Transversality Principles in Holomorphic Dynamics

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Overview

- Parameter spaces and moduli spaces :
  - The parameter space $\mathbf{Rat}_D$ of all degree D rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a smooth affine algebraic variety of dimension $2D + 1$.
  - The group of projective transformations Aut acts on $\mathbf{Rat}_D$ by conjugation, and for $D > 1$ the quotient moduli space $\mathbf{rat}_D$ is an orbifold of dimension $2D - 2$.
- These spaces have various dynamically significant subspaces, determined by such conditions as the existence of :
  - specified critical orbit relations,
  - points of specified period and multiplier,
  - parabolic points of specified degeneracy and index,
  - Herman rings of specified period and rotation number.
- Concerning these loci, we might ask :
  - **Local Questions** : Are they smooth ? Of what dimension ? Are their intersections transverse ?
  - **Global Questions** : Are they nonempty ? Are they connected ? How do they behave near infinity ?

Case Study

Milnor's inspiring paper

Geometry and dynamics of quadratic rational maps

makes a study of $\mathbf{rat}_2$ using elementary algebraic methods.

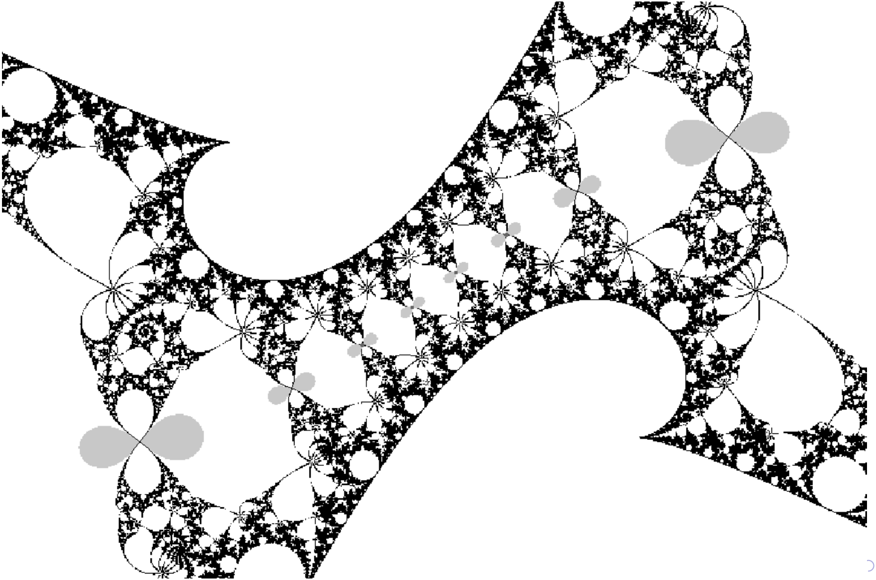
Consider the symmetric functions

$$X = \alpha\beta\gamma,\ Y = \alpha\beta + \beta\gamma + \alpha\gamma,\ Z = \alpha + \beta + \gamma$$

of the fixed point multipliers α, β, γ.

- The Holomorphic Index Formula yields the relation $Z = X + 2$.
- The map
  $$\mathbf{rat}_2 \ni [f] \mapsto (X, Y) \in \mathbb{C}^2$$
  is an isomorphism.
- For $n \geq 1$ and $\rho \in \mathbb{C}$, the locus $\mathbf{Per}_n(\rho) \subset \mathbf{rat}$ corresponding to maps which possess a (formal) n-cycle of multiplier ρ is an algebraic curve : in particular, $\mathbf{Per}_1(\rho)$ is a line.

$\mathbf{Per}_1(e^{2\pi i/10})$



Manifesto

We :

- Develop language for posing, and methodology for answering, such local questions ;
- Propose that aspects of this formalism may also be useful in the study of certain global questions ;
- Contend that the abstraction and generality reveal unexpected unity.

Unity

‘A un moment où la mode mathématique est au mépris de la généralité (assimilée à "des généralités" gratuites, voire à des bombinage), je puis constater que la force principale manifeste à travers toute mon oeuvre de mathématicien a bien été la quête du "général". Il est vrai que je préfère mettre l’accent sur "l‘unité", plutôt que sur "la généralité". Mais ce sont là pour moi deux aspects d’une seule et même quête. L’unité en représente l’aspect profond, et la généralité, l’aspect superficiel.’

Grothendieck, *Récoltes et Semailles*

Origins

Kodaira-Spencer's fundamental results in complex analytic geometry, concerning the relation between deformation theory and cohomology.

- The space of infinitesimal deformations of a compact complex manifold is canonically isomorphic to the first cohomology of the sheaf of germs of infinitesimal automorphisms.
- **Idea :** Variation of $h_{W,U} \circ h_{V,W} \circ h_{U,V} = I$ yields a 1-cocycle.

