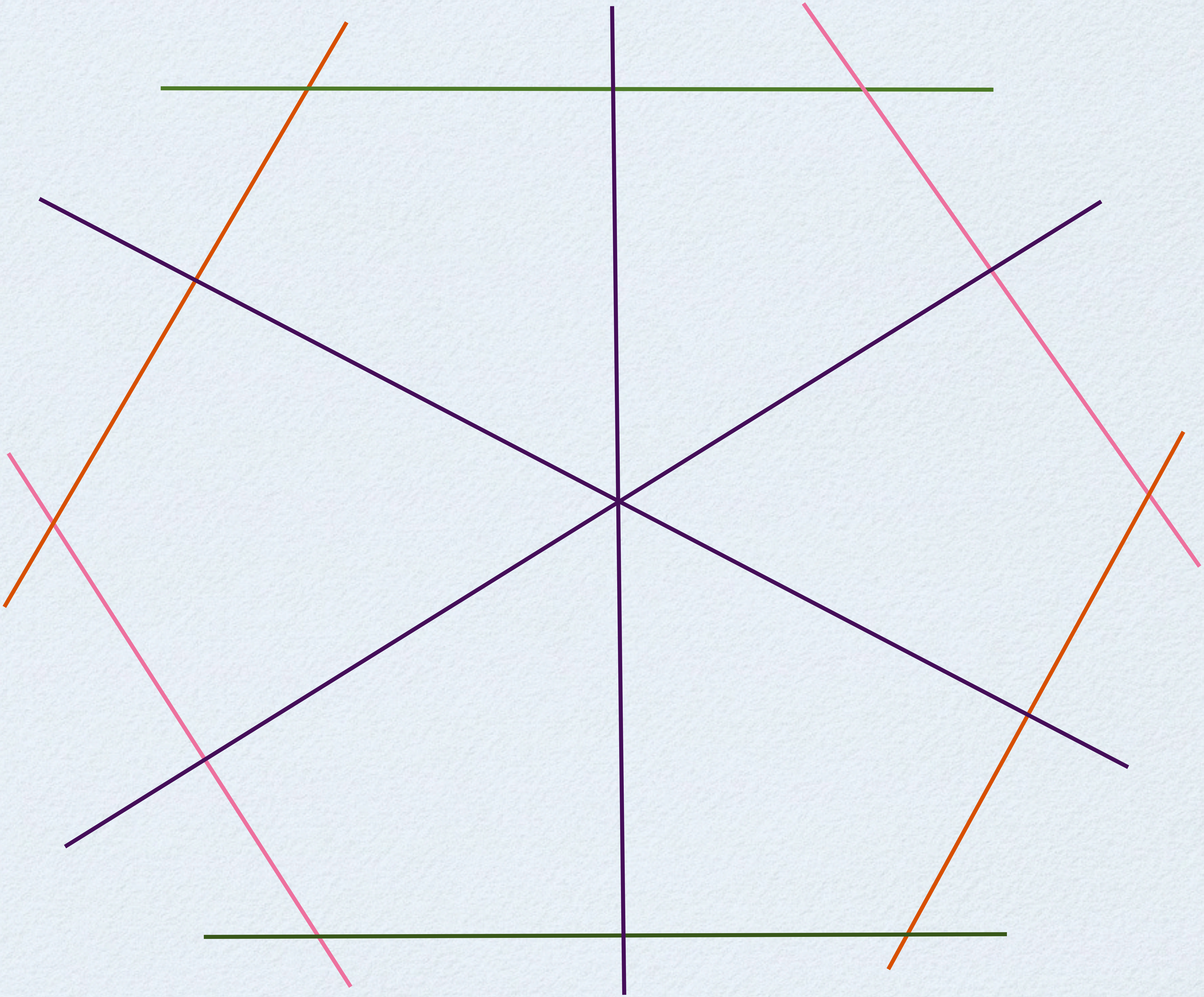
$1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots$

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^2,\ [x : y : z] \mapsto [yz : xz : xy]$$



Sequence of algebraic degrees:
$$1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots$$

This map is *not algebraically stable*.



Algebraic stability

Notion was introduced by Fornæss and Sibony.

Let X be a compact complex manifold. Let $F : X \dashrightarrow X$ be a meromorphic map.

F induces an action on $H^2(X)$ by pullback.

Algebraic stability

Notion was introduced by Fornæss and Sibony.

Let X be a compact complex manifold. Let $F : X \dashrightarrow X$ be a meromorphic map.

F induces an action on $H^2(X)$ by pullback.

The map F is algebraically stable iff functoriality holds:

$$(F^{\circ n})^* = (F^*)^{\circ n} \quad \forall\, n \geqslant 0,$$

Algebraic stability

Notion was introduced by Fornæss and Sibony.

Let X be a compact complex manifold. Let $F : X \dashrightarrow X$ be a meromorphic map.

F induces an action on $H^2(X)$ by pullback.

The map F is algebraically stable iff functoriality holds:

$$(F^{\circ n})^* = (F^*)^{\circ n} \quad \forall\, n \geqslant 0,$$

Let X be a compact complex manifold, and $F : X \dashrightarrow X$ a meromorphic map. Then F is algebraically stable if no hypersurface is contracted to something of higher codimension which is contained in $\mathcal{I}_F$.

Algebraic stability

Notion was introduced by Fornæss and Sibony.

Let X be a compact complex manifold. Let $F : X \dashrightarrow X$ be a meromorphic map.

F induces an action on $H^2(X)$ by pullback.

The map F is algebraically stable iff functoriality holds:

$$(F^{\circ n})^* = (F^*)^{\circ n} \quad \forall\, n \geqslant 0,$$

Let X be a compact complex manifold, and $F : X \dashrightarrow X$ a meromorphic map. Then F is algebraically stable if no hypersurface is contracted to something of higher codimension which is contained in $\mathcal{I}_F$.

$$\lambda_1 = \lim_{n\to\infty} \|(F^{\circ n})^*\|^{1/n}$$

Preliminary Preliminaries

Let S^2 be the unit sphere in $\mathbb{C} \times \mathbb{R}$, and let

$$P_1 : \mathbb{C} \to \mathbb{C}, \quad \text{and} \quad P_2 : \mathbb{C} \to \mathbb{C}$$

be monic polynomials of degree $d \geqslant 2$. The *formal mating* of P_1 and P_2 is the branched cover $f : S^2 \to S^2$ defined as follows.

Preliminary Preliminaries

Let S^2 be the unit sphere in $\mathbb{C} \times \mathbb{R}$, and let

$$P_1 : \mathbb{C} \to \mathbb{C}, \quad \text{and} \quad P_2 : \mathbb{C} \to \mathbb{C}$$

be monic polynomials of degree $d \geqslant 2$. The *formal mating* of P_1 and P_2 is the branched cover $f : S^2 \to S^2$ defined as follows.

Identify dynamical plane of P_1 to H^+, and identify the dynamical plane of P_2 to H^- via the projections

$$\rho_1 : \mathbb{C} \to H^+ \quad \text{and} \quad \rho_2 : \mathbb{C} \to H^-$$

$$\rho_1(z) = \frac{(z,1)}{\|(z,1)\|} \quad \text{and} \quad \rho_2(z) = \frac{(\overline{z},1)}{\|(\overline{z},1)\|}.$$

Preliminary Preliminaries

Let S^2 be the unit sphere in $\mathbb{C} \times \mathbb{R}$, and let

$$P_1 : \mathbb{C} \to \mathbb{C}, \quad \text{and} \quad P_2 : \mathbb{C} \to \mathbb{C}$$

be monic polynomials of degree $d \geqslant 2$. The *formal mating* of P_1 and P_2 is the branched cover $f : S^2 \to S^2$ defined as follows.

Identify dynamical plane of P_1 to H^+, and identify the dynamical plane of P_2 to H^- via the projections

$$\rho_1 : \mathbb{C} \to H^+ \quad \text{and} \quad \rho_2 : \mathbb{C} \to H^-$$

$$\rho_1(z) = \frac{(z,1)}{\|(z,1)\|} \quad \text{and} \quad \rho_2(z) = \frac{(\overline{z},1)}{\|(\overline{z},1)\|}.$$

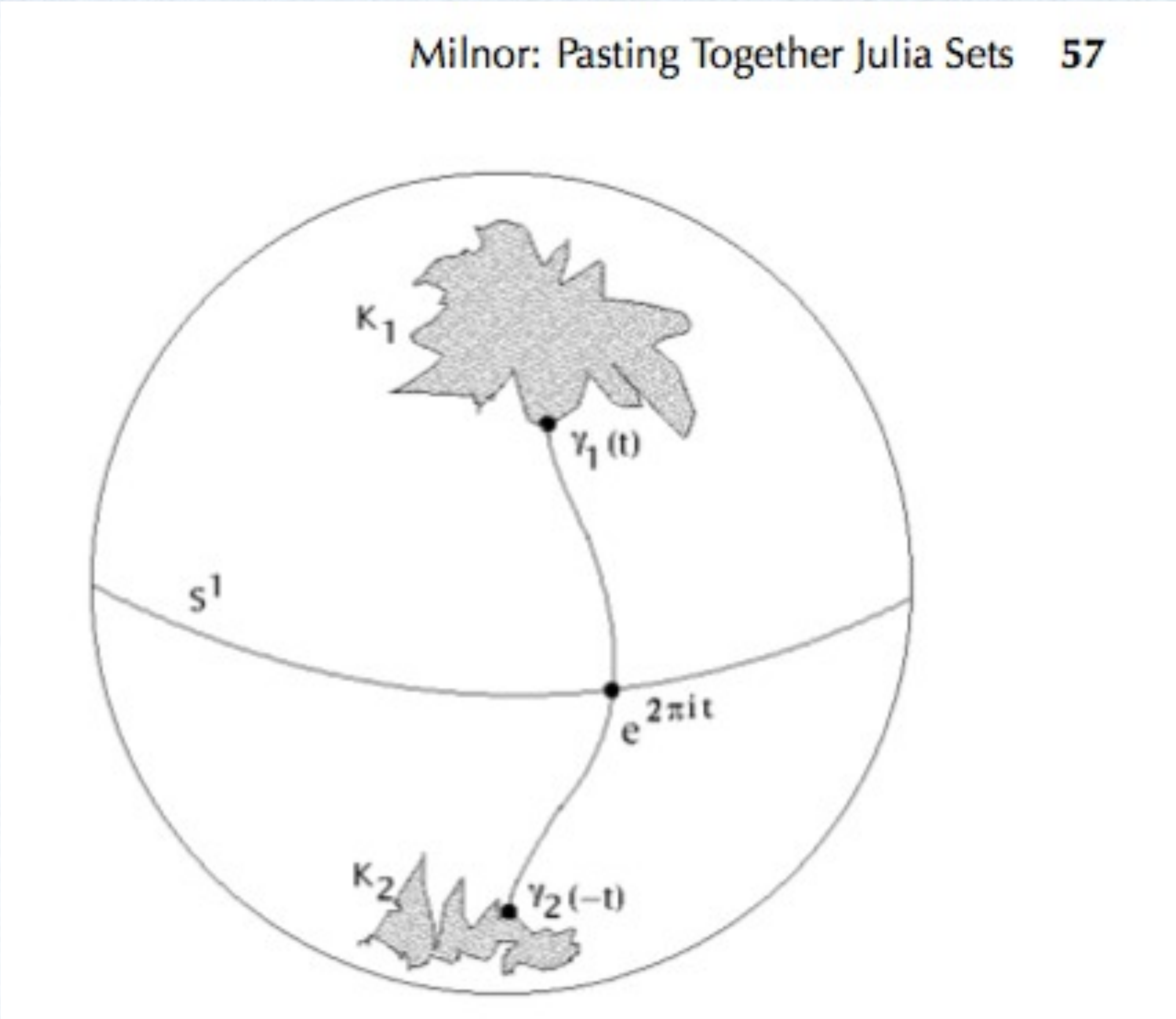
Since the polynomials have the same degree, the map $\rho_1 \circ P_1 \circ \rho_1^{-1}$ defined on H^+ and the map $\rho_2 \circ P_2 \circ \rho_2^{-1}$ defined on H^- extend continuously to the equator of S^2.

Form the quotient $S^2/\sim$ by collapsing along external rays. The rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ is a *geometric mating* of P_1 and P_2 if $S^2/\sim$ is homeomorphic to S^2 and if the formal mating $f : S^2 \to S^2$ induces a map

$$S^2/\sim \quad \to \quad S^2/\sim$$

which is topogically conjugate to

$$F : \mathbb{P}^1 \to \mathbb{P}^1.$$

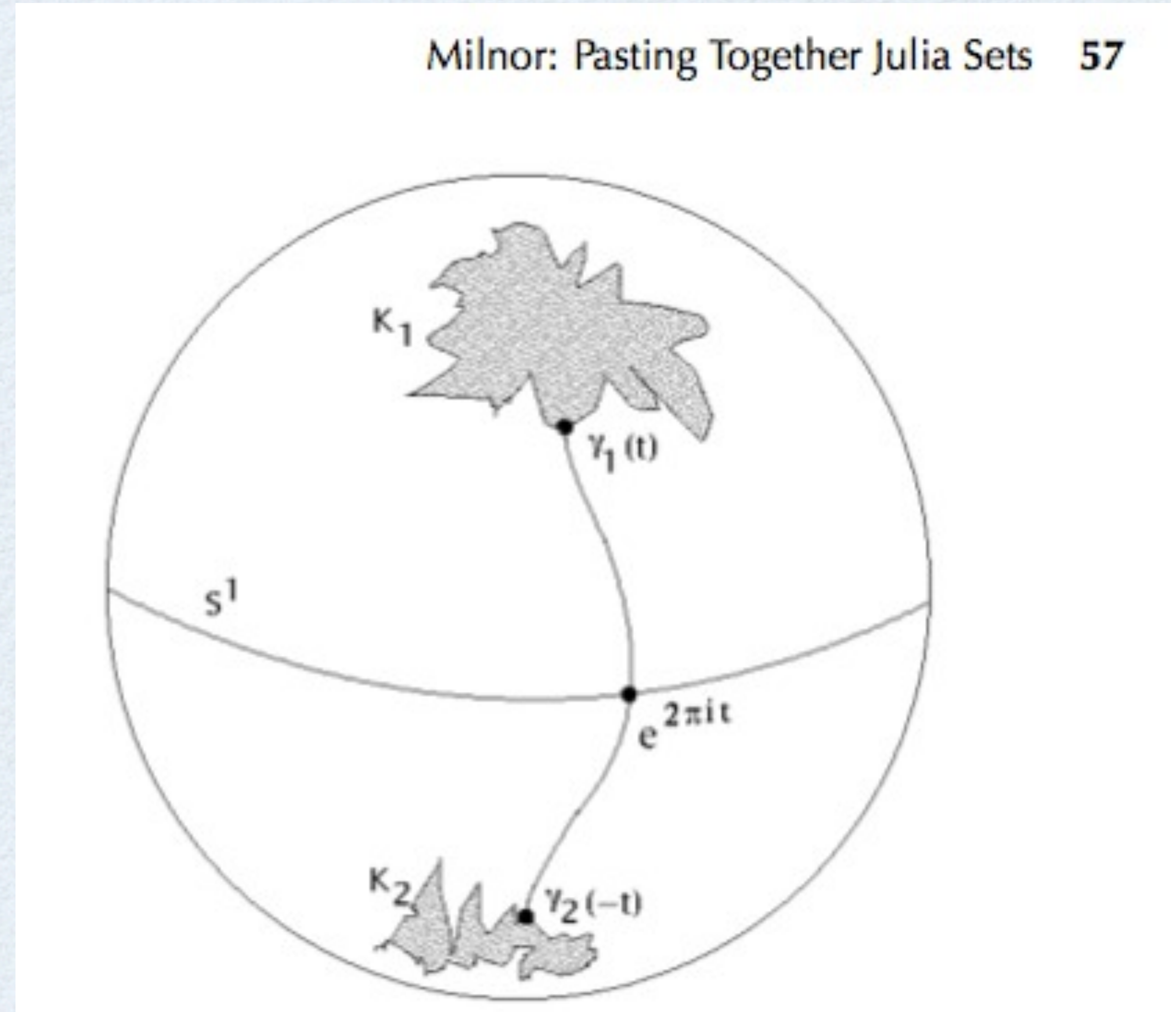


Form the quotient $S^2/\sim$ by collapsing along external rays. The rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ is a *geometric mating* of P_1 and P_2 if $S^2/\sim$ is homeomorphic to S^2 and if the formal mating $f : S^2 \to S^2$ induces a map

$$S^2/\sim \quad \to \quad S^2/\sim$$

which is topogically conjugate to

$$F : \mathbb{P}^1 \to \mathbb{P}^1.$$



Theorem. (Rees) Let P_1 and P_2 be critically finite hyperbolic polynomials. The formal mating of P_1 and P_2 is combinatorially equivalent to a rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ if and only if F is a geometric mating of P_1 and P_2.

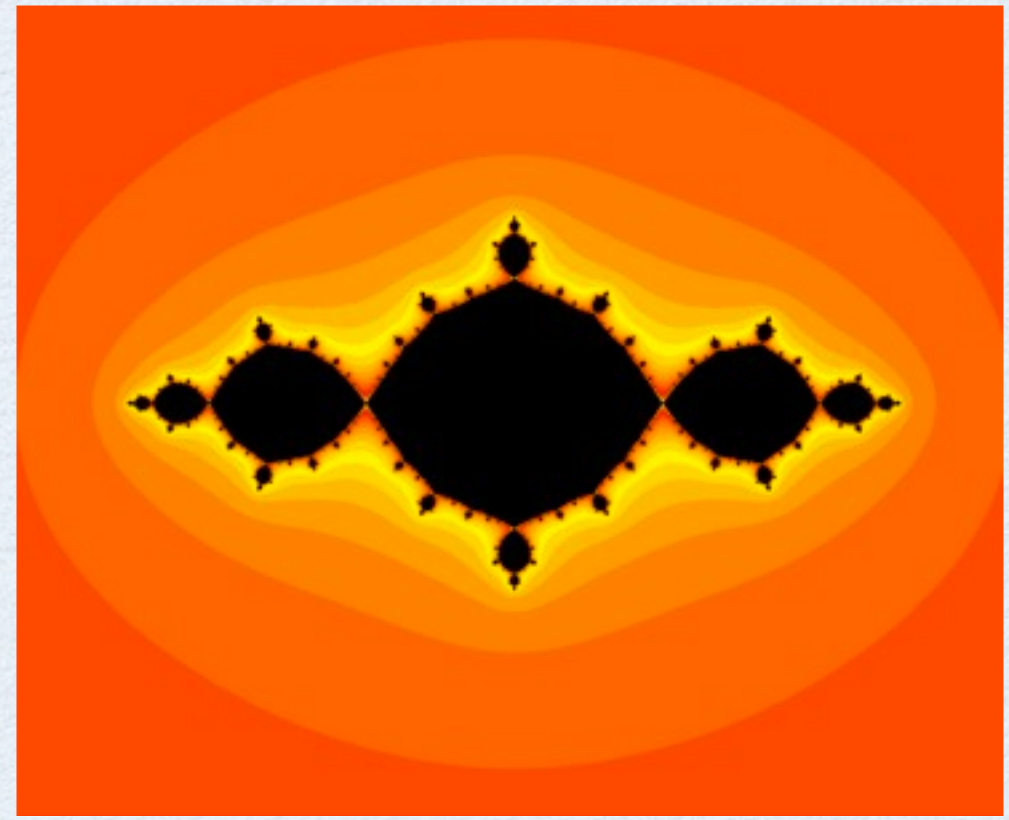
Example: Basilica mate Basilica

Example: Basilica mate Basilica

$$\begin{tikzcd} \star \arrow[r, "2"] & \cdot \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \infty \arrow[loop right, "2"] \end{tikzcd}$$

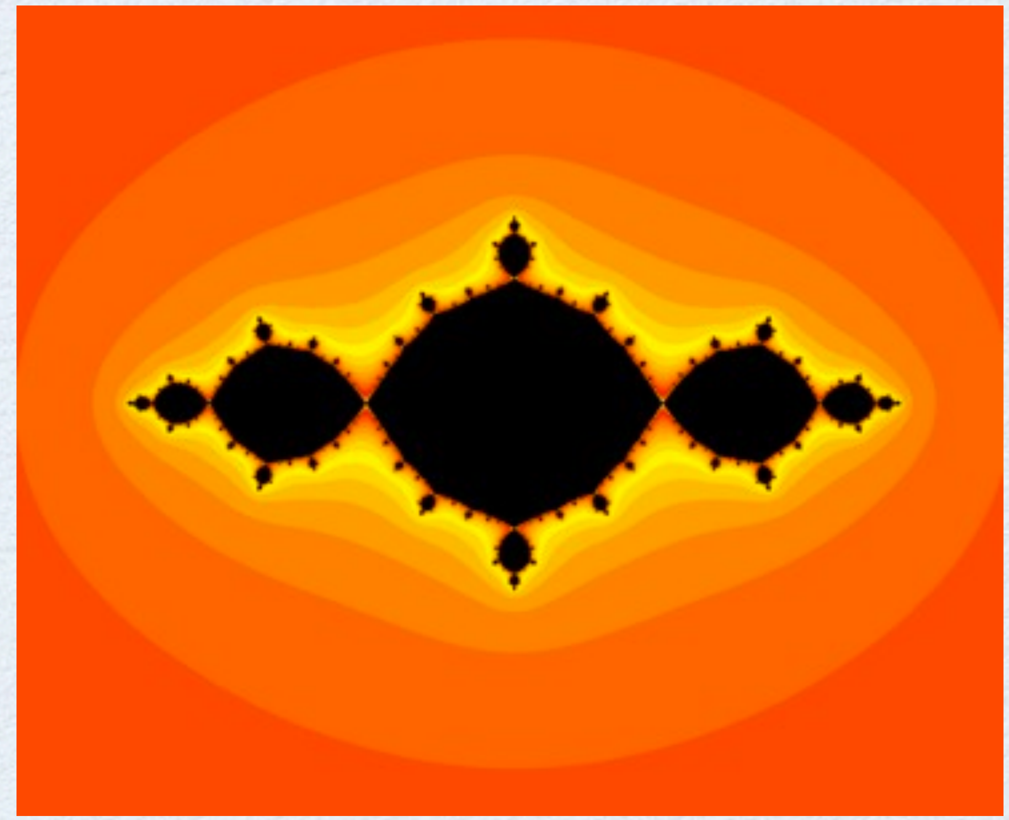
Example: Basilica mate Basilica

$$\begin{tikzcd} \star \arrow[r, "2"] & \cdot \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \infty \arrow[loop right, "2"] \end{tikzcd}$$



Example: Basilica mate Basilica

$$\begin{tikzcd} \star \arrow[r, "2"] & \cdot \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \infty \arrow[loop right, "2"] \end{tikzcd}$$



Formal mating:

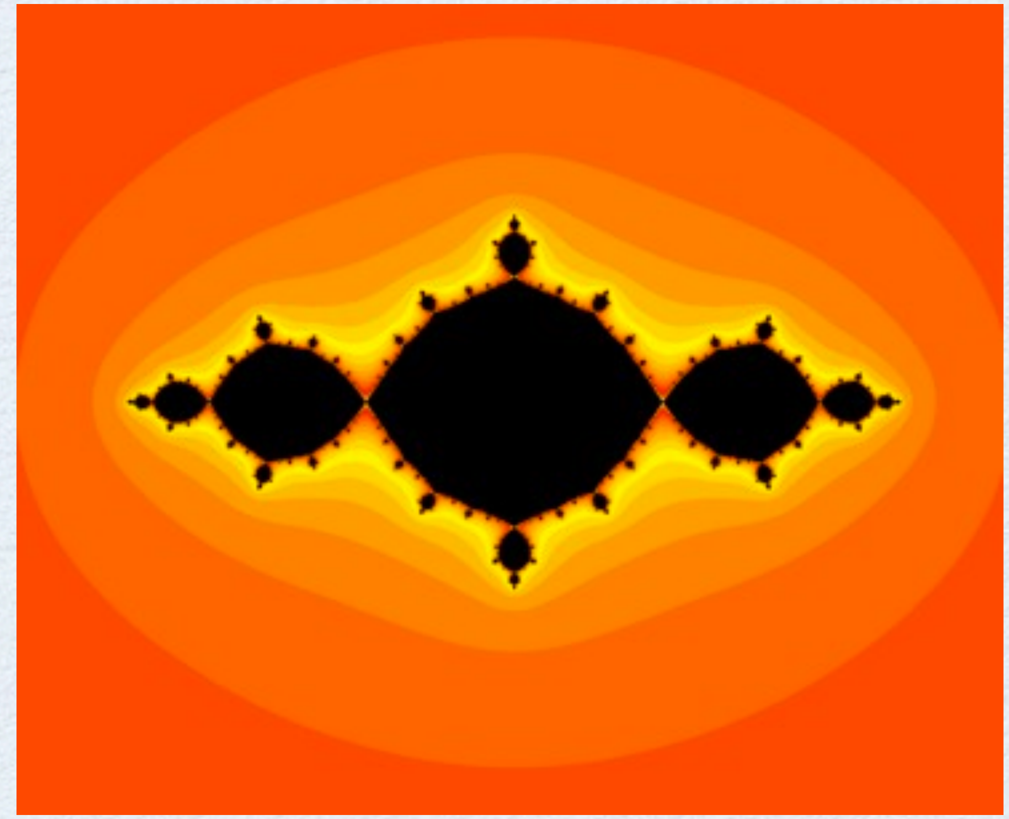
$$f : (S^2, P) \to (S^2, P)$$

$$\begin{tikzcd} \star \arrow[r, "2"] & \cdot \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \star' \arrow[r, "2"] & \cdot' \arrow[l, bend left] \end{tikzcd}$$

Example: Basilica mate Basilica

$$\begin{tikzcd} \star \arrow[r, "2"] & \cdot \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \infty \arrow[loop right, "2"] \end{tikzcd}$$



Formal mating:

$$f : (S^2, P) \to (S^2, P)$$

$$\begin{tikzcd} \star \arrow[r, "2"] & \cdot \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \star' \arrow[r, "2"] & \cdot' \arrow[l, bend left] \end{tikzcd}$$

No geometric mating exists; this mating is *obstructed*.

Twisted Matings

If P is a monic polynomial of degree $d \geqslant 2$, then the polynomial $T(P) : \mathbb{C} \to \mathbb{C}$ defined by

$$T(P)(z) = e^{-2\pi i/(d-1)} P(e^{2\pi i/(d-1)} z)$$

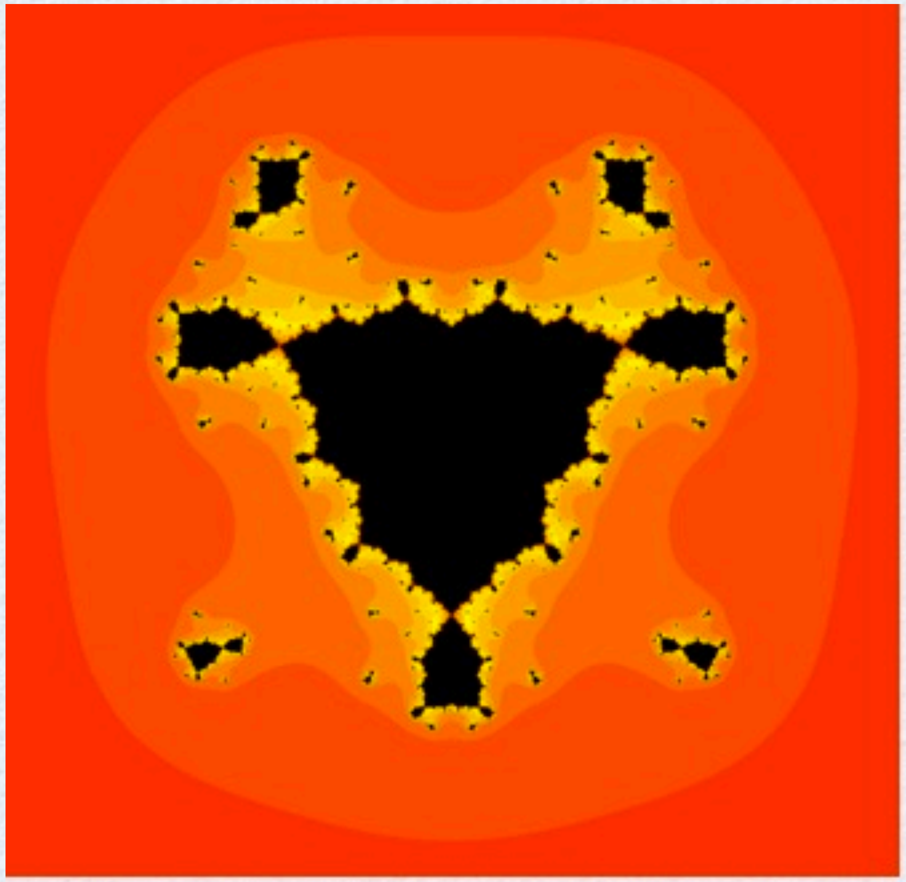
is also monic. The filled Julia set of $T(P)$ is the image of the Julia set of P by the rotation of angle $-1/(d-1)$ turns centered at 0.

Twisted Matings

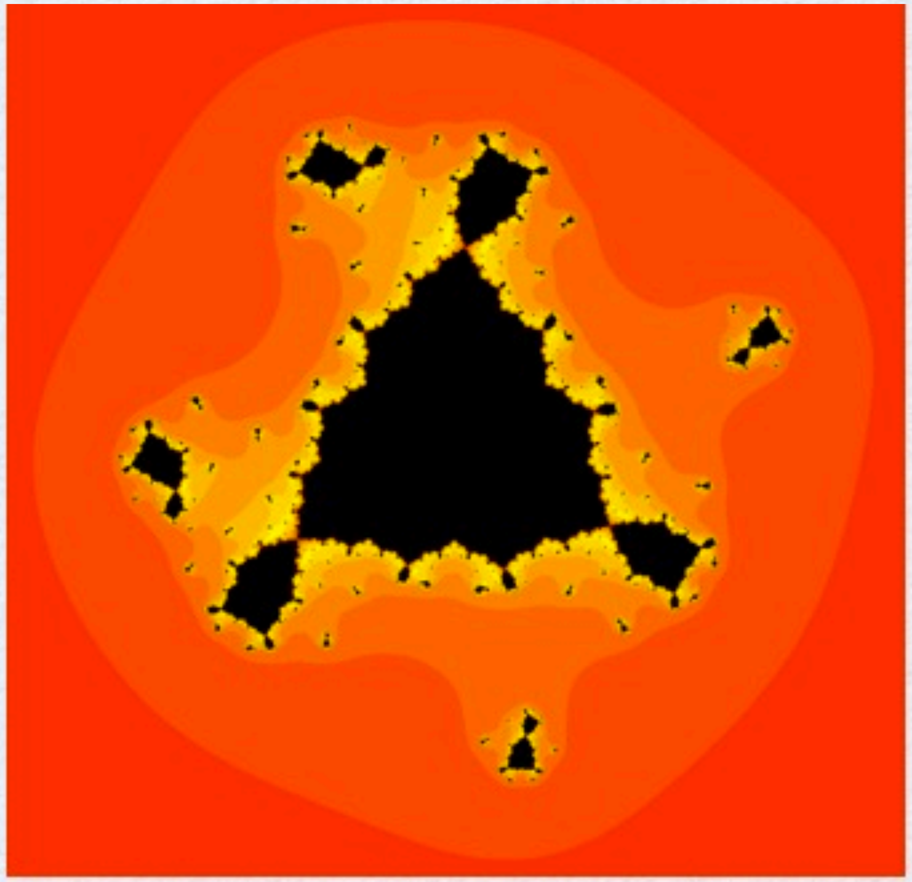
If P is a monic polynomial of degree $d \geqslant 2$, then the polynomial $T(P) : \mathbb{C} \to \mathbb{C}$ defined by

$$T(P)(z) = e^{-2\pi i/(d-1)} P(e^{2\pi i/(d-1)} z)$$

is also monic. The filled Julia set of $T(P)$ is the image of the Julia set of P by the rotation of angle $-1/(d-1)$ turns centered at 0.



$$p(z) = z^7 + z^3 - \frac{6}{7} i$$



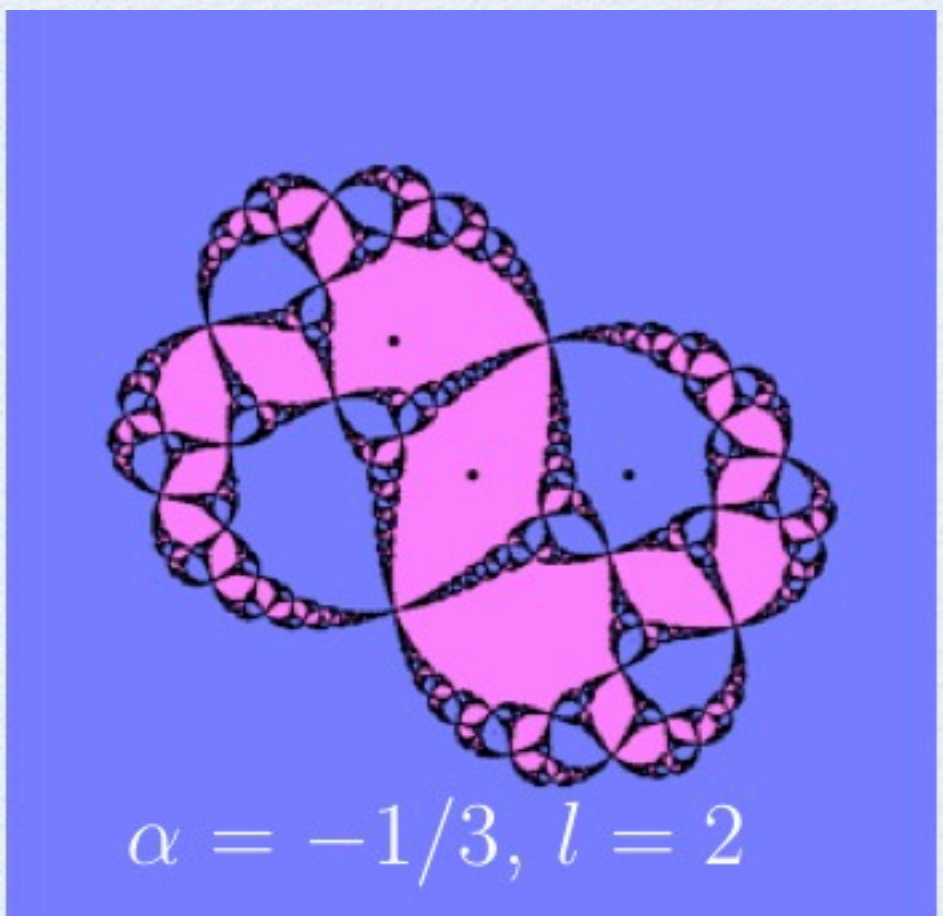
$$T(p)(z) = z^7 - e^{-\pi i/3} \left(z^3 + \frac{6}{7} i \right)$$

Construct the formal mating $f : S^2 \to S^2$, and form $S^2/\sim$ by identifying θ and $-k/(d-1) - \theta$.

Proposition. Let P_1 and P_2 be two monic polynomials of degree $d \geqslant 2$ which are critically finite. Let $f : (S^2, \mathcal{P}_f) \to (S^2, \mathcal{P}_f)$ be the formal mating of P_1 and P_2, and let $g : (S^2, \mathcal{P}_g) \to (S^2, \mathcal{P}_g)$ be the formal mating of P_1 and $T^{\circ k}(P_2)$ (the twisted mating of angle $k/(d-1)$). Let $D : S^2 \to S^2$ be the Dehn twist around the equator of $S^2 - \mathcal{P}_f$. Then g is combinatorially equivalent to $D^{\circ k} \circ f$.

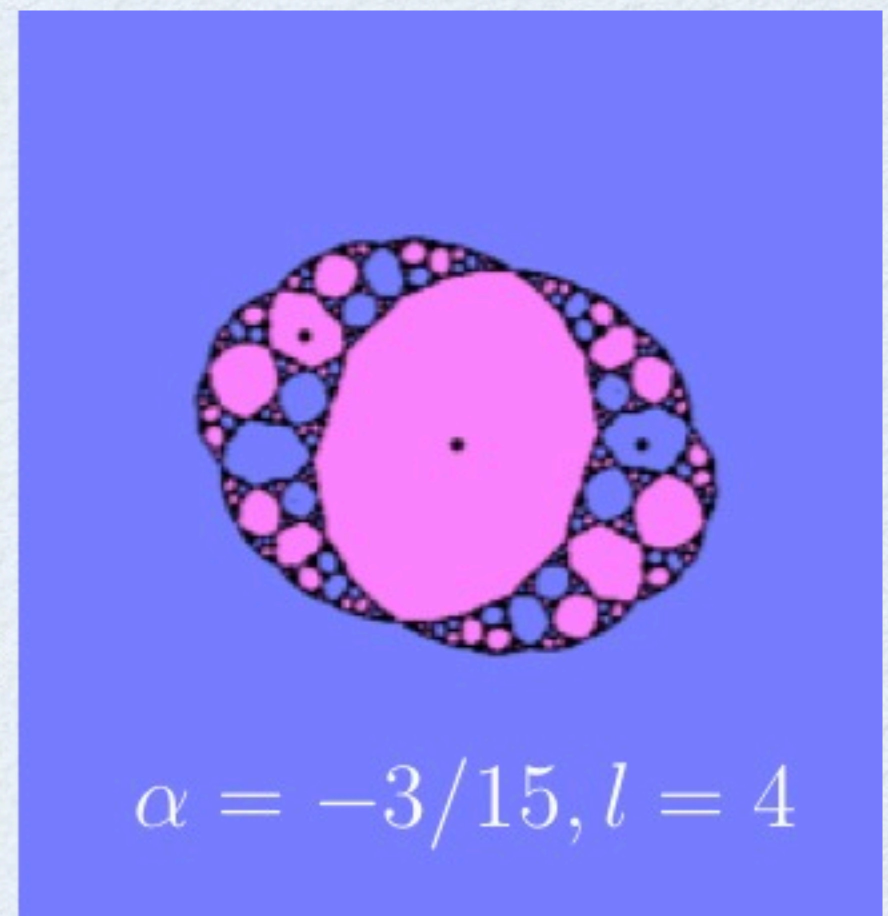
Construct the formal mating $f : S^2 \to S^2$, and form $S^2/\sim$ by identifying θ and $-k/(d-1)-\theta$.

Proposition. Let P_1 and P_2 be two monic polynomials of degree $d \geqslant 2$ which are critically finite. Let $f : (S^2, \mathcal{P}_f) \to (S^2, \mathcal{P}_f)$ be the formal mating of P_1 and P_2, and let $g : (S^2, \mathcal{P}_g) \to (S^2, \mathcal{P}_g)$ be the formal mating of P_1 and $T^{\circ k}(P_2)$ (the twisted mating of angle $k/(d-1)$). Let $D : S^2 \to S^2$ be the Dehn twist around the equator of $S^2 - \mathcal{P}_f$. Then g is combinatorially equivalent to $D^{\circ k} \circ f$.


$\alpha = -1/3, l = 2$

$P(z) = z^2 - 1$

geometric twisted mating of angle α of $P^{\circ l}$ with itself


$\alpha = -3/15, l = 4$

Preliminaries

Recall that if $f : (S^2, P) \to (S^2, P)$ is a critically finite branched cover, then there is an associated holomorphic endomorphism

$$\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$$

where $\mathcal{T}_P$ is the *Teichmüller space* of (S^2, P):

$$\phi : S^2 \to \mathbb{P}^1 \ : \ \phi_1 \sim \phi_2 \iff \exists \mu \in \mathrm{Aut}(\mathbb{P}^1) \text{ such that}$$

- $\phi_1|_P = (\mu \circ \phi_2)|_P,$ and
- ϕ_1 is isotopic to $\mu \circ \phi_2$ relative to P

The space $\mathcal{T}_P$ is the universal cover of the *moduli space*, $\mathcal{M}_P$:

$$\{\varphi : P \hookrightarrow \mathbb{P}^1 \text{ up to postcomposition by elements of } \mathrm{Aut}(\mathbb{P}^1)\}.$$

$$\pi : \mathcal{T}_P \to \mathcal{M}_P$$

$$\begin{tikzcd}
(S^2, P) \arrow[d, "f"] \\
(S^2, P)
\end{tikzcd}$$

$$\begin{tikzcd}
(S^2, P) \arrow[d, "f"] & \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"] & (\mathbb{P}^1, \psi(P)) \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F_\phi"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F_\phi"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$$\sigma_f : [\phi] \mapsto [\psi]$$

$$\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F_\phi"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$$\sigma_f : [\phi] \mapsto [\psi]$$

$$\begin{tikzcd}
\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \arrow[d, "\pi"] \\
\mathcal{M}_P & \mathcal{M}_P
\end{tikzcd}$$

$$\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F_\phi"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$$\sigma_f : [\phi] \mapsto [\psi]$$

$$\begin{tikzcd}
\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \arrow[d, "\pi"] \\
\mathcal{M}_P \arrow[r] & \mathcal{M}_P
\end{tikzcd}$$

$$\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F_\phi"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$$\sigma_f : [\phi] \mapsto [\psi]$$

$$\begin{tikzcd}
\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"] & \mathcal{T}_P \arrow[d, "\pi"] \\
\mathcal{M}_P & \mathcal{M}_P
\end{tikzcd}$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F_\phi"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$\sigma_f : [\phi] \mapsto [\psi]$$

$$
\begin{tikzcd}
\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \arrow[d, "\pi"] \\
\mathcal{M}_P & \mathcal{M}_P \arrow[l, dashed, "g_f"]
\end{tikzcd}
$$

Proposition. Let $f : (S^2, P) \to (S^2, P)$ be a critically finite branched cover which is a topological polynomial such that the critical points of f are contained in P. Then a moduli space map exists.

Corollary. Let $f : (S^2, P) \to (S^2, P)$ be a critically finite branched cover such that the critical points of f are contained in P, and there is a critical point of multiplicity $d-1$. Then a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ exists.

Application. Mating two critically finite hyperbolic polynomials of degree $d \geqslant 2$.

Proposition. Let $f : (S^2, P) \to (S^2, P)$ be a critically finite branched cover which is a topological polynomial such that the critical points of f are contained in P. Then a moduli space map exists.

Corollary. Let $f : (S^2, P) \to (S^2, P)$ be a critically finite branched cover such that the critical points of f are contained in P, and there is a critical point of multiplicity $d-1$. Then a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ exists.

Application. Mating two critically finite hyperbolic polynomials of degree $d \geqslant 2$.

$$\begin{tikzcd} \star \arrow[r, "d"] & \cdot \arrow[r] & \cdots\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[llll, bend left] \end{tikzcd}$$

Proposition. Let $f : (S^2, P) \to (S^2, P)$ be a critically finite branched cover which is a topological polynomial such that the critical points of f are contained in P. Then a moduli space map exists.

Corollary. Let $f : (S^2, P) \to (S^2, P)$ be a critically finite branched cover such that the critical points of f are contained in P, and there is a critical point of multiplicity $d-1$. Then a moduli space map $g_f : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$ exists.

Application. Mating two critically finite hyperbolic polynomials of degree $d \geqslant 2$.

$$\begin{tikzcd} \star \arrow[r, "d"] & \cdot \arrow[r] & \cdots \arrow[r] & \cdot \arrow[r] & \cdot \arrow[llll, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \star \arrow[r, "d"] & \cdot \arrow[r] & \cdots \arrow[r] & \cdot \arrow[r] & \cdot \arrow[llll, bend left] \end{tikzcd}$$

Example: Basilica mate Basilica

$P = \{\color{red}{\star}, \color{red}{p}, \color{blue}{\star}, \color{blue}{q}\}$ $\qquad f : (S^2, P) \to (S^2, P)$

$$\begin{tikzcd} \color{red}{\star} \arrow[r, red, "2"] & \color{red}{p} \arrow[l, red, bend left] \end{tikzcd} \qquad \begin{tikzcd} \color{blue}{\star} \arrow[r, blue, "2"] & \color{blue}{q} \arrow[l, blue, bend left] \end{tikzcd}$$

Example: Basilica mate Basilica

$P = \{\star, p, \star, q\}$ $\qquad f : (S^2, P) \to (S^2, P)$

$$\begin{tikzcd} \star \arrow[r, "2"] & p \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[r, "2"] & q \arrow[l, bend left] \end{tikzcd}$$

$\mathcal{M}_P$ is a 1-dimensional manifold.

