Rescaling limits of Complex Rational Maps

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Definition of Rescaling limits

Consider a sequence of maps $\{f_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\}$ of degree $d \geq 2$.

A *rescaling for* $\{f_n\}$ is a sequence of changes of coordinates

$$\{M_n\} \subset \mathrm{PSL}(2, \mathbb{C})$$

with the following property:

There exist an iterate $q \in \mathbb{N}$ such that

$$M_n^{-1} \circ f_n^q \circ M_n \to g$$

for some rational map g with $\deg g \geq 2$.

The convergence is uniform in compact subsets of $\overline{\mathbb{C}}$ with finitely many points removed.

g is called a *rescaling limit for* $\{f_n\}$,
q is called a *rescaling period for* $\{f_n\}$ *at* $\{M_n\}$.

Some remarks

Given a rescaling $\{M_n\}$ the periods for $\{f_n\}$ at $\{M_n\}$ are of the form $q \cdot \mathbb{N}$.

If $f_n \to f$ where f also has degree d, then rescaling limits are iterates of f.

Example: Lattès

$$f_t(w) = \frac{(w^2 - t)^2}{4w(w-1)(w-t)}$$

Consider a sequence $\{f_{t_n}\}$ with $t_n \to 0$.

Rescaling of period 1: Let $M_n(z) = z$. In $\mathbb{C} \setminus \{0\}$,

$$f_{t_n}(w) \to \frac{w^2}{4(w-1)},$$

which is $z \mapsto z^2 - 2$ modulo change of coordintes.

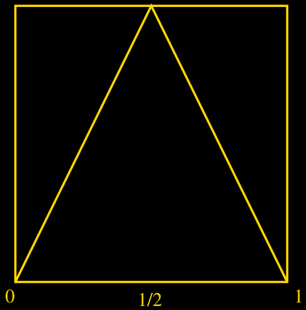
Higher periods

For $\alpha \in]0,1[$,

$$f_t(t^\alpha \cdot z) \sim t^{\mathrm{Tent}(\alpha)} \cdot \mathrm{monomial}(z).$$

Any periodic point α_0, say of period m, of Tent in $]0,1[$ determines a monomial rescaling limit of period m of $\{f_{t_n}\}$ since

$$f_t^m(t^{\alpha_0} z) = t^{\alpha_0} \cdot \mathrm{monomial}(z)$$



Lattès: continued

In fact,

$$\frac{f_t(t^\alpha \cdot z)}{t^{\mathrm{Tent}(\alpha)}} \to \begin{cases} -\dfrac{z^2}{4} & \text{if } 0 < \alpha < 1/2 \\[2ex] -\dfrac{z^{-2}}{4} & \text{if } 1/2 < \alpha < 1 \end{cases}$$

For example $\mathrm{Tent}^2(\frac{2}{5}) = \frac{2}{5}$. Take s_n such that $s_n^5 = t_n^2$. Then,

$$\frac{1}{s_n} \cdot f_{t_n}^2(s_n \cdot z) \to -4z^{-4}.$$

Example: Parabolic rescaling of quadratic rational maps

Quadratic rational maps with a period 3 critical point are parametrized by:

$$f_t(z) = t - \frac{1+t^2}{z} + \frac{t}{z^2}.$$

In fact, the critical point $\omega = \infty \mapsto t \mapsto 0 \mapsto \infty$.

Parabolic rescaling (Rees-Stimson):

For $M_t(z) = tz$,

$$M_t^{-1} \circ f_t^2 \circ M_t(z) \to g(z) = \frac{z^2 + z - 1}{z - 1}$$

as $t \to 0$, uniformly on compact subsets of $\mathbb{C} \setminus \{1\}$.

$g(z)$ has a parabolic (multiple) fixed point at $z = \infty$.

Same phenomena for all "periodic curves".

Related to: Petersen, Epstein, De Marco, K.-Rees.

(continued): Quadratic polynomial rescaling

A perturbation of maps with a period 3 critical point.
Given $a \in \mathbb{C}$, let

$$f_t(z) = t - \frac{1 + t^2}{z} + \frac{t}{z^2} - at^5.$$

For $L_t(z) = t^3 z$,

$$L_t^{-1} \circ f_t^3 \circ L_t(z) \to Q_a(z) = z^2 + a,$$

as $t \to 0$, uniformly on compact subsets of $\mathbb{C}$.