Thurston's fundamental results in complex analytic dynamics, concerning the relation between branched covers on topological spheres and rational maps on $\mathbb{P}^1$.

- A postcritically finite branched cover $F : \Sigma \to \Sigma$ is combinatorially equivalent to a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ if and only if there is no obstruction.
- **Idea :** Seek a fixed point in the deformation space of $(\Sigma, P(f))$.

Ideology

- For dynamically significant loci as above :
  - Local properties - smoothness and transversality - are manifestations of Thurston's Rigidity Theorem.
  - Global properties - nontriviality, irreducibility, homotopy type, ends - are manifestations of Thurston's Existence Theorem.
- Transversality is most naturally phrased, studied, and proved in deformation spaces obtained by a functorial construction from first principles in Teichmüller theory.
- These deformation spaces are finite dimensional, and there are explicit verifiable conditions for the nonsingularity of the canonical maps to moduli space.

Scope

- **Transcendental Dynamics.** The deformation space construction is available, and the transversality principles are valid, for *finite type* maps - exp, tan, $\wp$, λ, j, *parabolic renormalizations*, *skinning maps* . . . - some of which belong to evident finite dimensional parameter spaces, and others of which do not.
- **Arithmetic Dynamics ?** The transversality principles are largely algebraic, and the underlying cohomological formalism is available over any algebraically closed field of characteristic zero. The core infinitesimal rigidity principle is a striking example of a purely algebraic statement only known via transcendental techniques applied over $\mathbb{C}$.

Finite Type Maps

An analytic map of complex 1-manifolds

$$f : W \to X$$

is of *finite type* if :

- X is compact,
- f is open,
- f has no isolated removable singularities,
- $S(f)$ is finite.

Here $S(f)$ is the set of *singular values* : the points $x \in X$ such that no open neighborhood of x is *evenly covered*. For a finite type map, this set consists of the *critical values* and the *asymptotic values*.

Teichmüller Spaces

Let X be a compact oriented real 2-manifold, and let $E \subset X$ be finite.

- The Teichmüller space $\mathrm{Teich}(X, E)$ consists of all equivalence classes of complex structures on X, where structures are identified if they are related via pullback by a homeomorphism which is isotopic to the identity relative to E.
- $\mathrm{Teich}(X, E) \cong \prod_{Z \in \pi_0(X)} \mathrm{Teich}(Z, E \cap Z)$
- $\mathrm{Teich}(X, E)$ is a finite dimensional complex manifold. If X is connected of genus g then

$$\dim \mathrm{Teich}(X, E) = \begin{cases} \max(\#E - 3, 0) & \text{if} \quad g = 0 \\ \max(\#E, 1) & \text{if} \quad g = 1 \\ 3g - 3 + \#E & \text{if} \quad g \geq 2 \end{cases}$$

Serre Duality

If X is a complex 1-manifold then $\mathrm{Teich}(X, E)$ has a basepoint $\bullet$.
The cotangent and tangent spaces at $\bullet$ have canonical descriptions in terms of sheaf cohomology :

$$\begin{tikzcd}
T^*_\bullet \mathrm{Teich}(X,E) \times T_\bullet \mathrm{Teich}(X,E) \arrow[r] \arrow[d, "\cong \times \cong"'] & \mathbb{C} \\
H^0(X, \Omega \otimes \Omega \otimes \mathcal{O}_E) \times H^1(X, \Theta \otimes \mathcal{O}_{-E}) \arrow[r] & H^1(X, \Omega) \arrow[u, "\cong"']
\end{tikzcd}$$

- Ω is the sheaf of germs of holomorphic differential forms
- Θ is the sheaf of germs of holomorphic vector fields

The isomorphism $H^1(X, \Omega) \to \mathbb{C}$ is given in terms of a residue sum.

Such a cohomological discussion is available over any algebraically closed field of characteristic zero, for example $\overline{\mathbb{Q}}$.

Dolbeault Isomorphism

$$H^1(X, \Theta \otimes \mathcal{O}_{-E}) \cong \mathrm{Bel}(X)/\mathrm{bel}_E(X)$$

where

$$\begin{aligned} \mathrm{Bel}(X) &= \{(-1,1)\text{-forms on } X\} \\ \mathrm{bel}_E(X) &= \bar{\partial}\{\text{vector fields on } X \text{ which vanish on } E\} \end{aligned}$$

In terms of this description, the pairing

$$H^0(X, \Omega \otimes \Omega \otimes \mathcal{O}_E) \times H^1(X, \Theta \otimes \mathcal{O}_{-E}) \to \mathbb{C}$$

takes the form

$$(q, [\mu]_E) \mapsto \langle q, \mu \rangle = \frac{1}{2\pi i} \int_X q \cdot \mu$$

Quadratic Differentials

We denote by $\mathcal{Q}(X)$ the $\mathbb{C}$-linear space of all meromorphic quadratic differentials on X with at worst simple poles :

$$\mathcal{Q}(X) = \bigcup_{\text{finite} E \subset X} \mathcal{Q}(X, E)$$

where

$$\mathcal{Q}(X, E) = H^0(X, \Omega \otimes \Omega \otimes \mathcal{O}_E).$$

- $\mathcal{Q}(X)$ consists of all meromorphic quadratic differentials q on X such that
$$\|q\| = \int_X |q|$$
is finite.