Example: Basilica mate Basilica

$P = \{\color{red}{\star}, \color{red}{p}, \color{blue}{\star}, \color{blue}{q}\}$ $\qquad f : (S^2, P) \to (S^2, P)$

$$\color{red}{\begin{tikzcd} \star \arrow[r, "2"] & p \arrow[l, bend left] \end{tikzcd}} \qquad \color{blue}{\begin{tikzcd} \star \arrow[r, "2"] & q \arrow[l, bend left] \end{tikzcd}}$$

$\mathcal{M}_P$ is a 1-dimensional manifold.

$$\varphi \in \mathcal{M}_P,$$

$$\varphi(\color{blue}{\star}) = \color{blue}{\infty},$$

$$\varphi(\color{red}{\star}) = \color{red}{0},$$

$$\varphi(\color{blue}{q}) = \color{blue}{1}$$

Example: Basilica mate Basilica

$P = \{\star, p, \star, q\}$ $\qquad f : (S^2, P) \to (S^2, P)$

$$\begin{tikzcd} \star \arrow[r, "2"] & p \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[r, "2"] & q \arrow[l, bend left] \end{tikzcd}$$

$\mathcal{M}_P$ is a 1-dimensional manifold.

$$\varphi \in \mathcal{M}_P,$$

$$\varphi(\star) = \infty,$$

$$\varphi(\star) = 0,$$

$$\varphi(q) = 1$$

Then φ is determined by $x := \varphi(p)$

$x \in \mathbb{C} - \{0, 1\}, \quad \mathcal{M}_P \approx \mathbb{P}^1 - \{0, 1, \infty\}$

Example: Basilica mate Basilica

$P = \{{\color{red}\star}, {\color{red}p}, {\color{blue}\star}, {\color{blue}q}\}$ $\qquad f : (S^2, P) \to (S^2, P)$

$$\begin{tikzcd} \star \arrow[r, "2"] & p \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \star \arrow[r, "2"] & q \arrow[l, bend left] \end{tikzcd}$$

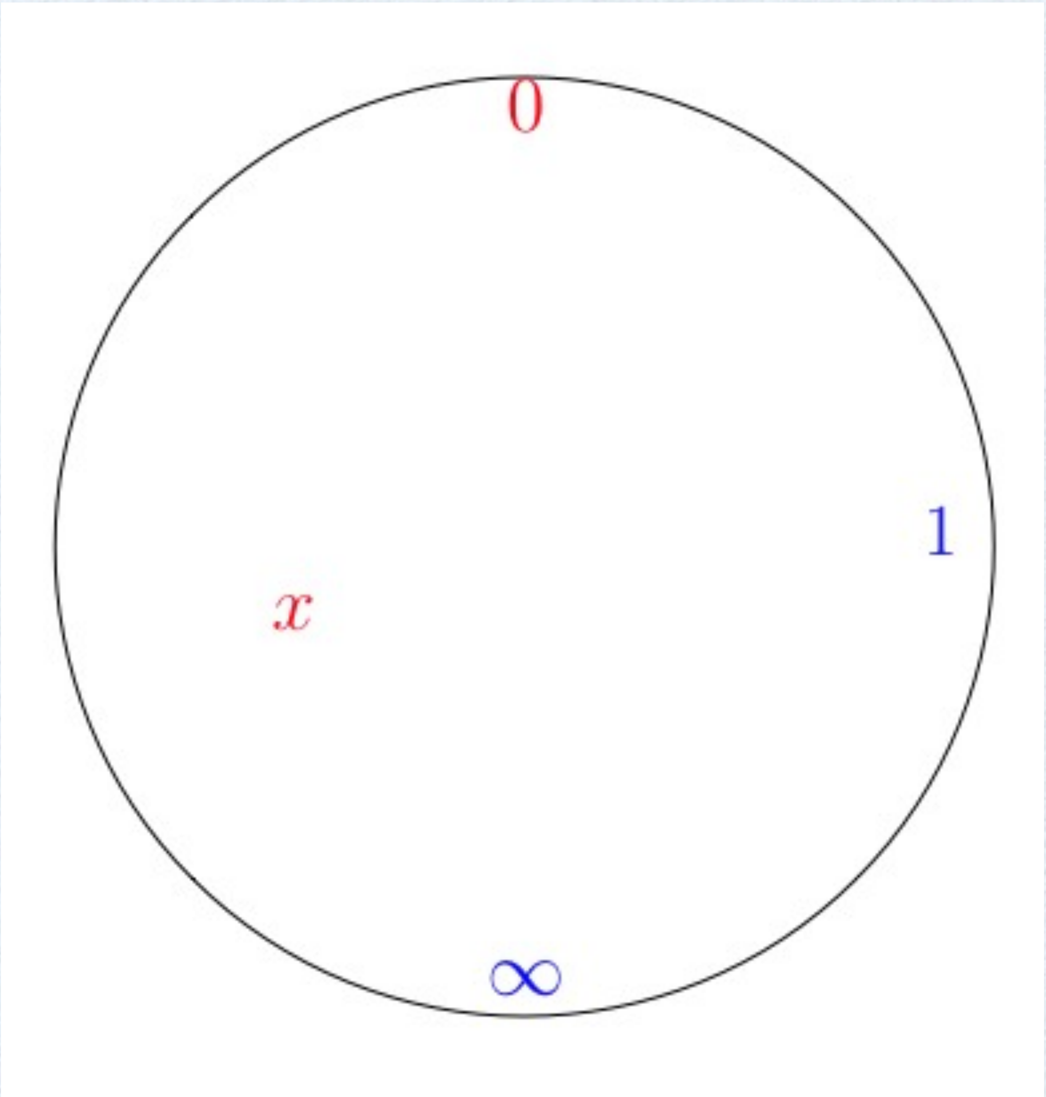
$\mathcal{M}_P$ is a 1-dimensional manifold.

$$\varphi \in \mathcal{M}_P,$$

$$\varphi({\color{blue}\star}) = {\color{blue}\infty},$$

$$\varphi({\color{red}\star}) = {\color{red}0},$$

$$\varphi({\color{blue}q}) = {\color{blue}1}$$



Then φ is determined by ${\color{red}x} := \varphi({\color{red}p})$

${\color{red}x} \in \mathbb{C} - \{0, 1\}, \quad \mathcal{M}_P \approx \mathbb{P}^1 - \{0, 1, \infty\}$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$\phi({\color{red}\star}) = 0$$

$$\phi({\color{blue}\star}) = \infty$$

$$\phi({\color{blue}q}) = 1$$

$$\phi({\color{red}p}) = y$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$\phi(\star) = 0$ $\qquad$ $\psi(\star) = 0$

$\phi(\star) = \infty$ $\qquad$ $\psi(\star) = \infty$

$\phi(q) = 1$ $\qquad$ $\psi(q) = 1$

$\phi(p) = y$ $\qquad$ $\psi(p) = x$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$
\begin{aligned}
\phi(\star) &= 0 & \qquad \psi(\star) &= 0 \\
\phi(\star) &= \infty & \qquad \psi(\star) &= \infty \\
\phi(q) &= 1 & \qquad \psi(q) &= 1 \\
\phi(p) &= y & \qquad \psi(p) &= x
\end{aligned}
$$

$$
\begin{tikzcd}
\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \arrow[d, "\pi"] \\
\mathcal{M}_P & \mathcal{M}_P
\end{tikzcd}
$$

$$\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$\phi(\star) = 0$ $\qquad$ $\psi(\star) = 0$

$\phi(\star) = \infty$ $\qquad$ $\psi(\star) = \infty$

$\phi(q) = 1$ $\qquad$ $\psi(q) = 1$

$\phi(p) = y$ $\qquad$ $\psi(p) = x$

ϕ

$$\begin{tikzcd}
\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \arrow[d, "\pi"] \\
\mathcal{M}_P & \mathcal{M}_P
\end{tikzcd}$$

$$\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$$\phi(\star) = 0 \qquad \psi(\star) = 0$$

$$\phi(\star) = \infty \qquad \psi(\star) = \infty$$

$$\phi(q) = 1 \qquad \psi(q) = 1$$

$$\phi(p) = y \qquad \psi(p) = x$$

$$\begin{tikzcd}
\phi \quad \mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \quad \psi \arrow[d, "\pi"'] \\
\mathcal{M}_P & \mathcal{M}_P
\end{tikzcd}$$

$$\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$$\phi(\star) = 0 \qquad \psi(\star) = 0$$

$$\phi(\star) = \infty \qquad \psi(\star) = \infty$$

$$\phi(q) = 1 \qquad \psi(q) = 1$$

$$\phi(p) = y \qquad \psi(p) = x$$

$$\begin{tikzcd}
\phi \quad \mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \quad \psi \arrow[d, "\pi"'] \\
y \quad \mathcal{M}_P & \mathcal{M}_P
\end{tikzcd}$$

$$\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}$$

$\phi(\star) = 0$ $\qquad$ $\psi(\star) = 0$

$\phi(\star) = \infty$ $\qquad$ $\psi(\star) = \infty$

$\phi(q) = 1$ $\qquad$ $\psi(q) = 1$

$\phi(p) = y$ $\qquad$ $\psi(p) = x$

$$\begin{tikzcd}
{}^{\phi}\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P{}^{\psi} \arrow[d, "\pi"'] \\
{}_{y}\mathcal{M}_P & \mathcal{M}_P{}_{x}
\end{tikzcd}$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$\phi({\color{red}\star}) = 0 \qquad \psi({\color{red}\star}) = 0$$

$$\phi({\color{blue}\star}) = \infty \qquad \psi({\color{blue}\star}) = \infty$$

$$\phi({\color{blue}q}) = 1 \qquad \psi({\color{blue}q}) = 1$$

$$\phi({\color{red}p}) = y \qquad \psi({\color{red}p}) = x$$

$$
\begin{tikzcd}
1 \arrow[d] & \infty \arrow[d, "2"] & x \arrow[d] & 0 \arrow[d, "2"] \\
\infty & 1 & 0 & y
\end{tikzcd}
$$

$$
\begin{tikzcd}
{\color{magenta}\phi} & & & {\color{magenta}\psi} \\
& \mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \arrow[d, "\pi"'] & \\
& \mathcal{M}_P & \mathcal{M}_P & \\
{\color{magenta}y} & & & {\color{magenta}x}
\end{tikzcd}
$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$
\begin{array}{ll}
\phi(\star) = 0 & \psi(\star) = 0 \\
\phi(\star) = \infty & \psi(\star) = \infty \\
\phi(q) = 1 & \psi(q) = 1 \\
\phi(p) = y & \psi(p) = x
\end{array}
$$

$$
\begin{tikzcd}
1 \arrow[d] & \infty \arrow[d, "2"] & x \arrow[d] & 0 \arrow[d, "2"] \\
\infty & 1 & 0 & y
\end{tikzcd}
$$

$$F(t) = \frac{t^2 - x^2}{t^2 - 1}$$

$$
\begin{tikzcd}
{}^{\phi}\,\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P\,{}^{\psi} \arrow[d, "\pi"] \\
{}_{y}\,\mathcal{M}_P & \mathcal{M}_P\,{}_{x}
\end{tikzcd}
$$

$$\begin{array}{ccc} (S^2, P) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(P)) \\ \downarrow f & & \downarrow F \\ (S^2, P) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(P)) \end{array}$$

$$\phi(\color{red}{\star}) = 0 \qquad \psi(\color{red}{\star}) = 0$$

$$\phi(\color{blue}{\star}) = \infty \qquad \psi(\color{blue}{\star}) = \infty$$

$$\phi(\color{blue}{q}) = 1 \qquad \psi(\color{blue}{q}) = 1$$

$$\phi(\color{red}{p}) = y \qquad \psi(\color{red}{p}) = x$$

$$\begin{array}{cccc} 1 & \infty & x & 0 \\ \downarrow & \downarrow 2 & \downarrow & \downarrow 2 \\ \infty & 1 & 0 & y \end{array}$$

$$F(t) = \frac{t^2 - x^2}{t^2 - 1}$$

$$y = x^2$$

$$\begin{array}{ccc} \phi\ \mathcal{T}_P & \xrightarrow{\sigma_f} & \mathcal{T}_P\ \psi \\ \downarrow \pi & & \downarrow \pi \\ y\ \mathcal{M}_P & & \mathcal{M}_P\ x \end{array}$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$\phi(\star) = 0$ $\quad$ $\psi(\star) = 0$

$\phi(\star) = \infty$ $\quad$ $\psi(\star) = \infty$

$\phi(q) = 1$ $\quad$ $\psi(q) = 1$

$\phi(p) = y$ $\quad$ $\psi(p) = x$

$$
\begin{tikzcd}
1 \arrow[d] & \infty \arrow[d, "2"] & x \arrow[d] & 0 \arrow[d, "2"] \\
\infty & 1 & 0 & y
\end{tikzcd}
$$

$$
F(t) = \frac{t^2 - x^2}{t^2 - 1}
$$

$$
y = x^2
$$

$$
\begin{tikzcd}
{}^{\phi}\mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P^{\;\psi} \arrow[d, "\pi"] \\
{}_{y}\mathcal{M}_P & \mathcal{M}_P{}_{x} \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(S^2, P) \arrow[r, "\psi"] \arrow[d, "f"'] & (\mathbb{P}^1, \psi(P)) \arrow[d, "F"] \\
(S^2, P) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(P))
\end{tikzcd}
$$

$$\phi({\color{red}\star}) = 0 \qquad \psi({\color{red}\star}) = 0$$

$$\phi({\color{blue}\star}) = \infty \qquad \psi({\color{blue}\star}) = \infty$$

$$\phi({\color{blue}q}) = 1 \qquad \psi({\color{blue}q}) = 1$$

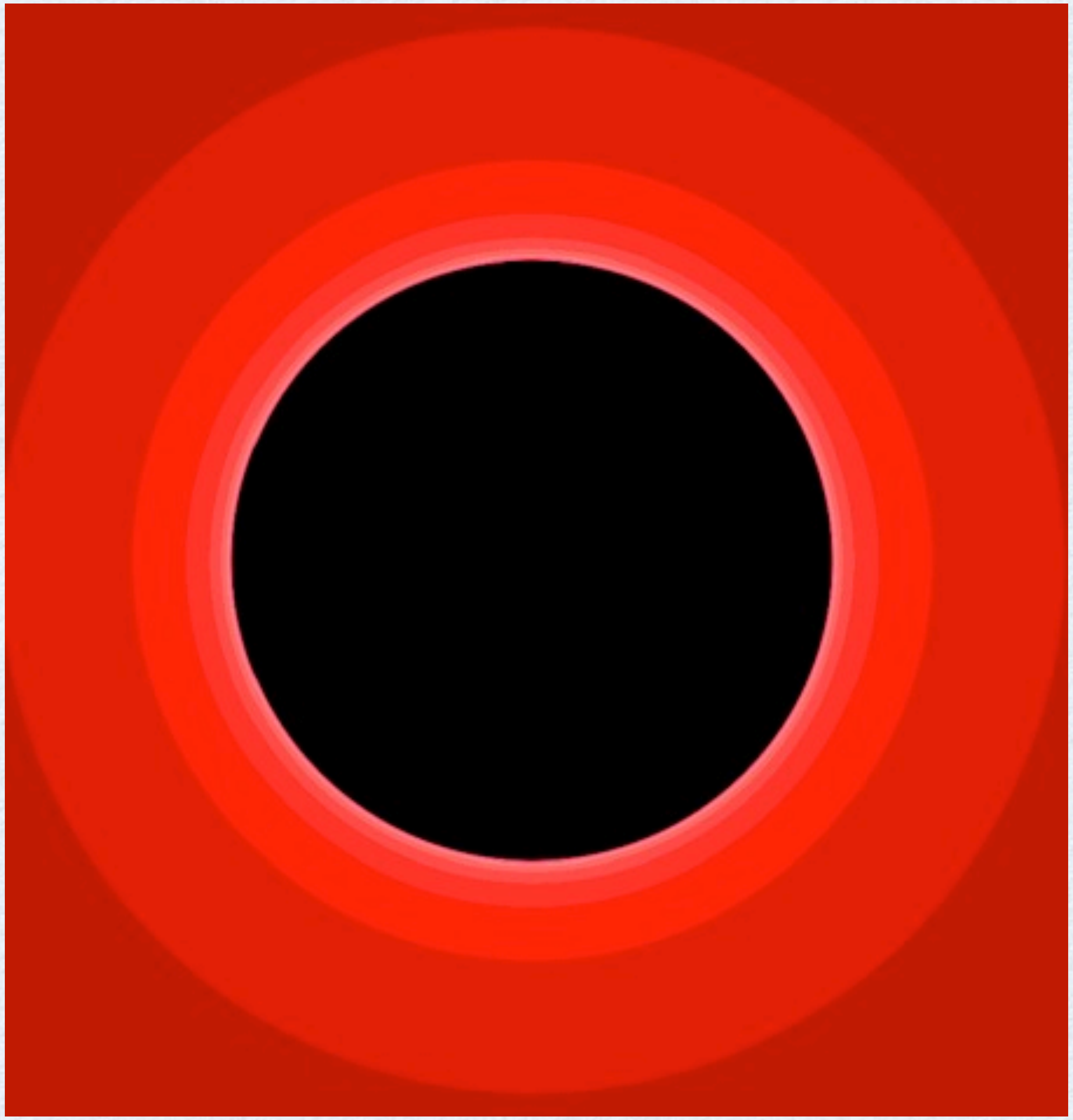
$$\phi({\color{red}p}) = y \qquad \psi({\color{red}p}) = x$$

$$
\begin{tikzcd}
1 \arrow[d] & \infty \arrow[d, "2"] & x \arrow[d] & 0 \arrow[d, "2"] \\
\infty & 1 & 0 & y
\end{tikzcd}
$$

$$F(t) = \frac{t^2 - x^2}{t^2 - 1}$$

$$y = x^2$$

$$
\begin{tikzcd}
{\color{magenta}\phi} & \mathcal{T}_P \arrow[r, "\sigma_f"] \arrow[d, "\pi"'] & \mathcal{T}_P \arrow[d, "\pi"] & {\color{magenta}\psi} \\
{\color{magenta}y} & \mathcal{M}_P & \mathcal{M}_P \arrow[l, "g"] & {\color{magenta}x}
\end{tikzcd}
$$

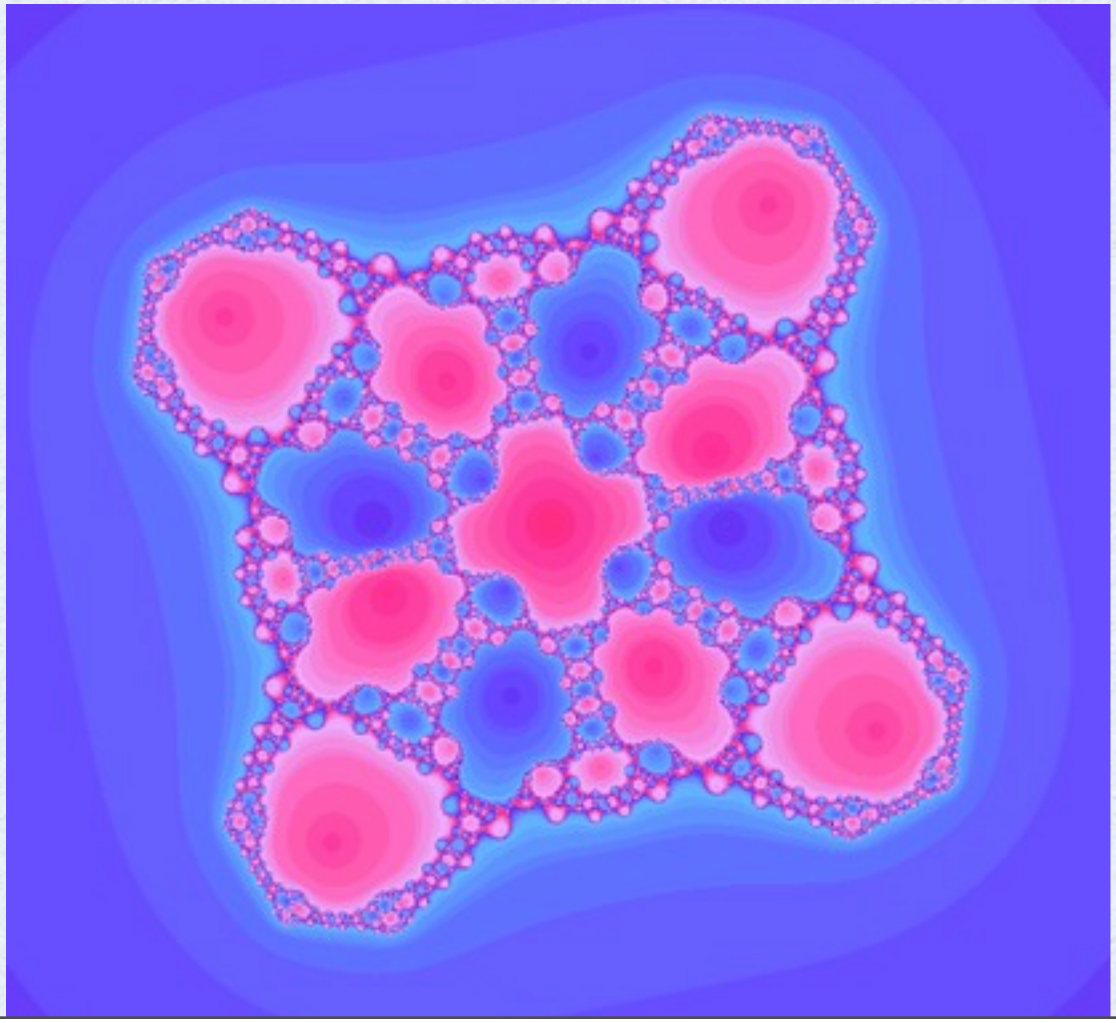


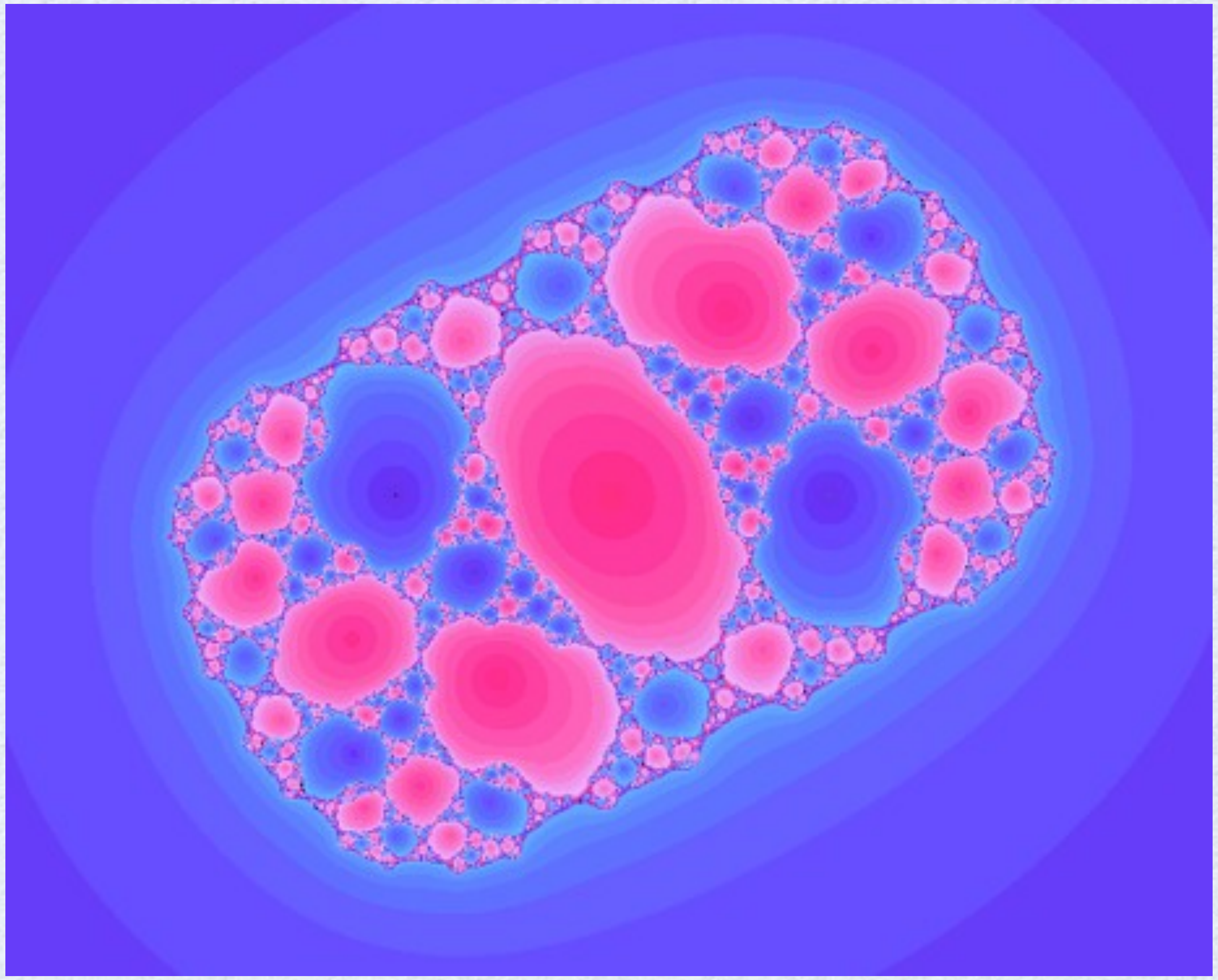
The skew product

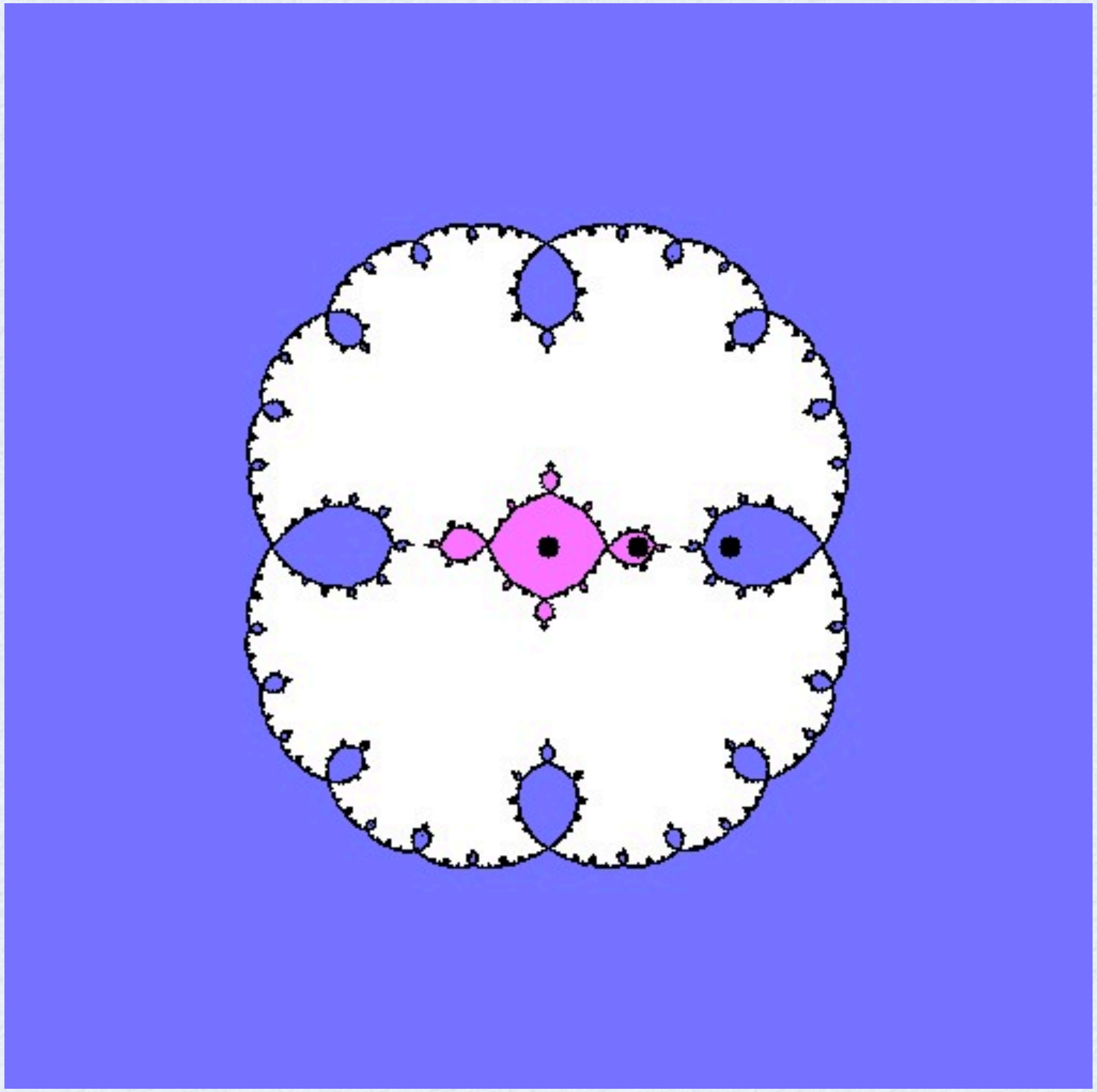
$$G : \mathbb{C}^2 \to \mathbb{C}^2 \quad \text{given by} \quad G : \begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} F_x(t) \\ g(x) \end{pmatrix}$$

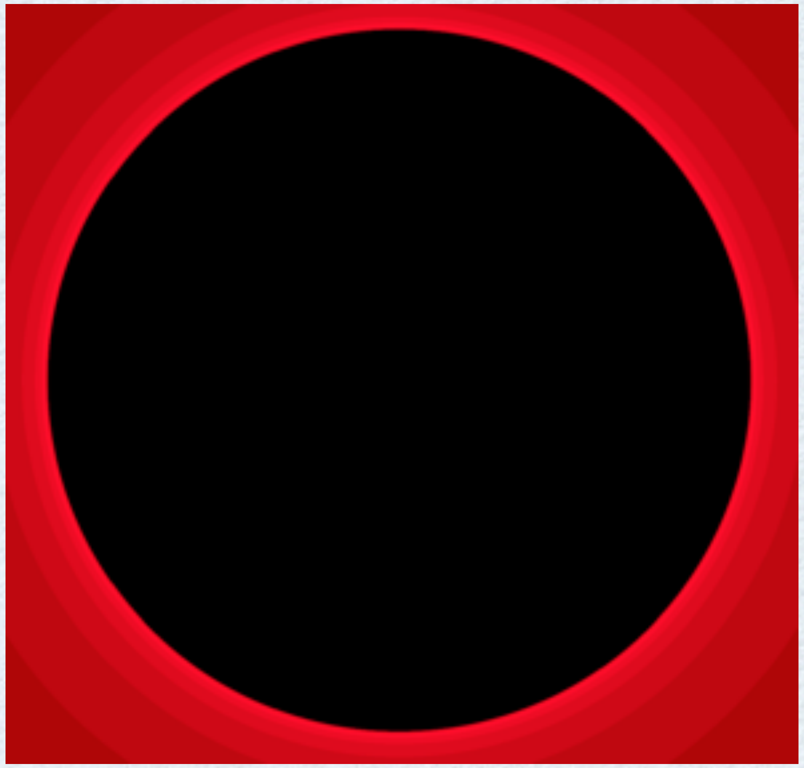
where $F_x(t) = (t^2 - x^2)/(t^2 - 1)$, and $g(x) = x^2$

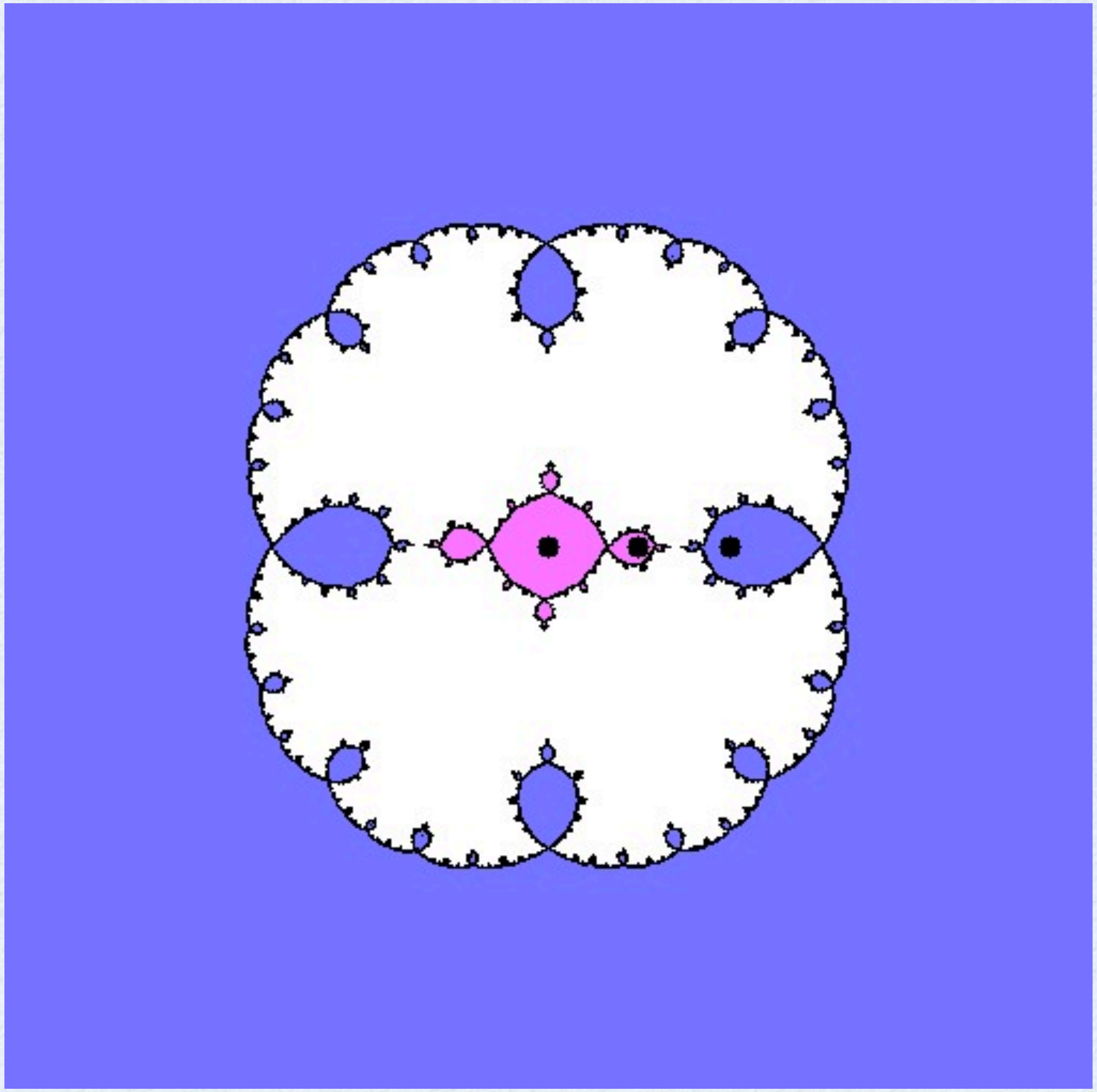
Proposition. Let $\lambda = e^{2\pi i \alpha}$ be a periodic point of g, hence $\alpha = -k/(2^l - 1)$ for some l. If $k \neq 0$, the rational map $F_\lambda^{\circ l}$ is a geometric twisted mating of angle α of $P^{\circ l}$ with itself.

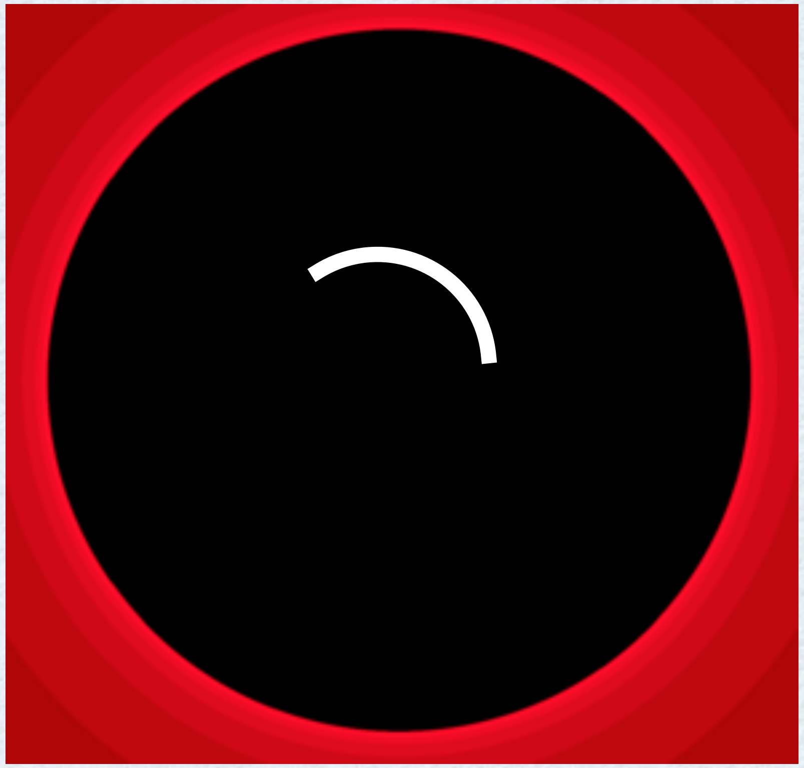


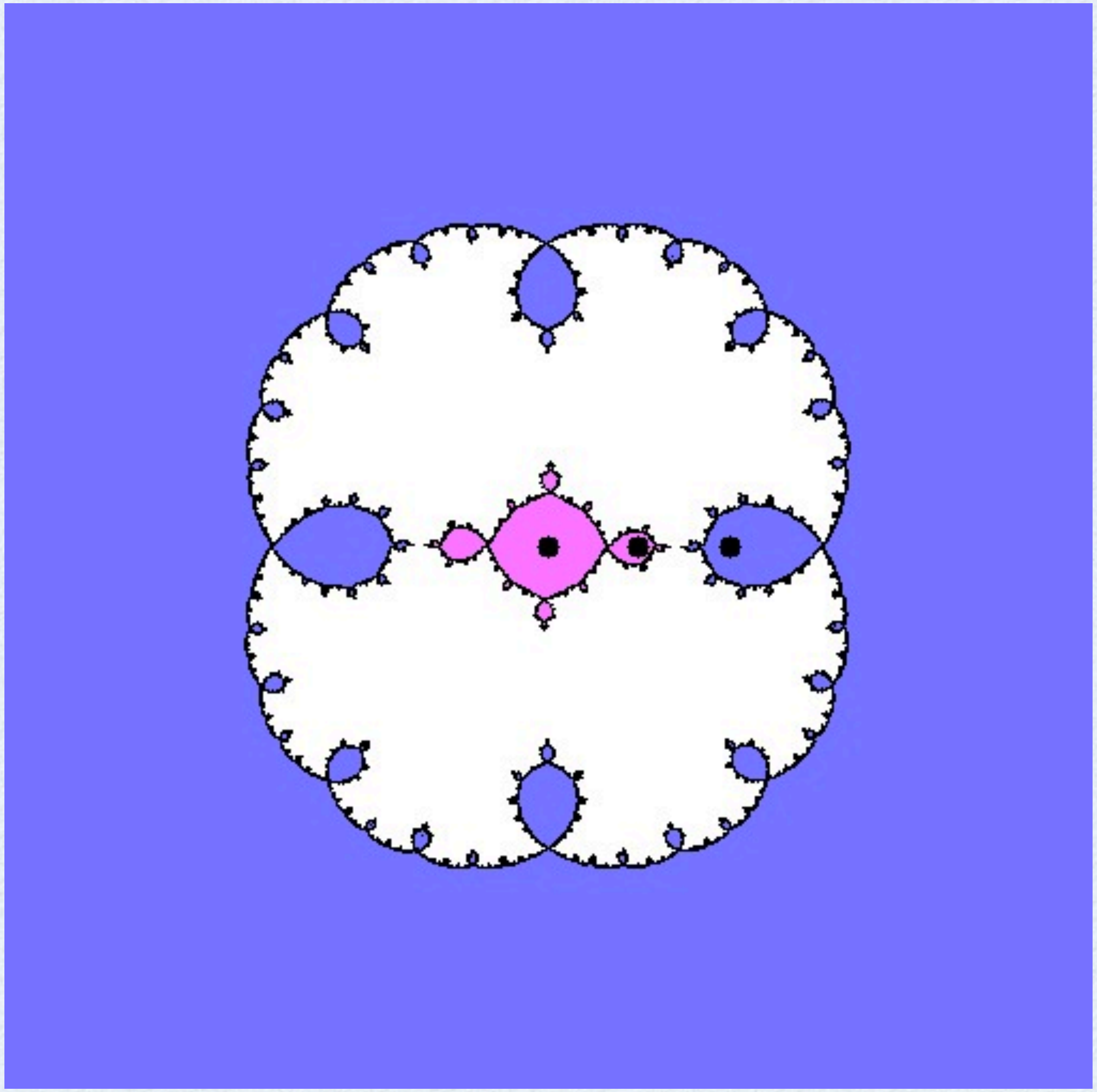


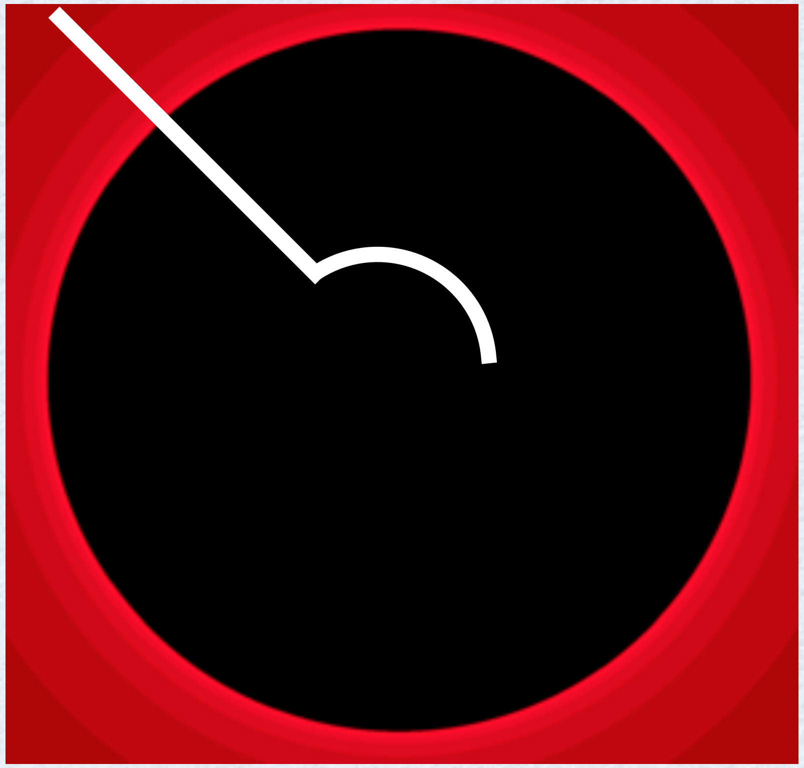












Compactifying

$$G : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [t : x : z] \mapsto [z^2(t^2 - x^2) : x^2(t^2 - z^2) : z^2(t^2 - z^2)]$$

$$G = \mu \circ s \quad \text{where} \quad s : [t : x : z] \mapsto [t^2 : x^2 : z^2],$$

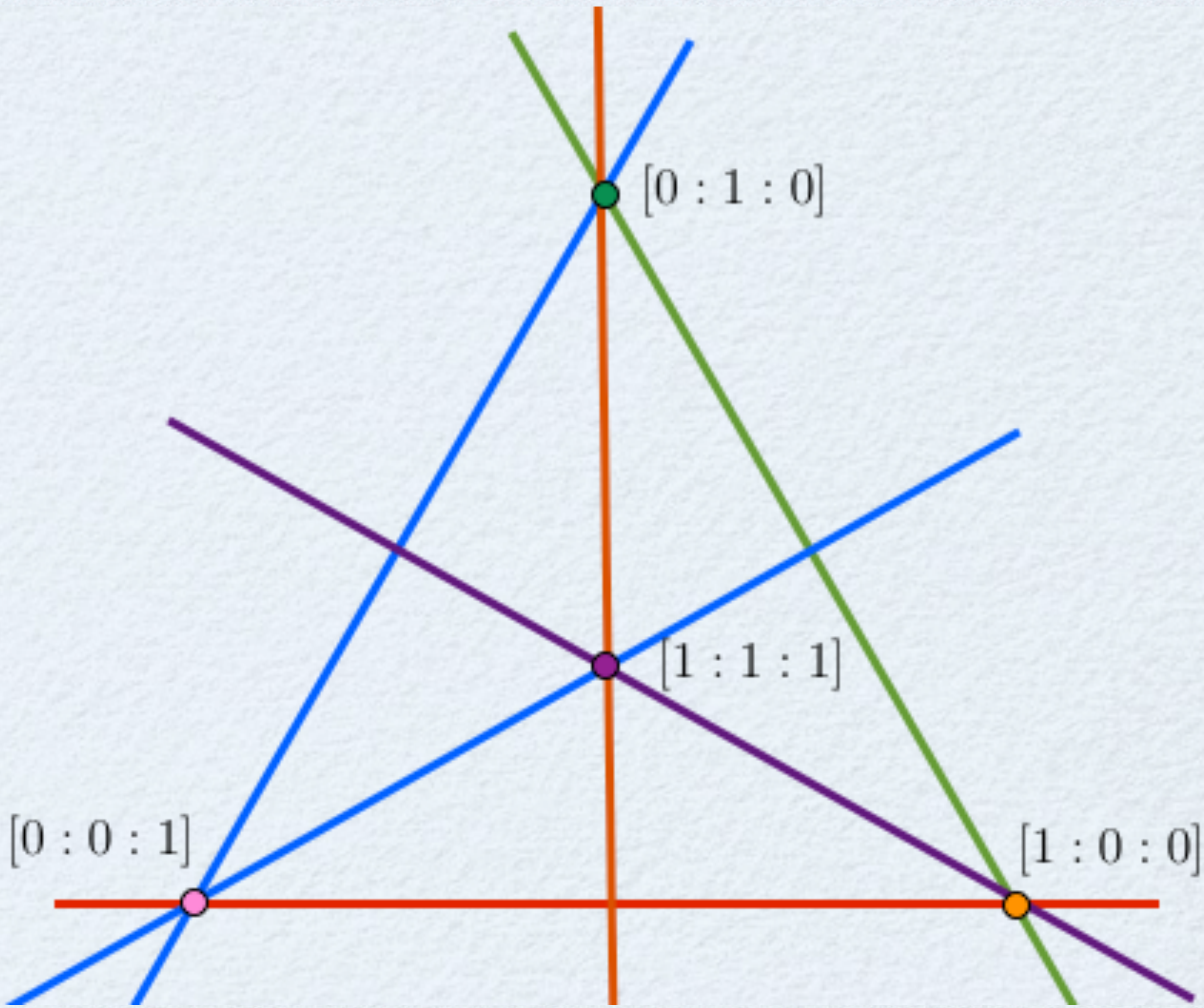
$$\mu : [t, x, z] \mapsto [z(t - x) : x(t - z) : z(t - z)]$$