Similar phenomena "around" all periodic curves...also for cubic polynomials (Bonifant, Milnor)

All known rescalings of quadratic rational have a parabolic fixed point or are quadratic polynomials.

Question

How many distinct and meaningful rescaling limits can a sequence have?

"At most $2d - 2$ rescalings are not postcritically finite."

"At most 2 for quadratic rational maps: one quadratic rational map with a parabolic fixed point, one a quadratic polynomial."

The space of rational maps

$$\begin{array}{rcccl} & & \mathrm{Rat}^d_{\mathbb{C}} & \longrightarrow & \mathbb{P}^{2d+1}_{\mathbb{C}} \setminus \{\text{Resultant} = 0\} \\ f(z) & = & \dfrac{P(z)}{Q(z)} & \longmapsto & [a_0 : \cdots : a_d : b_0 : \cdots : b_d] \\ & = & \dfrac{a_0 z^d + \cdots + a_d}{b_0 z^d + \cdots + b_d}. & & \end{array}$$

$f \in \mathrm{Rat}^d_{\mathbb{C}} \Leftrightarrow \text{Resultant}\,(P, Q) \neq 0.$

Algebraic rescalings: holomorphic families

Consider a small disk

$$\mathbb{D}_\varepsilon = \{z \in \mathbb{C} \mid |z| < \varepsilon\}.$$

A *degenerate holomorphic family f_t of degree $d \geq 1$* is a holomorphic map:

$$\begin{array}{ccc} \mathbb{D}_\varepsilon & \to & \mathbb{P}_{\mathbb{C}}^{2d+1} \\ t & \mapsto & f_t \end{array}$$

such that $f_t \in \mathrm{Rat}_{\mathbb{C}}^d \not\ni f_0$ for all $t \neq 0$.

Algebraic rescaling: definition

Consider a holomorphic family f_t of degree $d \geq 2$.

An *algebraic rescaling for f_t* is a holomorphic family of coordinate changes M_t (i.e. degree 1) with the following property:

There exists an iterate q such that

$$M_t^{-1} \circ f_t^q \circ M_t(z) \to g(z)$$

for some rational map g with $\deg g \geq 2$.

The convergence is uniform on compact subsets of $\overline{\mathbb{C}}$ with finitely many points removed.

g is called a *rescaling limit for f_t*,
q is called a *rescaling period for f_t at M_t*.

Algebraic rescalings: equivalence and action.

Two rescalings M_t and L_t are *equivalent* if:

$$M_t^{-1} \circ L_t \to M \in \mathrm{PSL}(2, \mathbb{C}).$$

A degree 1 holomorphic family F_t acts on the set of equivalence classes:

$$F_t : [M_t] \mapsto [F_t \circ M_t].$$

Higher degrees

(Rivera-Letelier) A holomorphic family f_t acts on the set of equivalence classes of algebraic rescalings:

> Given a rescaling class $[M_t]$ there exists a unique rescaling class $[L_t] = f_t([M_t])$ such that:
>
> $$L_t^{-1} \circ f_t \circ M_t(z) \to \varphi(z)$$
>
> for some φ with $\deg \varphi \geq 1$.

Rescaling and orbits

$[M_t]$ is a rescalings for f_t if and only if:

$[M_t]$ is a periodic point, say of period q, for the action of f_t on

$$\{[M_t] \mid M_t \text{ is a holomorphic family of degree } 1\},$$

such that

$$M_t^{-1} \circ f_t^q \circ M_t(z) \to \varphi(z)$$

for some φ with $\deg \varphi \geq 2$.

Results for holomorphic families

Theorem A. (K.) *Given a holomorphic family f_t of degree $d \geq 2$, there exists at most $2d-2$ distinct periodic rescaling orbits with rescaling limits which are not postcritically finite.*

Theorem B. (K.) *Given a holomorphic family f_t of degree $d = 2$, there exists at most 2 distinct rescaling orbits.*

> If there exists one, then the rescaling limit is parabolic.
> If there exists two, then one rescaling is parabolic and the other one is a quadratic polynomial rescaling limit.