Forgetful and Pullback Maps

Let A and B be finite subsets of a compact complex 1-manifold X.

- For $A \subseteq B$ there is a *forgetful map*

$$p : \mathrm{Teich}(X, B) \to \mathrm{Teich}(X, A)$$

with coderivative the inclusion

$$\mathcal{Q}(X, A) \hookrightarrow \mathcal{Q}(X, B)$$

- For finite type f on X, if $f(A) \cup S(f) \subseteq B$ there is a *pullback map*

$$\sigma_f : \mathrm{Teich}(X, B) \to \mathrm{Teich}(X, A)$$

with coderivative the pushforward operator

$$f_* : \mathcal{Q}(X, A) \to \mathcal{Q}(X, B)$$

given by

$$f_* q = \sum_{\text{branches } h \text{ of } f^{-1}} h^* q$$

Deformation Spaces

If $A \cup f(A) \cup S(f) \subseteq B$ then p and σ_f share domain and codomain.

$$\mathrm{Def}_A^B(f) \dashrightarrow \mathrm{Teich}(X, B) \rightrightarrows \mathrm{Teich}(X, A)$$

$$\mathrm{Def}_A^B(f) = \left\{\tau \in \mathrm{Teich}(X, B) \;:\; \sigma_f(\tau) = p(\tau)\right\}.$$

Theorem

Assume that

$$(\star)\begin{cases} f \text{ is a finite type analytic map on a compact Riemann surface } X, \\ f \text{ is not an automorphism, Lattès example or toral endomorphism,} \\ \#A \geq 3 \text{ if } X \text{ has genus 0, and } \#A \geq 1 \text{ if } X \text{ has genus 1,} \\ A \cup f(A) \cup S(f) \subseteq B. \end{cases}$$

Then $\mathrm{Def}_A^B(f)$ *is a* $\#(B \setminus A)$*-dimensional* $\mathbb{C}$*-analytic manifold.*

Contraction Principle

By the Implicit Function Theorem, the above follows from

Proposition (Thurston, Douady-Hubbard, McMullen, E)

Let X and Y be compact Riemann surfaces, and $f : W \to Y$ a finite type analytic map where $W \subseteq X$. For $q \in \mathcal{Q}(X)$:

- $\|f_* q\| \leq \|q\|$.
- *Equality holds if and only if $f^* f_* q = (\deg f) \cdot q$, whence $\deg f < \infty$.*

$$\nabla_f = I - f_*$$

Corollary

Under assumptions $(\star)$:

- $\nabla_f : \mathcal{Q}(X) \to \mathcal{Q}(X)$ *is injective,*
- $T^*_\bullet \mathrm{Def}_A^B(f)$ *is canonically isomorphic to* $\mathcal{Q}(X, B)/\nabla_f \mathcal{Q}(X, A)$.

Global Properties

- If $\dim \mathcal{Q}(X, A) = 0$ then $\mathrm{Def}_A^B(f) \cong \mathrm{Teich}(X, B)$.
- If $\operatorname{codim} \mathcal{Q}(X, A) = 0$ then $\mathrm{Def}_A^B(f)$ is a point.

In these extremal cases $\mathrm{Def}_A^B(f)$ is contractible.

- Is $\mathrm{Def}_A^B(f)$ always contractible ?
- Is $\mathrm{Def}_A^B(f)$ always connected ?

Transversality Principles

Theorem

Under assumptions $(\star)$ the loci in $\mathrm{Def}_A^B(f)$ corresponding to

- *specified singular orbit relations,*
- *points of specified period and nonrepelling multiplier,*
- *parabolic points of specified degeneracy and nonrepelling index,*
- *Herman rings of specified period and Brjuno rotation number,*

are smooth and pairwise transverse.

Variations

- $[\nabla_f q]$ where q has simple poles, some of which lie outside A. These describe the breaking of orbit relations.
- $[\nabla_f q]$ where q has appropriate multiple poles along cycles in A. These describe the variation of multipliers, and further quantities associated to parabolic cycles.
- $[\nabla_f q]$ where q has rotationally invariant discontinuities along Herman ring cycles. These describe the annihilation of such ring cycles.