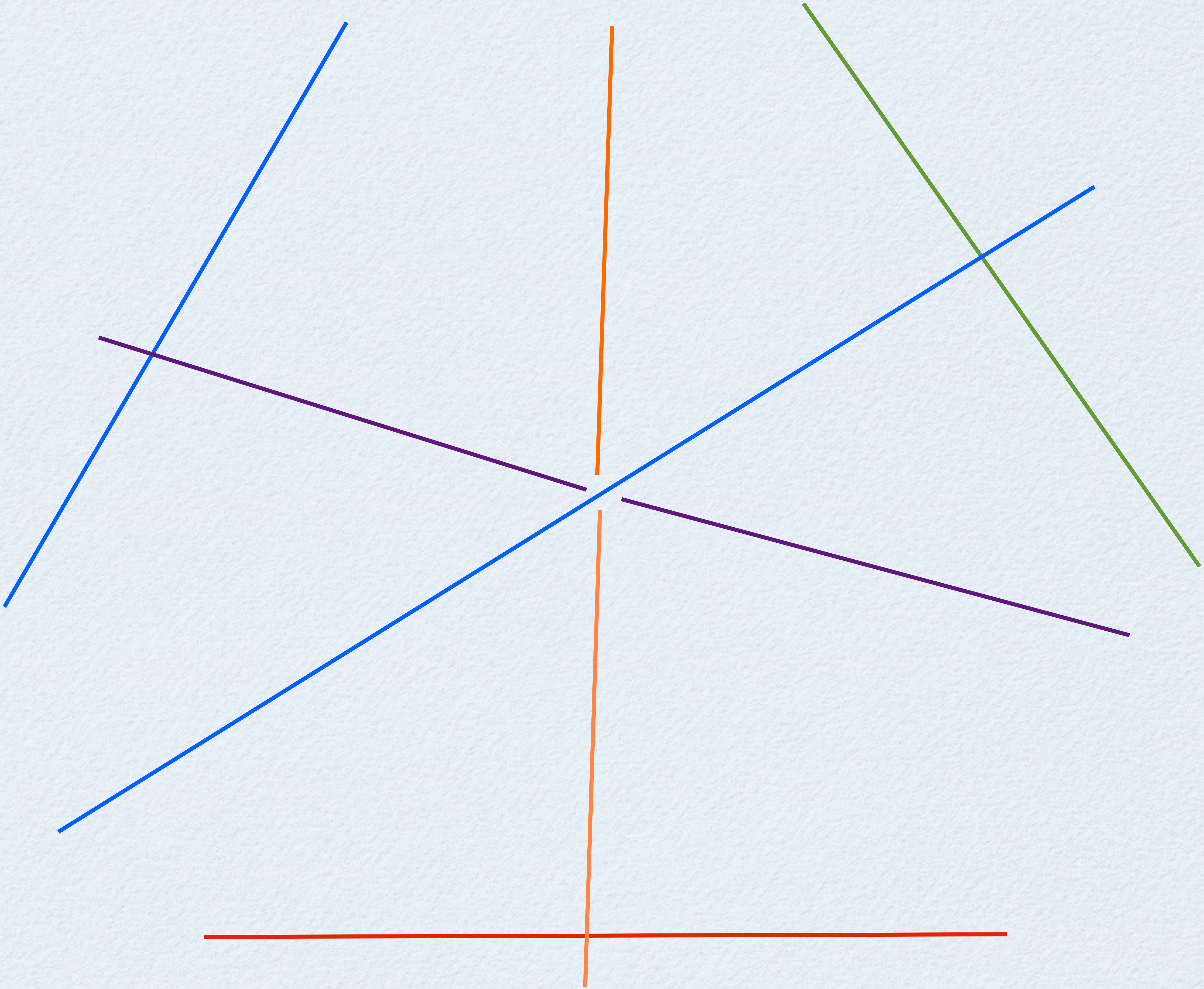
Compactifying

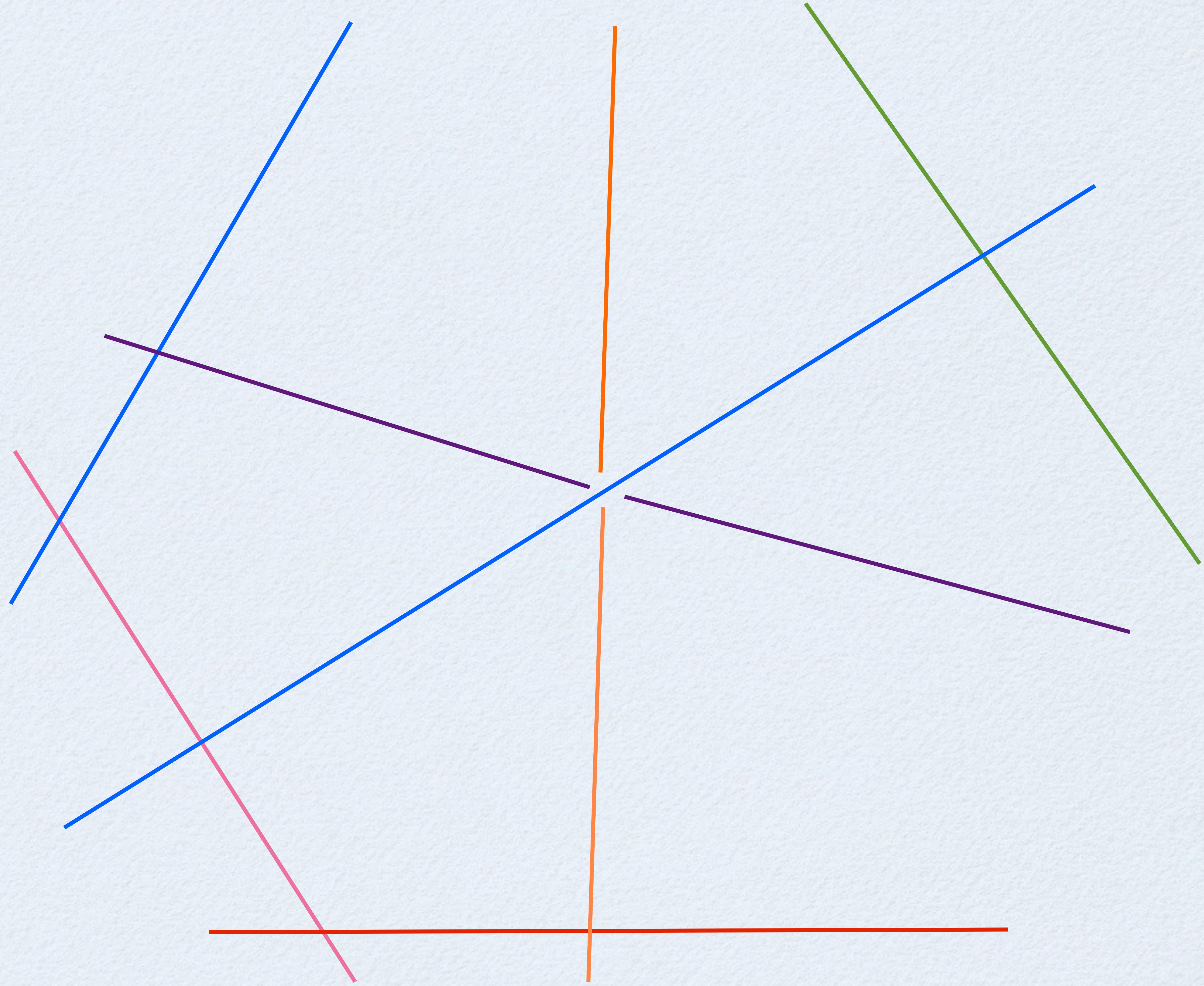
$$G : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [t : x : z] \mapsto [z^2(t^2 - x^2) : x^2(t^2 - z^2) : z^2(t^2 - z^2)]$$

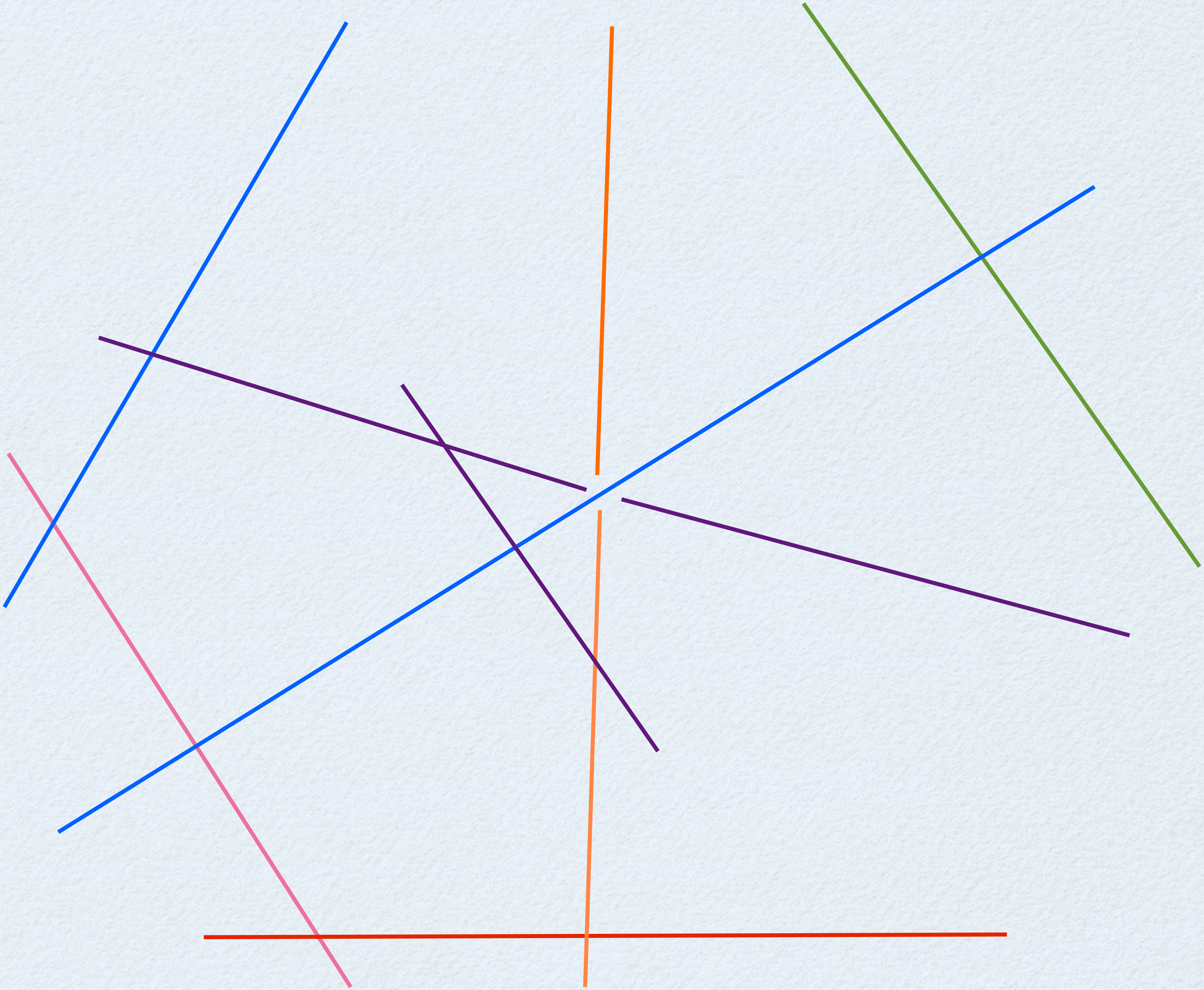
$$G = \mu \circ s \quad \text{where} \quad s : [t : x : z] \mapsto [t^2 : x^2 : z^2],$$

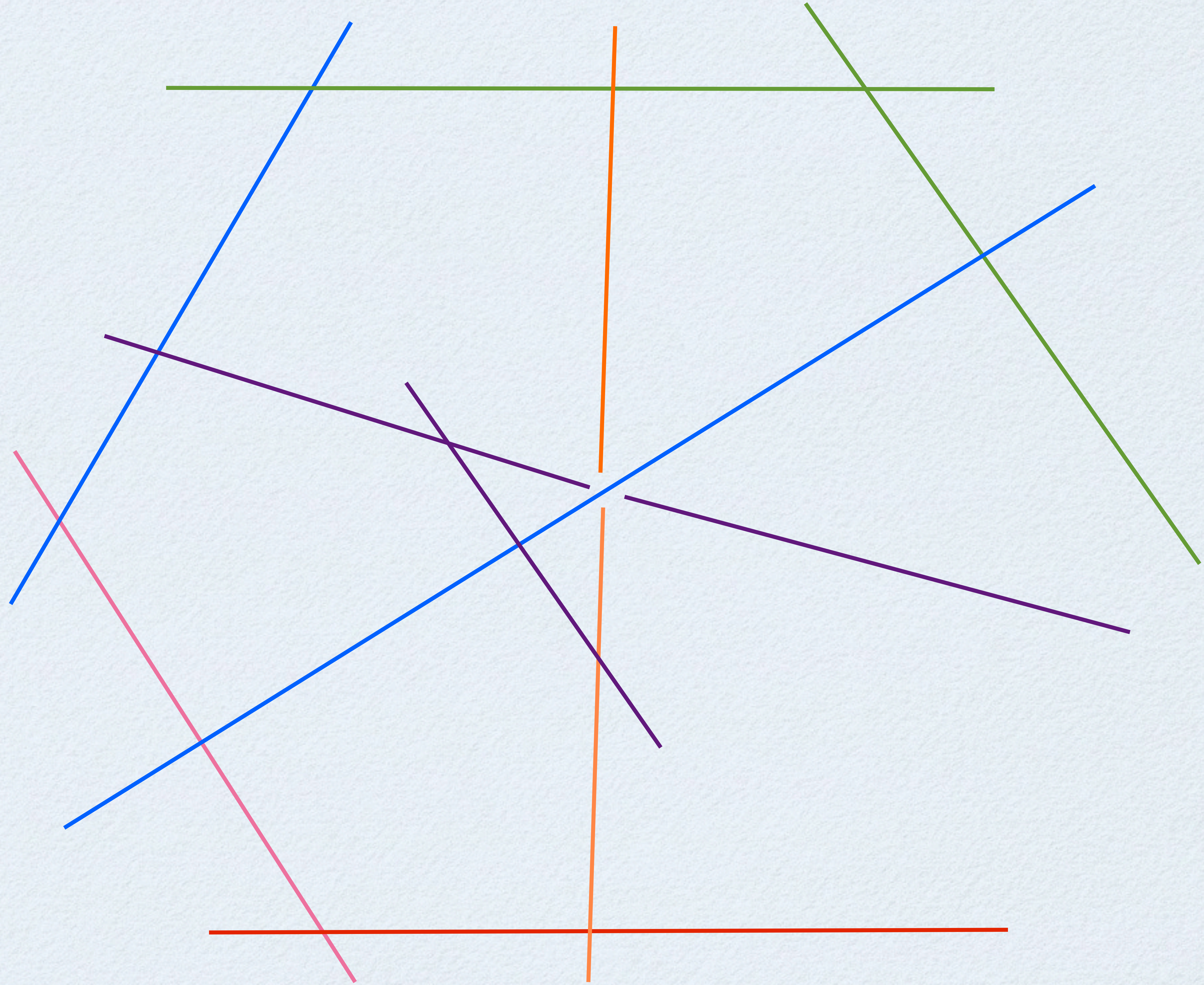
$$\mu : [t, x, z] \mapsto [z(t - x) : x(t - z) : z(t - z)]$$

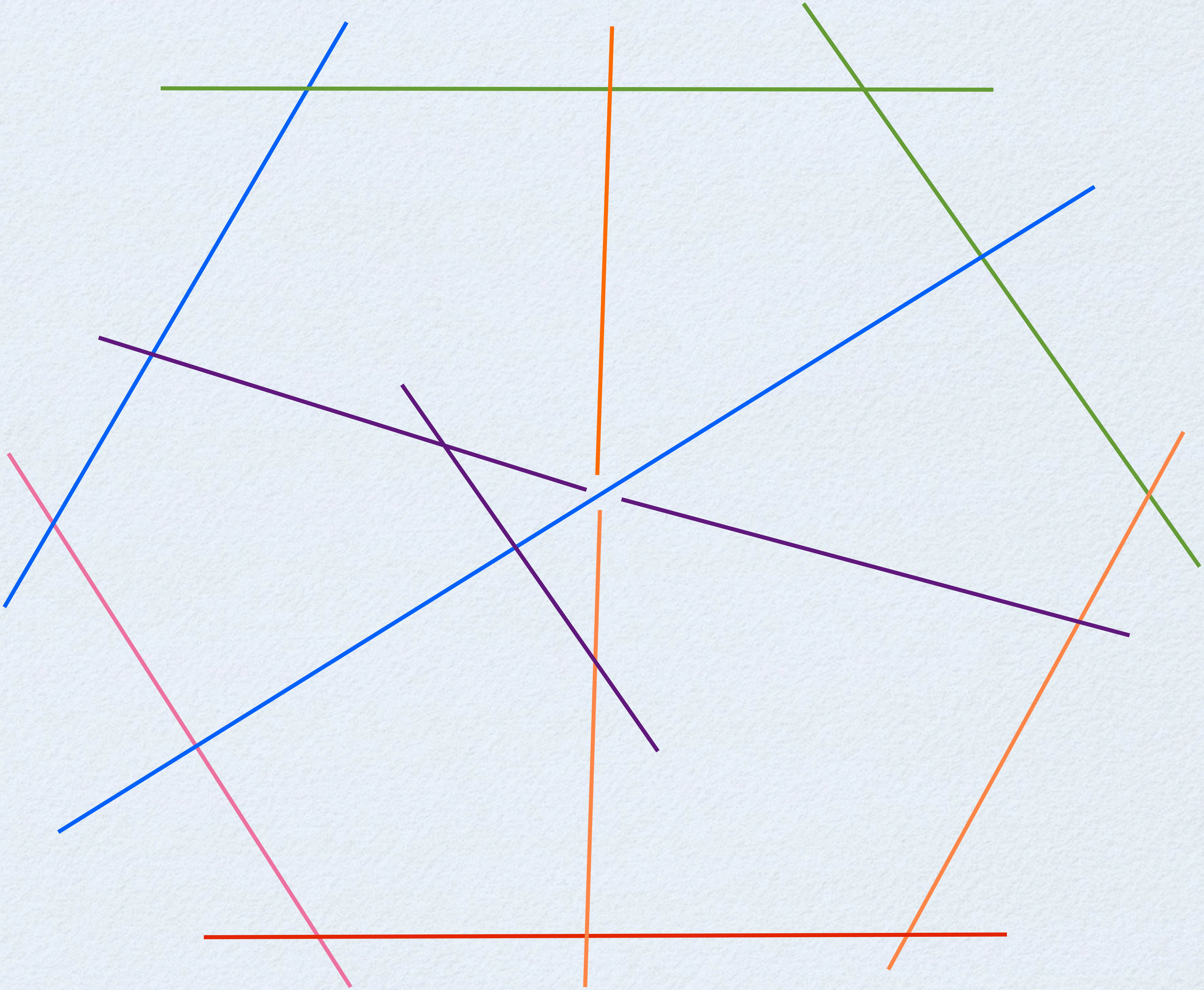


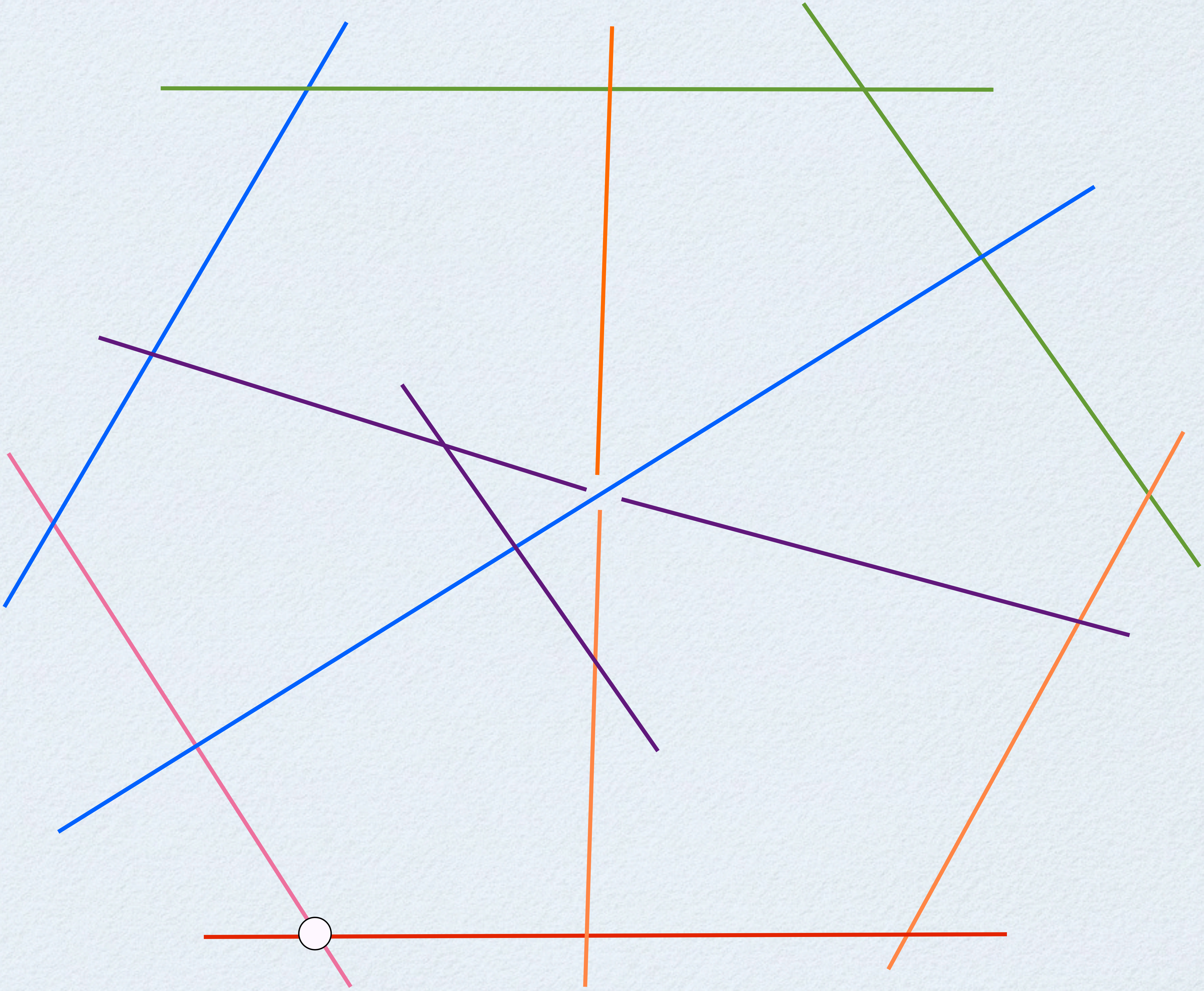


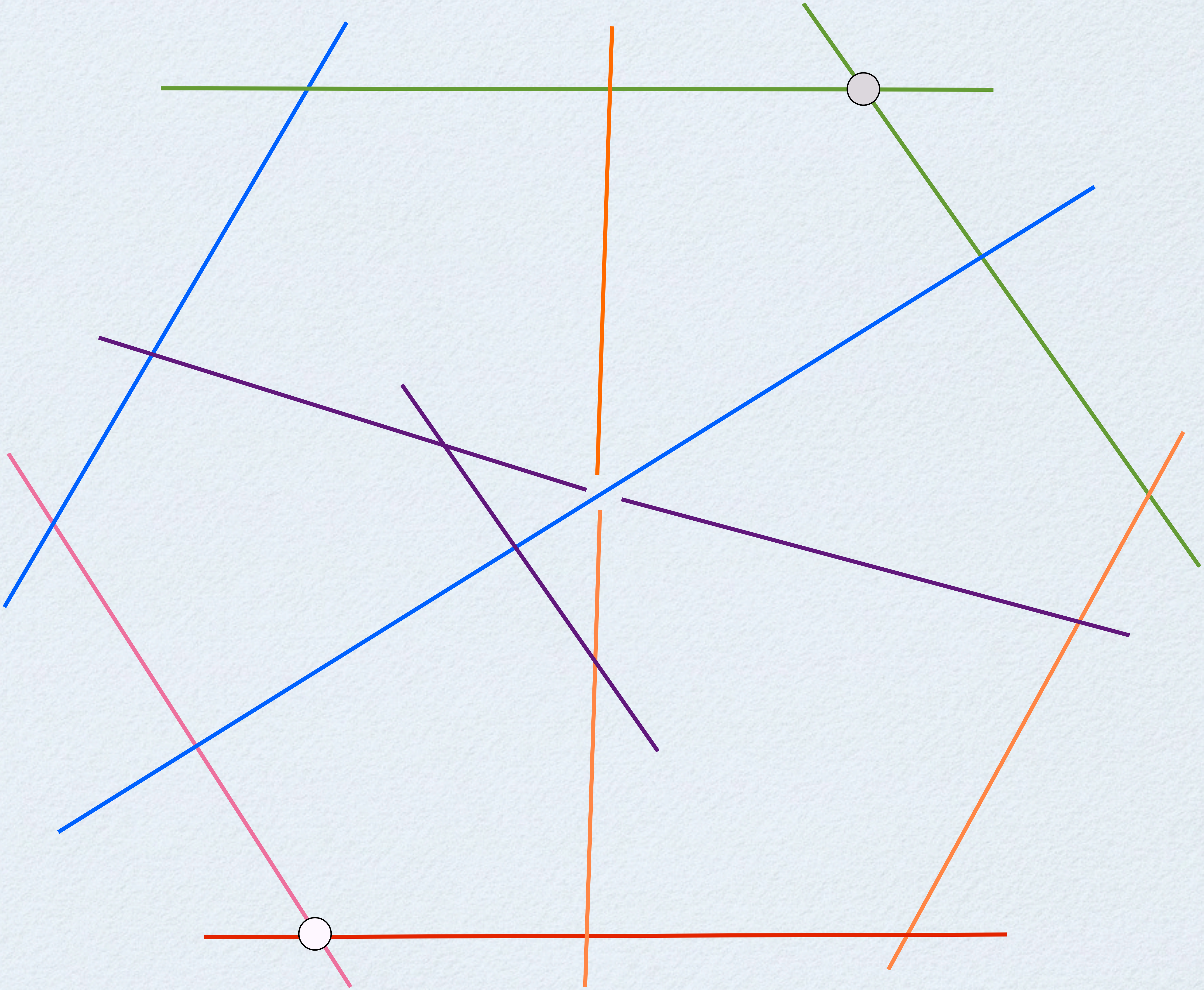


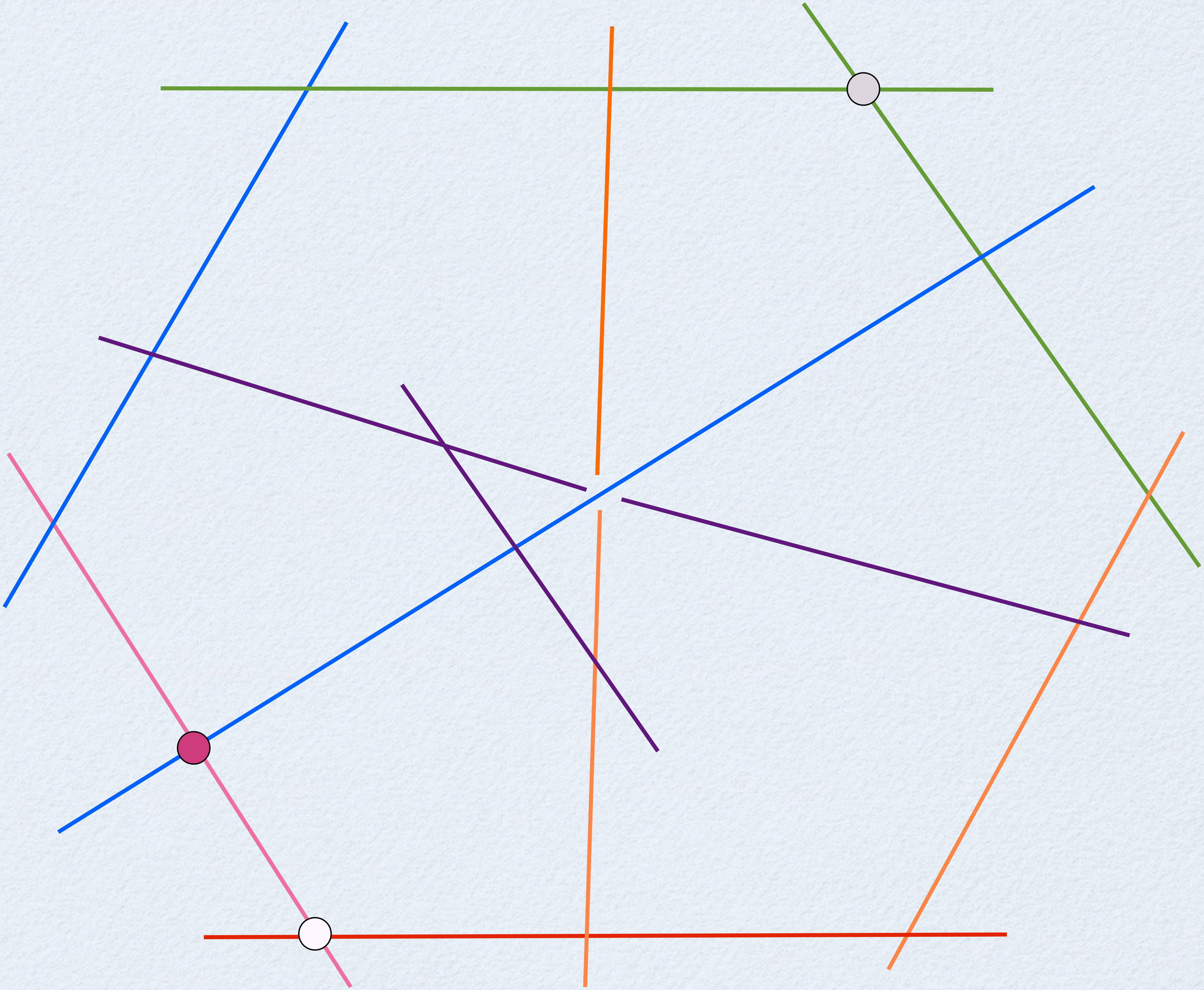


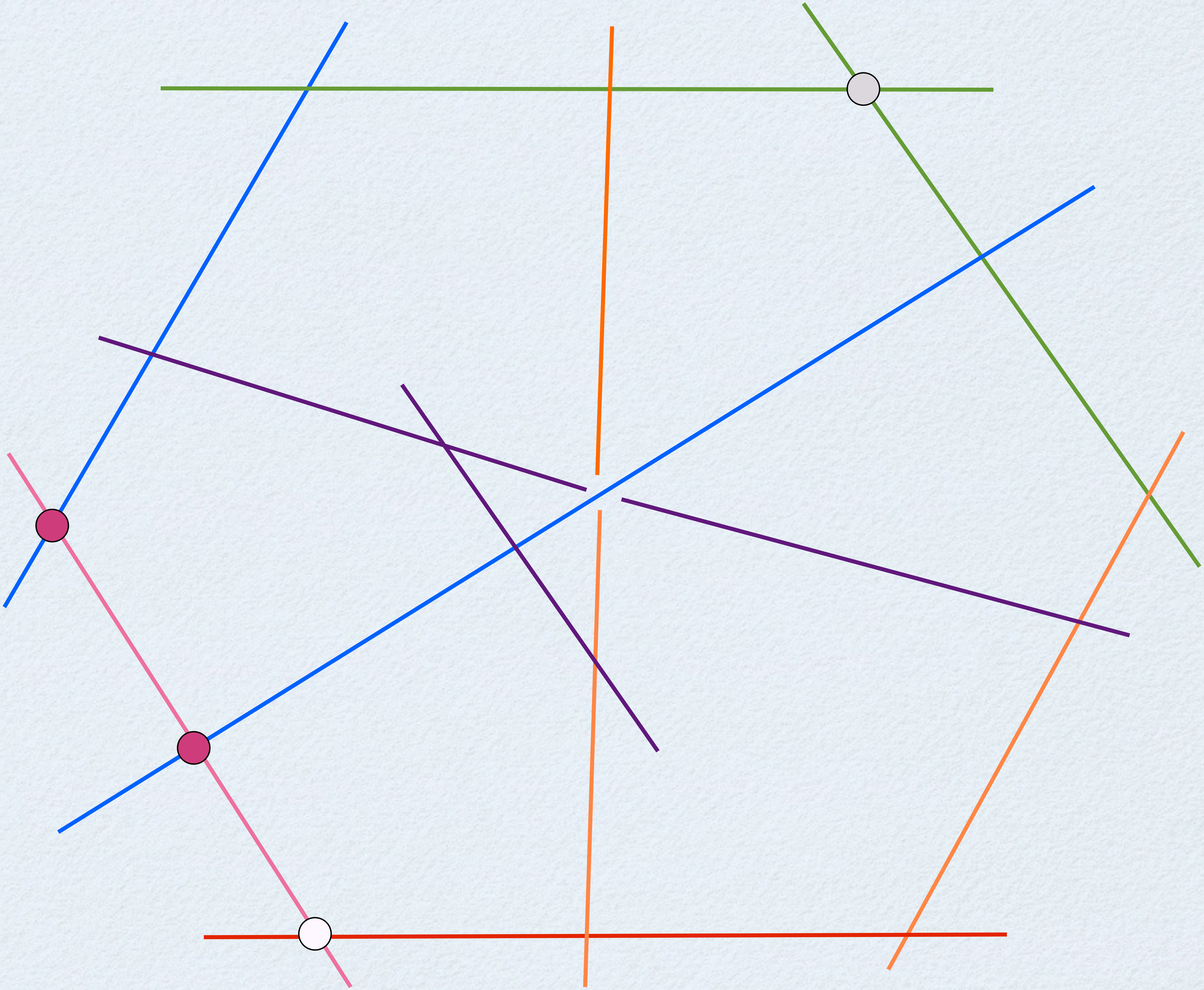


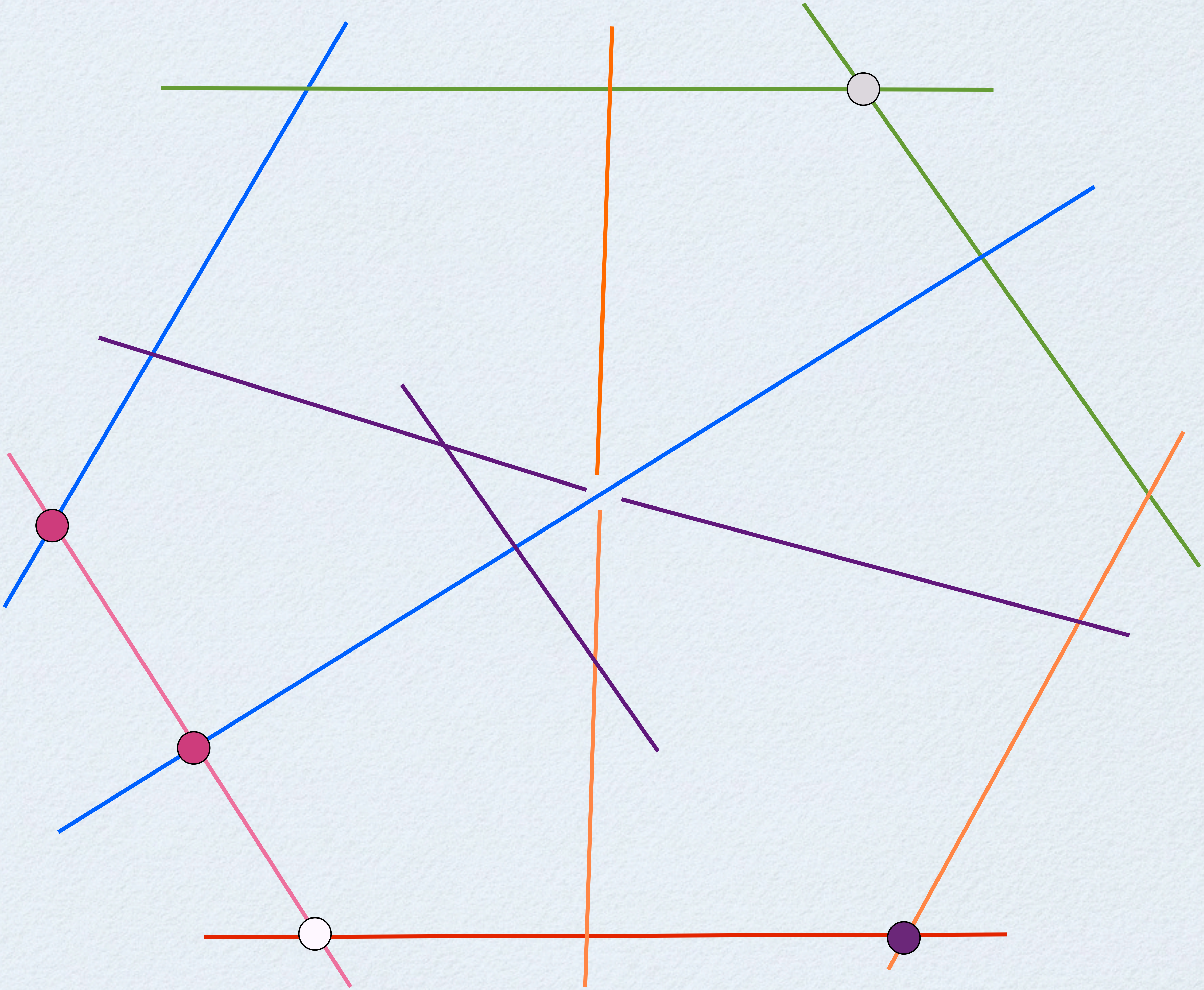


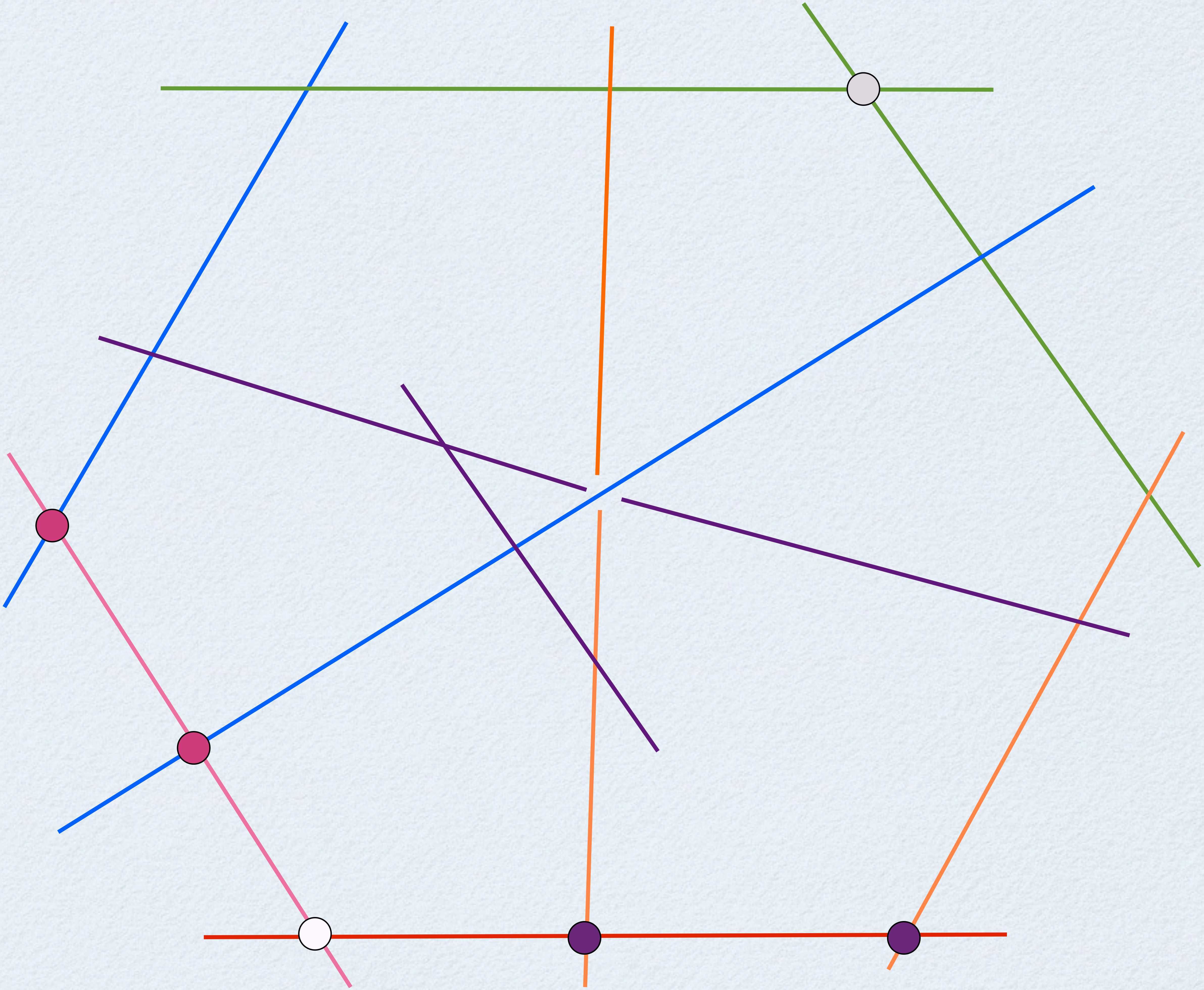


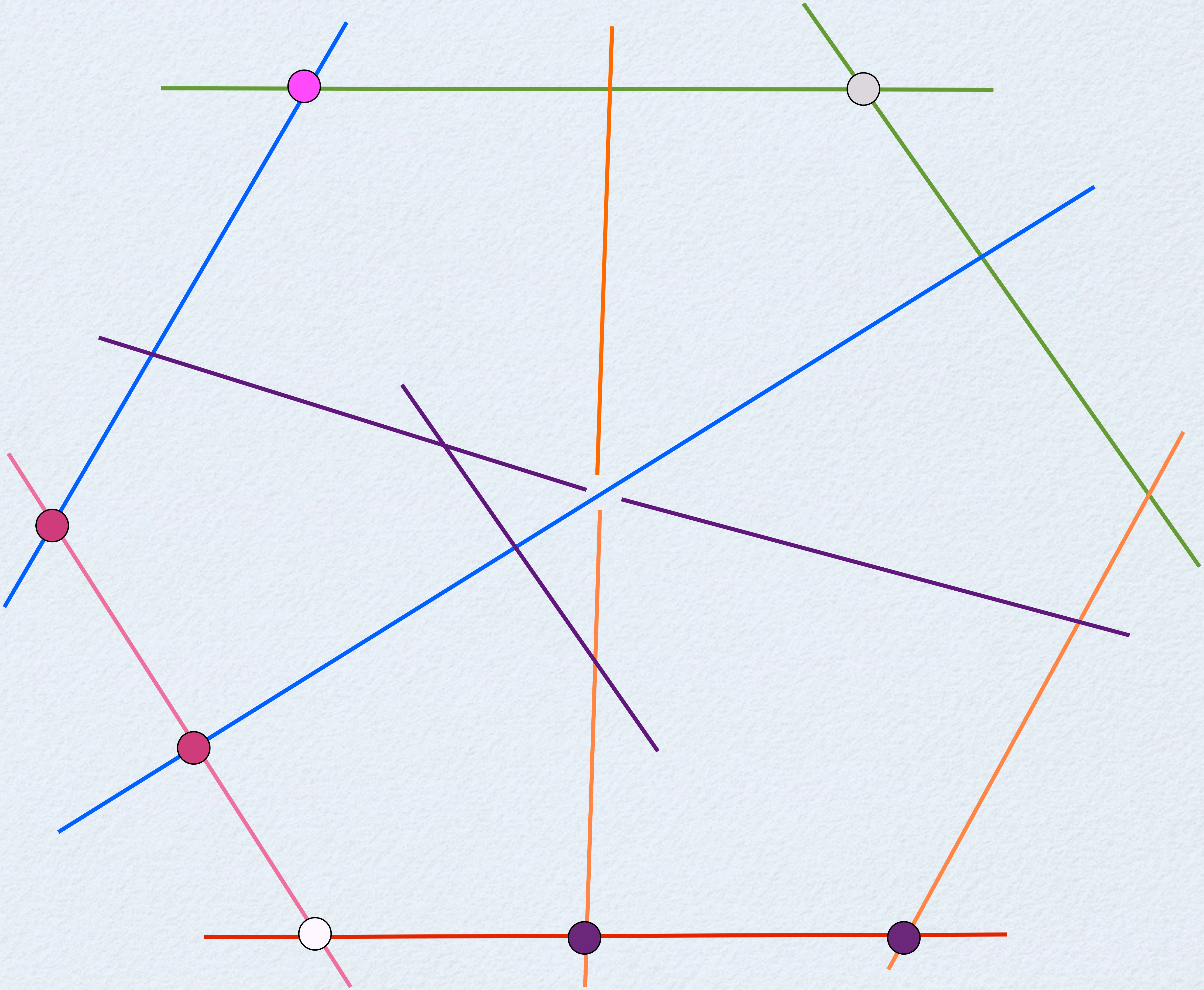


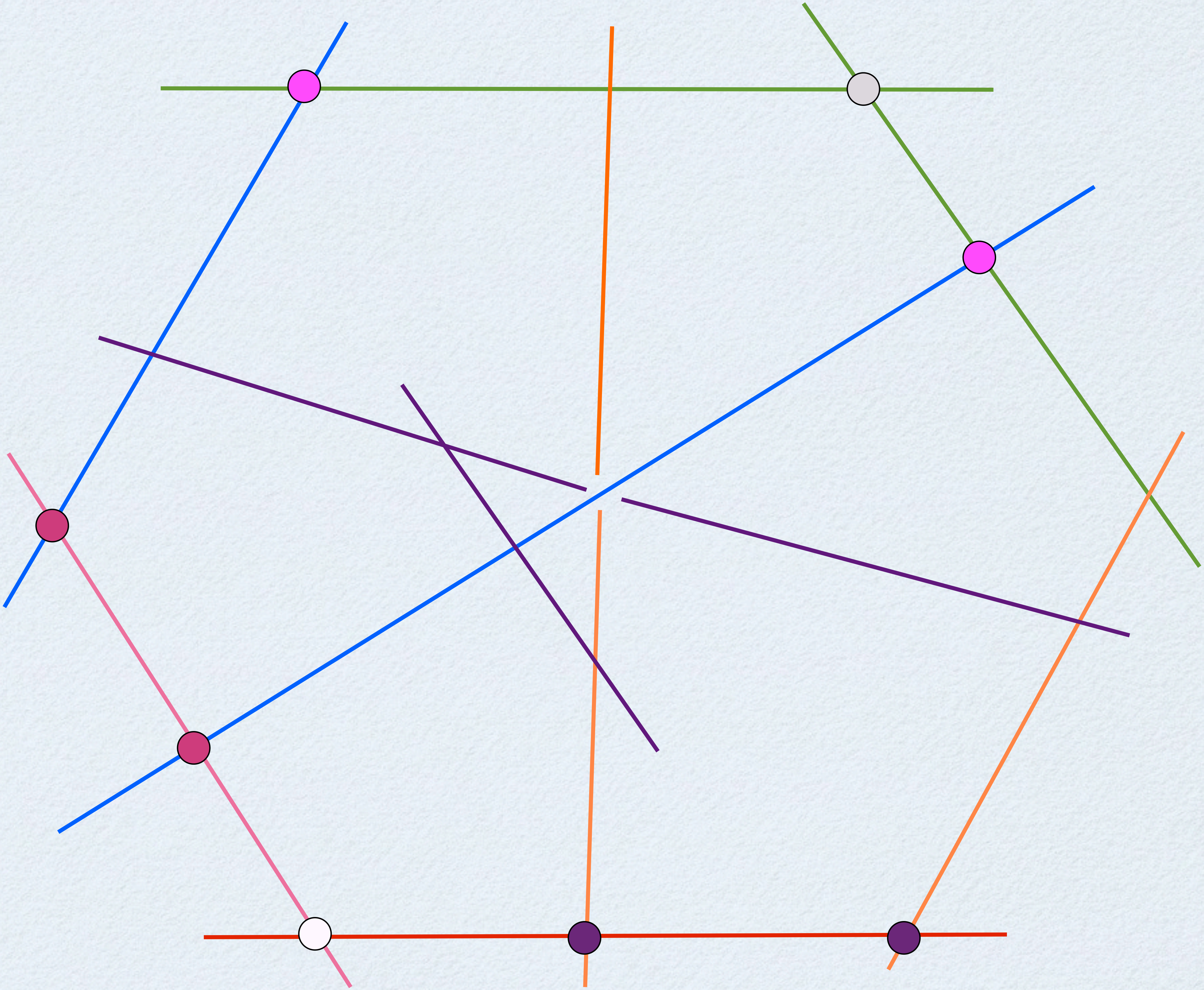


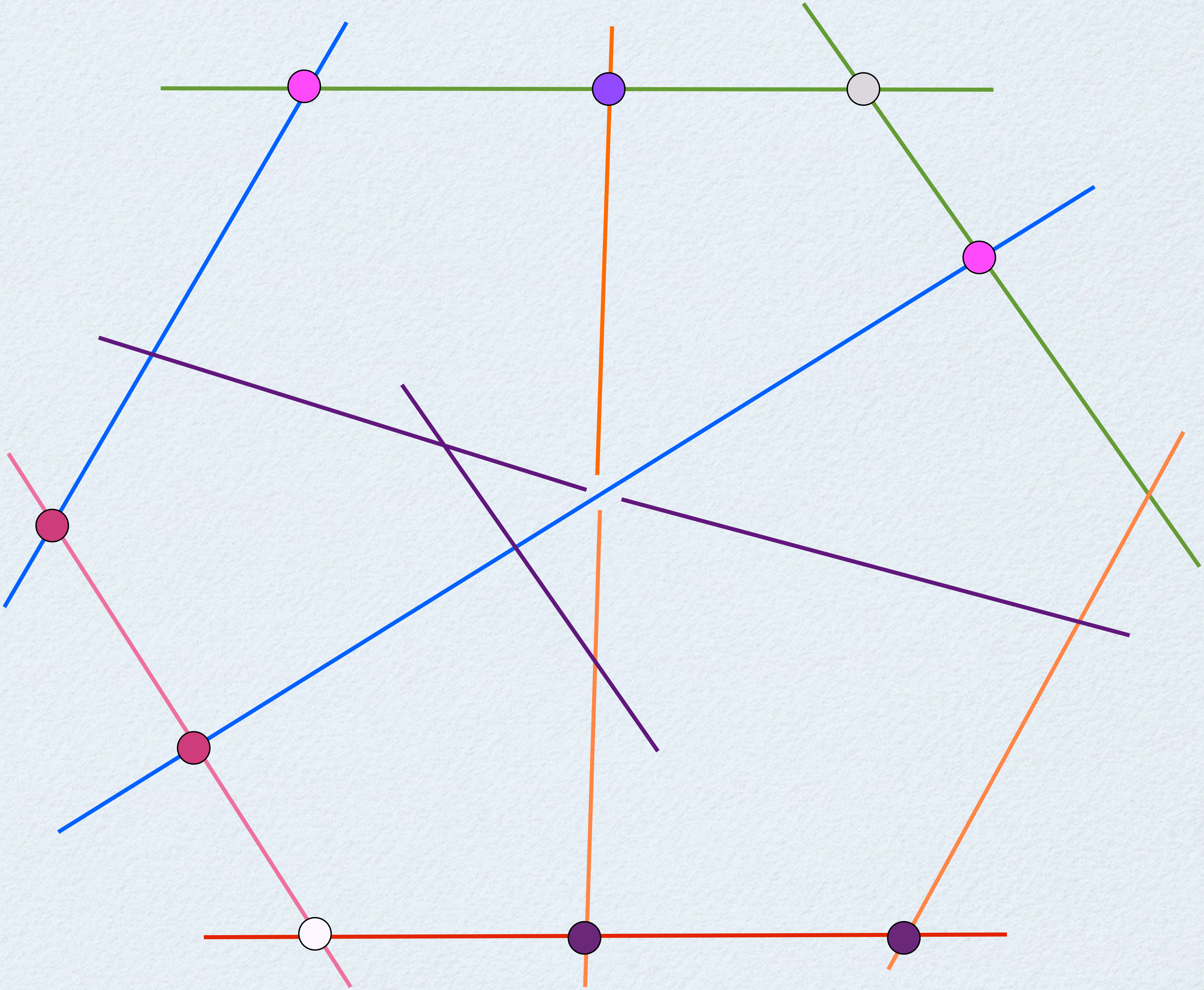


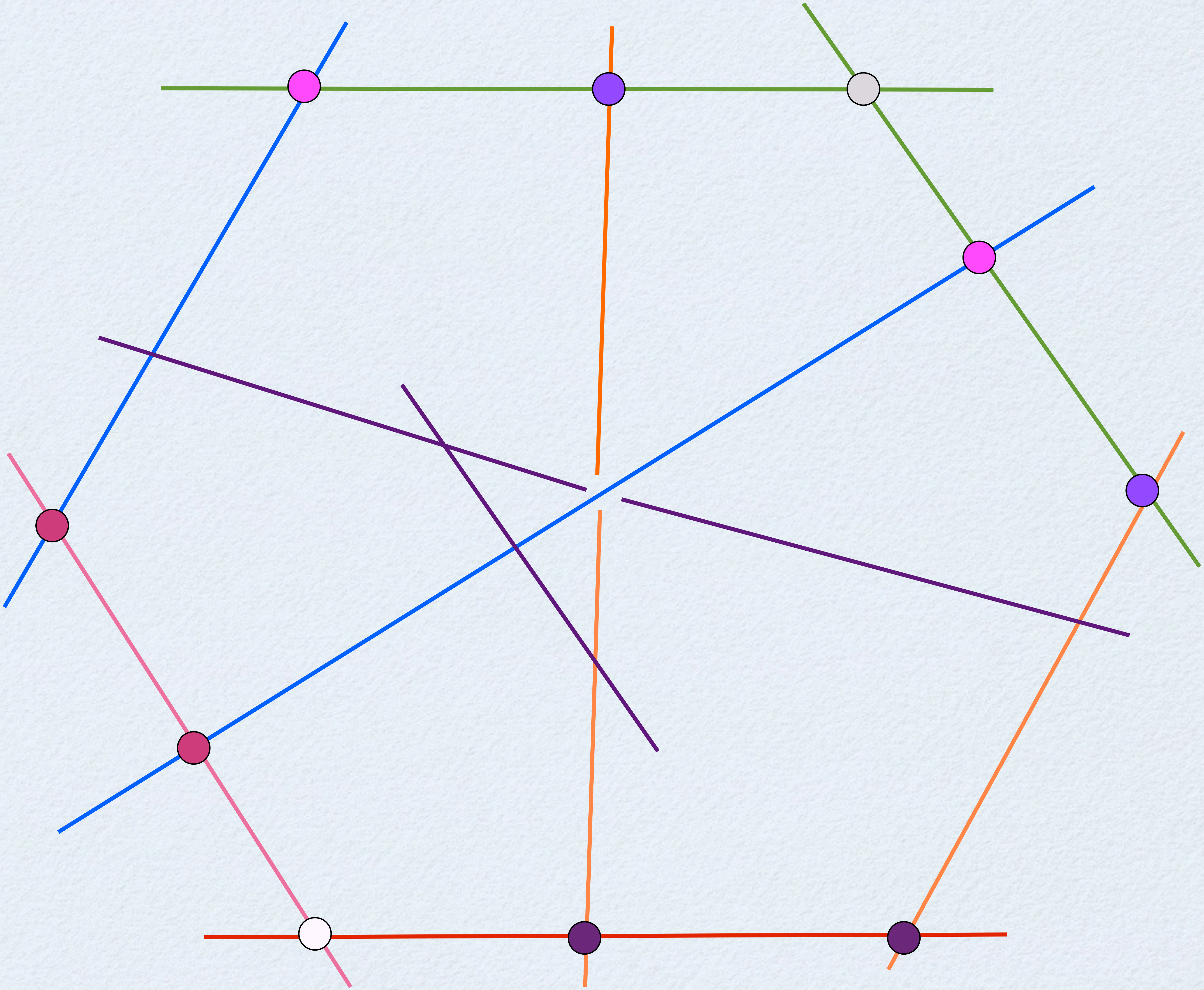












In general....