Can one replace postcritically finite by monomial, maybe with a bound depending on f_t?, maybe depending on d?

Berkovich space.

$$\{[M_t] \mid M_t \text{ is a holomorphic family of degree 1}\}$$

is contained in a space $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$, called the *Berkovich projective line.*

- $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ is a compact (infinite) tree.
- $[M_t]$ are vertices of this tree.
- Directions at $[M_t]$ are parametrized by $\overline{\mathbb{C}}$.

Action on $\mathbb{P}^1_{\mathbb{L}}$

$$f_t(z) = \frac{a_d(t)z^d + \cdots a_0(t)}{b_d(t)z^d + \cdots b_0(t)} \in \mathbb{C}((t))(z)$$

where $\mathbb{C}((t))$ is the field of formal Laurent series which has a (natural) non-Archimedean valuation.

Extend to the completion $\mathbb{L}$ of an algebraic closure of $\mathbb{C}((t))$. Thus,

$$f_t(z) \in \mathbb{L}(z)$$

is a rational map of degree d.

f_t acts on $\mathbb{P}^1_{\mathbb{L}}$ where it has $2d-2$ critical points.

$\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ also contains $\mathbb{P}^1_{\mathbb{L}}$.

Action on $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$

The action is better understood extending it from $\mathbb{P}^1_{\mathbb{L}}$ to $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$:

$$f_t : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$$

is a "degree d piecewise linear map".

The action on rescalings $[M_t] \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ is the given by Rivera-Letelier.

Fatou set, Julia set and repelling orbits

For

$$f_t : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$$

the notions of Julia set, Fatou set and components "generalize"

e.g., the Julia set of Lattès family is an interval with the tent dynamics.

> $\mathbb{P}^1_{\mathbb{L}} \subset \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ are the *rigid or classical points*.
>
> $\mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \setminus \mathbb{P}^1_{\mathbb{L}}$ is an $\mathbb{R}$-tree (hyperbolic space).

If $[M_t]$ is a rescaling for f_t, then $[M_t]$ is a non-rigid repelling periodic point of $f_t : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$.

The converse is also true, modulo passing to a finite extension (i.e. replacing f_t by f_{t^ℓ}).

Strategy of the proof.

Given a non-rigid repelling periodic orbit $\mathcal{O}$ of $f_t : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ with associated rescaling limit g such that g is not postcritical finite contains a critical point $c(t)$ in its basin of attraction.

Directions

At the Gauss point

$[id] \in \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ is called the *Gauss point*.

If $[A_t] \neq [id]$, then $A_t \to a \in \overline{\mathbb{C}}$, with a independent of choice of representative in $[A_t]$.

If $z(t) \in \mathbb{C}((t))$, then $z(0) \in \overline{\mathbb{C}}$.

Given $a \in \overline{\mathbb{C}}$, all $[M_t] \neq [id]$ such that $M_t \to a$ and all $z(t)$ such that $z(0) = a$ lie in the direction $D(a)$ at the Gauss point.

At any $[M_t]$

$M_t : \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$ maps directions at $[id]$ onto directions at $[M_t]$.

Action on directions

Assume that $f_t[M_t] = [L_t]$ and

$$L_t^{-1} \circ f_t \circ M_t(z) \to \varphi(z)$$

in $\text{Good} = \overline{\mathbb{C}} \setminus \{\text{Bad}\}$.

- If $z \in \text{Good}$, then $f_t(D(z)) = D(\varphi(z))$.
- If $z \in \text{Bad}$, then $f_t(D(z)) = \mathbb{P}^{1,\mathrm{an}}_{\mathbb{L}}$.

Moreover:

- If $\dfrac{d\varphi}{dz}(\omega) = 0$, then $D(\omega)$ contains a critical point of f_t.
- If $z \in \text{Bad}$, then $D(z)$ contains a critical point of f_t.
- If a critical point of f_t belongs to $D(z)$, then a critical value belongs to $D(\varphi(z))$.