Let $f : (S^2, P) \to (S^2, P)$ be the formal mating of two critically finite hyperbolic polynomials which are unicritical.

$$\begin{tikzcd} \star \arrow[r, "d"] & \cdot \arrow[r] & \cdots \arrow[r] & \cdot \arrow[r] & \cdot \arrow[llll, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \star \arrow[r, "d"] & \cdot \arrow[r] & \cdots \arrow[r] & \cdot \arrow[r] & \cdot \arrow[llll, bend left] \end{tikzcd}$$

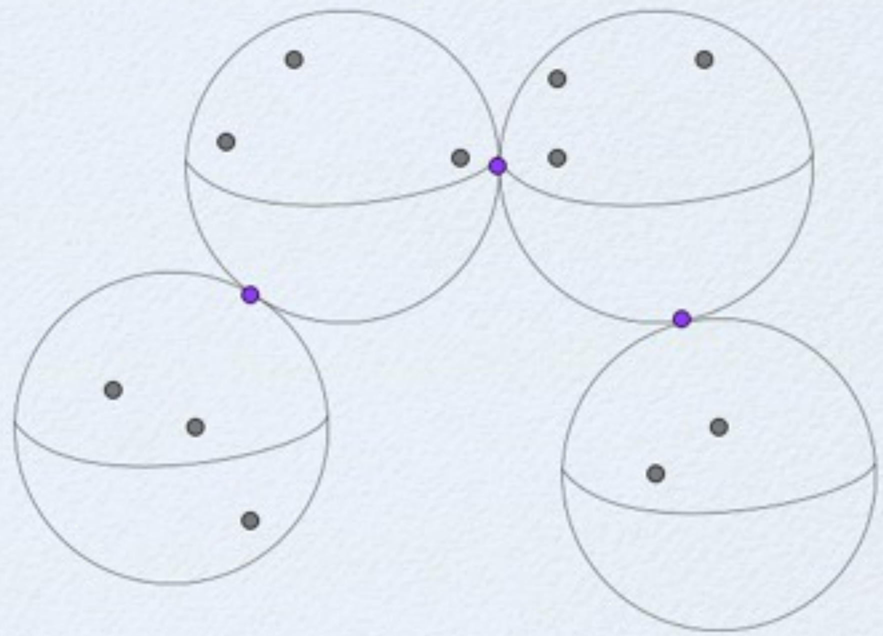
Then a moduli space map exists $g : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$, and we examine the associated skew product

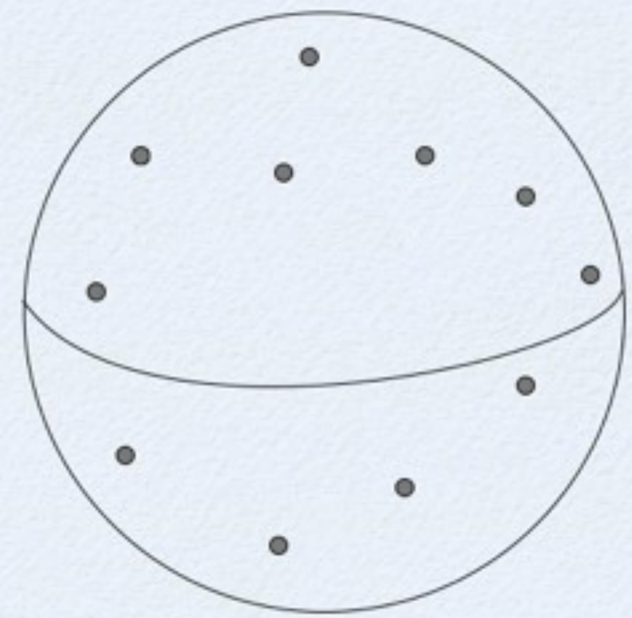
$$G : \mathbb{C}^n \to \mathbb{C}^n \quad \text{given by} \quad G : \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \mapsto \begin{pmatrix} F_{\mathbf{x}}(t) \\ g(\mathbf{x}) \end{pmatrix}$$

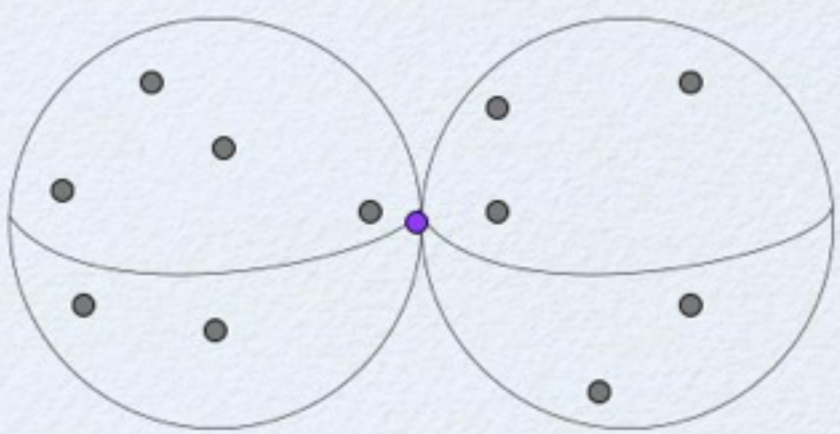
Proposition. The map $G = \mu \circ s$, where $s : \mathbb{P}^n \to \mathbb{P}^n$ is the dth power map, and $\mu : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is a birational transformation of $\mathbb{P}^n$ induced by the permutation of P coming from the ramification portrait.

A sufficient compactification

Suppose $|P| = n$, and consider $\mathcal{M}_P$.





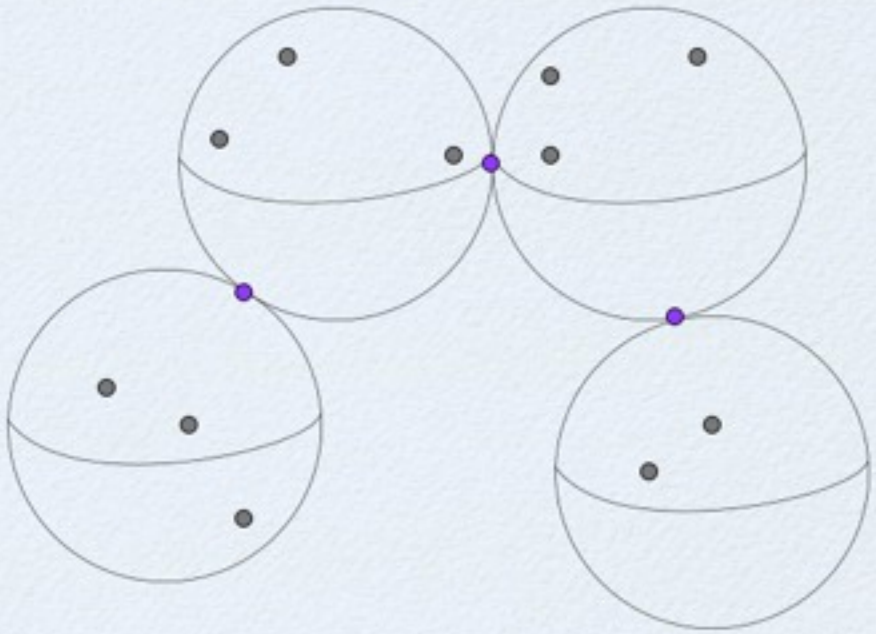


Compactify $\mathcal{M}_P$ so that the points on the boundary correspond to different ways to collapse curves to nodes.

Instead of a $\mathbb{P}^1$ with marked points, we have a *stable curve* with marked points and nodes.

Compactify $\mathcal{M}_P$ so that the points on the boundary correspond to different ways to collapse curves to nodes.

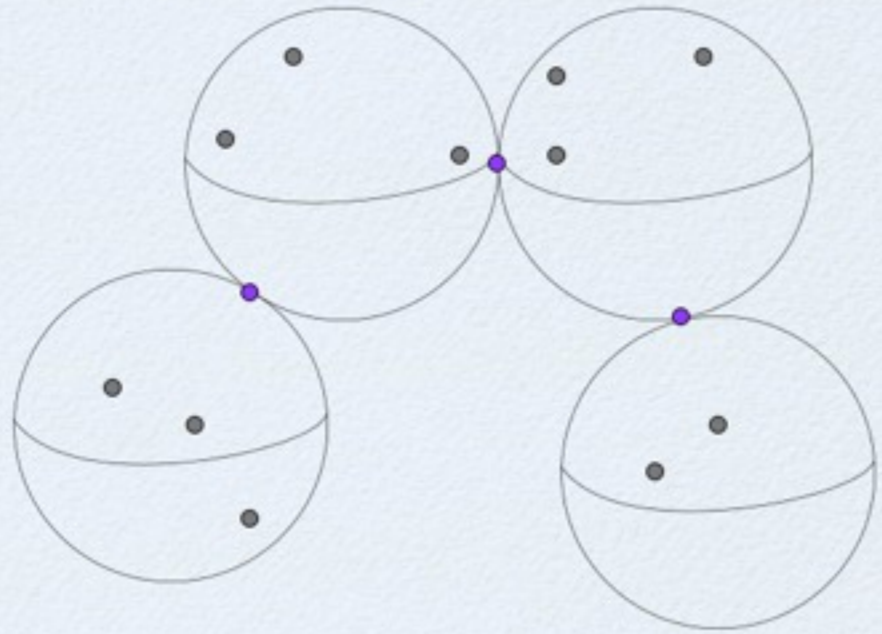
Instead of a $\mathbb{P}^1$ with marked points, we have a *stable curve* with marked points and nodes.



Compactify $\mathcal{M}_P$ so that the points on the boundary correspond to different ways to collapse curves to nodes.

Instead of a $\mathbb{P}^1$ with marked points, we have a *stable curve* with marked points and nodes.

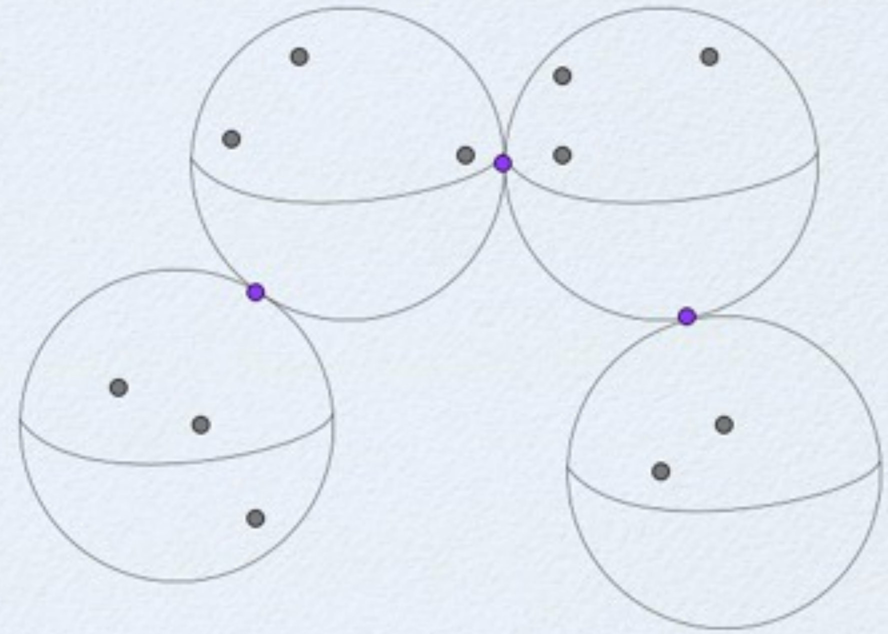
This is known as the Deligne-Mumford compactification of $\mathcal{M}_P$, which we denote as $\overline{\mathcal{M}}_n$.



Compactify $\mathcal{M}_P$ so that the points on the boundary correspond to different ways to collapse curves to nodes.

Instead of a $\mathbb{P}^1$ with marked points, we have a *stable curve* with marked points and nodes.

This is known as the Deligne-Mumford compactification of $\mathcal{M}_P$, which we denote as $\overline{\mathcal{M}}_n$.



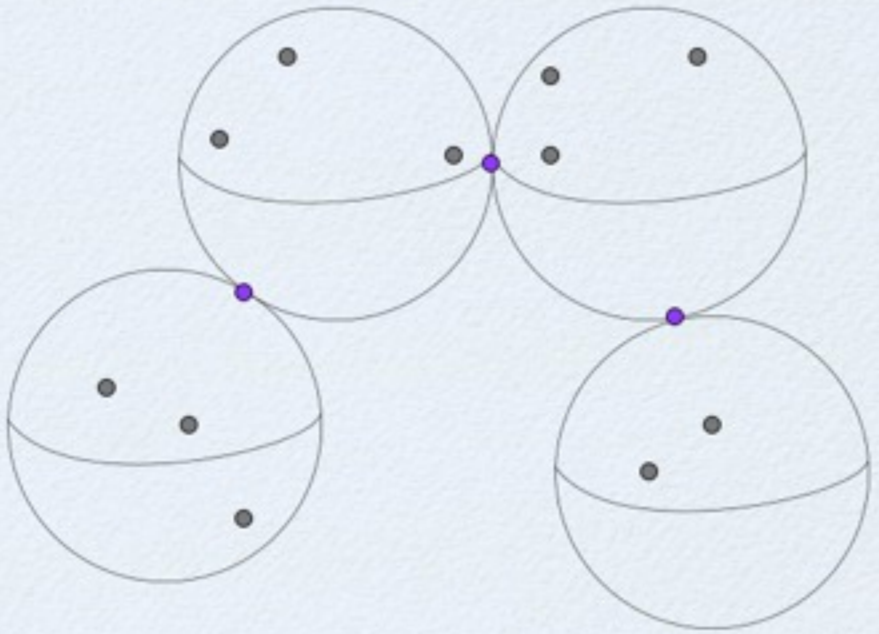
Example: Suppose $|P| = 4$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^1 - \Delta,$$

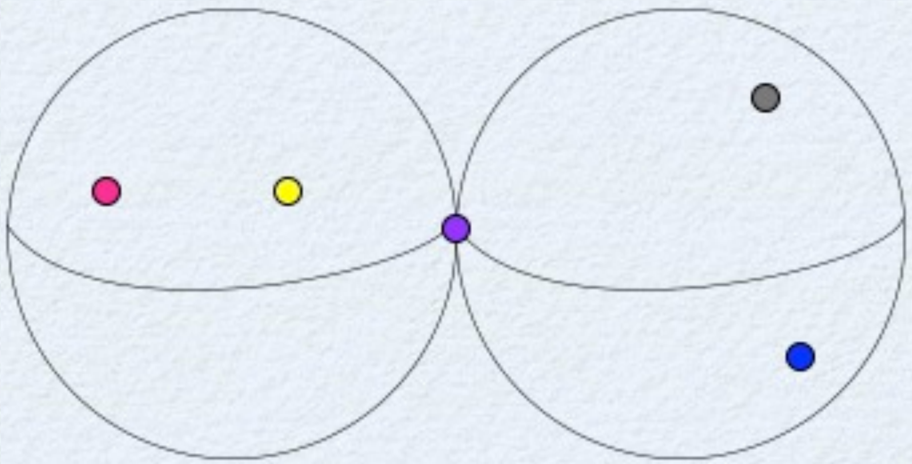
$\Delta = \{0, 1, \infty\}$

Compactify $\mathcal{M}_P$ so that the points on the boundary correspond to different ways to collapse curves to nodes.

Instead of a $\mathbb{P}^1$ with marked points, we have a *stable curve* with marked points and nodes.



This is known as the Deligne-Mumford compactification of $\mathcal{M}_P$, which we denote as $\overline{\mathcal{M}}_n$.



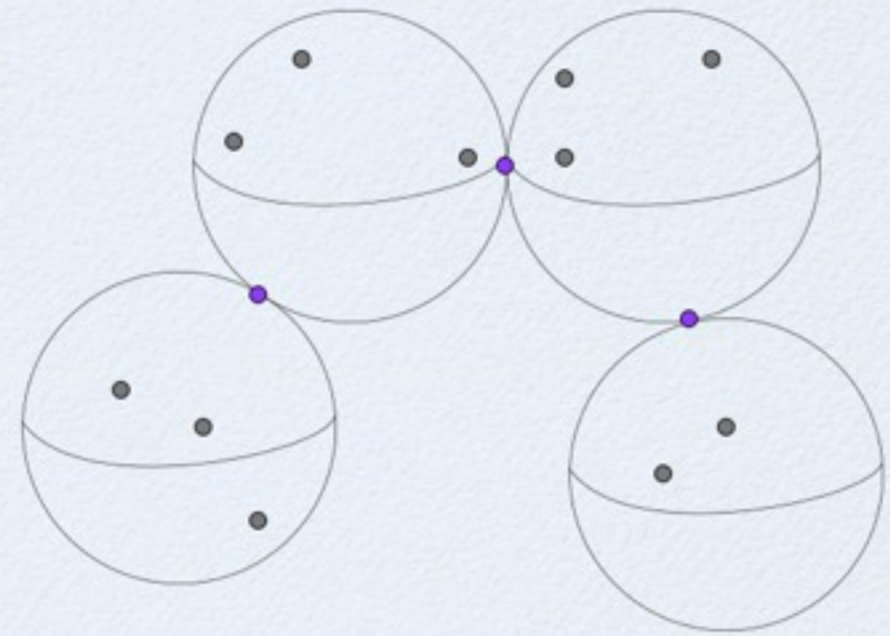
Example: Suppose $|P| = 4$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^1 - \Delta,$$

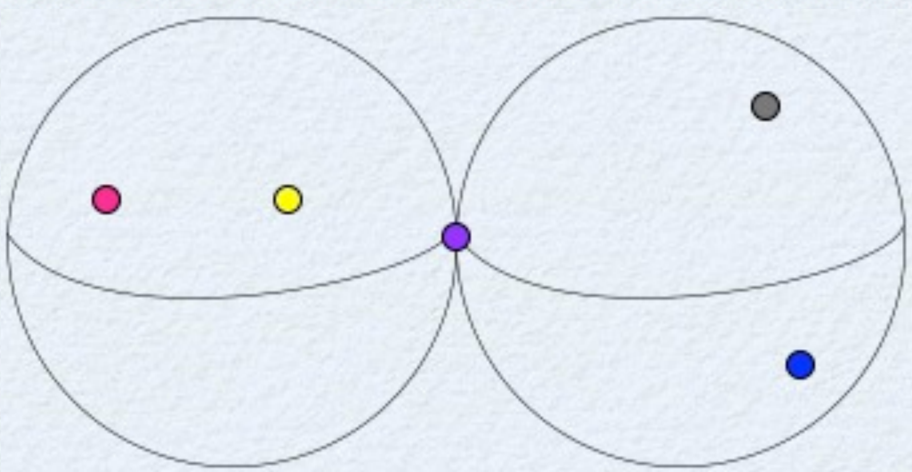
$$\Delta = \{0, 1, \infty\}$$

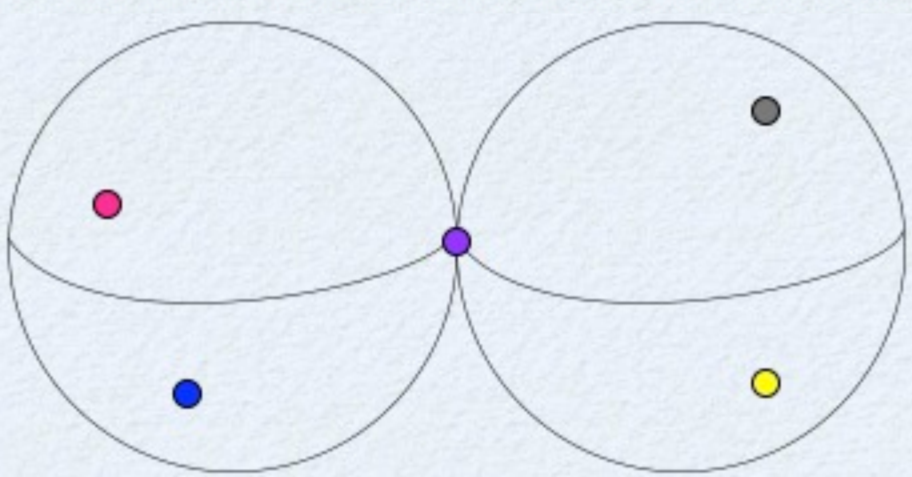
Compactify $\mathcal{M}_P$ so that the points on the boundary correspond to different ways to collapse curves to nodes.

Instead of a $\mathbb{P}^1$ with marked points, we have a *stable curve* with marked points and nodes.



This is known as the Deligne-Mumford compactification of $\mathcal{M}_P$, which we denote as $\overline{\mathcal{M}}_n$.





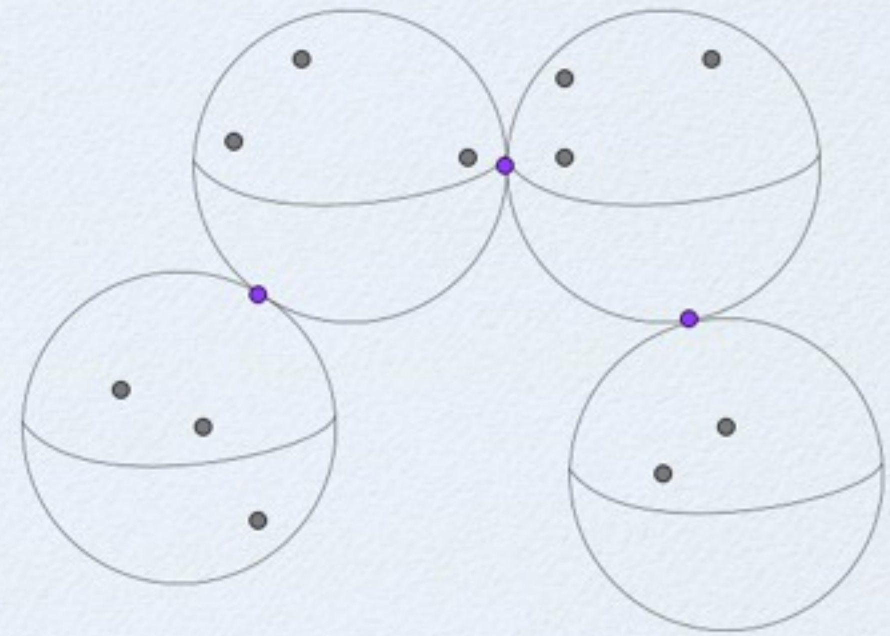
Example: Suppose $|P| = 4$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^1 - \Delta,$$

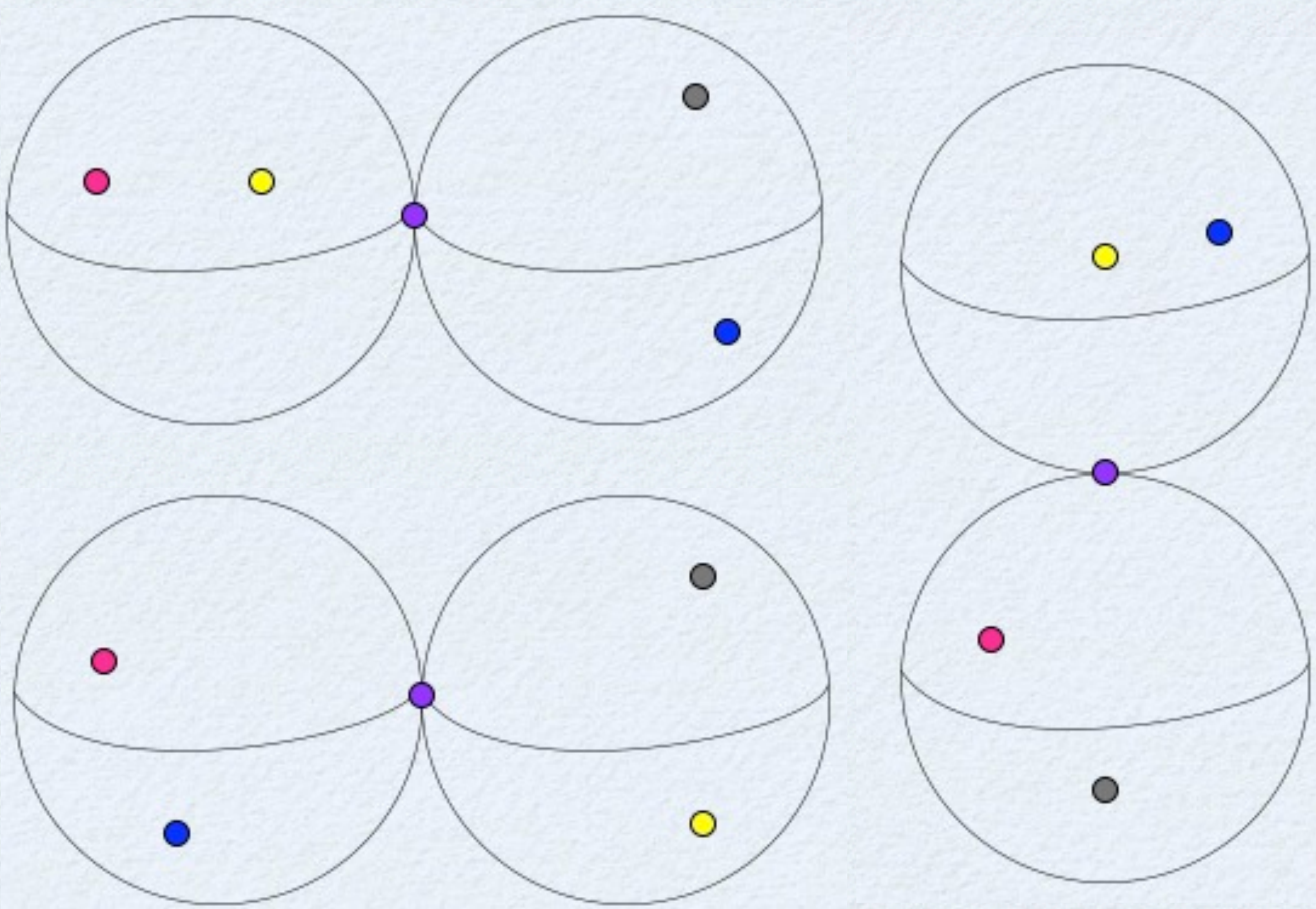
$$\Delta = \{0, 1, \infty\}$$

Compactify $\mathcal{M}_P$ so that the points on the boundary correspond to different ways to collapse curves to nodes.

Instead of a $\mathbb{P}^1$ with marked points, we have a *stable curve* with marked points and nodes.



This is known as the Deligne-Mumford compactification of $\mathcal{M}_P$, which we denote as $\overline{\mathcal{M}}_n$.



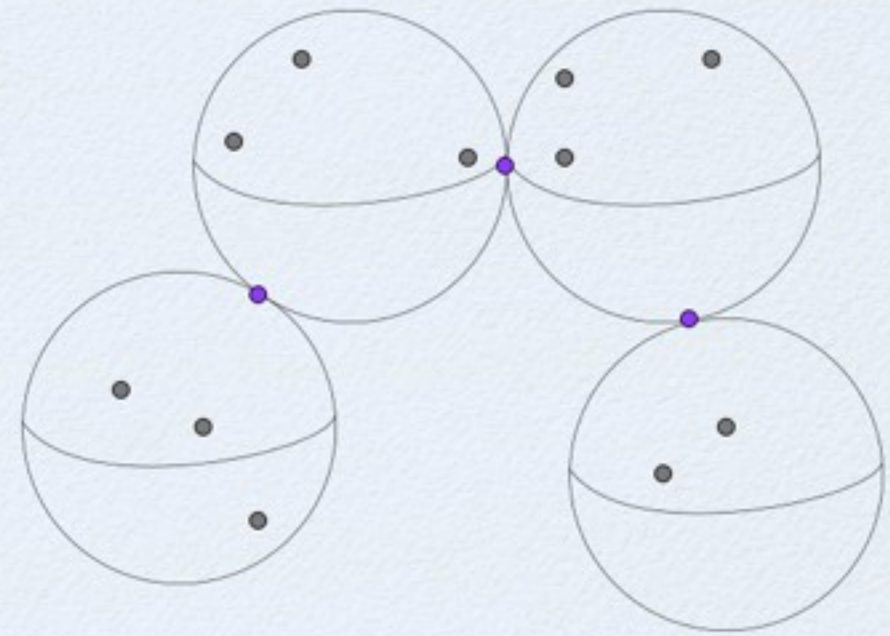
Example: Suppose $|P| = 4$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^1 - \Delta,$$

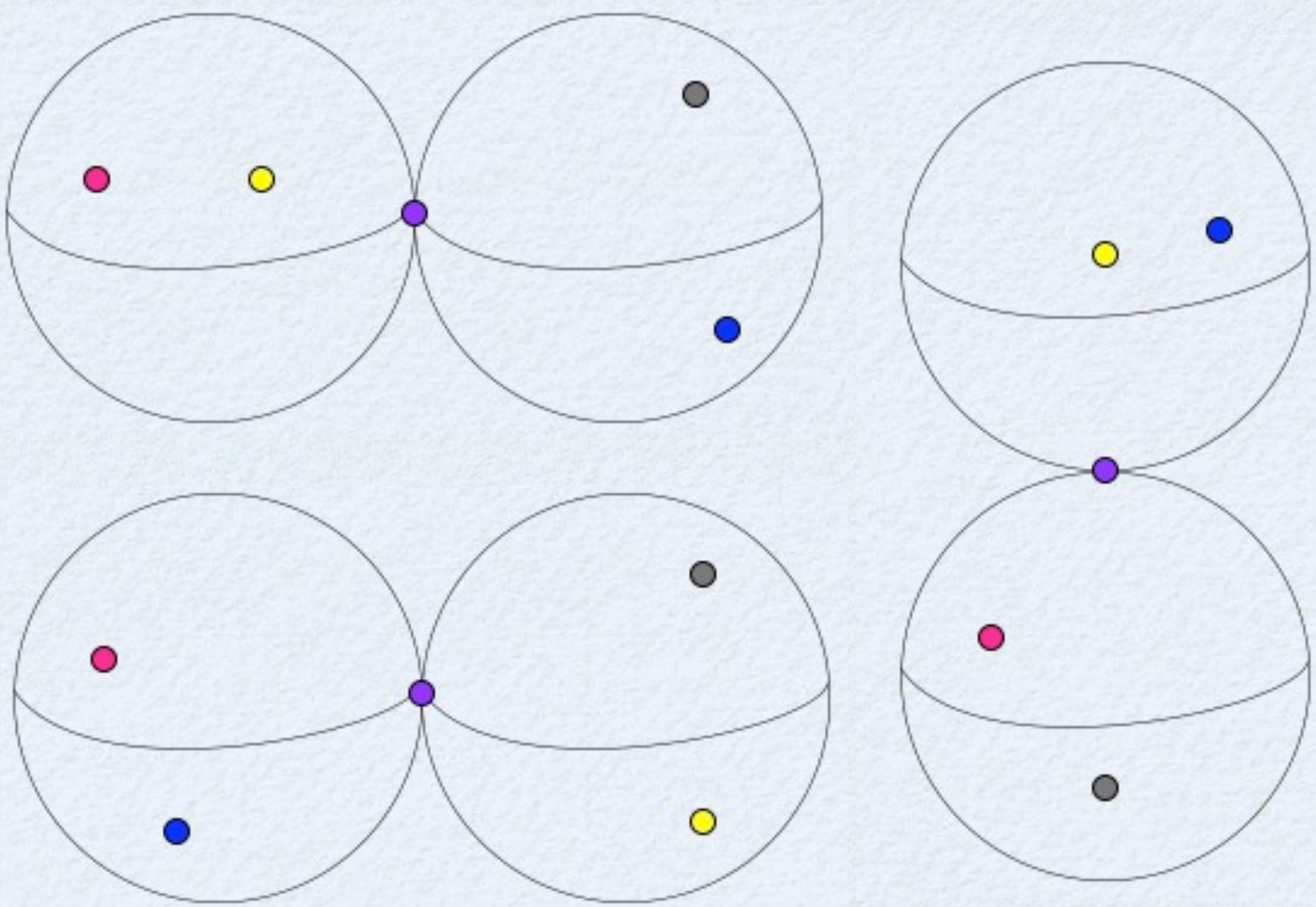
$\Delta = \{0, 1, \infty\}$

Compactify $\mathcal{M}_P$ so that the points on the boundary correspond to different ways to collapse curves to nodes.

Instead of a $\mathbb{P}^1$ with marked points, we have a *stable curve* with marked points and nodes.



This is known as the Deligne-Mumford compactification of $\mathcal{M}_P$, which we denote as $\overline{\mathcal{M}}_n$.

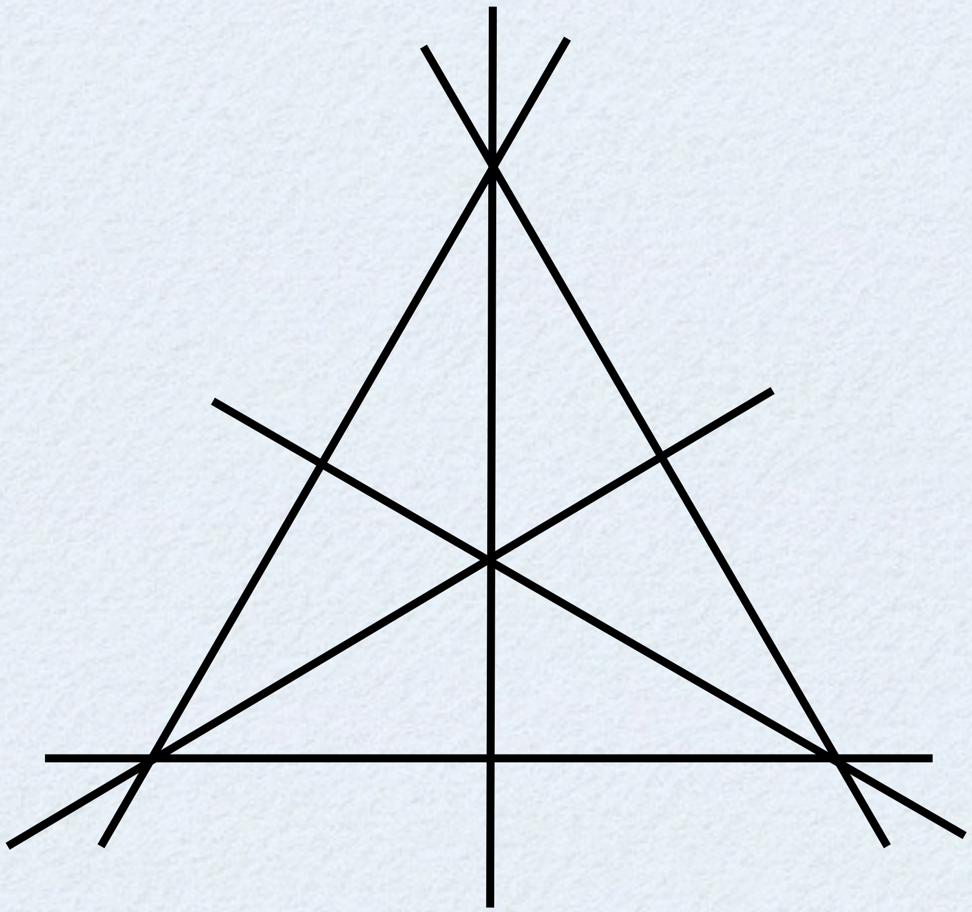


Example: Suppose $|P| = 4$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^1 - \Delta,$$

$\Delta = \{0, 1, \infty\}$

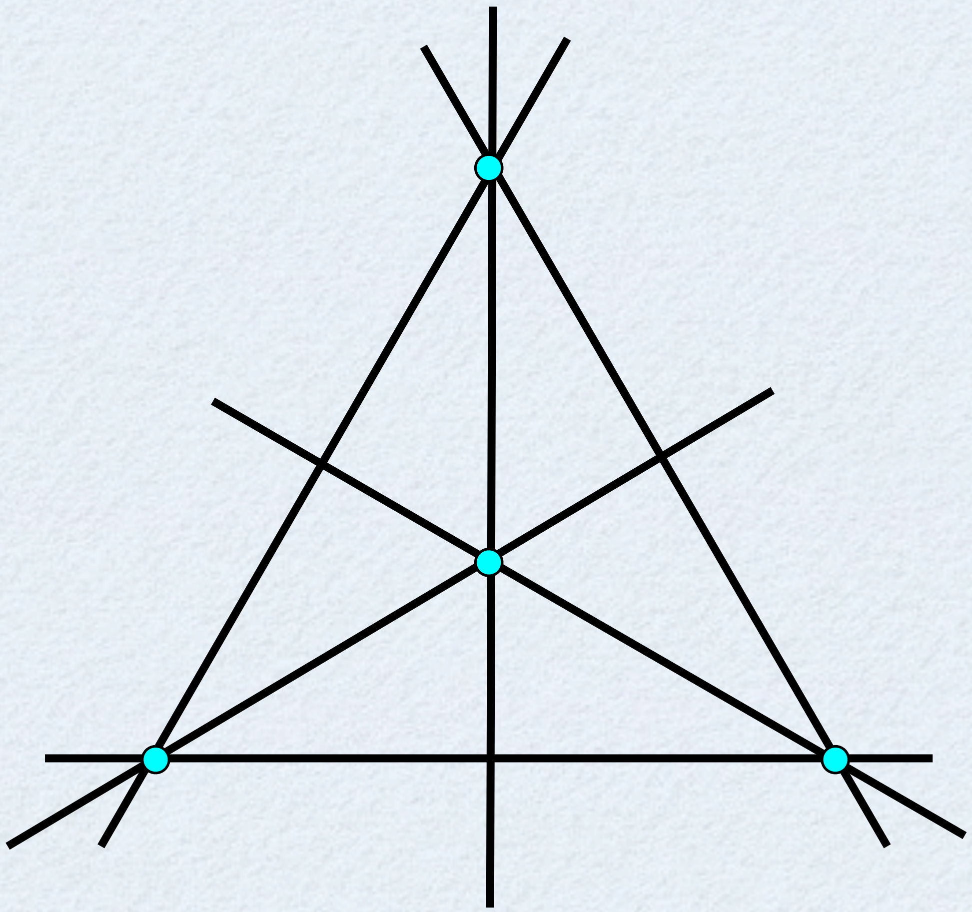
$$\overline{\mathcal{M}}_4 \approx \mathbb{P}^1$$



Example: Suppose $|P| = 5$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^2 - \Delta,$$

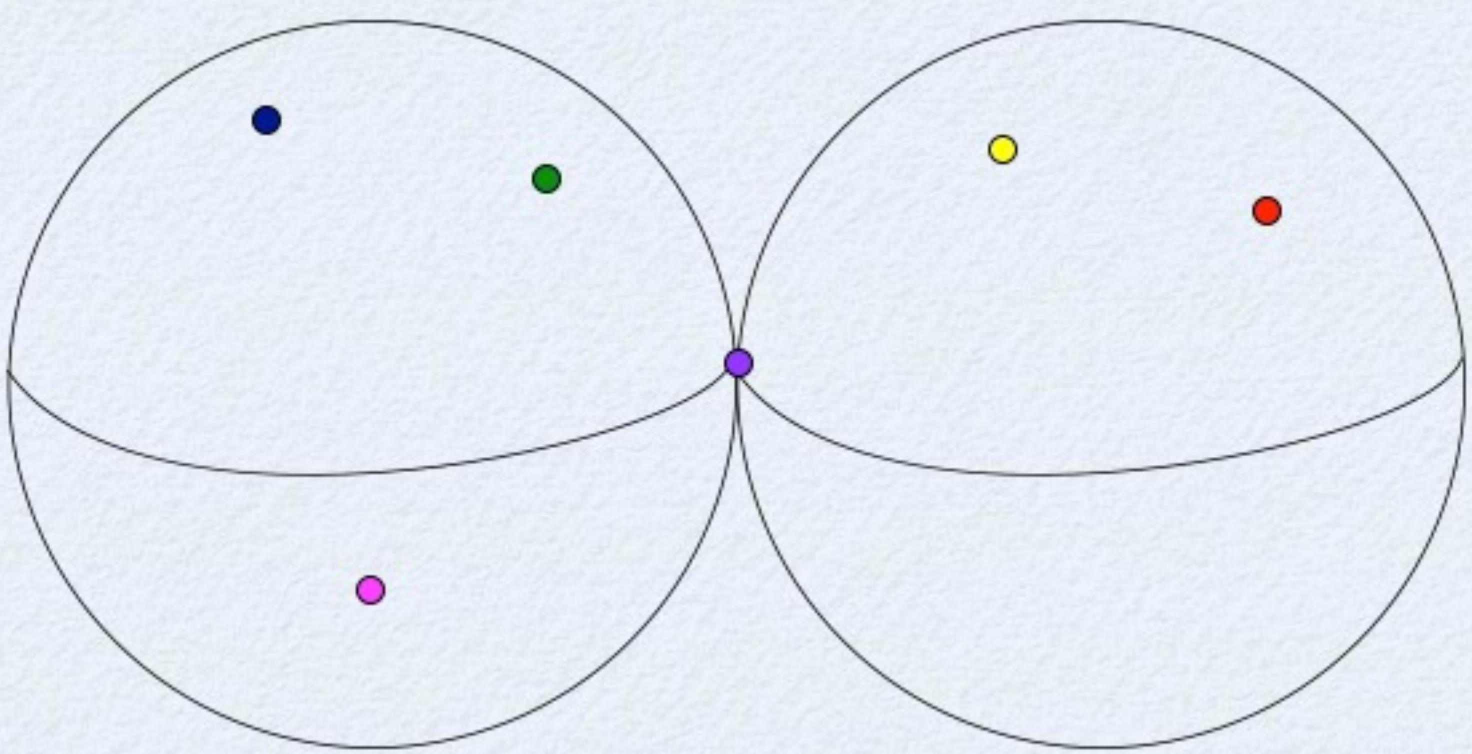
where Δ is the locus where points coalesce.

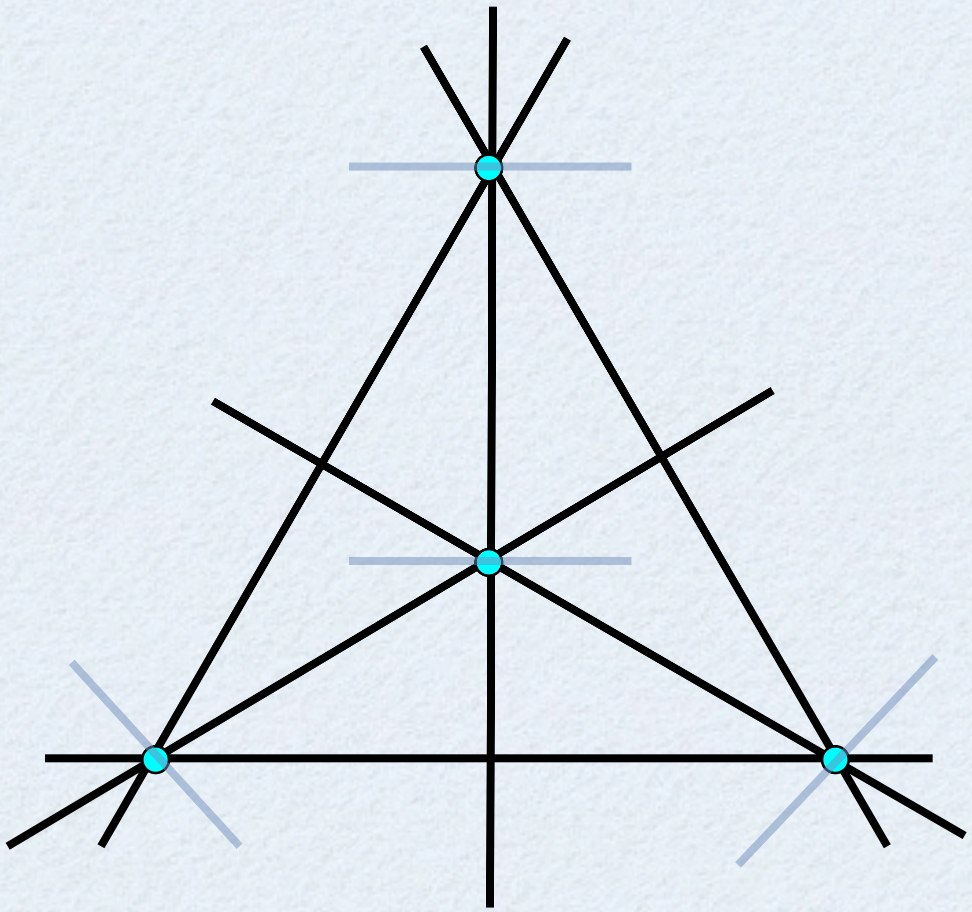


Example: Suppose $|P| = 5$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^2 - \Delta,$$

where Δ is the locus where points coalesce.

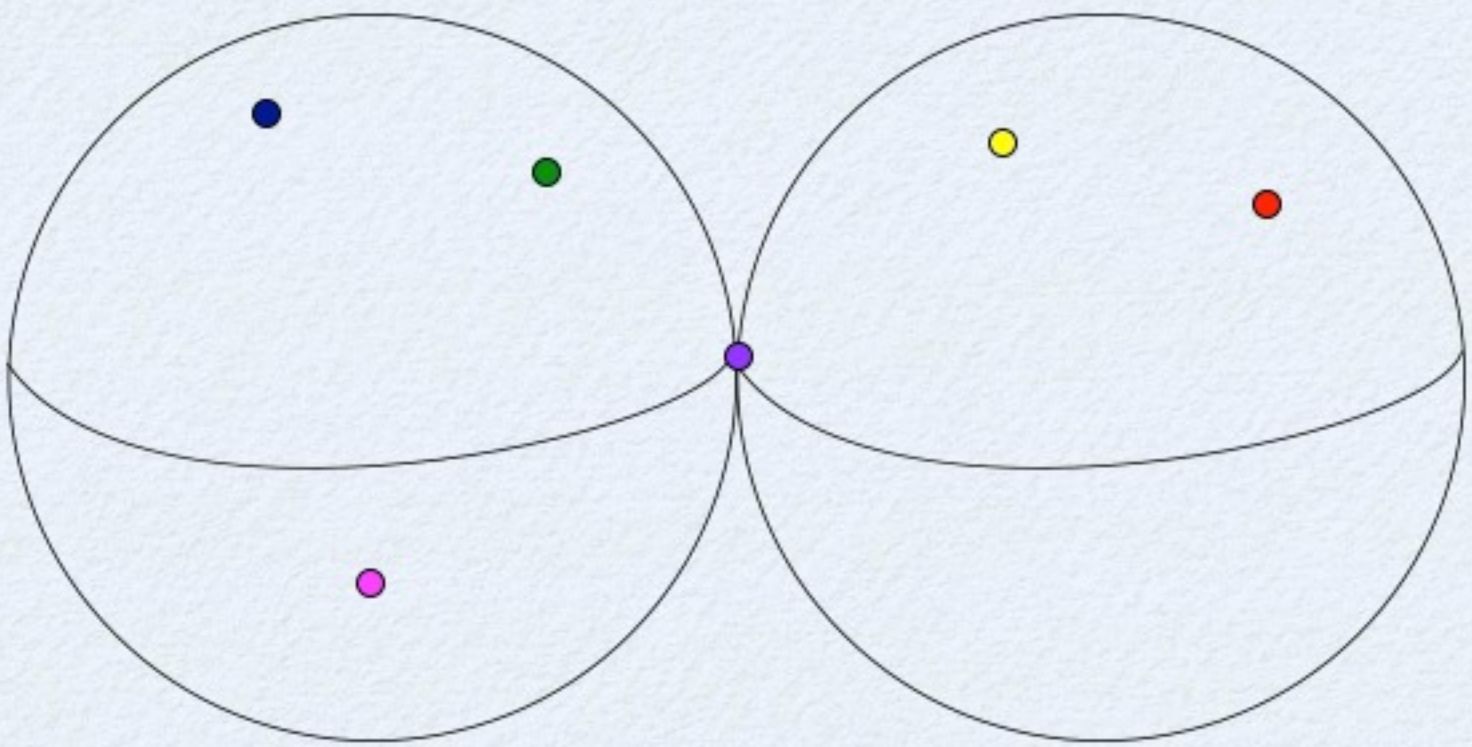


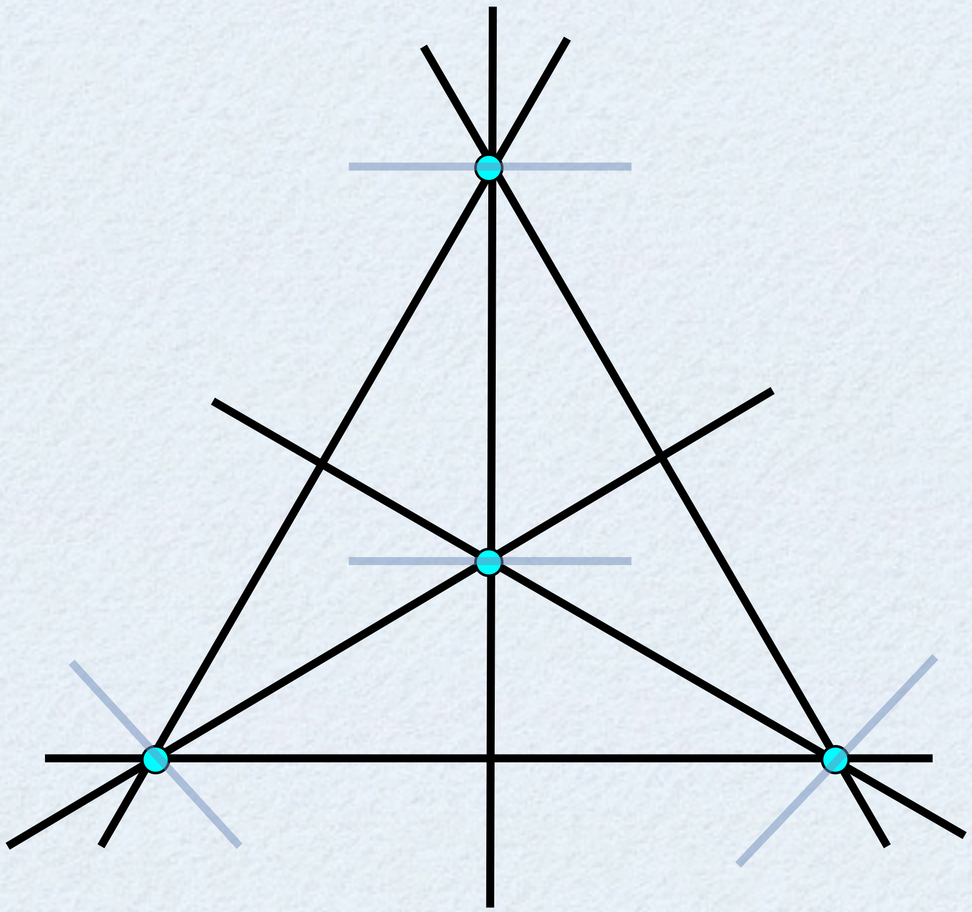


Example: Suppose $|P| = 5$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^2 - \Delta,$$

where Δ is the locus where points coalesce.

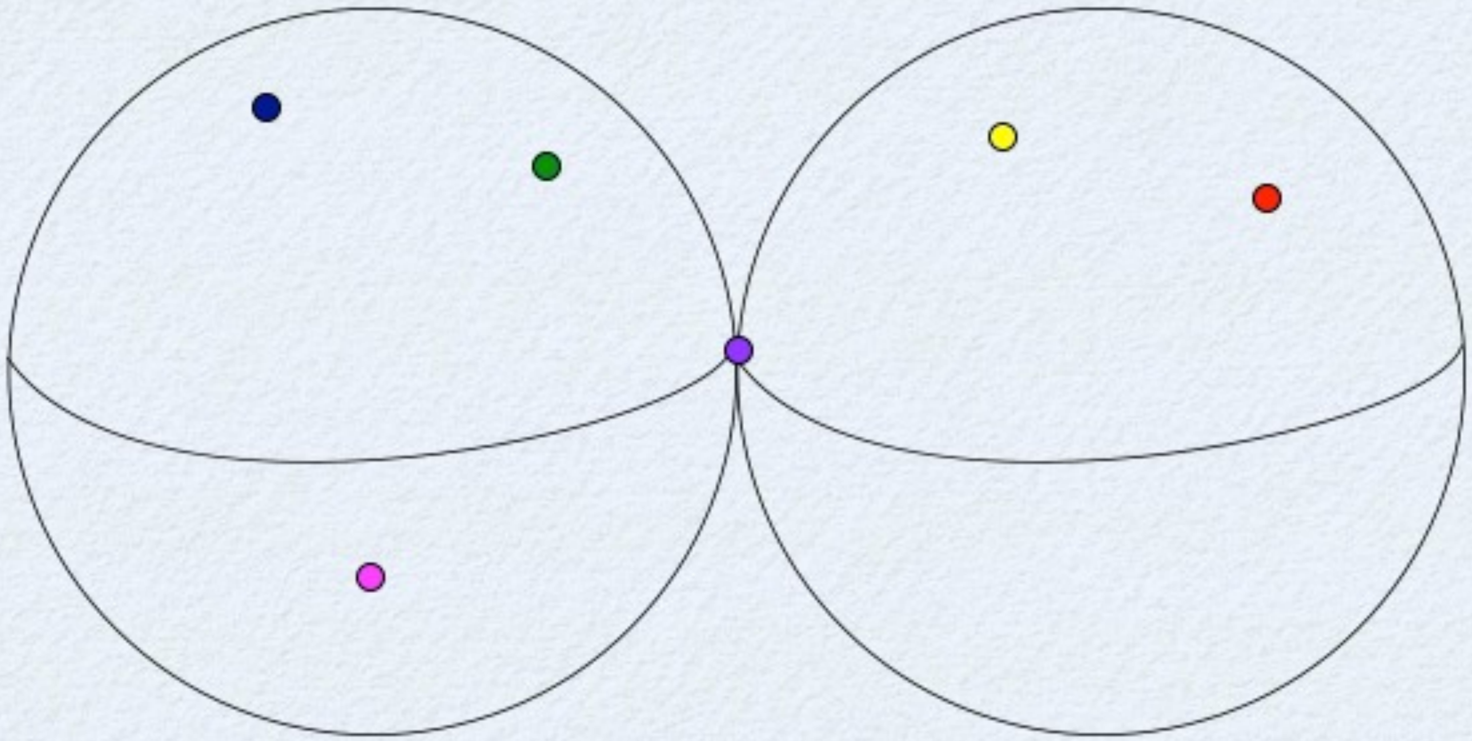




Example: Suppose $|P| = 5$, and normalize, identifying

$$\mathcal{M}_P \approx \mathbb{P}^2 - \Delta,$$

where Δ is the locus where points coalesce.



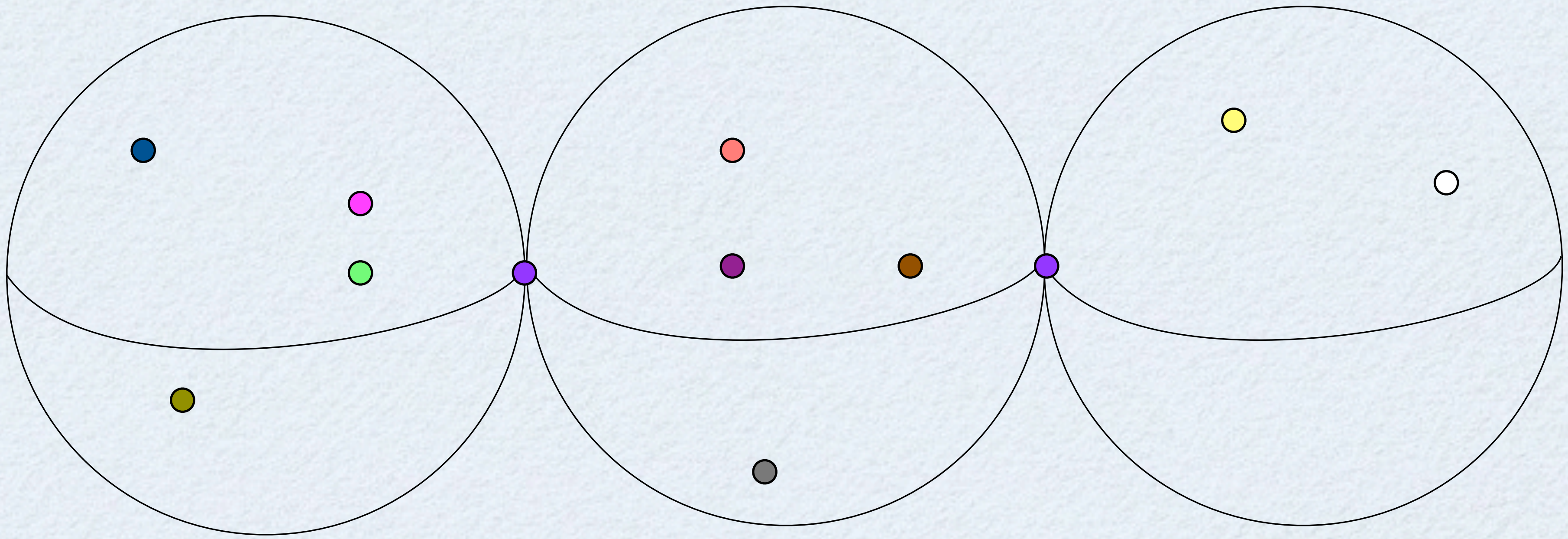
$$\overline{\mathcal{M}_5} \approx \widetilde{\mathbb{P}^2}$$

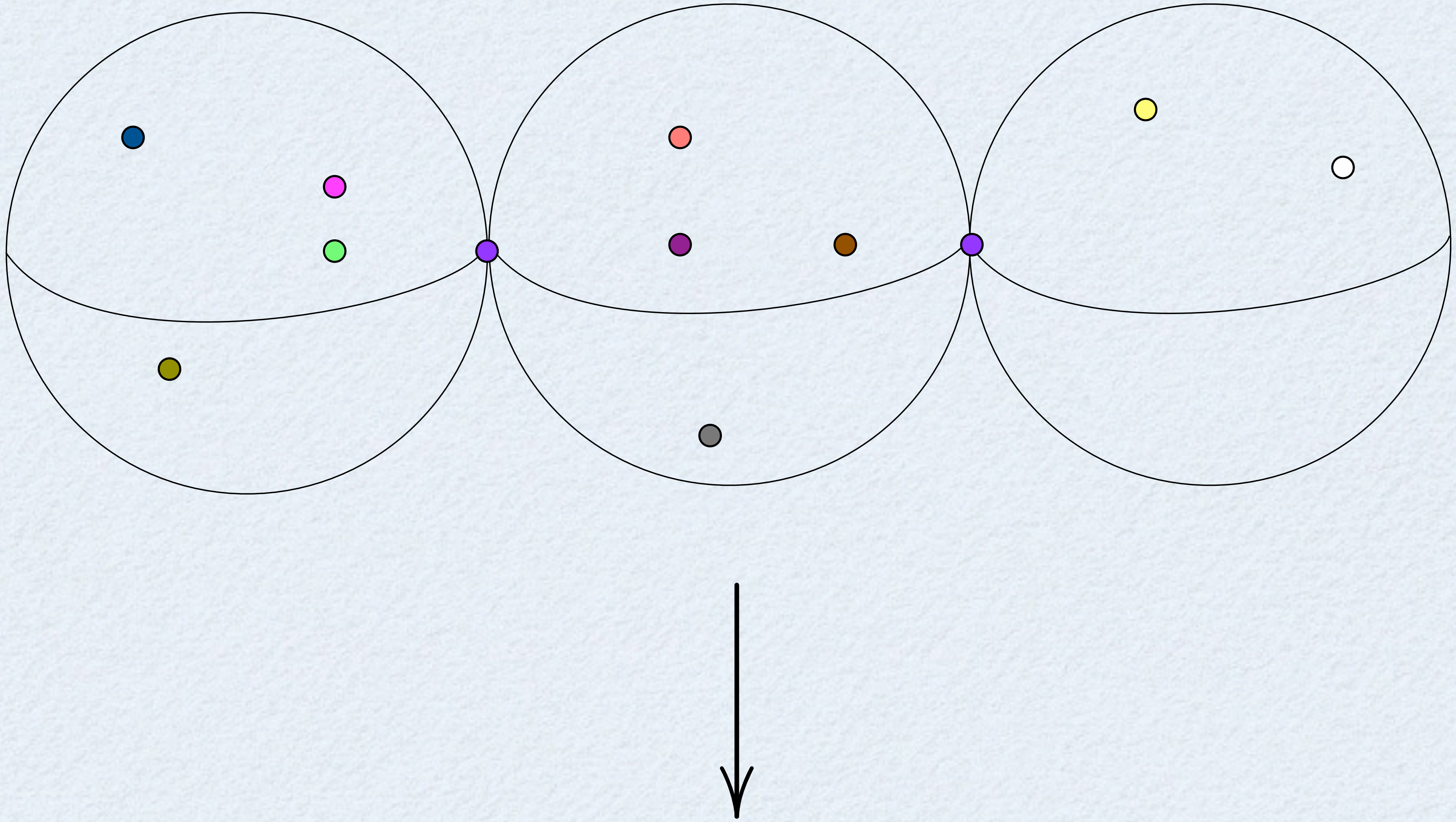
Proposition. $\overline{\mathcal{M}}_6$ is isomorphic to the space obtained by

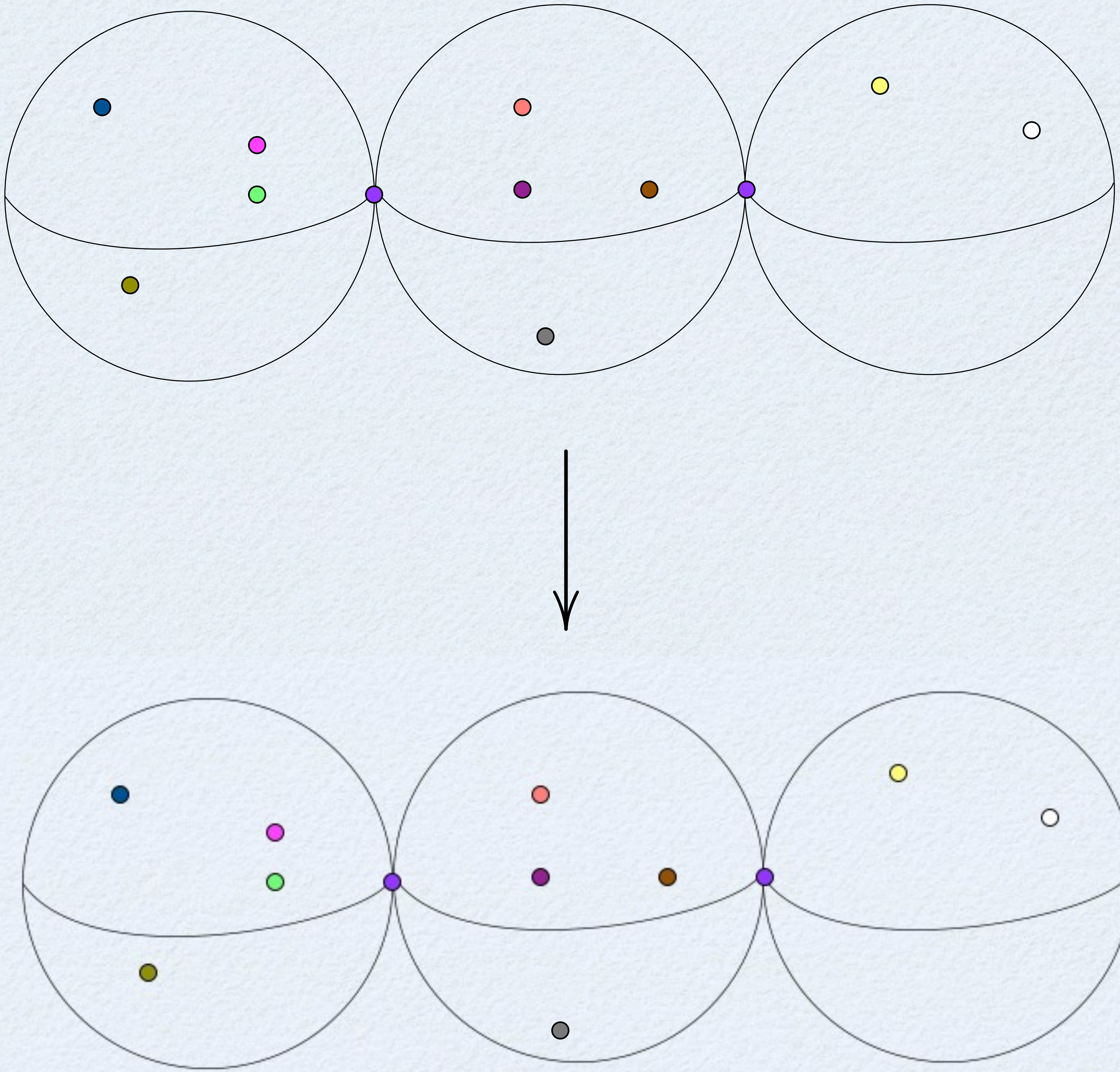
- first blowing up 5 points in $\mathbb{P}^3$ in general position
- then blowing up the proper transforms of the 10 lines between pairs of these points.

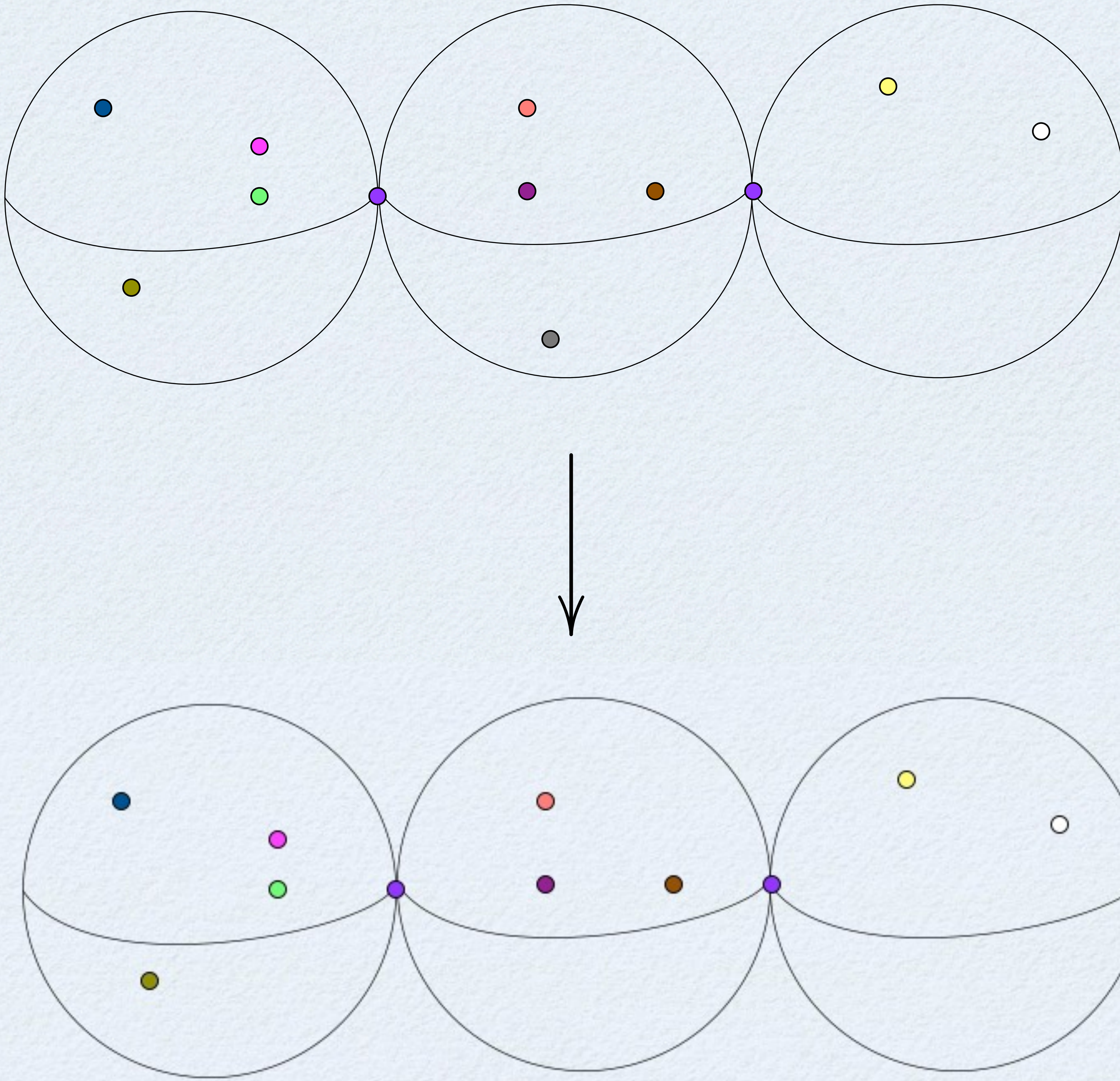
Consider the general case: $|P| = n$. Take $n-1$ points in general position in $\mathbb{P}^{n-3}$. Blow these points up, then blow up the proper transforms of the lines between pairs of these points, and continue...

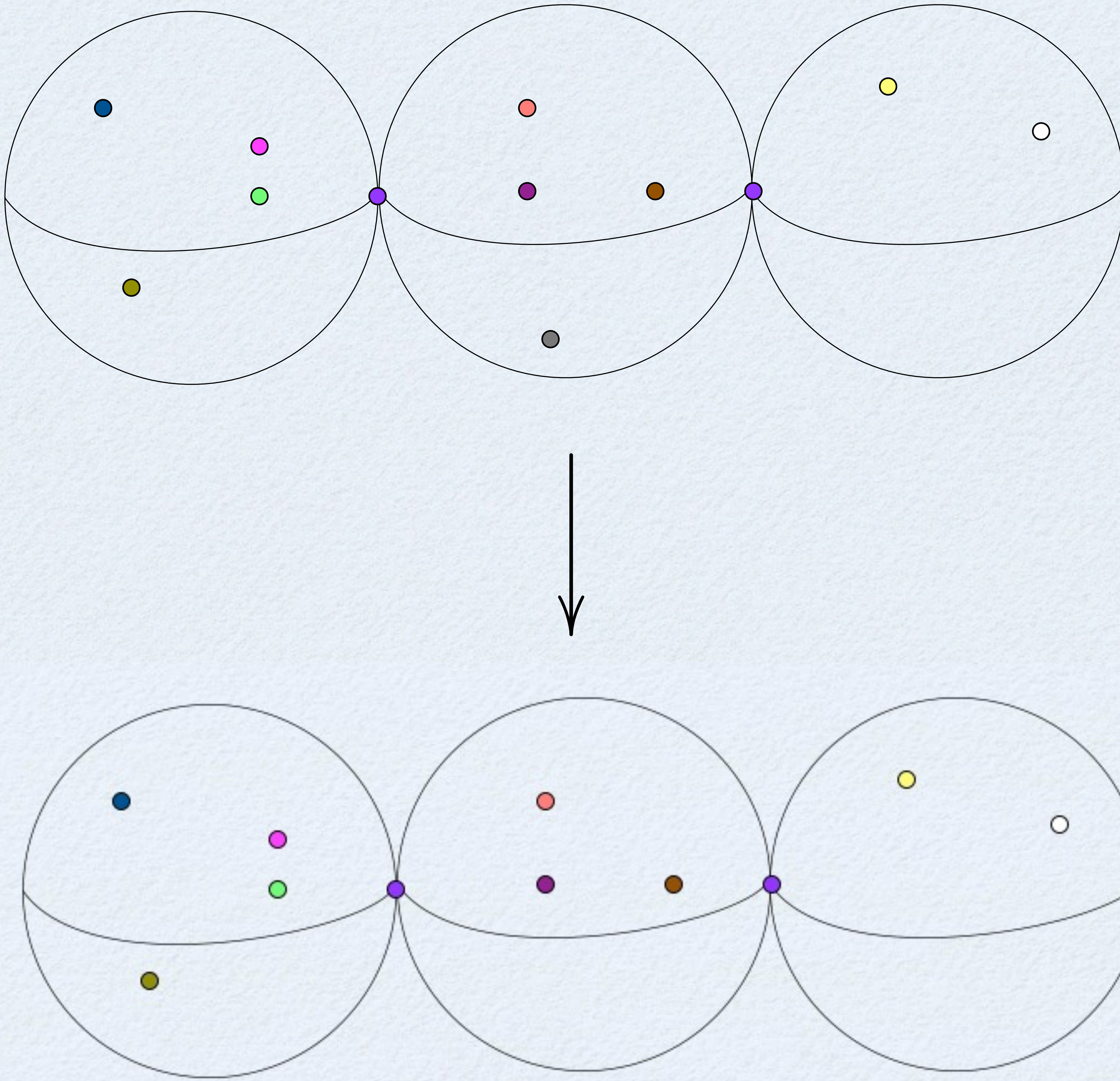
Theorem. (Lloyd-Philipps) The space $\overline{\mathcal{M}}_n$ is isomorphic to the "sequential blow up" of $\mathbb{P}^{n-3}$ described above.











$$\overline{\mathcal{M}}_{n+1} \longrightarrow \overline{\mathcal{M}}_n$$

$$\begin{tikzcd}[color=red] \star \arrow[r, "d"] & \cdot \arrow[r] & \cdots\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[llll, bend left] \end{tikzcd}$$

$$\begin{tikzcd}[color=blue] \star \arrow[r, "d"] & \cdot \arrow[r] & \cdots\cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[llll, bend left] \end{tikzcd}$$

Then a moduli space map exists $g : \mathcal{M}_P \dashrightarrow \mathcal{M}_P$, and we examine the associated skew product

$$G : \mathbb{C}^n \to \mathbb{C}^n \quad \text{given by} \quad G : \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \mapsto \begin{pmatrix} F_{\mathbf{x}}(t) \\ g(\mathbf{x}) \end{pmatrix}$$

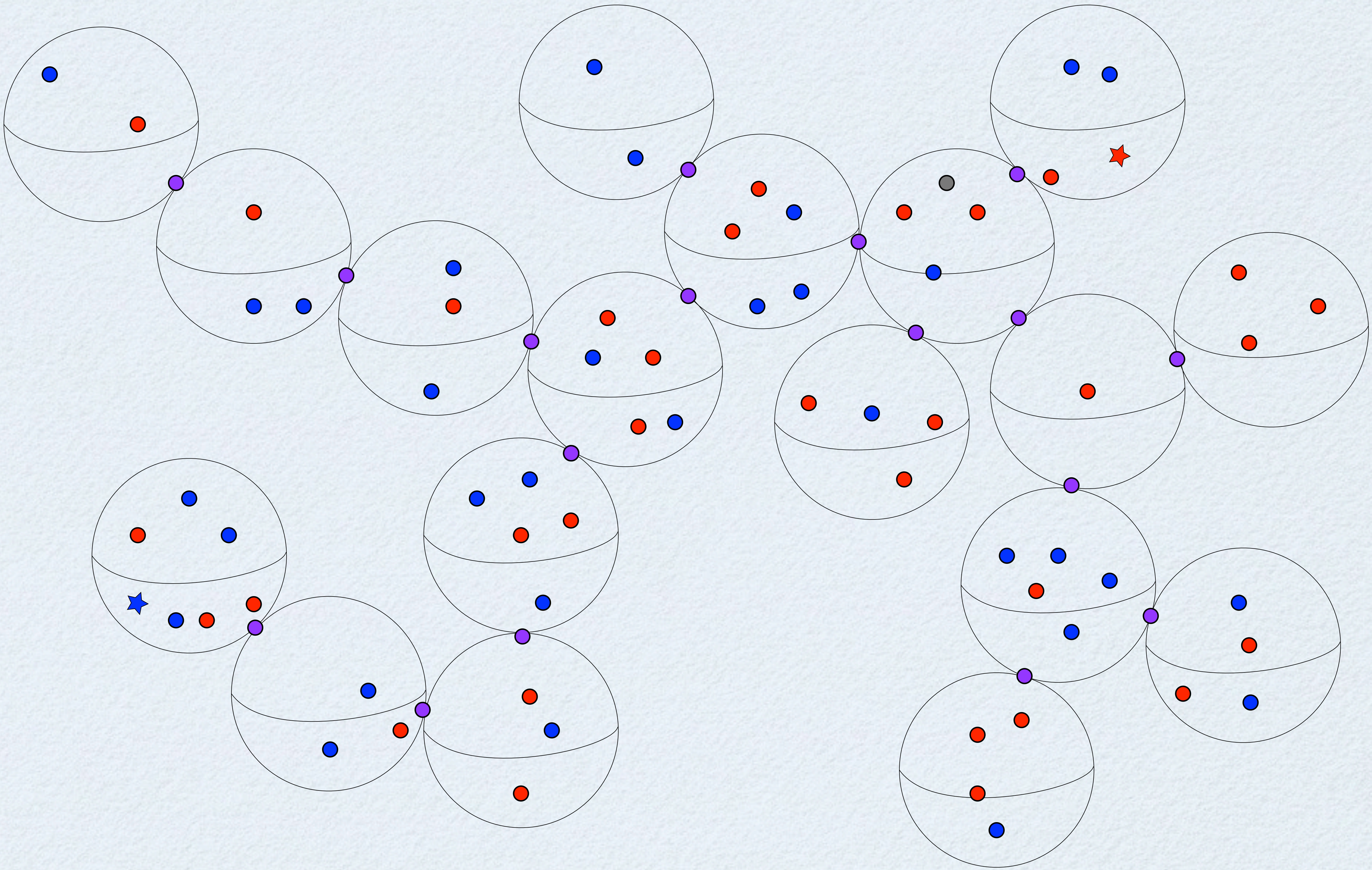
$$G : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \qquad\qquad \widetilde{G} : \widetilde{\mathbb{P}}^n \dashrightarrow \widetilde{\mathbb{P}}^n$$

This map $\widetilde{G}$ preserves this fibration

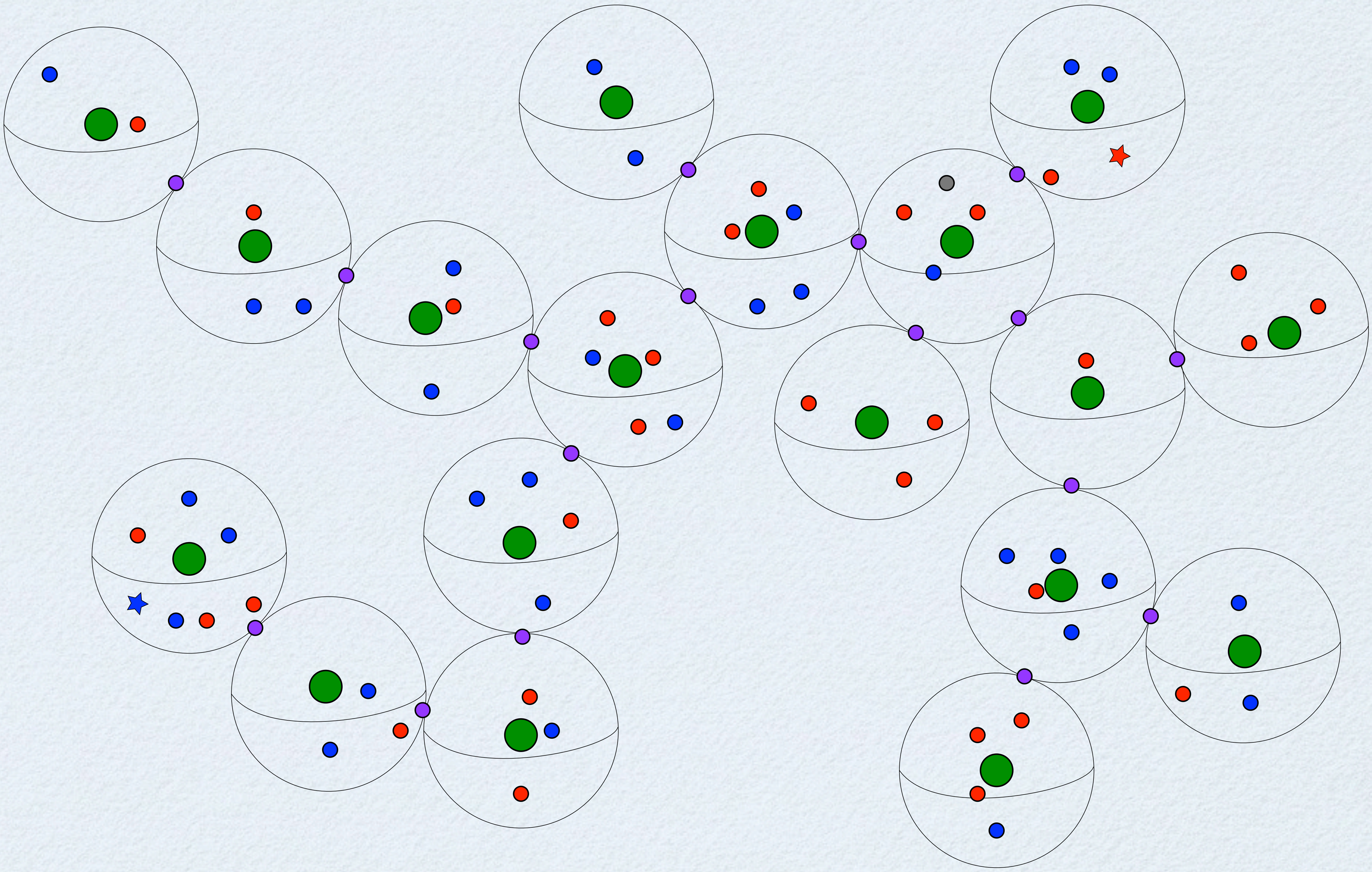
$$\overline{\mathcal{M}}_{n+3} \to \overline{\mathcal{M}}_{n+2}, \qquad \widetilde{\mathbb{P}}^n \to \widetilde{\mathbb{P}}^{n-1}$$

$$\begin{tikzcd} \widetilde{\mathbb{P}}^n \arrow[r, dashed, "\widetilde{G}"] \arrow[d] & \widetilde{\mathbb{P}}^n \arrow[d] \\ \widetilde{\mathbb{P}}^{n-1} \arrow[r, dashed, "\widetilde{g}"] & \widetilde{\mathbb{P}}^{n-1} \end{tikzcd}$$

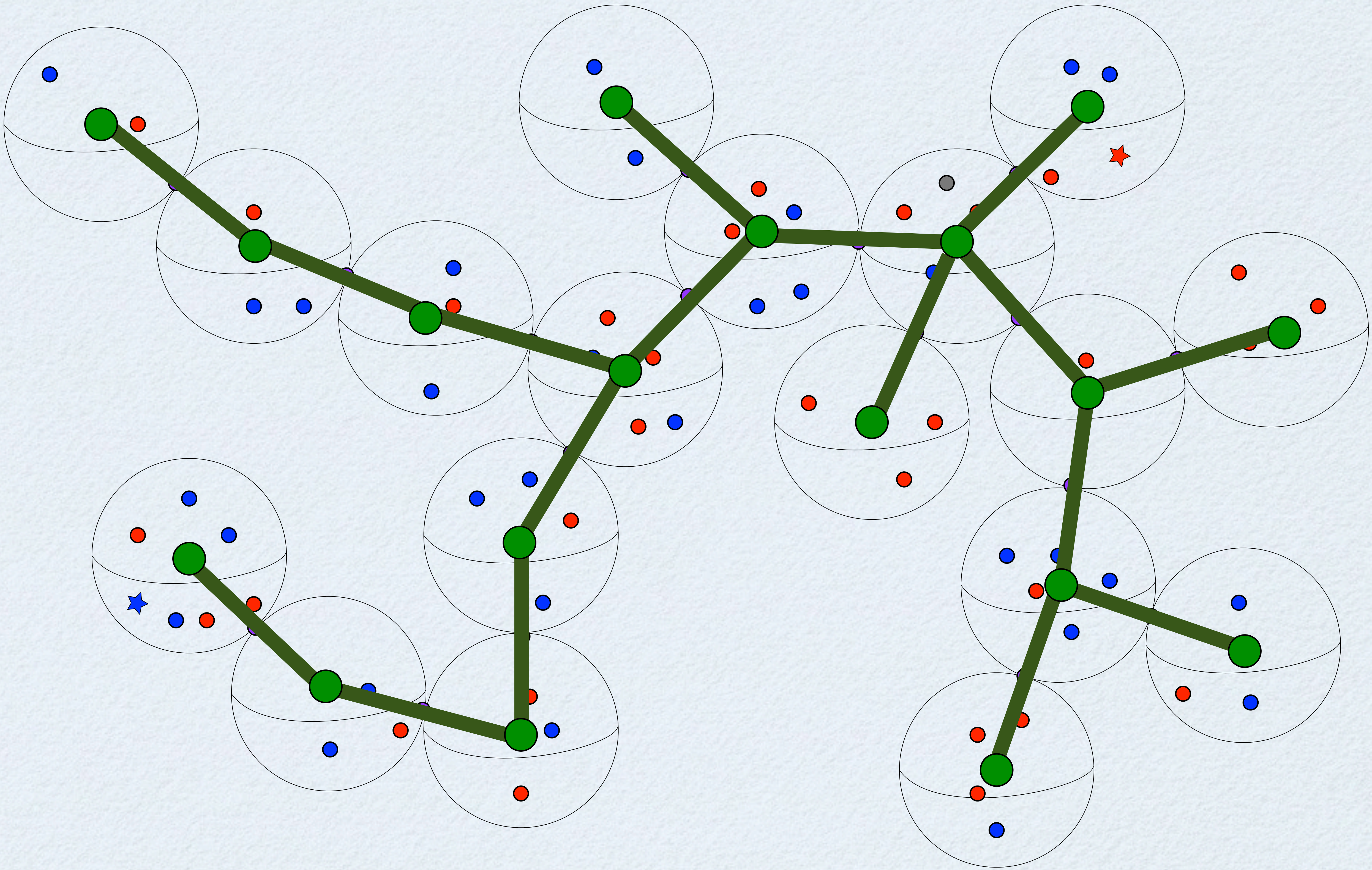
Trees of spheres in $\widetilde{\mathbb{P}}^n$



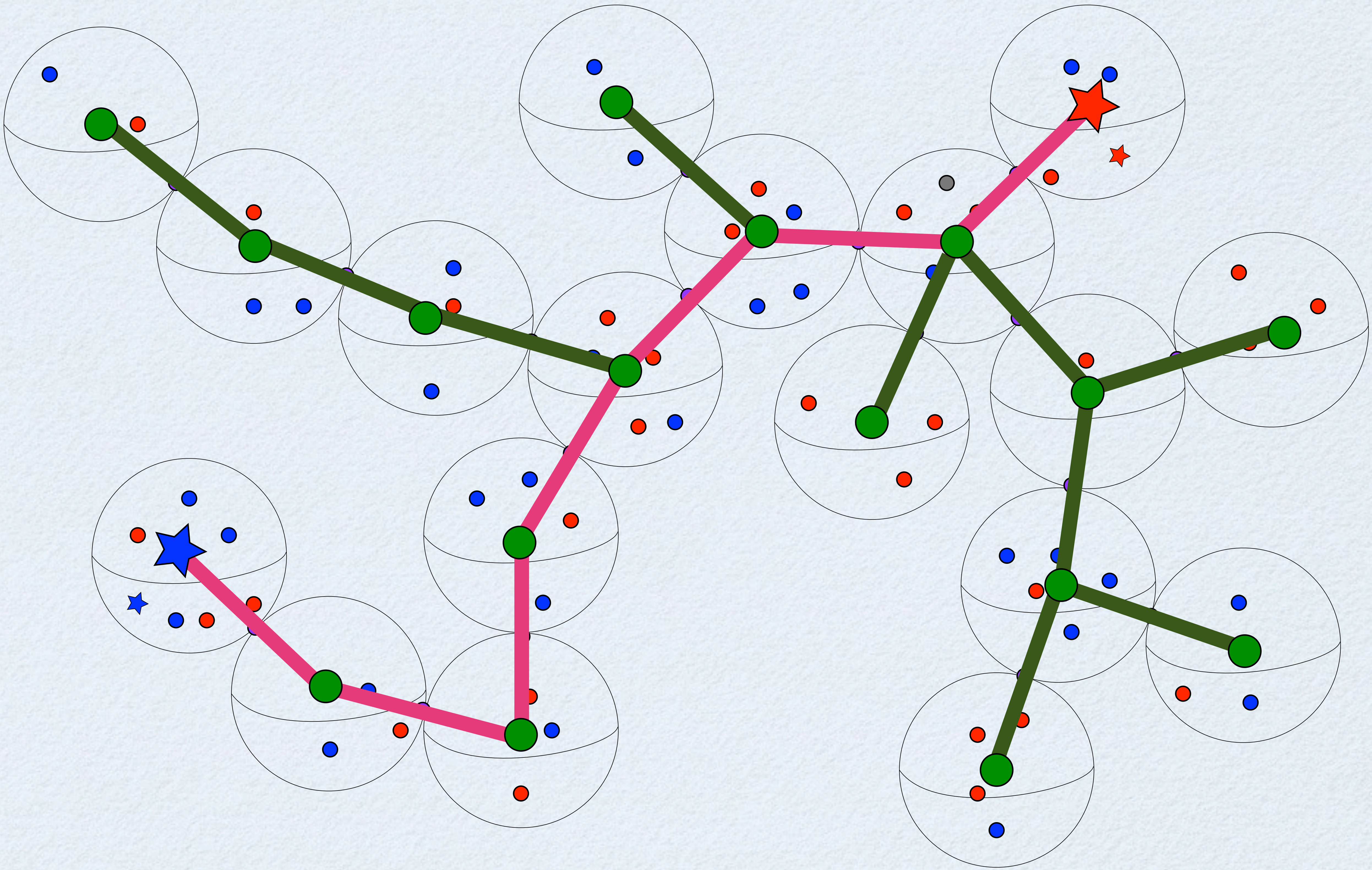
Trees of spheres in $\widetilde{\mathbb{P}}^n$



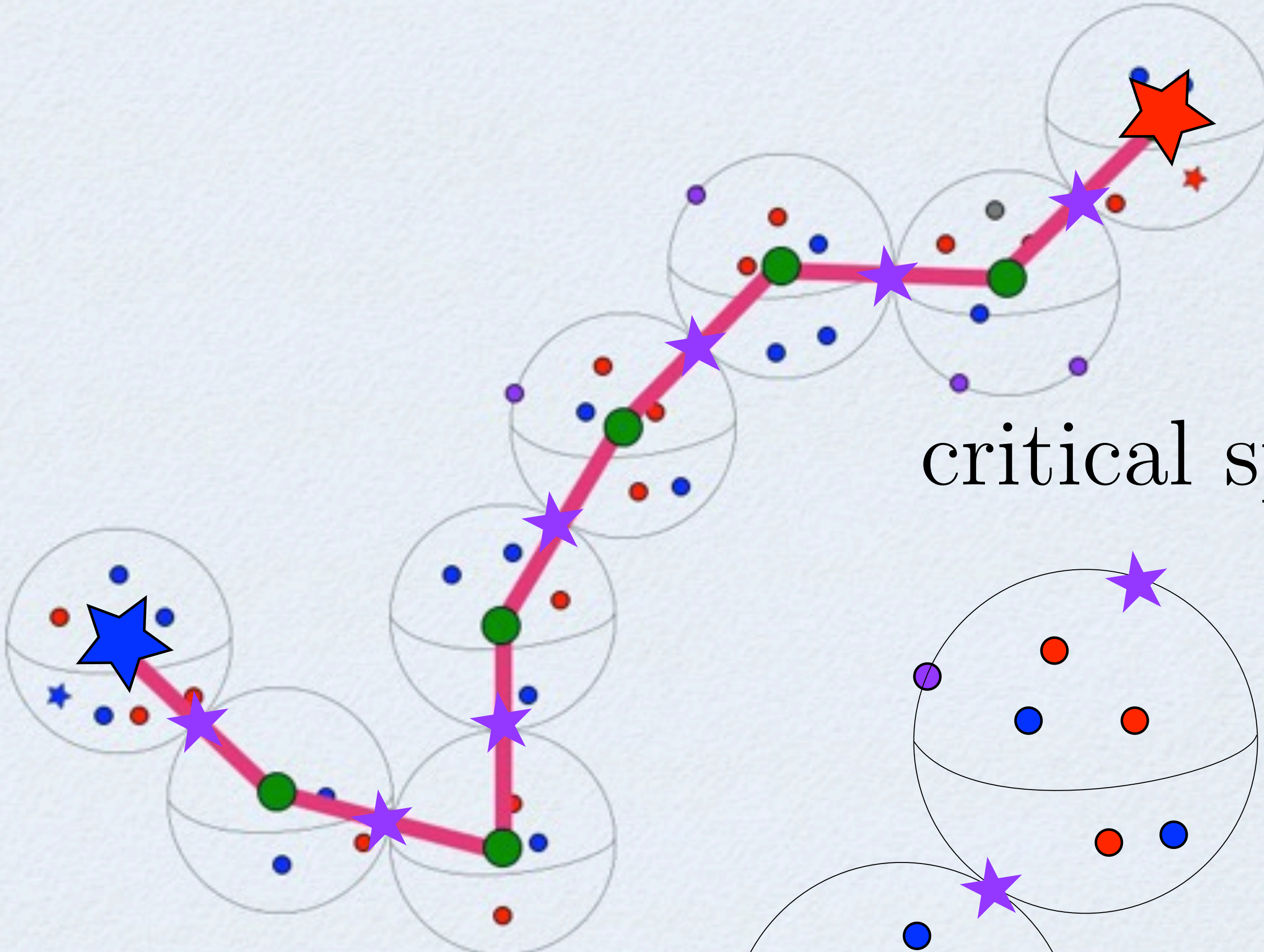
Trees of spheres in $\widetilde{\mathbb{P}}^n$



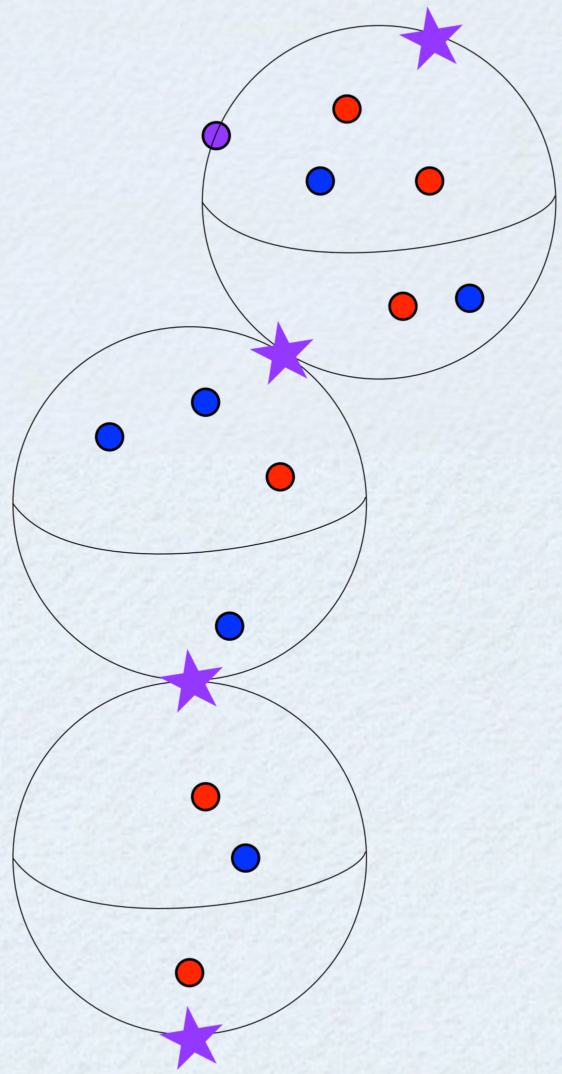
Define *critical trunk*:



Every sphere in the critical trunk is called a *critical sphere*.