Proof of **Theorem A.**

Assume that $f_t[M_t] = [M_t]$,

$$M_t^{-1} \circ f_t \circ M_t(z) \to \varphi(z)$$

with $\varphi(z)$ not postcritically finite.

Let c be a critical point of φ with infinite forward orbit.

Then $D(c)$ contains a critical point $c(t)$ of f_t.

If $\varphi^n(\omega) \in$ Good for all $n \geq 0$, then $f_t^n(c(t)) \in D(\varphi^n(c))$.
Thus, $f_t^n(c(t)) \to [M_t]$.

Otherwise, there exists n_0 such that: $\varphi^n(\omega) \in$ Good for all $n > n_0$ and $\varphi^{n_0}(\omega) \in$ Bad.
So there exists a critical value $v(t) \in D(\varphi^{n_0+1}(\omega))$. Hence, the iterates of $v(t)$ converge to $[M_t]$.

$\square$

Back to sequences. Rescaling limits: dependence

$\{M_n\}$ and $\{L_n\}$ are *independent rescalings* if

$$M_n^{-1} \circ L_n \to \infty \in PSL(2, \mathbb{C}).$$

If $\{M_n\}$ and $\{L_n\}$ are *dependent*, then there exists $M \in PSL(2, \mathbb{C})$ such that, passing to a subsequence,

$$M_n^{-1} \circ L_n \to M.$$

In particular, if $\{M_n\}$ leads to a rescaling g of period q, then $\{L_n\}$ leads to a rescaling $M \circ g \circ M^{-1}$.

Dynamical Independence.

$\{M_n\}$ and $\{L_n\}$ are rescalings of period p for $\{f_n\}$. They are *dynamically dependent*, if passing to a subsequence, there exists $0 \le p' \le p$ such that

$$M_n^{-1} \circ f_n^{p'} \circ L_n \to g_1 \in \mathrm{Rat}_{\mathbb{C}}^{d_1}$$

for some $d_1 \ge 1$,

$$L_n^{-1} \circ f_n^{p-p'} \circ M_n \to g_2 \in \mathrm{Rat}_{\mathbb{C}}^{d_2}$$

for some $d_2 \ge 1$.

Example: in Lattès $M_n(z) = s_n \cdot z$, $L_n(z) = s_n^2 \cdot z$ are dynamically dependent.

Counting rescalings

Theorem C.

Consider a sequence $\{f_n\} \subset \mathrm{Rat}^d_{\mathbb{C}}$. For $j = 1, \ldots, N$, assume that

$$\{M_n^{(1)}\}, \ldots, \{M_n^{(N)}\}$$

are pairwise dynamically independent rescalings for $\{f_n\}$ with rescaling limits $g_1, \ldots, g_N$, respectively.

If g_j is not postcritically finite for all j, then $N \leq 2d - 2$.

Rescaling of quadratic rational maps

Theorem D.

Consider a sequence $\{f_n\} \subset \mathrm{Rat}^2_{\mathbb{C}}$. For $j = 1, \ldots, N$, assume that

$$\{M_n^{(1)}\}, \ldots, \{M_n^{(N)}\}$$

are pairwise dynamically independent rescalings for $\{f_n\}$ with rescaling limits $g_1, \ldots, g_N$, respectively.

Then $N \leq 2$ and one of the following occur:

(a) $N = 1$ and g_1 is a quadratic rational map with a parabolic fixed point.
(b) $N = 2$, g_1 is as above, g_2 is a quadratic polynomial.

From theorems A-B to theorems C-D.

Given $\{f_n\} \subset \mathrm{Rat}^d_{\mathbb{C}}$. Assume that for $j = 1, \ldots, N$,

$$\{M_n^{(1)}\}, \ldots, \{M_n^{(N)}\}$$

are pairwise dynamically independent rescalings of $\{f_n\}$ with rescaling limit

$$g_1, \ldots, g_N.$$

Then, there exist a holomorphic family f_t and algebraic rescalings

$$\{M_t^{(1)}\}, \ldots, \{M_t^{(N)}\}$$

for f_t with rescaling limits

$$g_1, \ldots, g_N.$$

Happy birthday Jack!!!

Gracias!!!