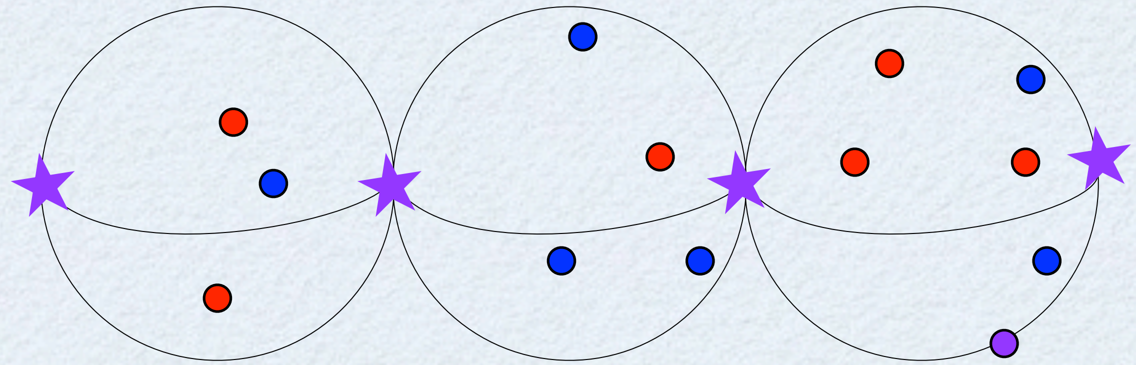


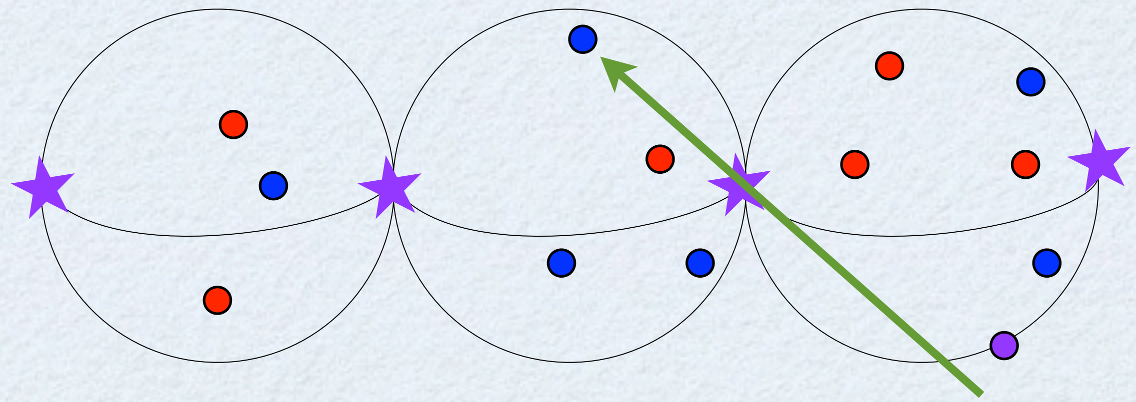
critical spheres: two distinguished points

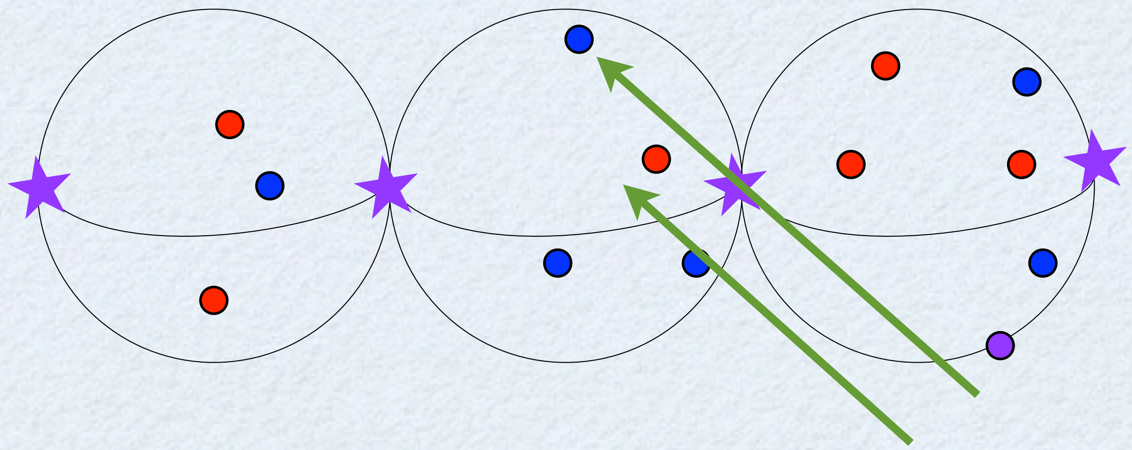


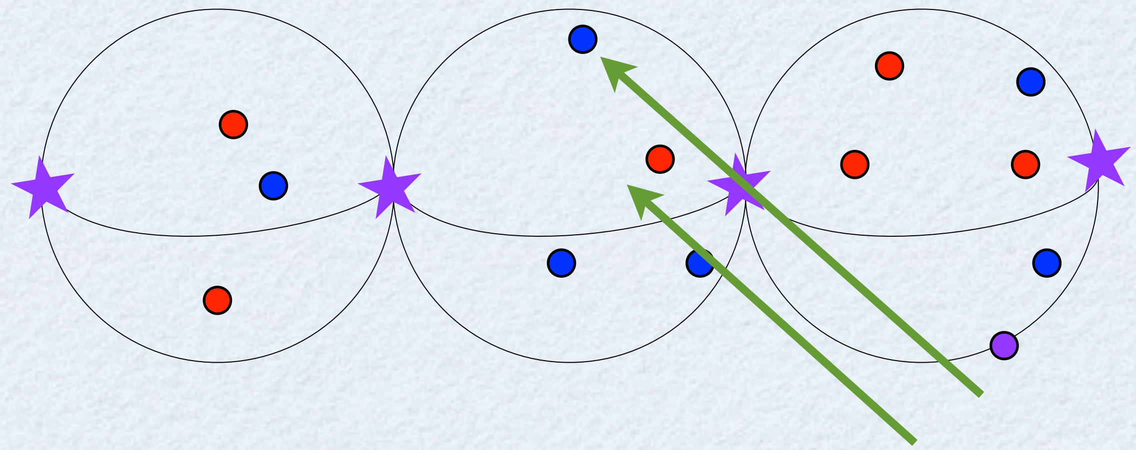
order d automorphism, σ_d

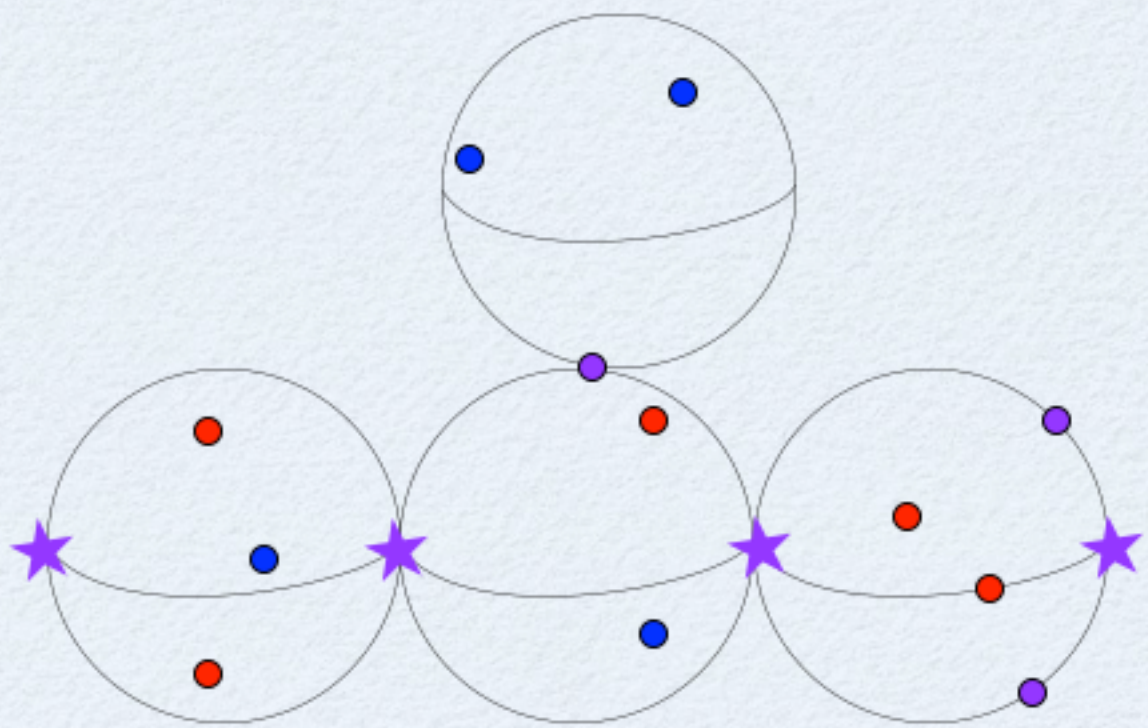
take quotient (critical trunk)/σ_d

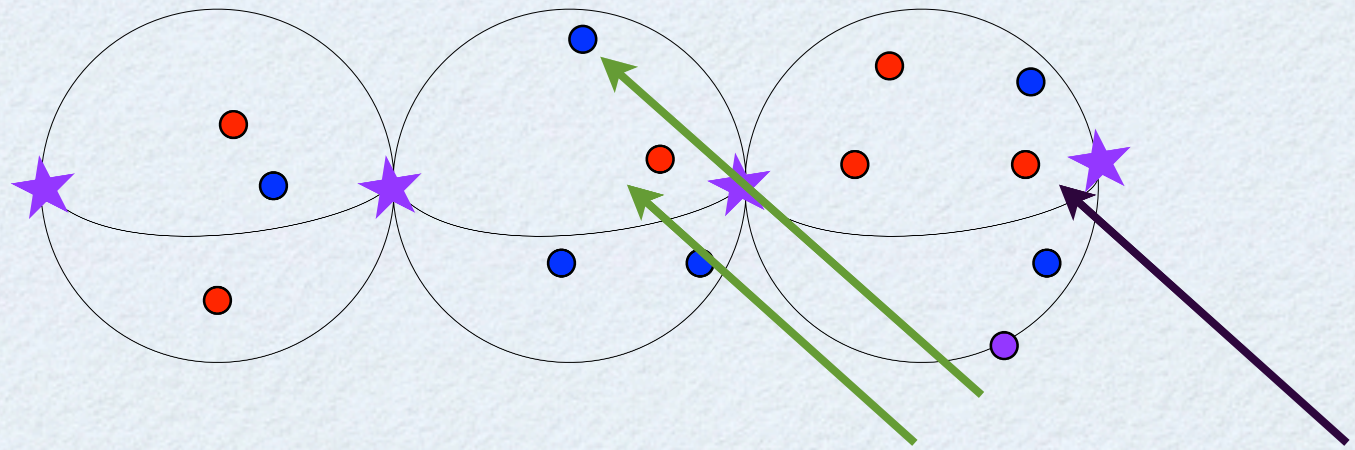


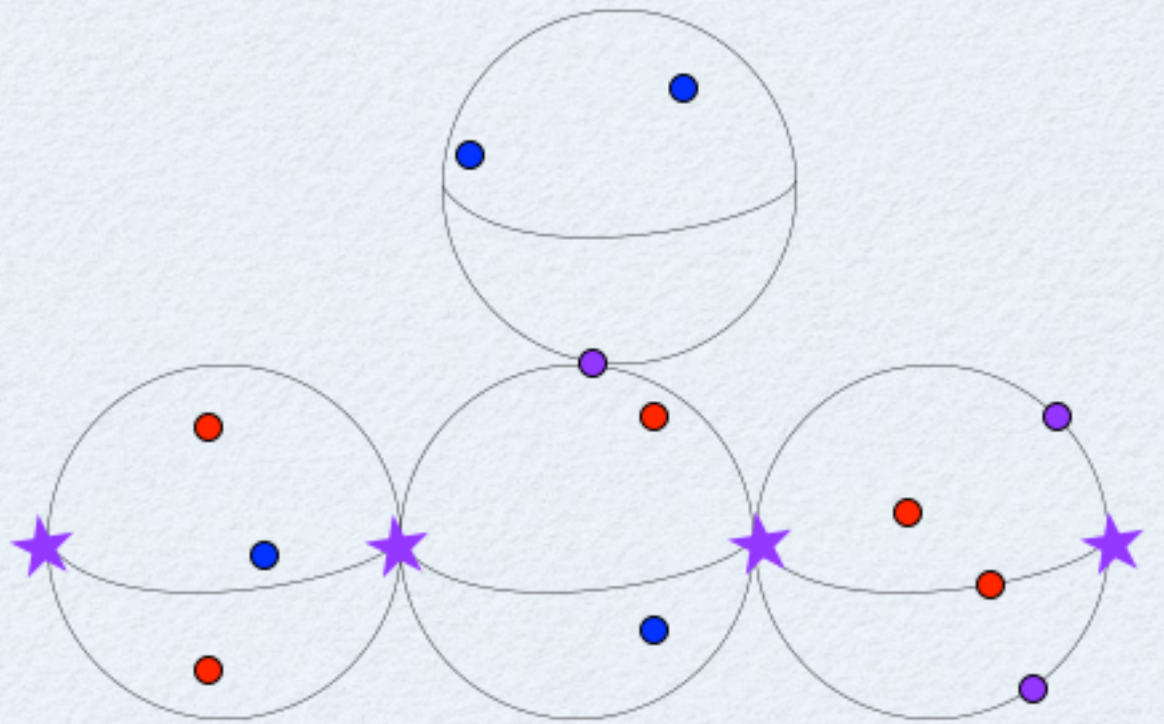


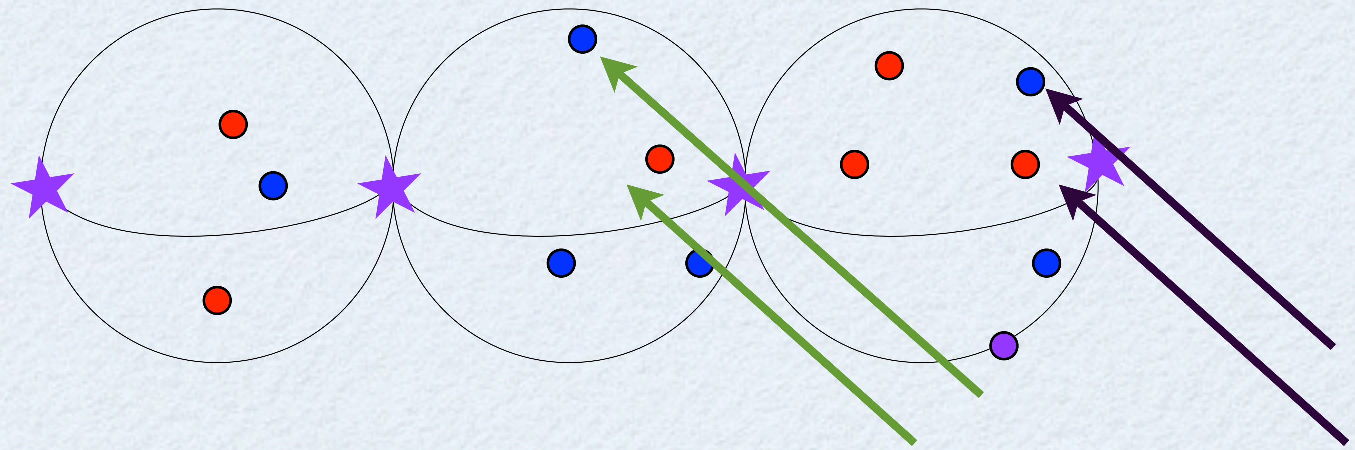


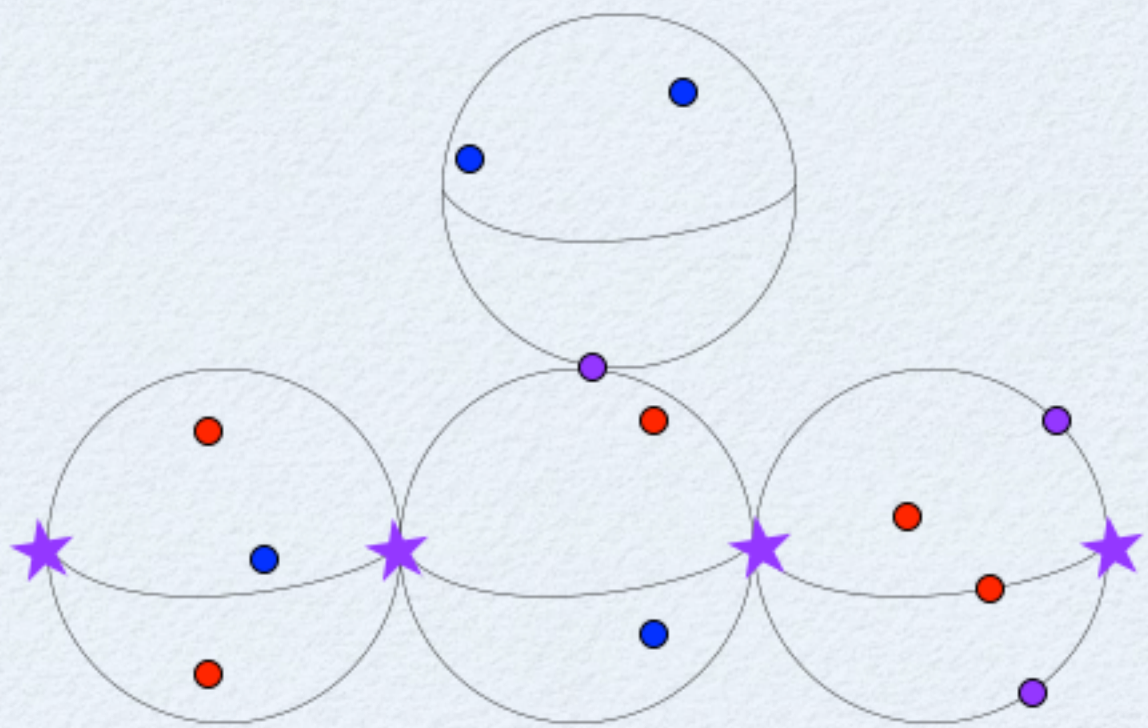


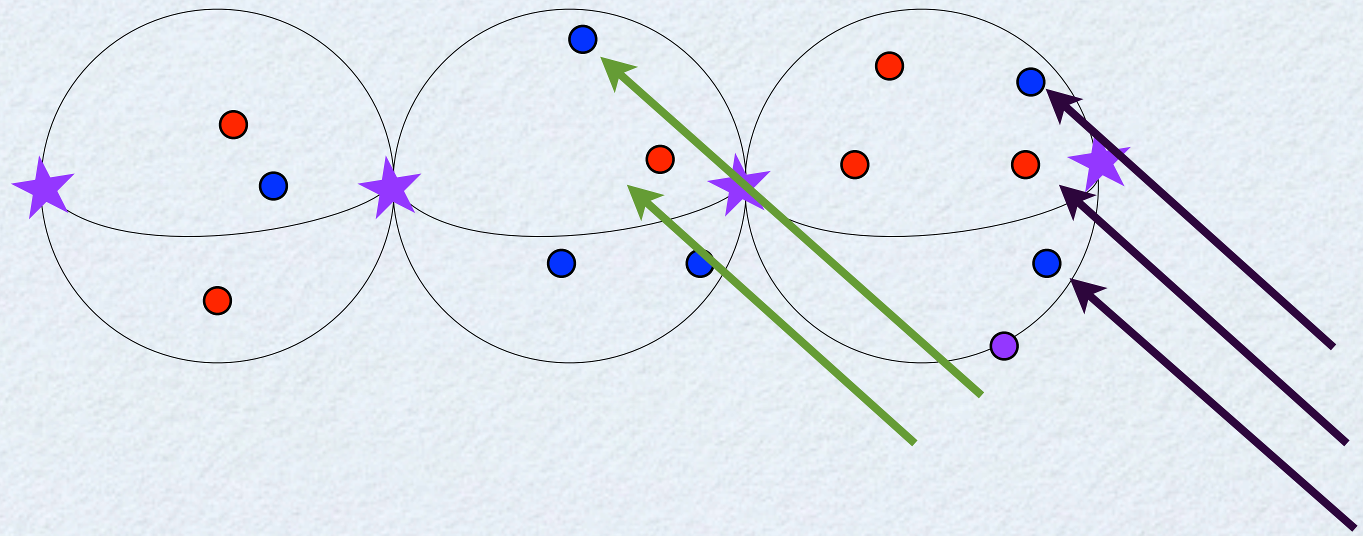


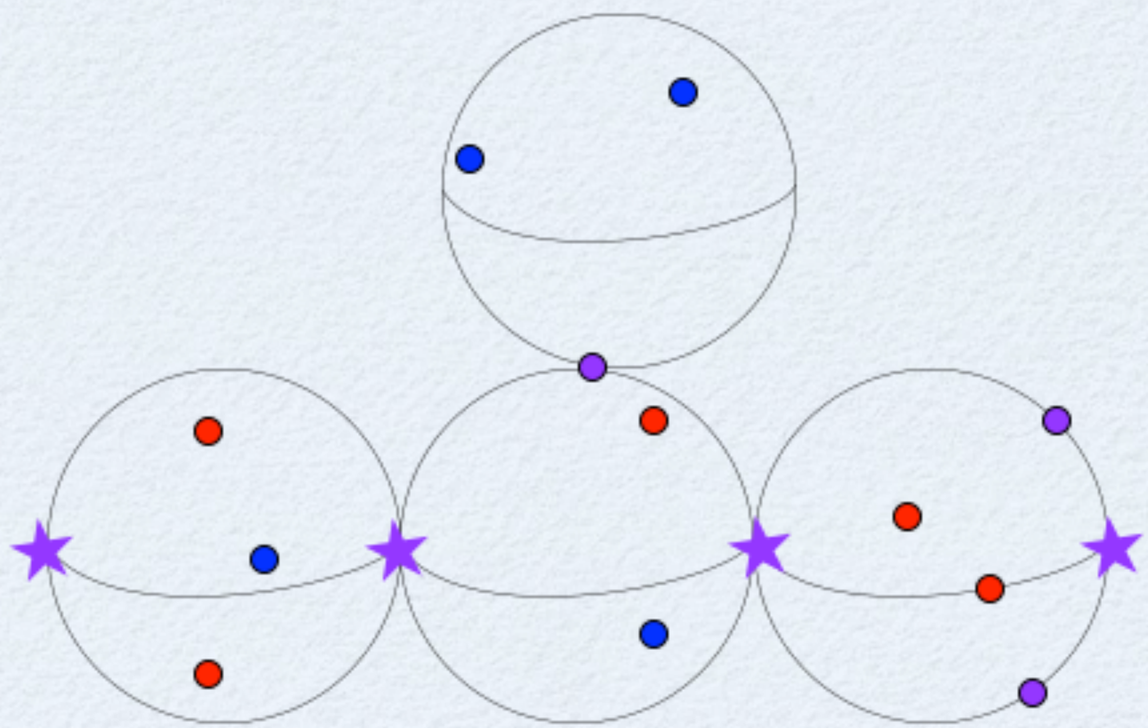


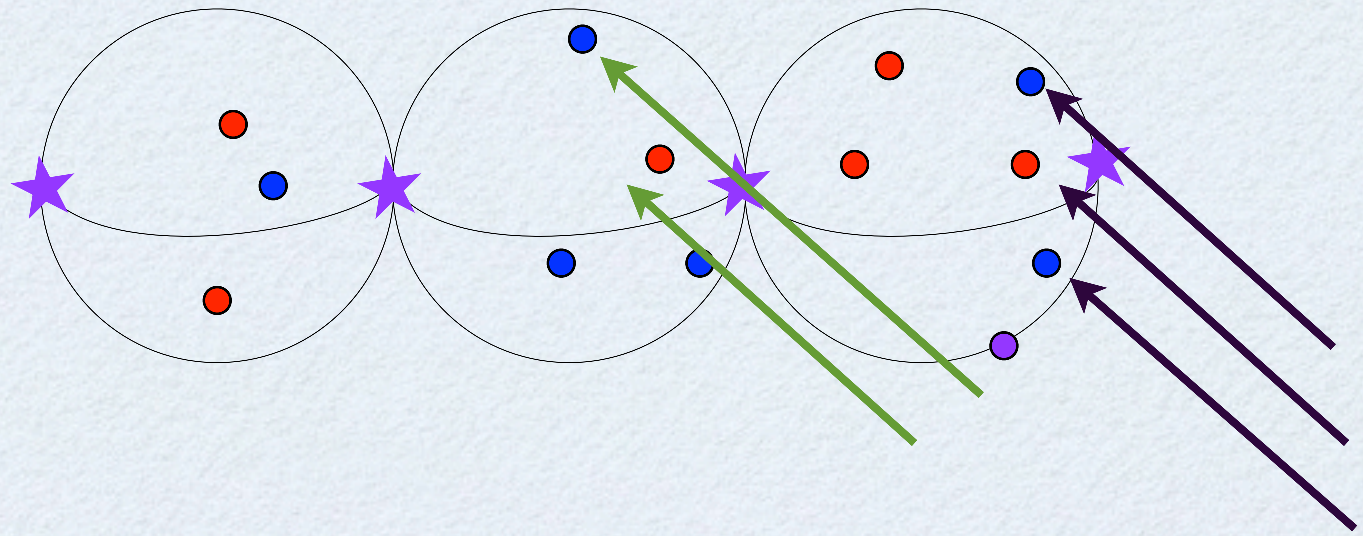


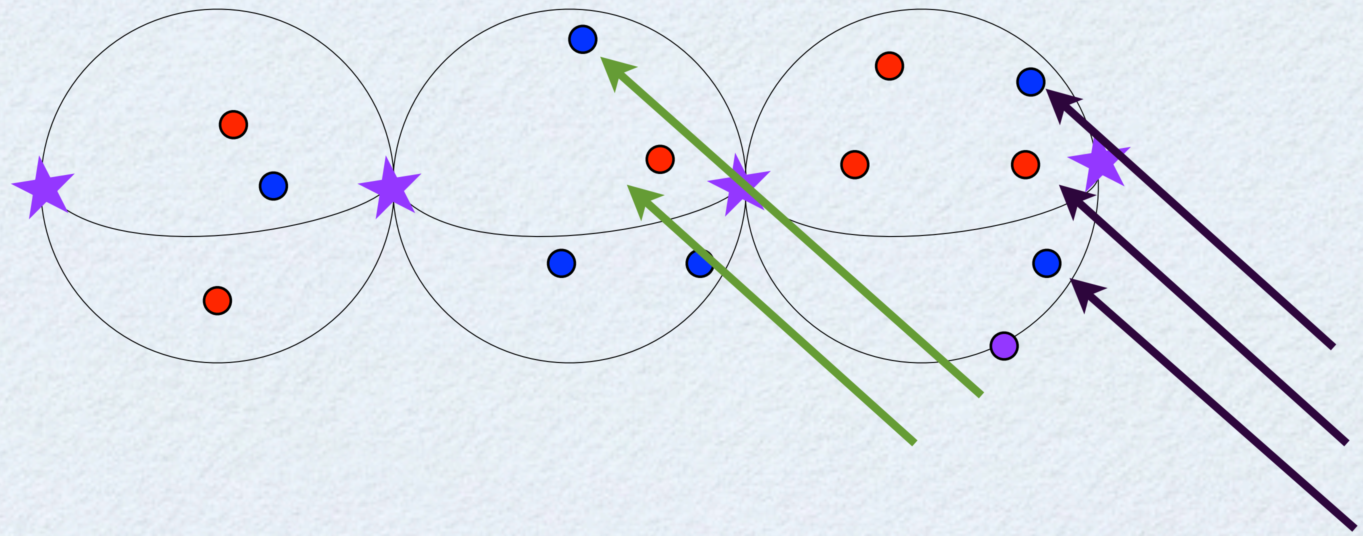


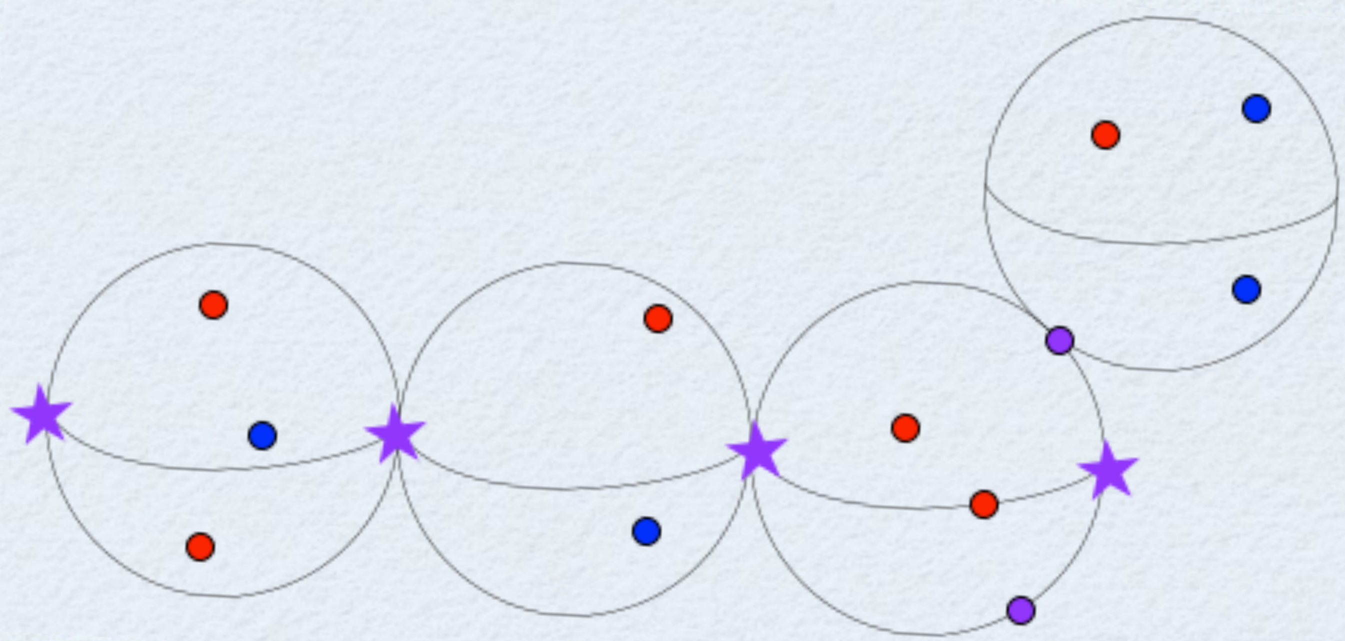


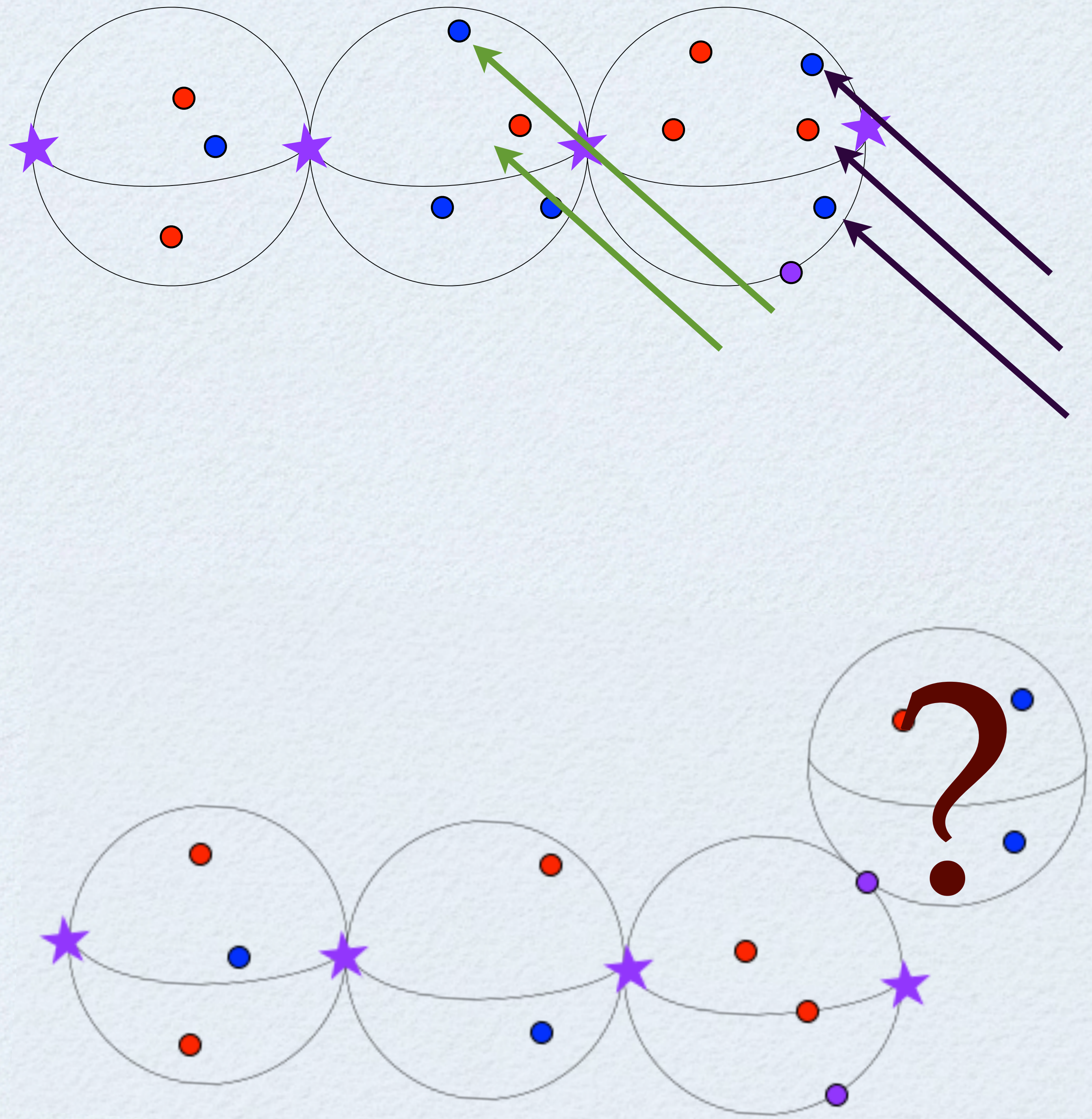


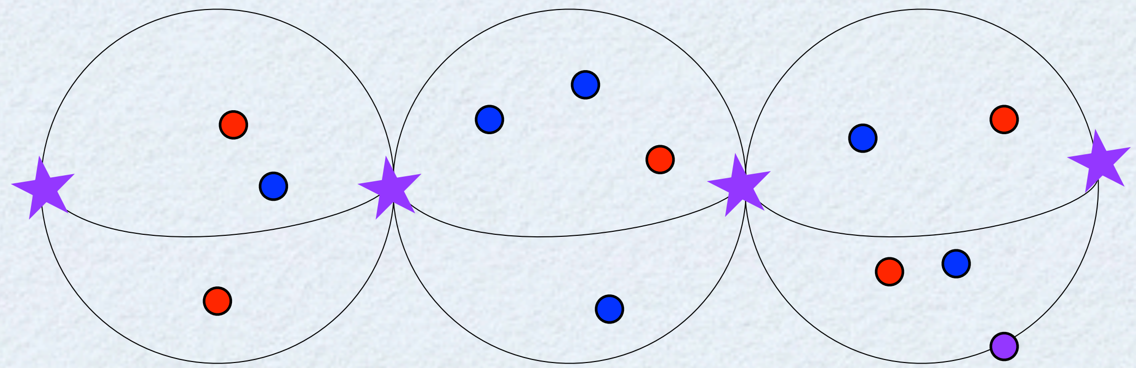


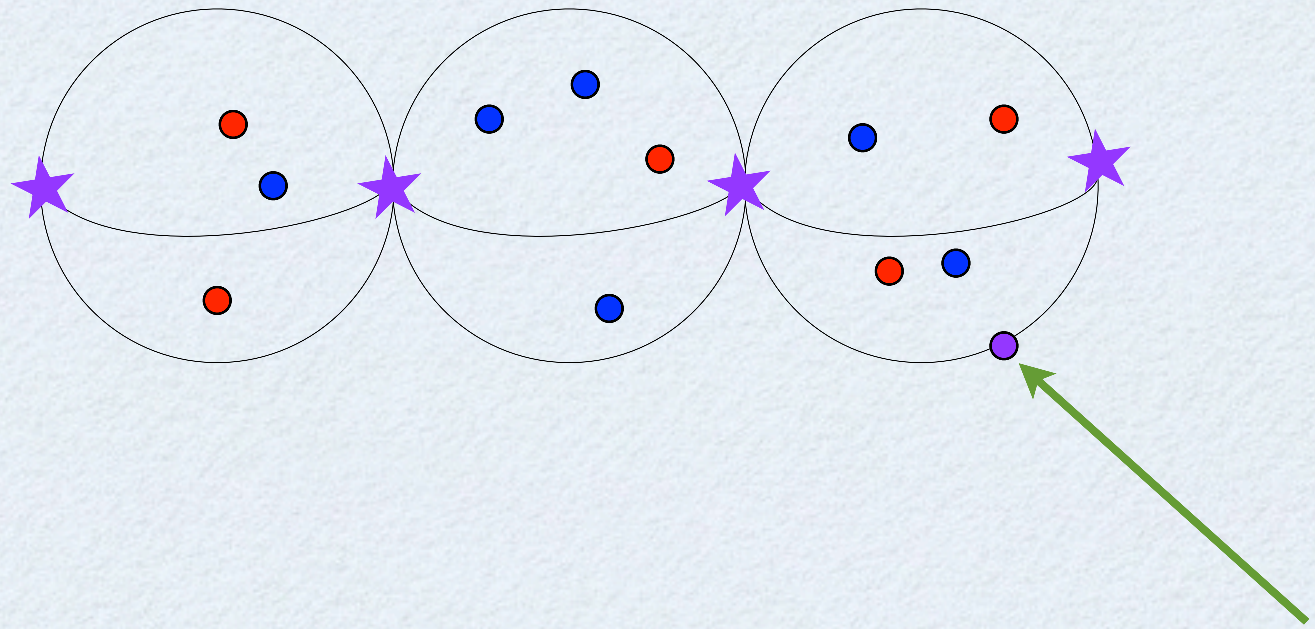


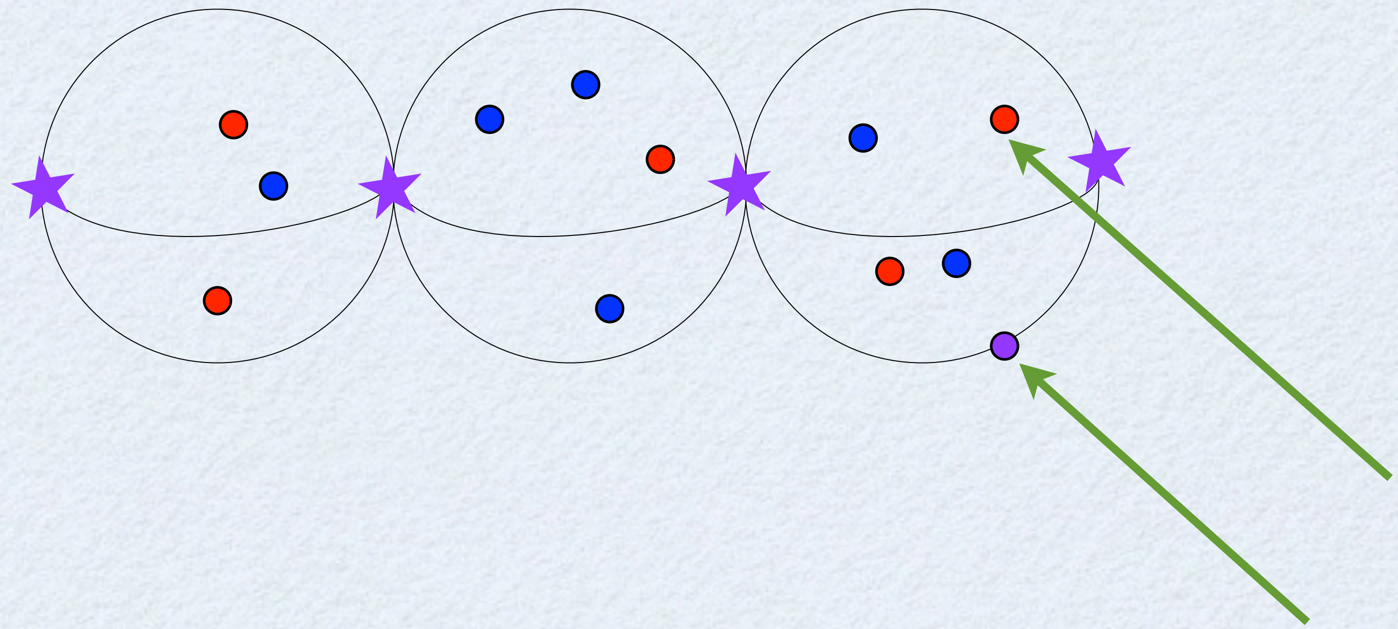


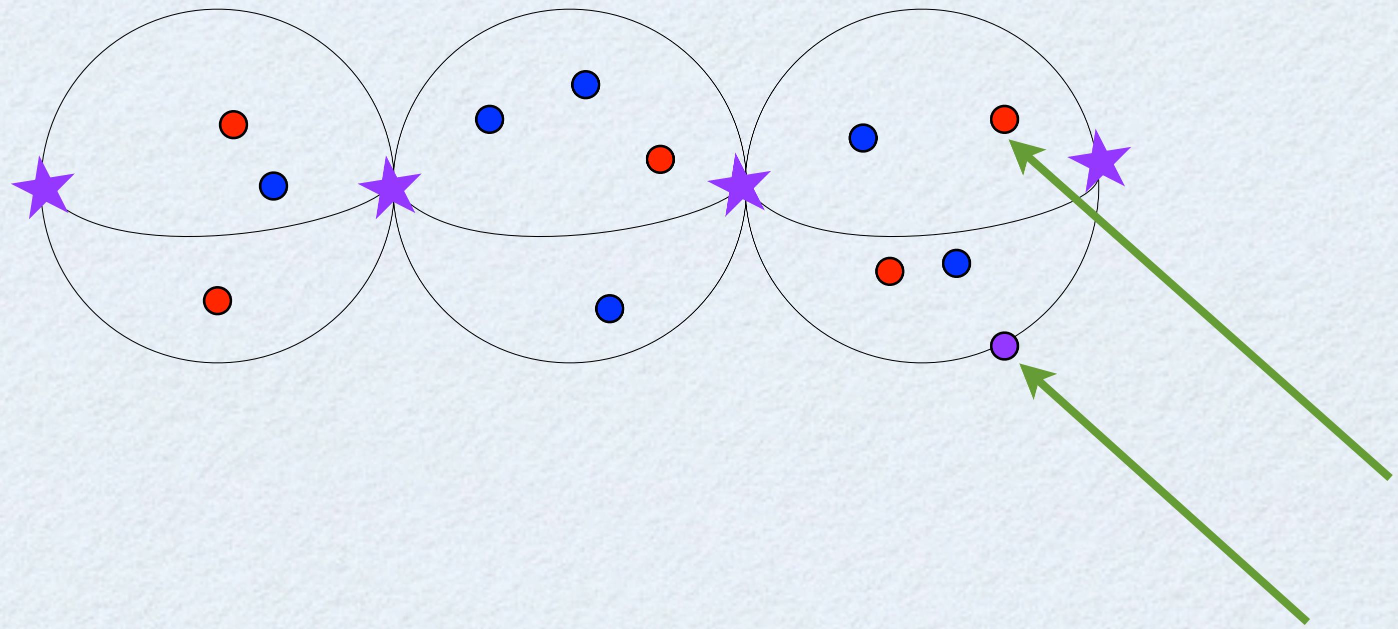


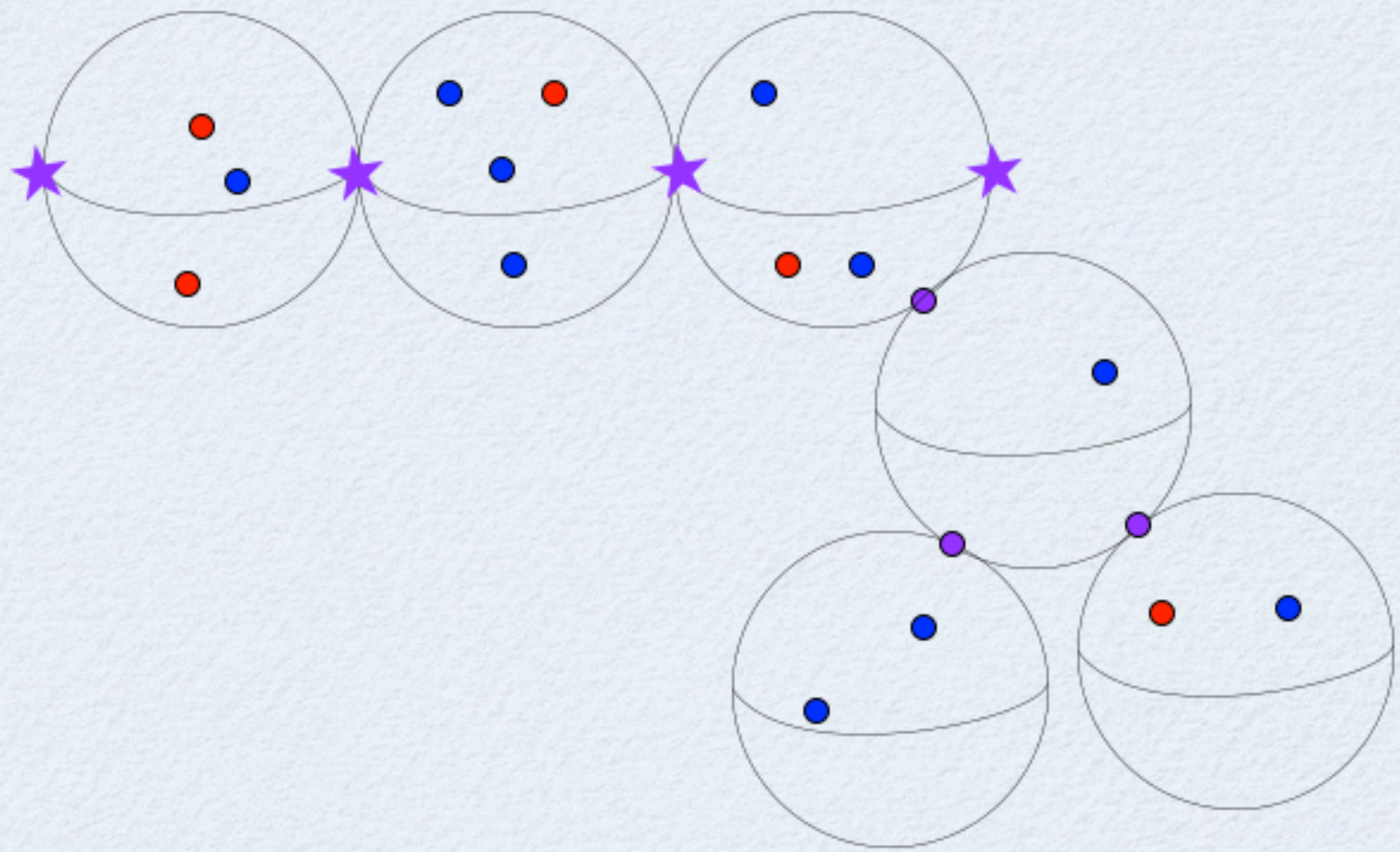


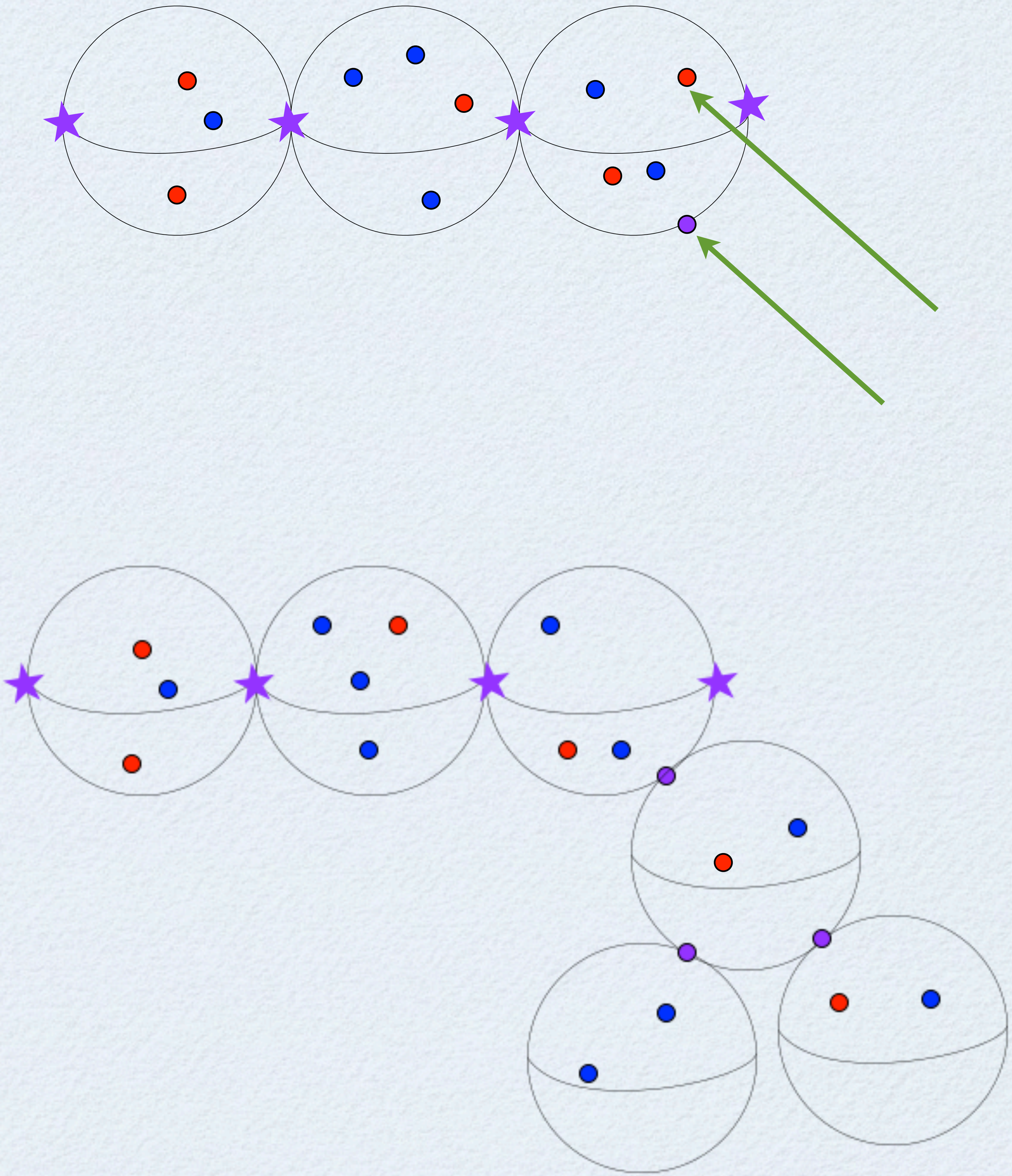


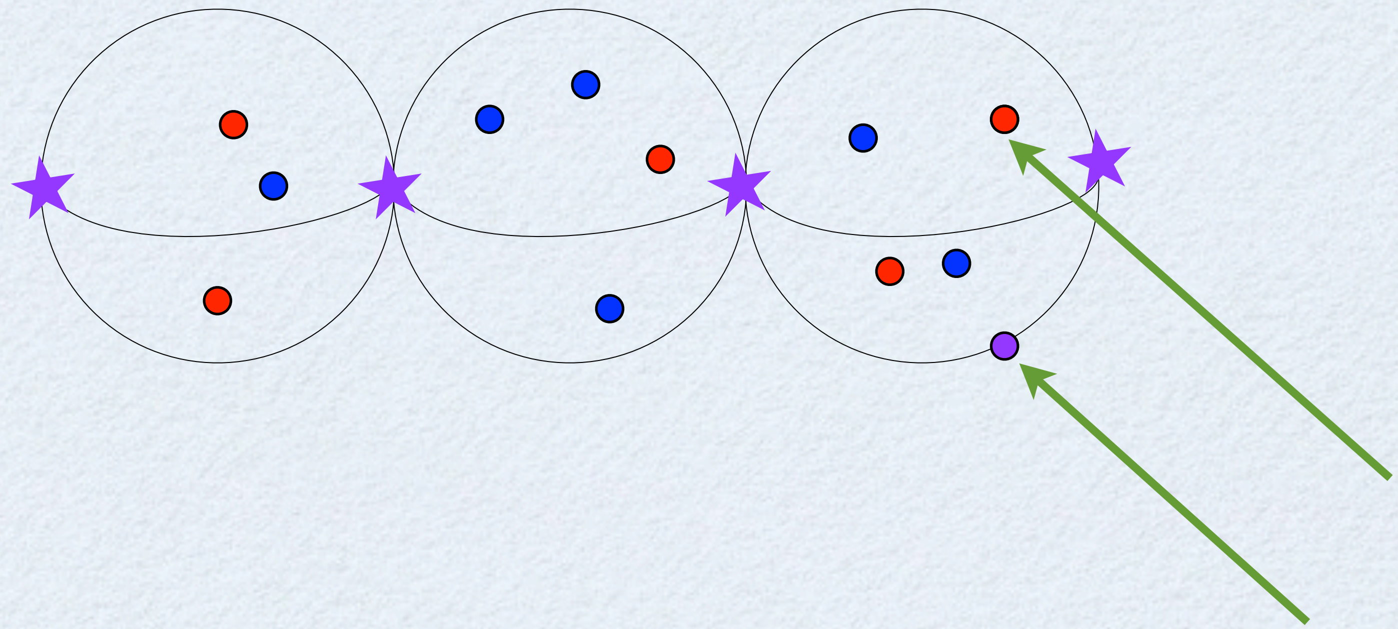


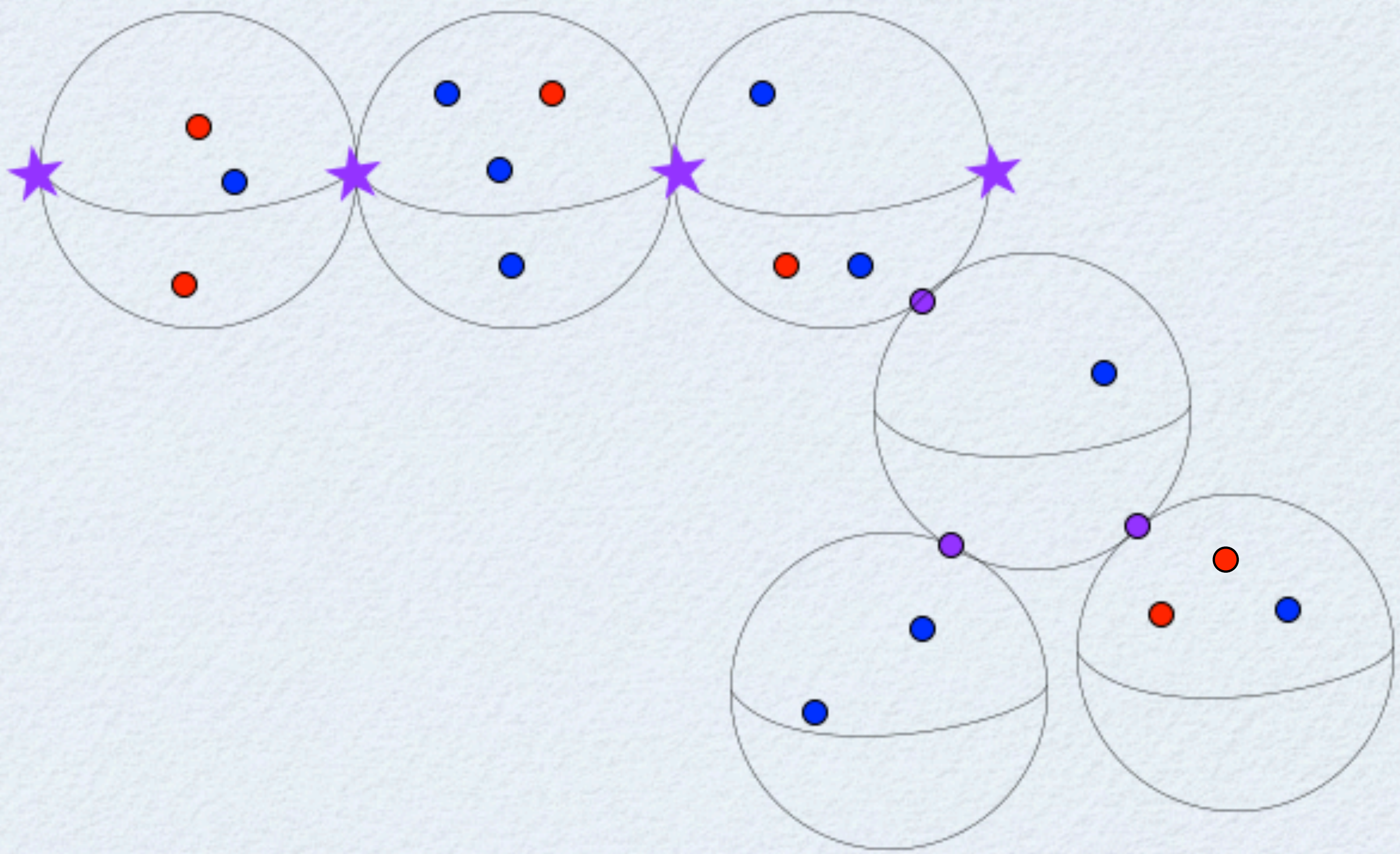


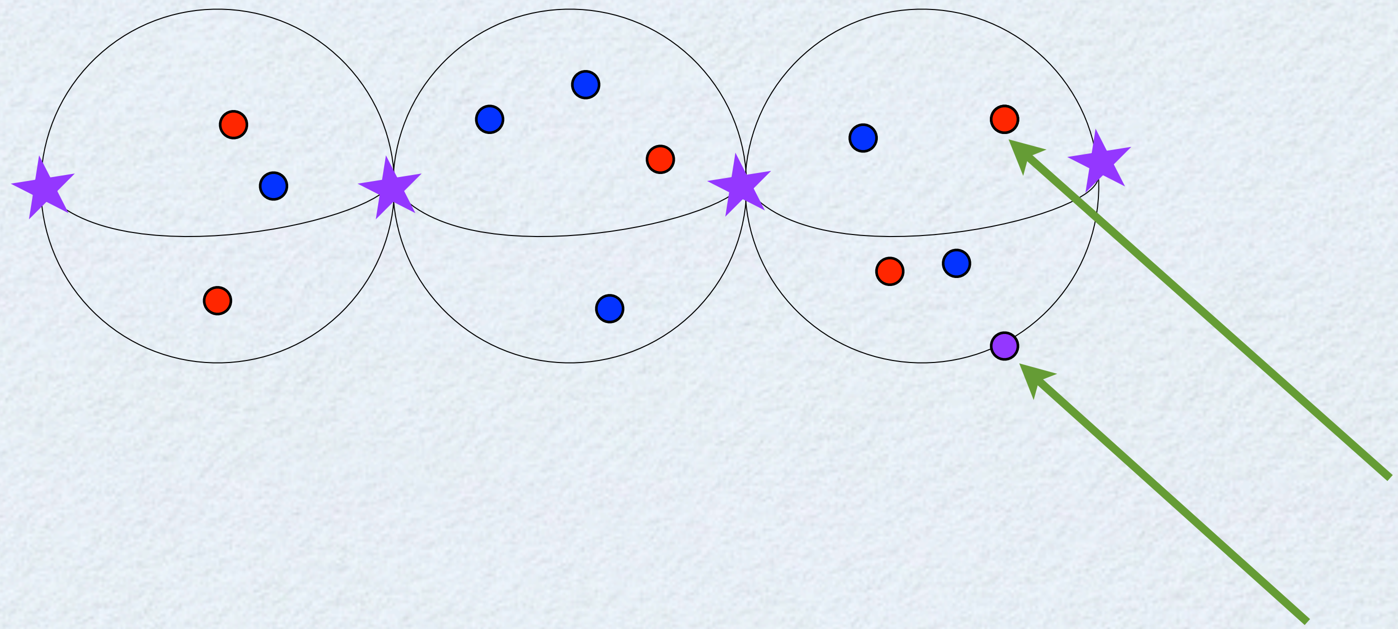


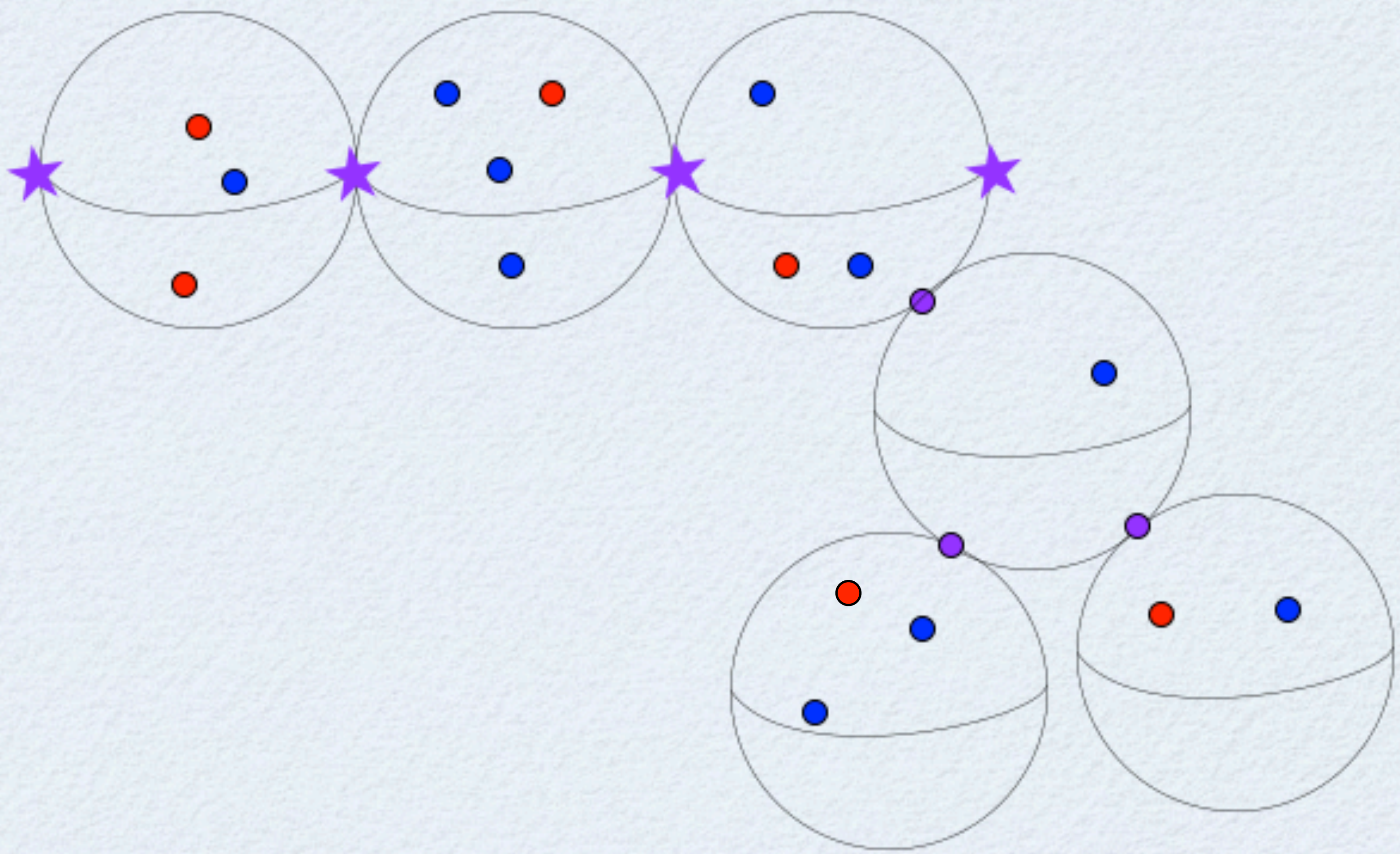


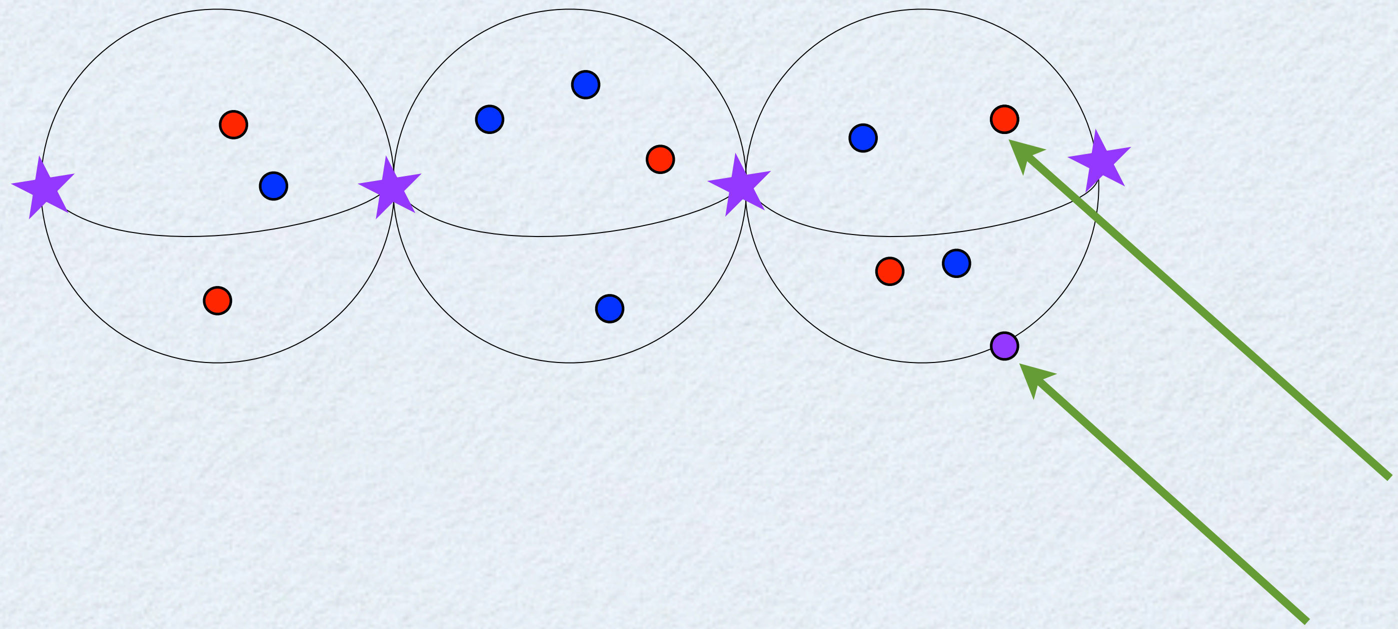


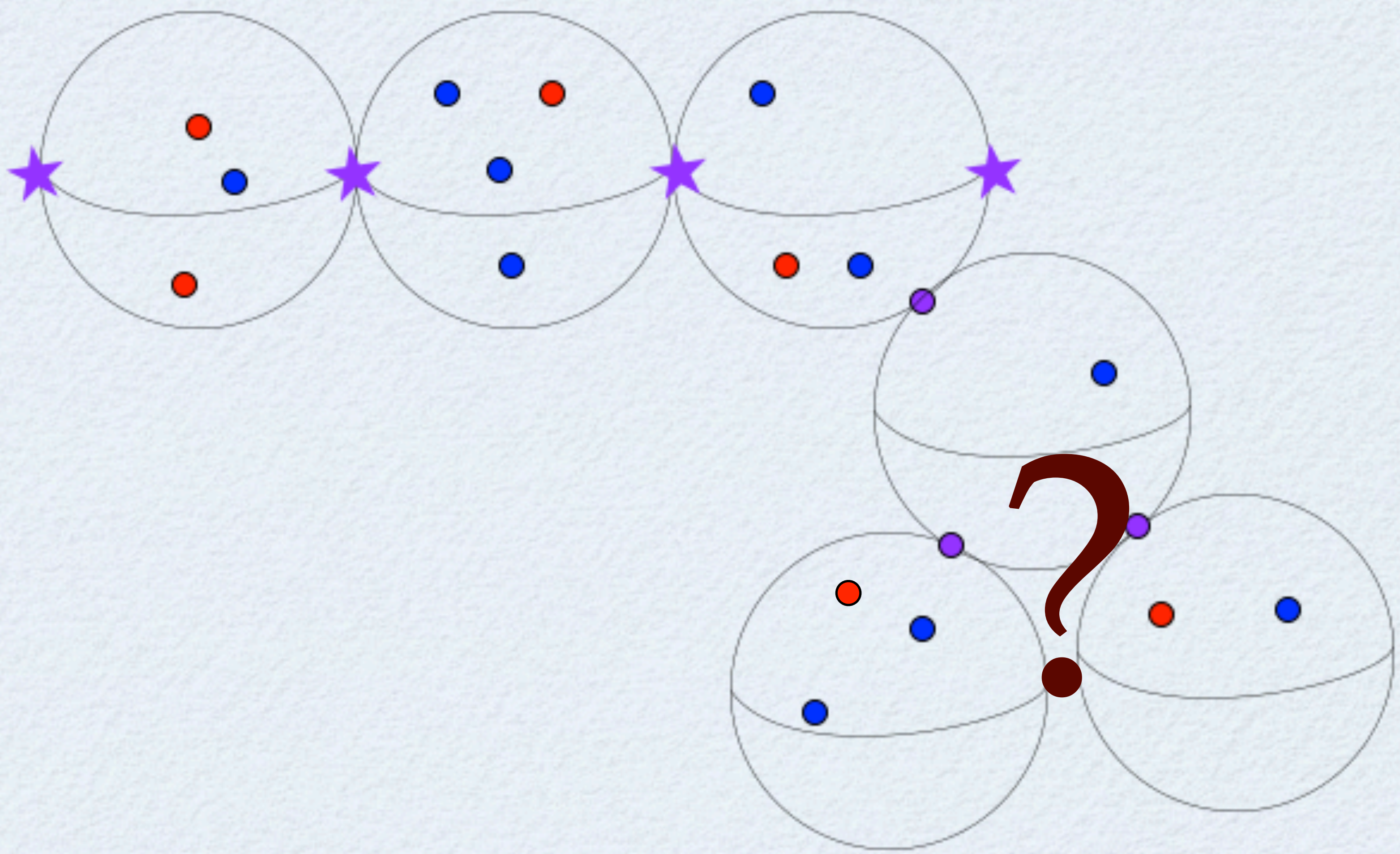


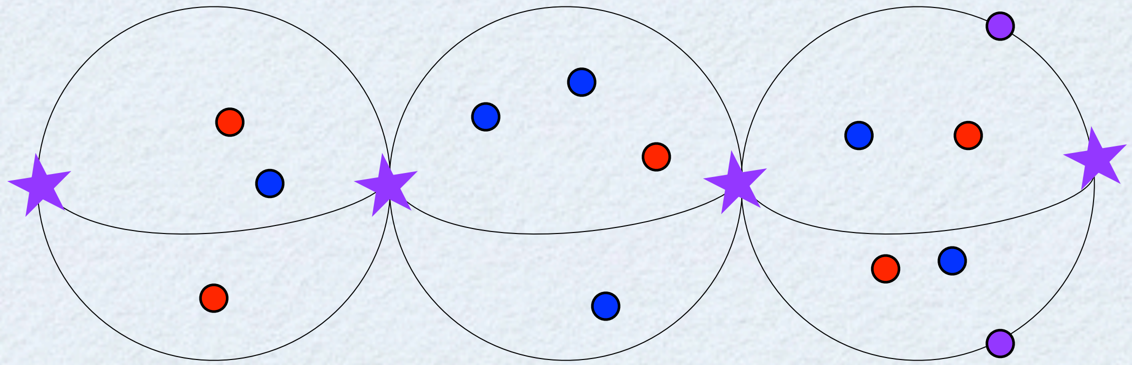


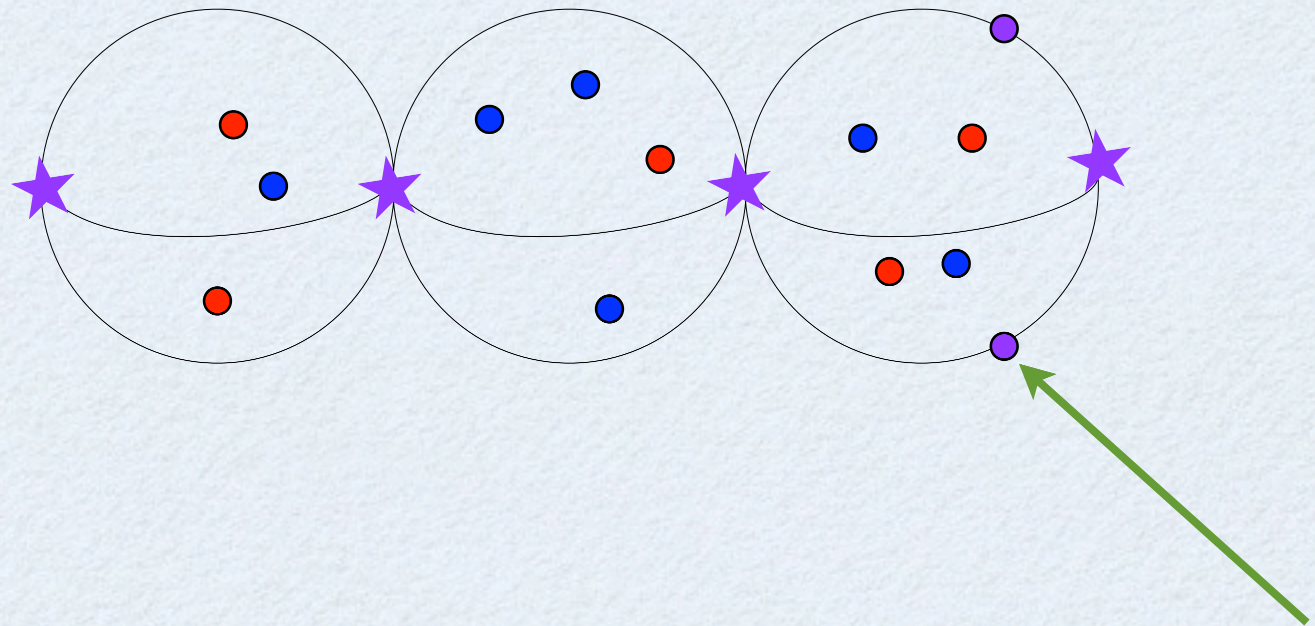


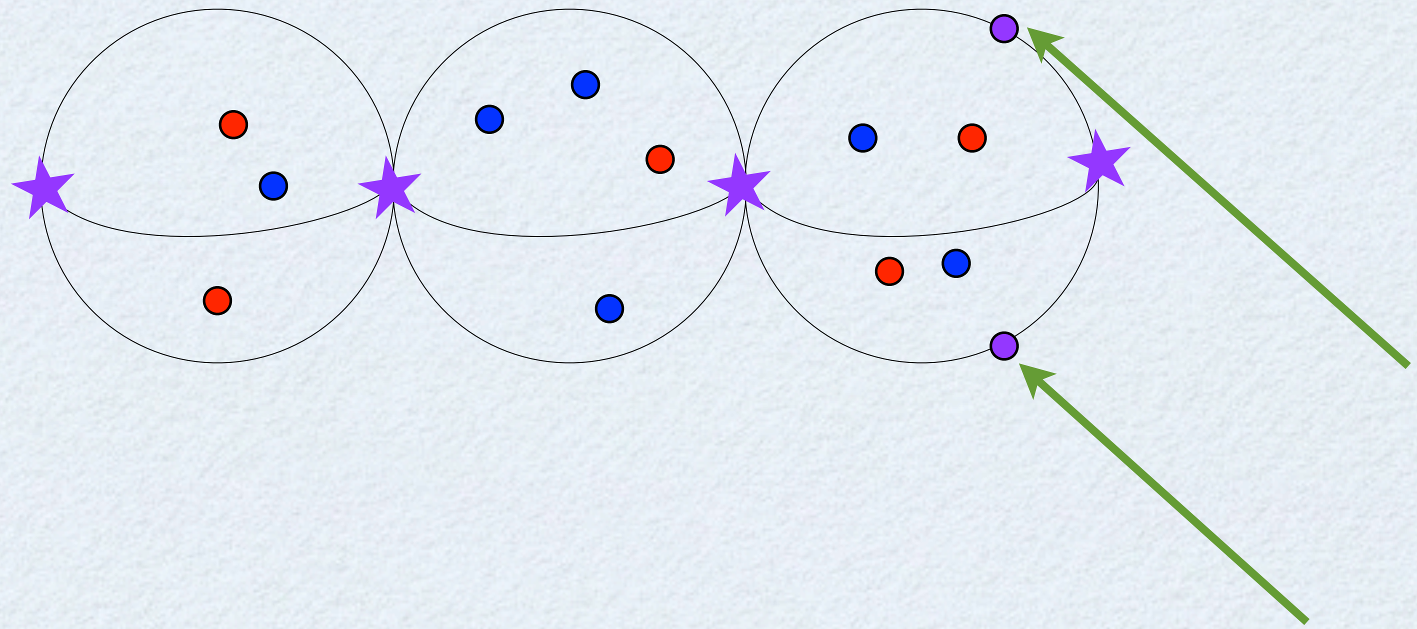


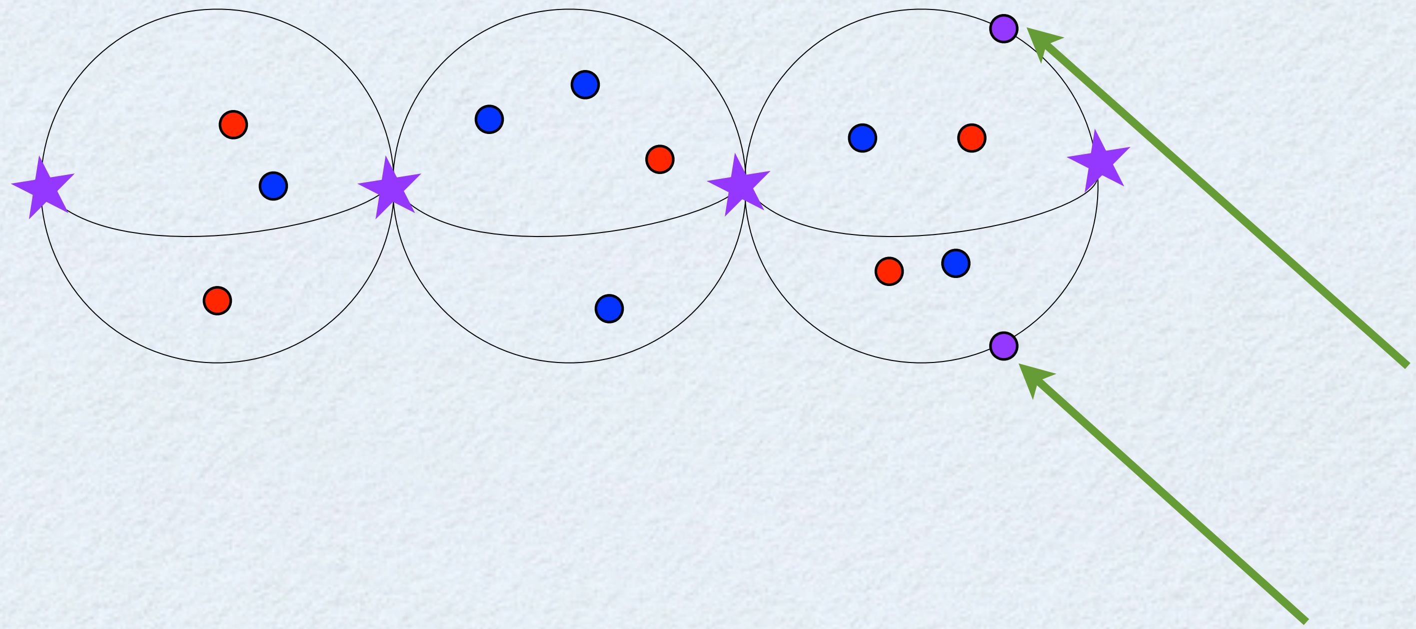


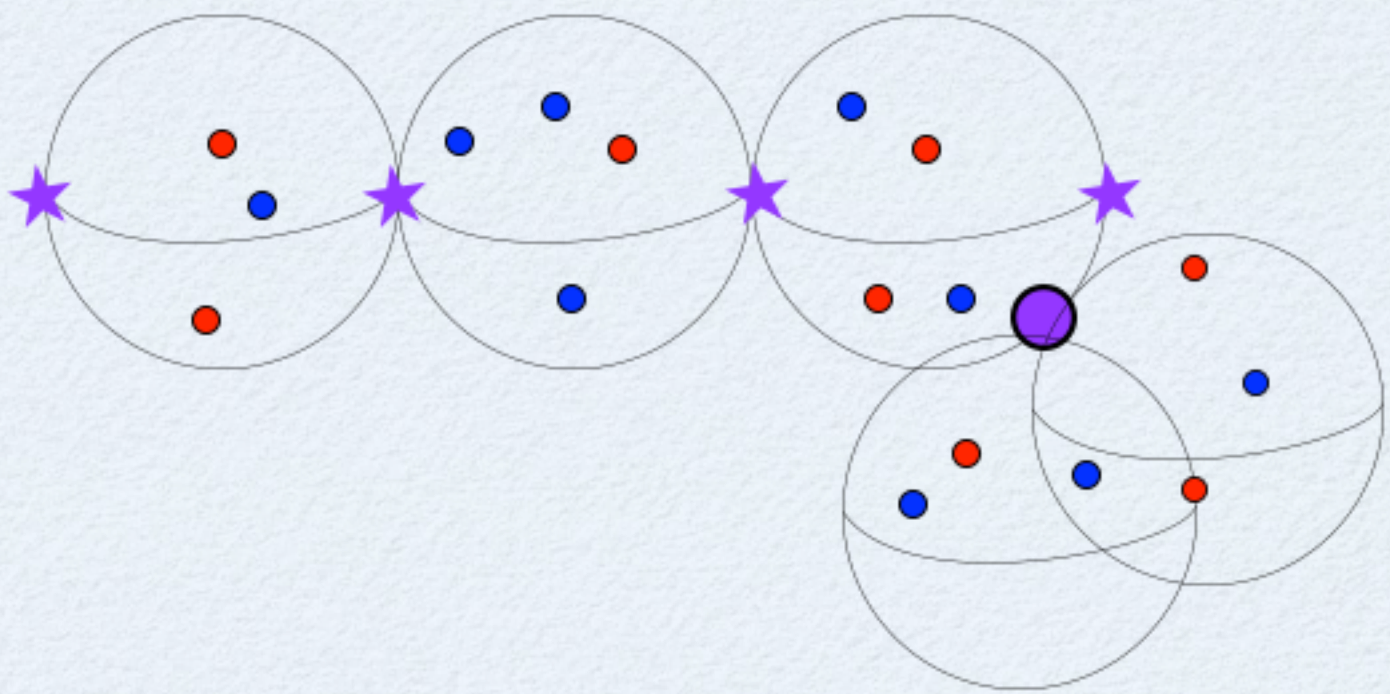


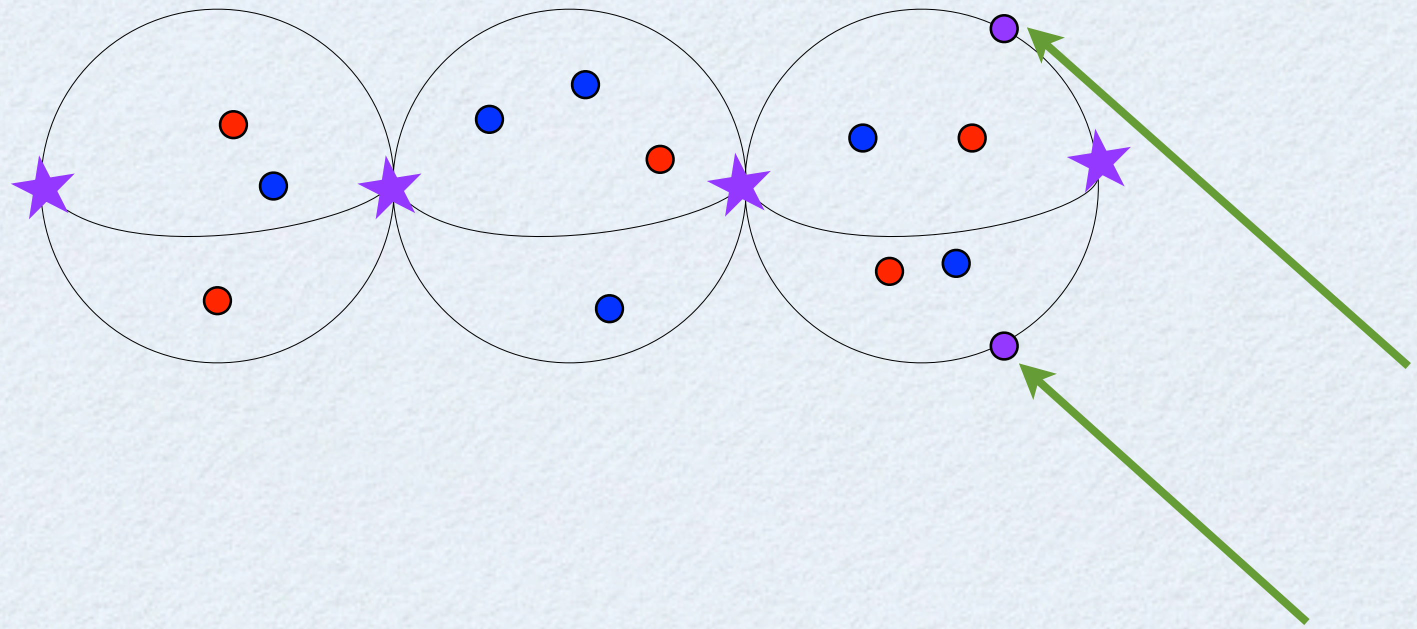


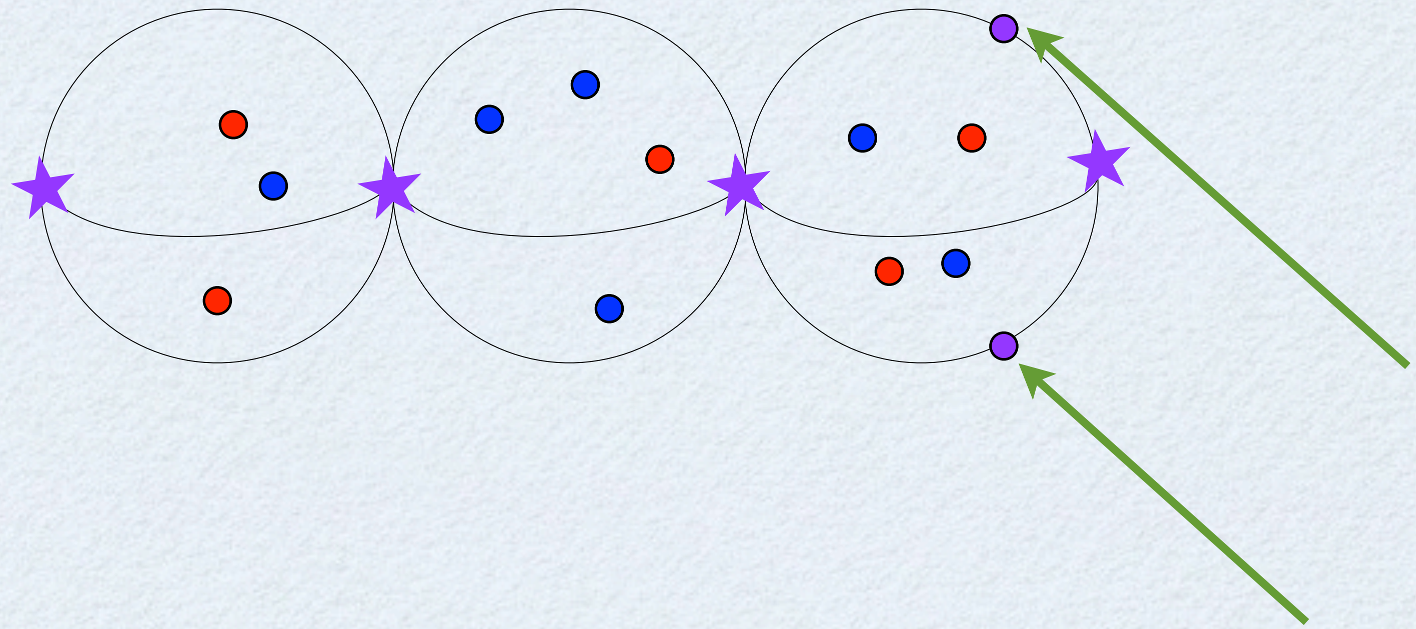


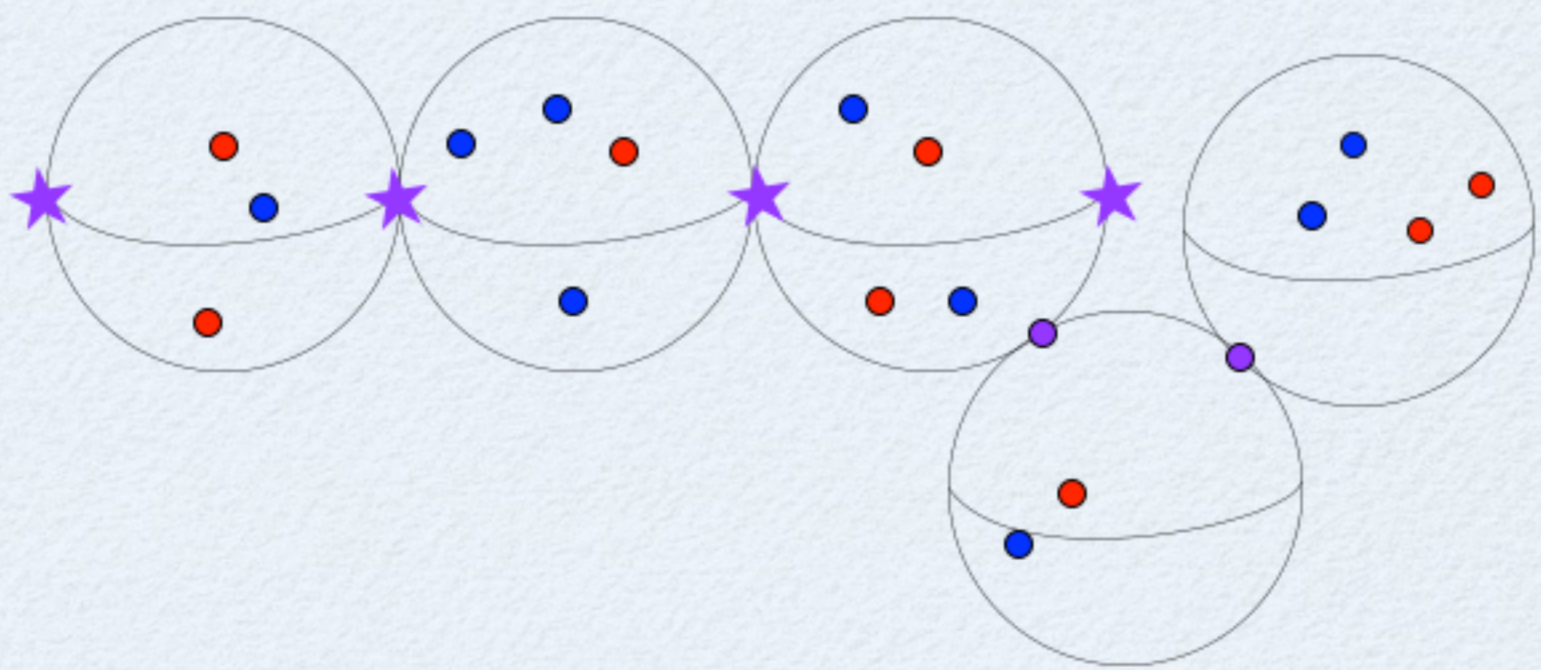


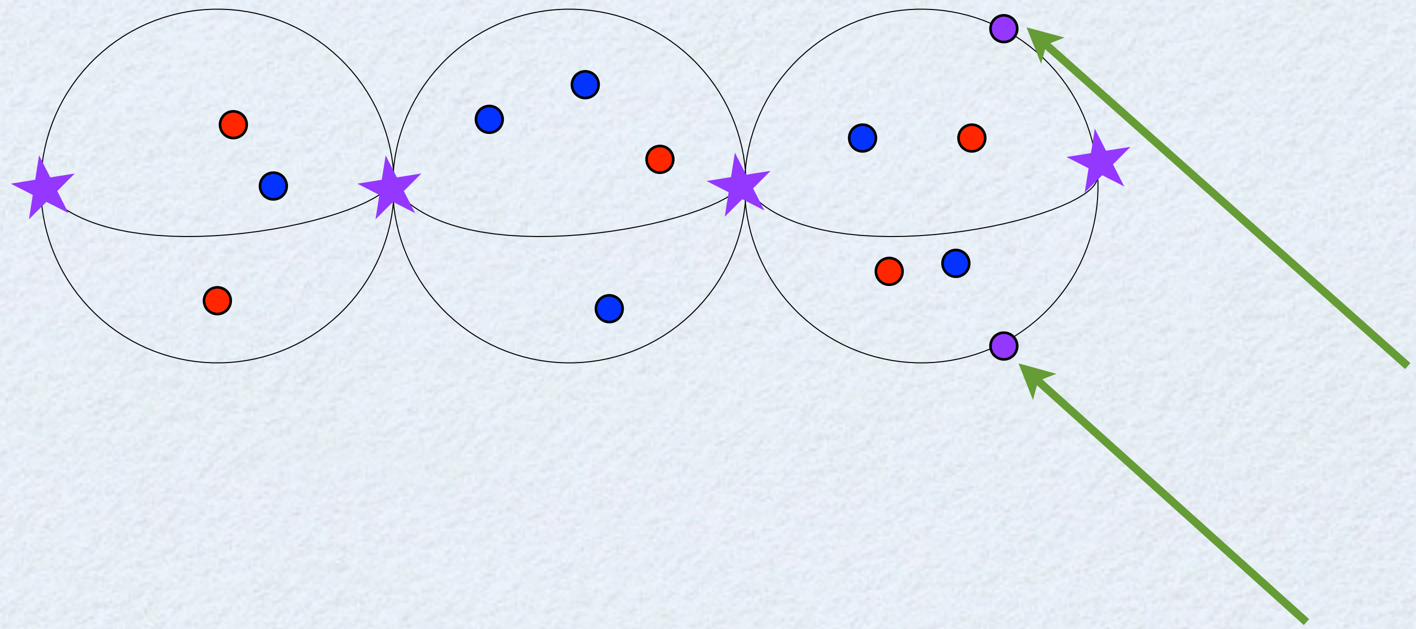


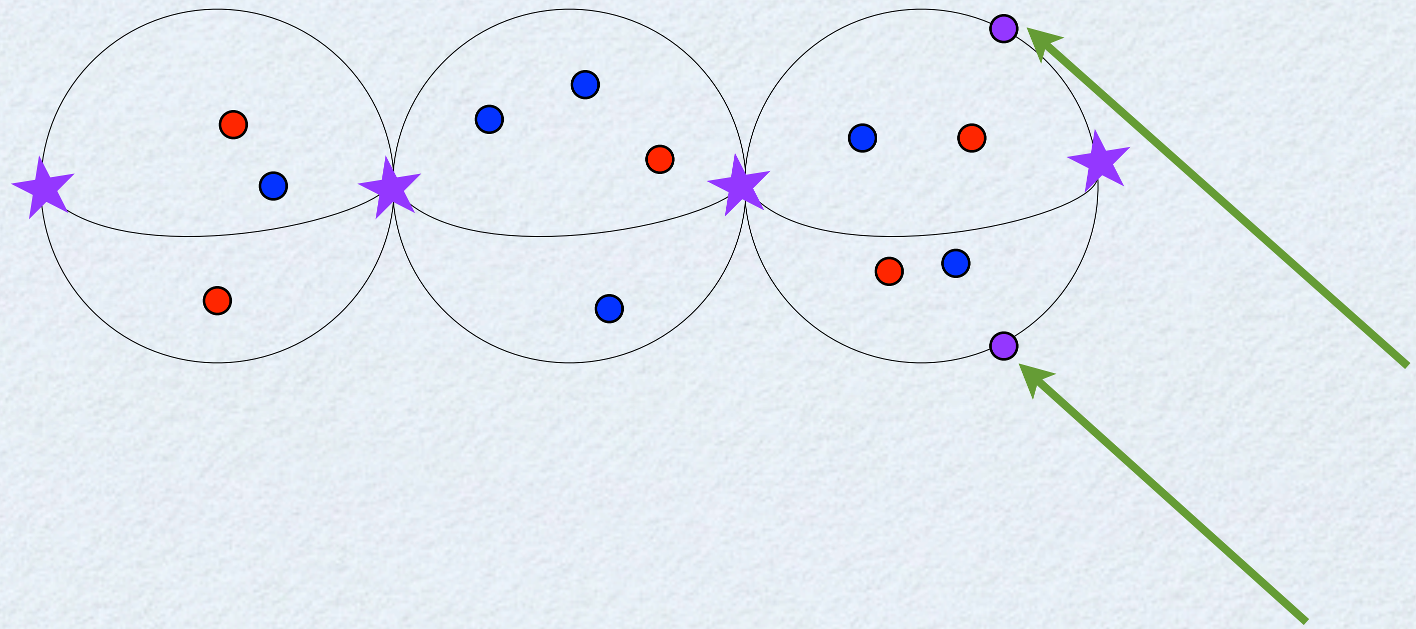


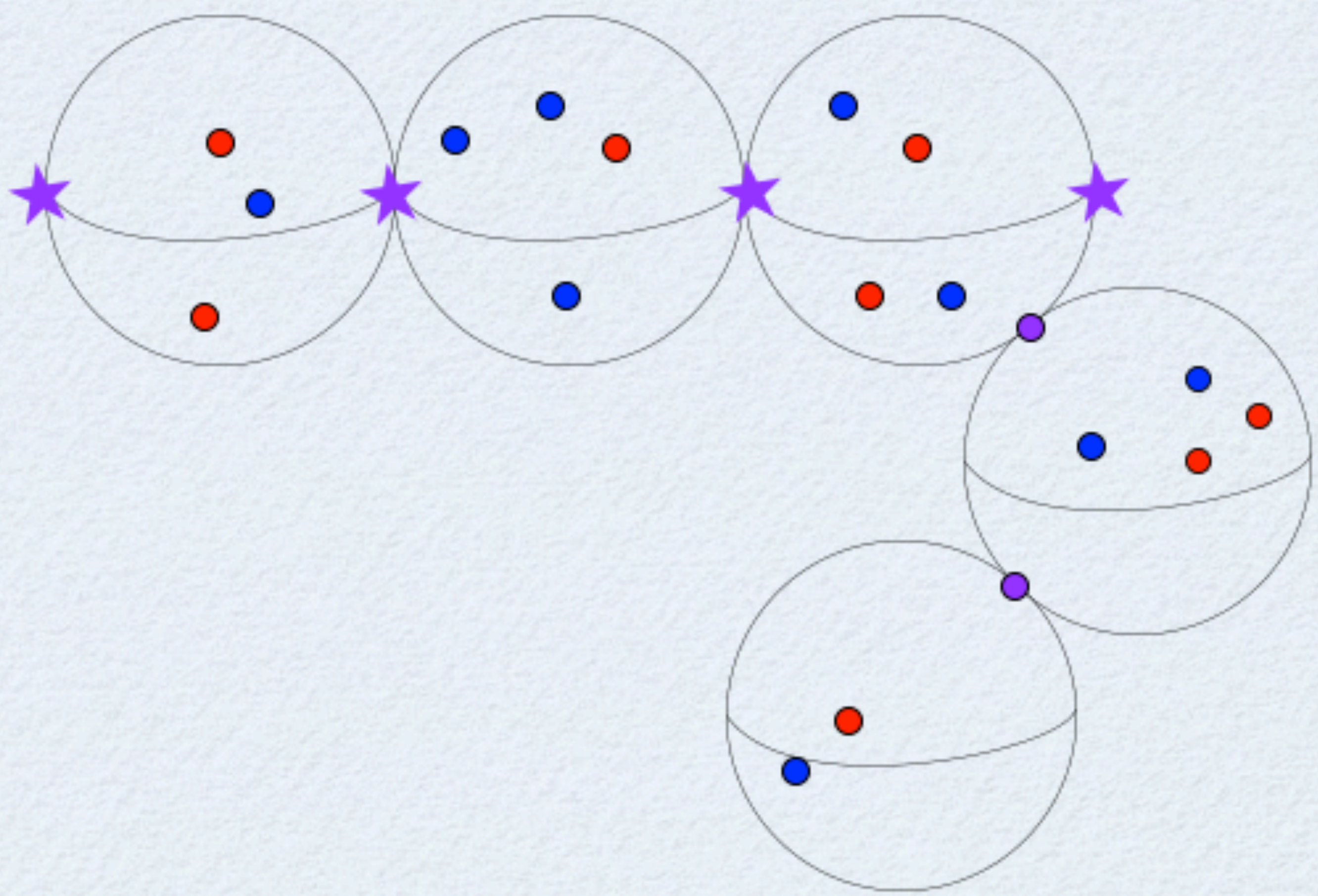


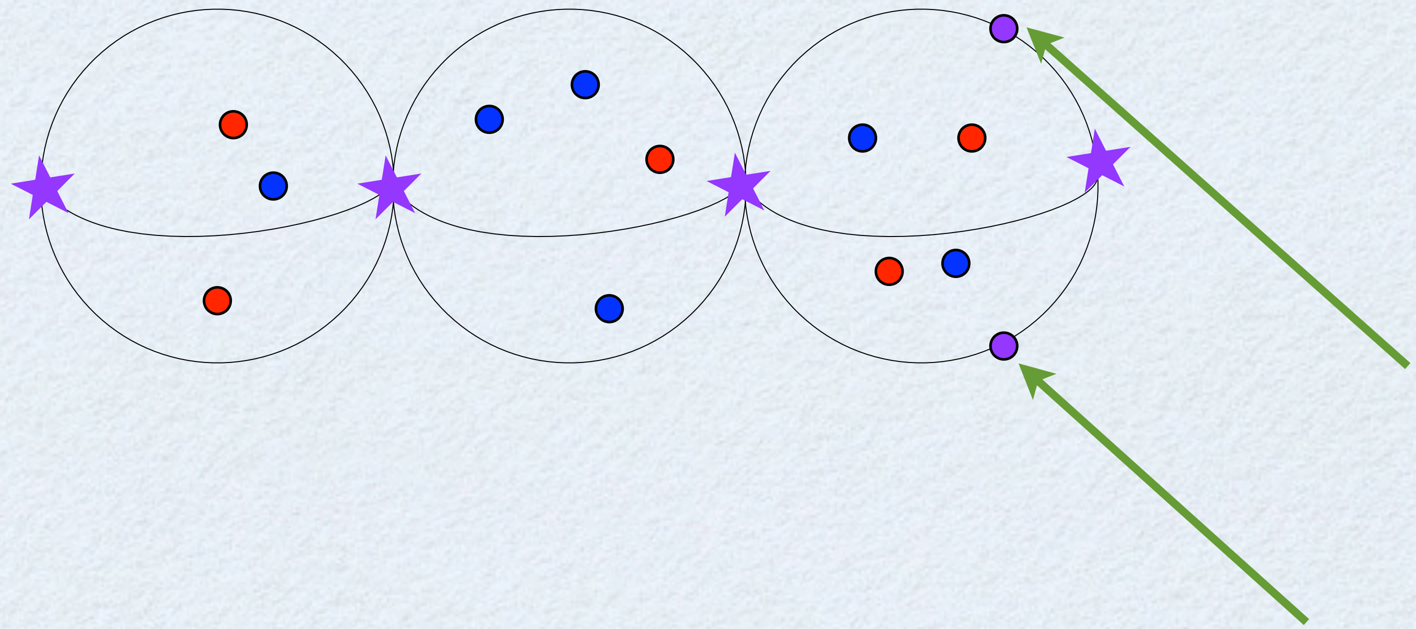


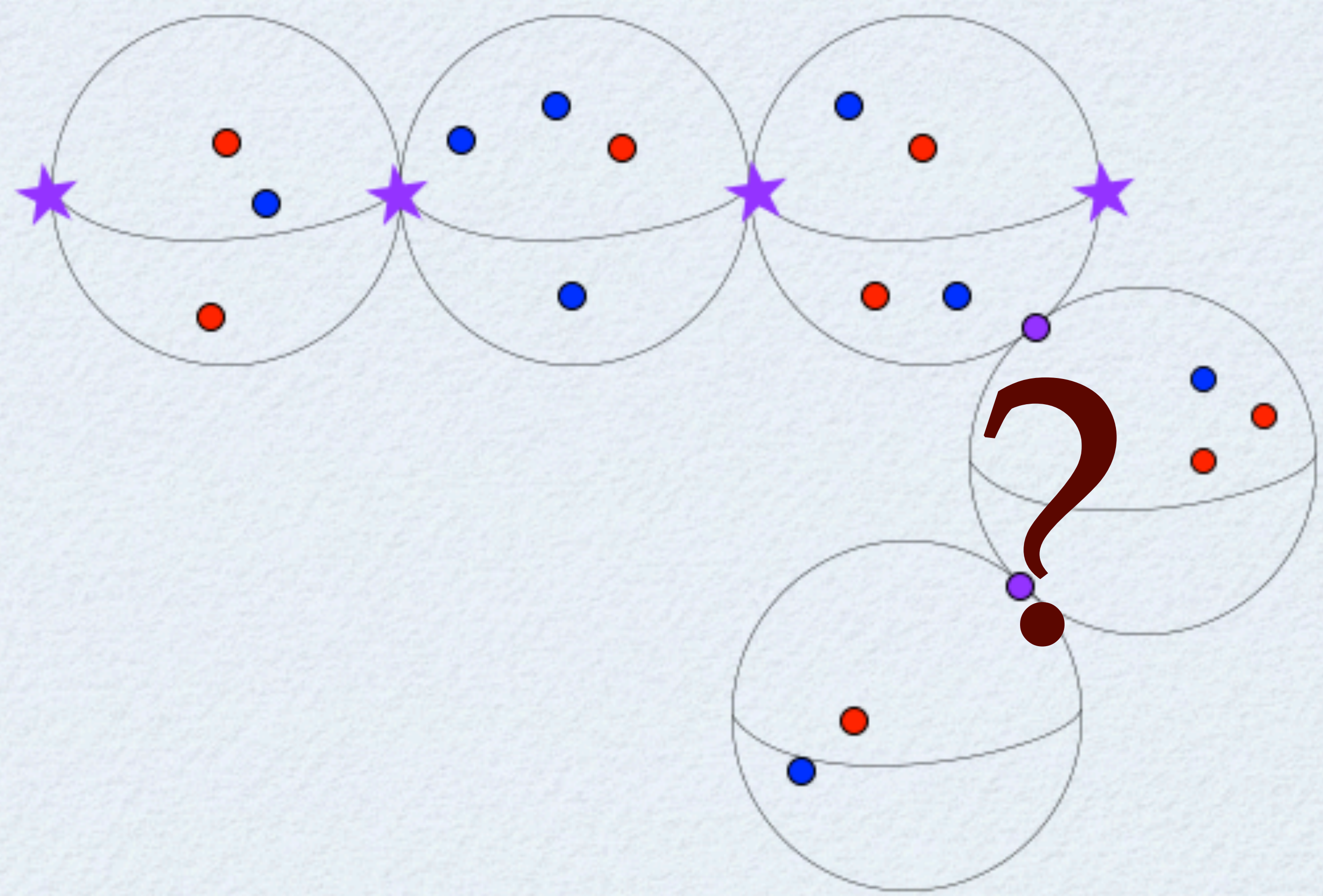












In summary...

critical trunk

two distinguished points

(critical trunk)/σ_d

points of indeterminacy

relative positions

Finally, $\mu : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$

In summary...

critical trunk

two distinguished points

(critical trunk)$/\sigma_d$

points of indeterminacy

relative positions

Finally, $\mu : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$

Proposition. This procedure defines a rational map $\widetilde{G} : \widetilde{\mathbb{P}}^n \dashrightarrow \widetilde{\mathbb{P}}^n$ which extends the map $G = \mu \circ s : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$

Lemma. The map $\widetilde{G} : \widetilde{\mathbb{P}}^n \dashrightarrow \widetilde{\mathbb{P}}^n$ has finite fibers.

Proof. Let η be a tree of spheres in the image of $\widetilde{G}$. Only the critical trunk is relevant as the preimages of all noncritical spheres are just these spheres themselves.

Critical trunk: Marked points and nodes.

Marked points: There are only d possibilities for the inverse image of each.

Lemma. The map $\widetilde{G} : \widetilde{\mathbb{P}}^n \dashrightarrow \widetilde{\mathbb{P}}^n$ has finite fibers.

Proof. Let η be a tree of spheres in the image of $\widetilde{G}$. Only the critical trunk is relevant as the preimages of all noncritical spheres are just these spheres themselves.

Critical trunk: Marked points and nodes.

Marked points: There are only d possibilities for the inverse image of each.

Nodes: 1) there are d possibilities for the inverse image of each node or 2) the node is a result of 2 marked points in the critical trunk upstairs which are symmetric; there are $d(d-1)/2$ possible configurations for this.

Lemma. The map $\widetilde{G} : \widetilde{\mathbb{P}}^n \dashrightarrow \widetilde{\mathbb{P}}^n$ has finite fibers.

Proof. Let η be a tree of spheres in the image of $\widetilde{G}$. Only the critical trunk is relevant as the preimages of all noncritical spheres are just these spheres themselves.

Critical trunk: Marked points and nodes.

Marked points: There are only d possibilities for the inverse image of each.

Nodes: 1) there are d possibilities for the inverse image of each node or 2) the node is a result of 2 marked points in the critical trunk upstairs which are symmetric; there are $d(d-1)/2$ possible configurations for this.

Theorem. The map $\widetilde{G} : \widetilde{\mathbb{P}}^n \dashrightarrow \widetilde{\mathbb{P}}^n$ is algebraically stable.

Lemma. The map $\widetilde{G} : \widetilde{\mathbb{P}}^n \dashrightarrow \widetilde{\mathbb{P}}^n$ has finite fibers.

Proof. Let η be a tree of spheres in the image of $\widetilde{G}$. Only the critical trunk is relevant as the preimages of all noncritical spheres are just these spheres themselves.

Critical trunk: Marked points and nodes.

Marked points: There are only d possibilities for the inverse image of each.

Nodes: 1) there are d possibilities for the inverse image of each node or 2) the node is a result of 2 marked points in the critical trunk upstairs which are symmetric; there are $d(d-1)/2$ possible configurations for this.

Theorem. The map $\widetilde{G} : \widetilde{\mathbb{P}}^n \dashrightarrow \widetilde{\mathbb{P}}^n$ is algebraically stable.

We can compute dynamical degrees.

Questions.

Questions.

Basilica-Basilica, $\lambda_1 = 2$

Basilica-Rabbit, λ_1 is the largest real root of

$$p(\lambda) = \lambda^4 + \lambda^3 - 2\lambda^2 - 8\lambda - 8,$$

$\lambda_1 \approx 2.229209$

Questions.

Basilica-Basilica, $\lambda_1 = 2$

Basilica-Rabbit, λ_1 is the largest real root of

$$p(\lambda) = \lambda^4 + \lambda^3 - 2\lambda^2 - 8\lambda - 8,$$

$\lambda_1 \approx 2.229209$

What do these numbers mean for the mating maps? What about $\sigma_f : \overline{\mathcal{T}}_P \to \overline{\mathcal{T}}_P$?

Questions.

Basilica-Basilica, $\lambda_1 = 2$

Basilica-Rabbit, λ_1 is the largest real root of

$$p(\lambda) = \lambda^4 + \lambda^3 - 2\lambda^2 - 8\lambda - 8,$$

$\lambda_1 \approx 2.229209$

What do these numbers mean for the mating maps? What about $\sigma_f : \overline{\mathcal{T}}_P \to \overline{\mathcal{T}}_P$?

What does the corresponding picture look like for us?

Questions.

Basilica-Basilica, $\lambda_1 = 2$

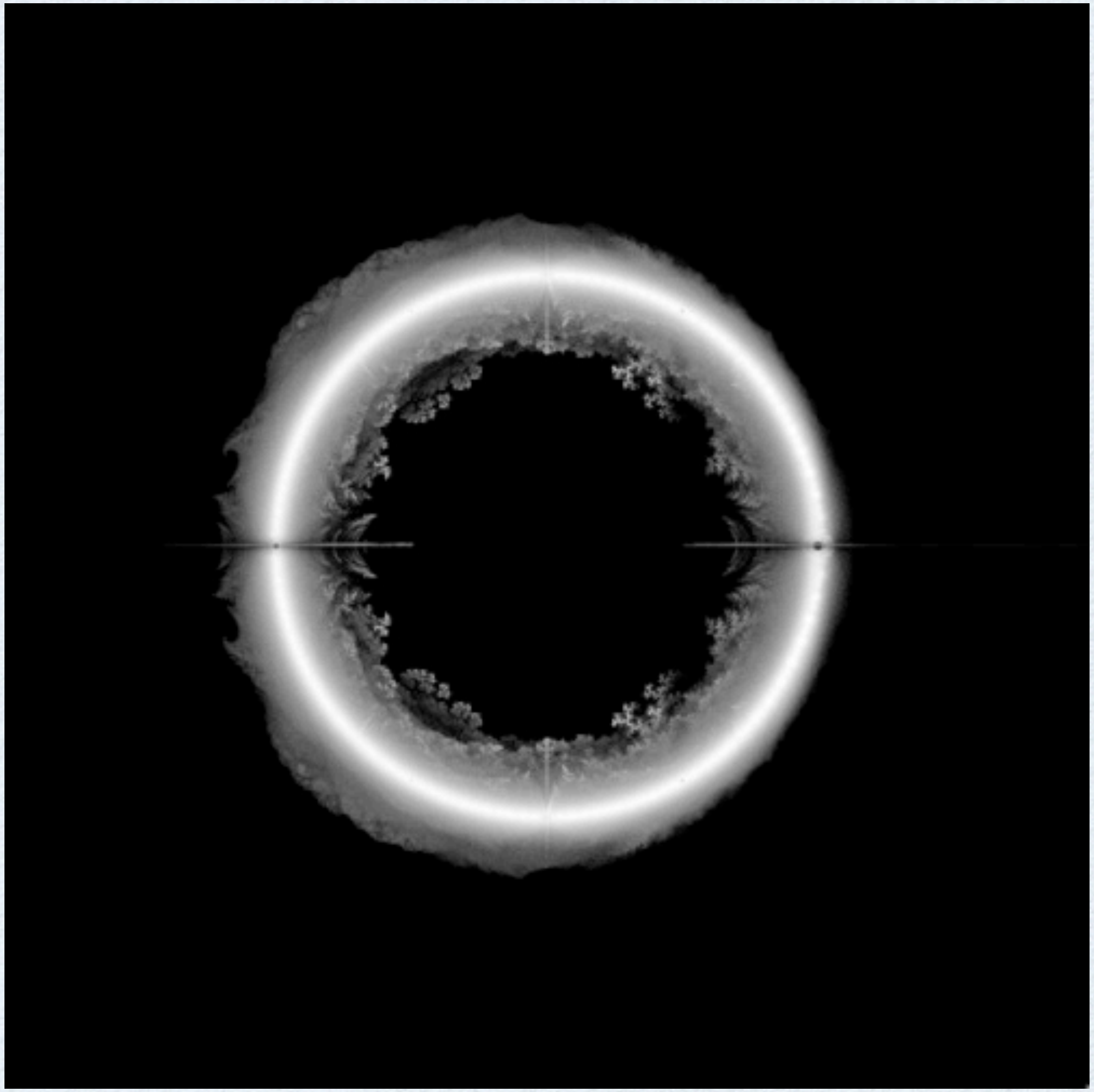
Basilica-Rabbit, λ_1 is the largest real root of

$$p(\lambda) = \lambda^4 + \lambda^3 - 2\lambda^2 - 8\lambda - 8,$$

$\lambda_1 \approx 2.229209$

What do these numbers mean for the mating maps? What about $\sigma_f : \overline{\mathcal{T}}_P \to \overline{\mathcal{T}}_P$?

What does the corresponding picture look like for us?



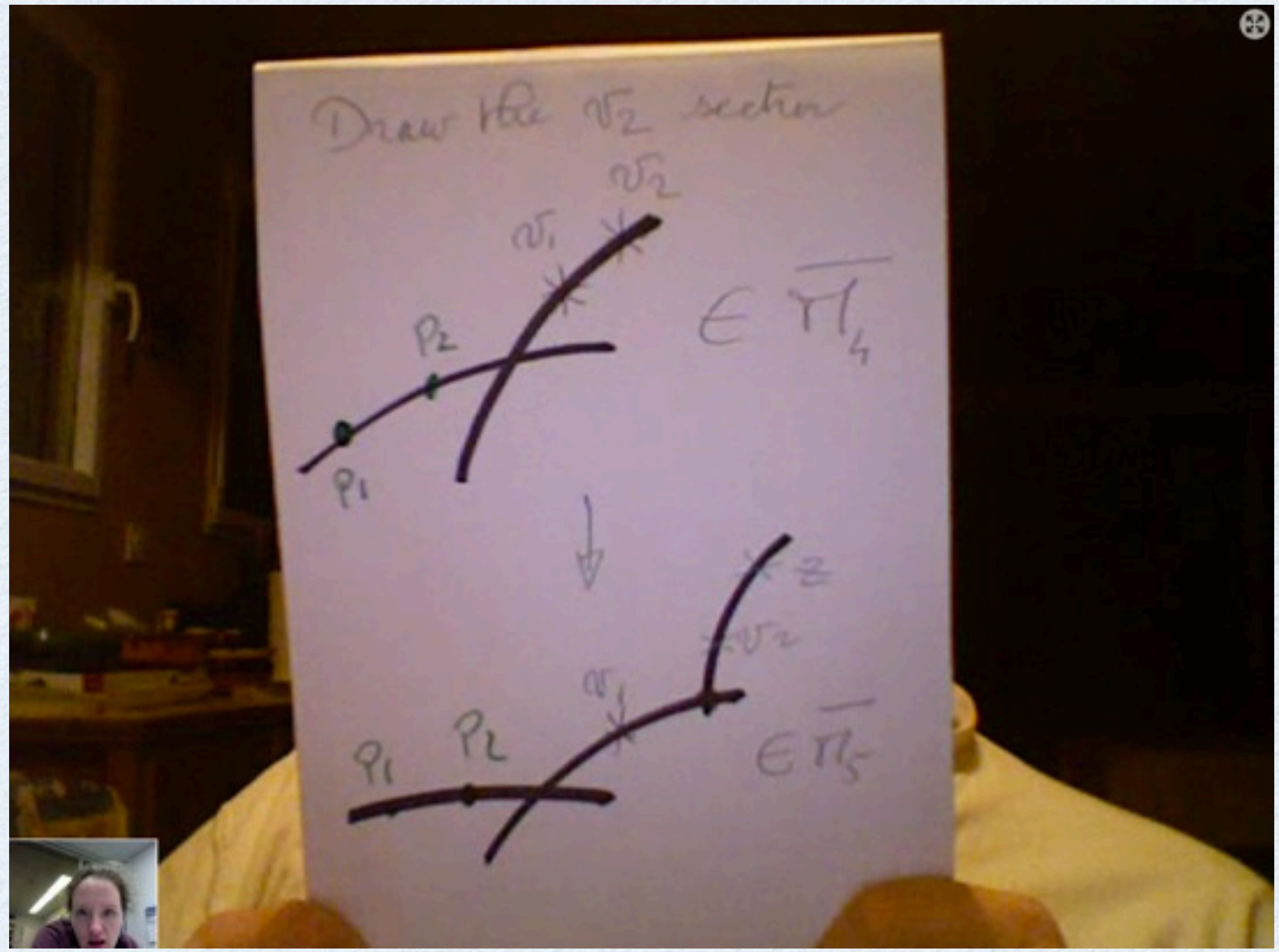
Is $\overline{\mathcal{M}}_n$ the minimal compactification for which these maps are algebraically stable?

Is $\overline{\mathcal{M}}_n$ the minimal compactification for which these maps are algebraically stable?

Conjecture. No.

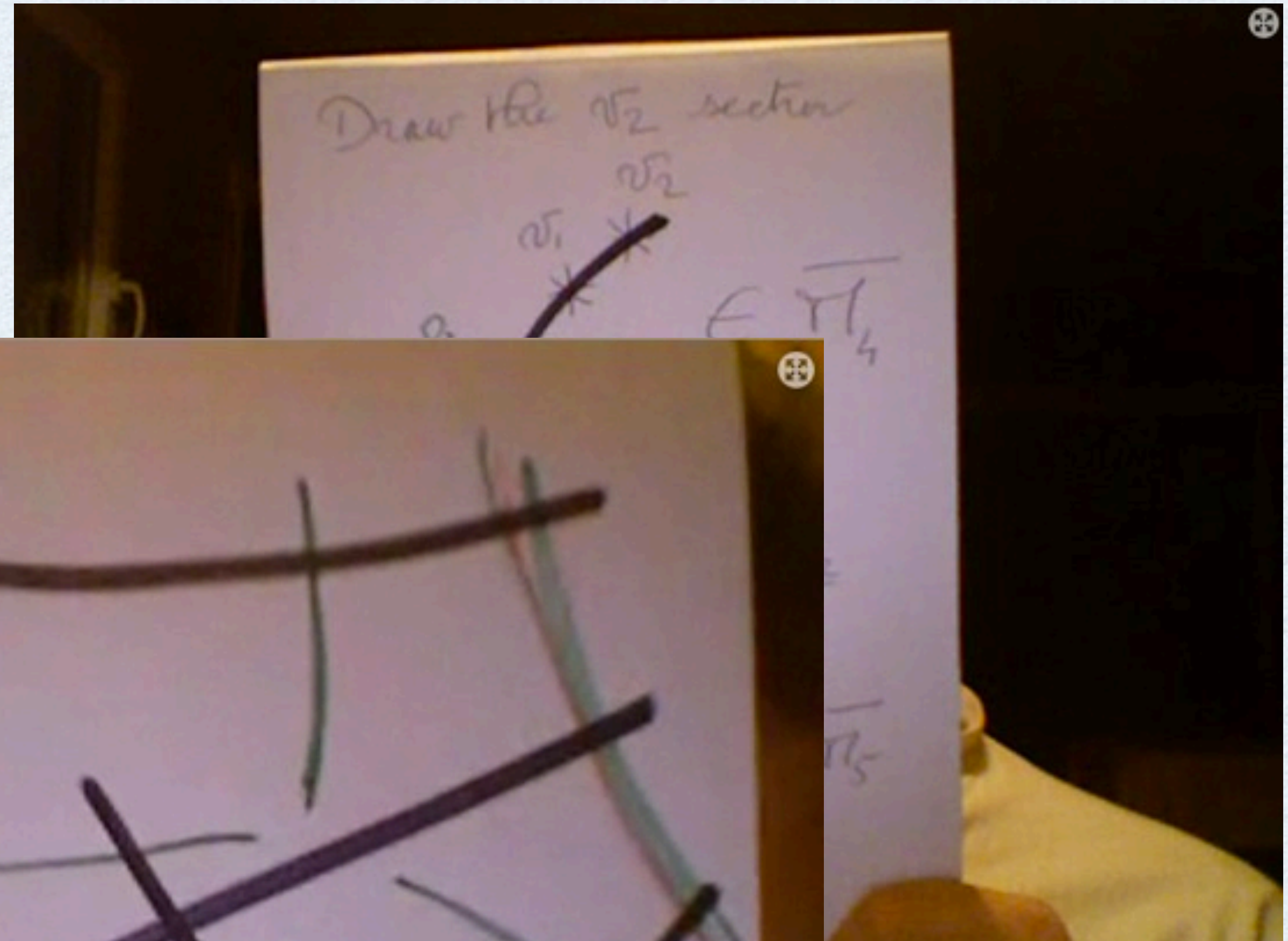
Is $\overline{\mathcal{M}}_n$ the minimal compactification for which these maps are algebraically stable?

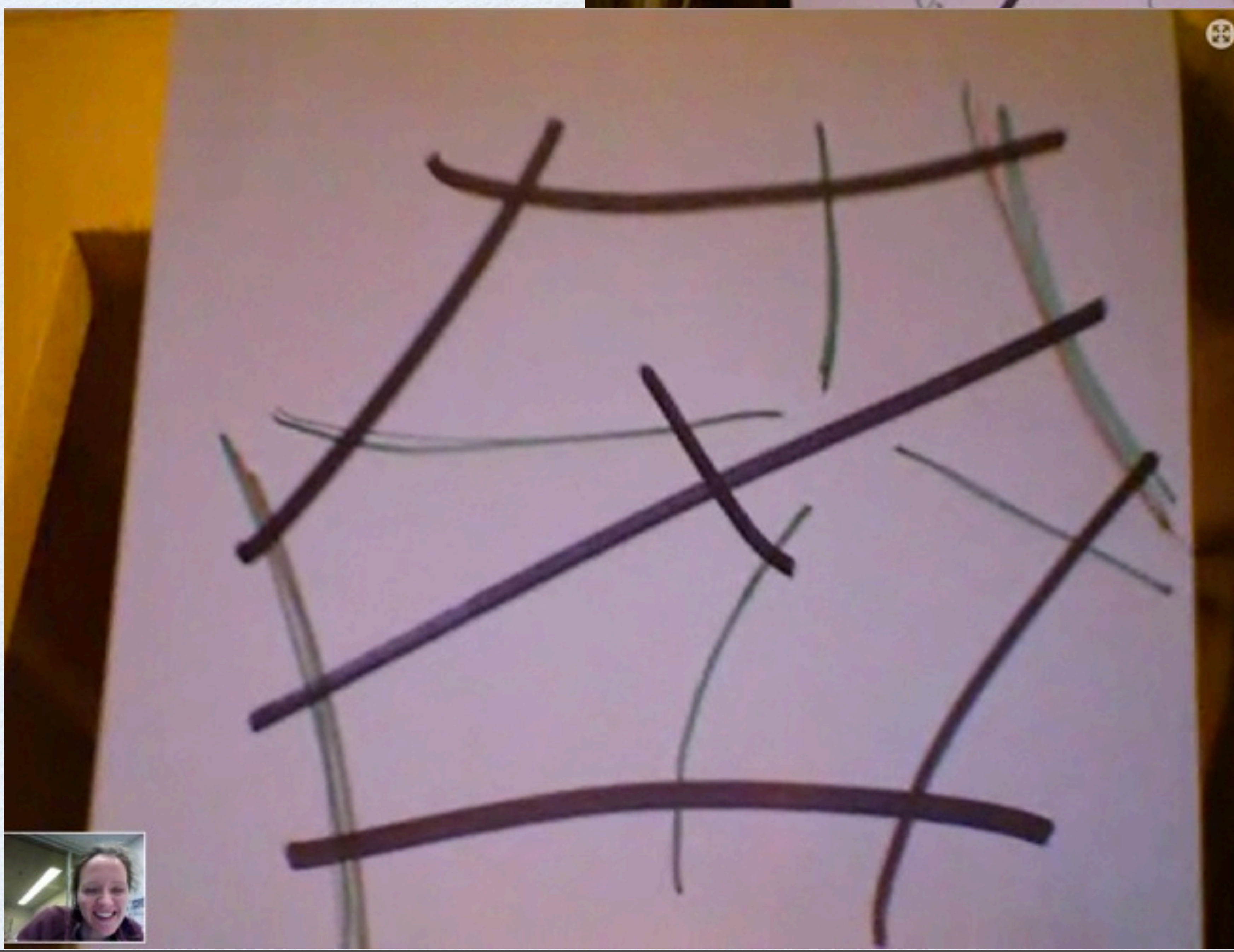
Conjecture. No.



Is $\overline{\mathcal{M}}_n$ the minimal compactification for which these maps are algebraically stable?

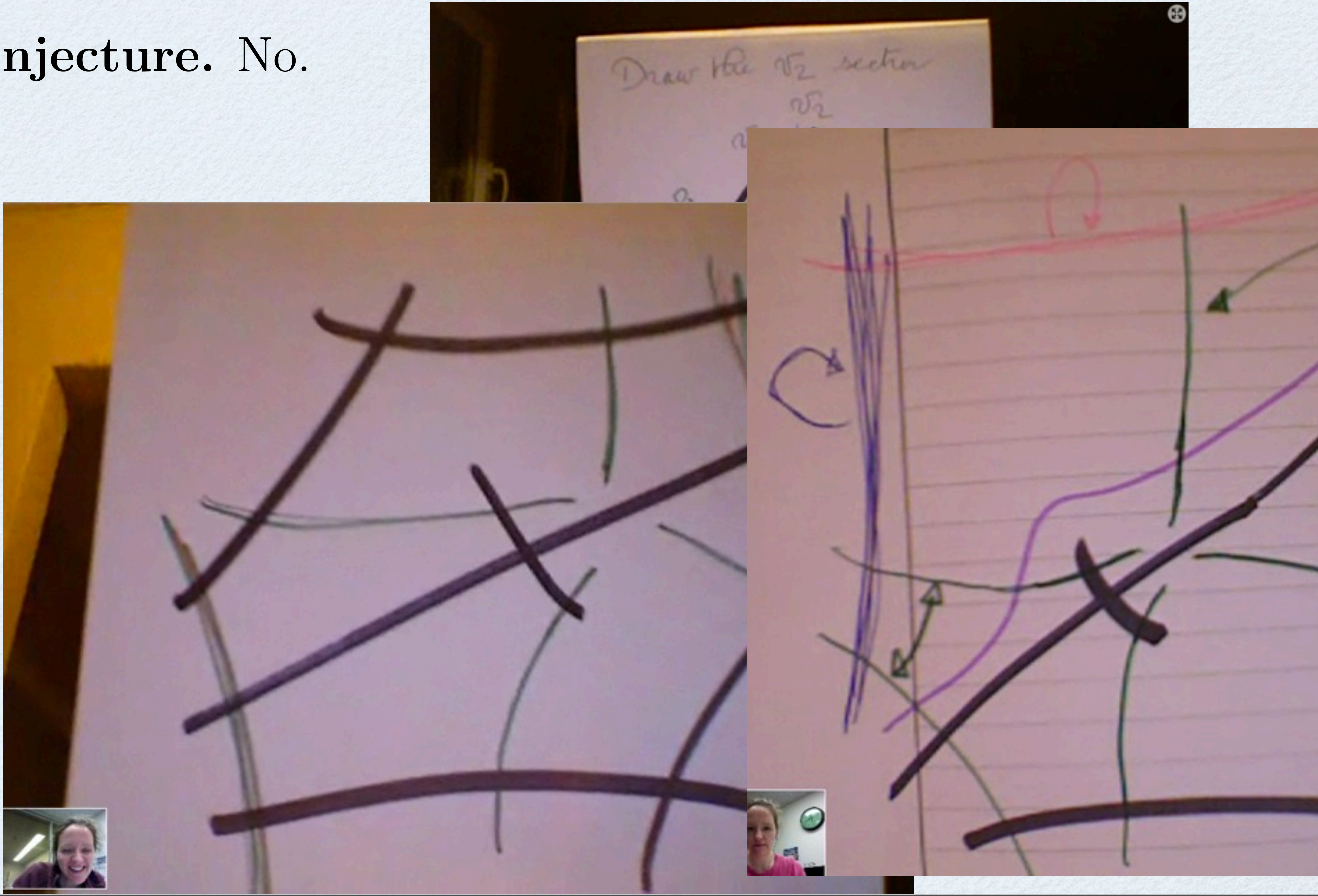
Conjecture. No.





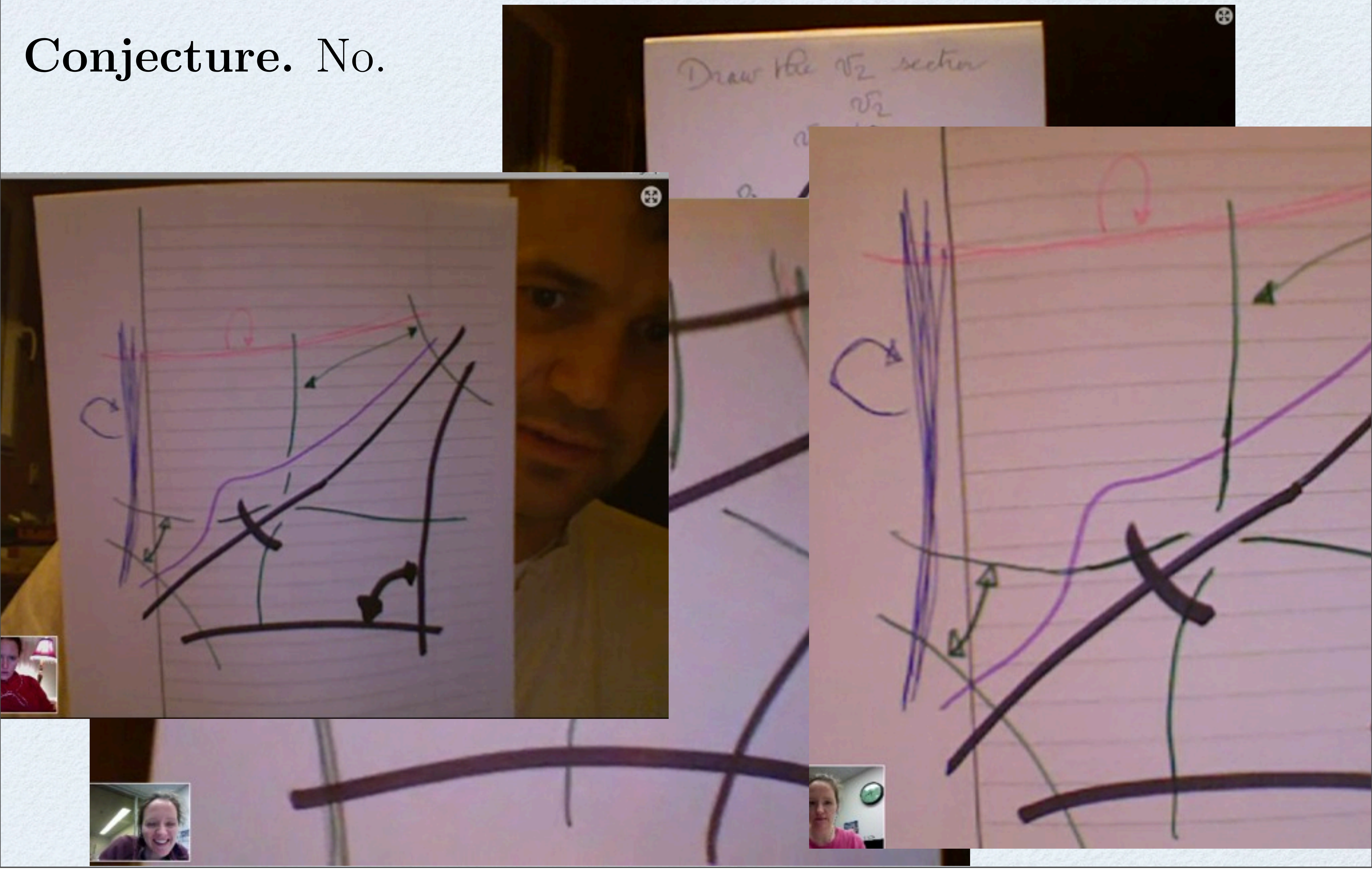
Is $\overline{\mathcal{M}}_n$ the minimal compactification for which these maps are algebraically stable?

Conjecture. No.



Is $\overline{\mathcal{M}}_n$ the minimal compactification for which these maps are algebraically stable?

Conjecture. No.



thank you for your attention!