Invariant Divergences

$$0 \to \mathcal{Q} \to \mathcal{M} \to \mathcal{D} \to 0$$

$$\begin{tikzcd}
 & \mathcal{Q}(X)^f \arrow[d] & \mathcal{M}(X)^f \arrow[d] & \mathcal{D}(X)^f \arrow[d] & \\
0 \arrow[r] & \mathcal{Q}(X) \arrow[r] \arrow[d, "\nabla_f"] & \mathcal{M}(X) \arrow[r] \arrow[d, "\nabla_f"] & \mathcal{D}(X) \arrow[r] \arrow[d, "\nabla_f"] & 0 \\
0 \arrow[r] & \mathcal{Q}(X) \arrow[r] \arrow[d] & \mathcal{M}(X) \arrow[r] \arrow[d] & \mathcal{D}(X) \arrow[r] \arrow[d] & 0 \\
 & \mathcal{Q}(X)_f & \mathcal{M}(X)_f & \mathcal{D}(X)_f &
\end{tikzcd}$$

$$0 \to \mathcal{Q}(X)^f \to \mathcal{M}(X)^f \to \mathcal{D}(X)^f \xrightarrow{\blacktriangledown_f} \mathcal{Q}(X)_f \to \mathcal{M}(X)_f \to \mathcal{D}(X)_f \to 0$$

Dictionary

- If x is superattracting then $\mathcal{D}_x^f$ is 0-dimensional.
- If x is attracting, repelling or irrationally indifferent then $\mathcal{D}_x^f$ is the 1-dimensional space spanned by $\left[\frac{\mathrm{d}\zeta^2}{\zeta^2}\right]_x$ where ζ is any local coordinate vanishing at x.
- If x is parabolic with multiplier a primitive n-th root of unity then $\mathcal{D}_x^f$ is the direct sum of the ν-dimensional subspace $\mathbf{D}_x^f$ spanned by

$$\left[\frac{\mathrm{d}\zeta^2}{\zeta^2}\right]_x, \dots, \left[\frac{\mathrm{d}\zeta^2}{\zeta^{\ell n+2}}\right]_x, \dots, \left[\frac{\mathrm{d}\zeta^2}{\zeta^{N-n+2}}\right]_x$$

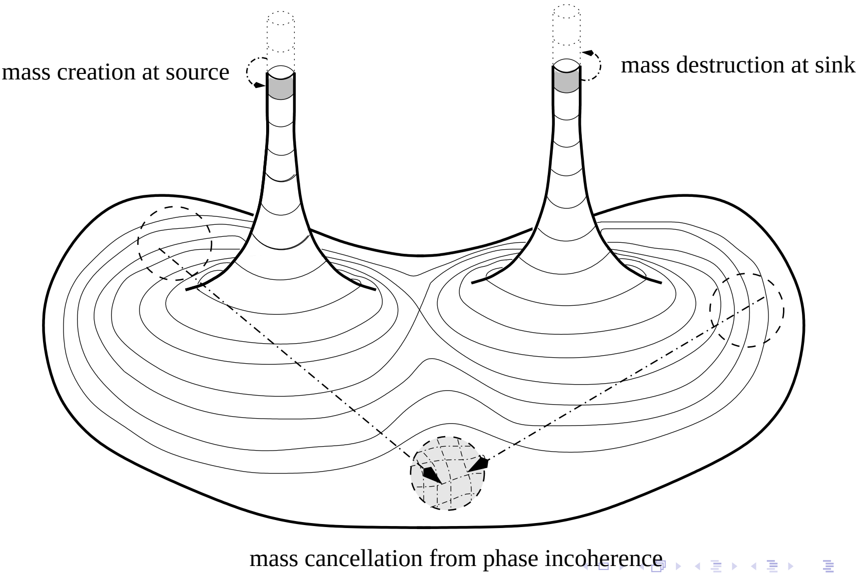
and the 1-dimensional subspace spanned by

$$\left[\frac{\mathrm{d}\zeta^2}{(\zeta^{N+1}-\beta\zeta^{2N+1})^2}\right]_x = \left[\frac{\mathrm{d}\zeta^2}{\zeta^{2N+2}} + 2\beta\frac{\mathrm{d}\zeta^2}{\zeta^{N+2}} + 3\beta^2\frac{\mathrm{d}\zeta^2}{\zeta^2} + O\left(\frac{\mathrm{d}\zeta^2}{\zeta}\right)\right]_x$$

where ζ is any local coordinate in which f is given by

$$\zeta \mapsto \rho\zeta\left(1 + \zeta^N + \left(\frac{N+1}{2} - \beta\right)\zeta^{2N} + O(\zeta^{2N+1})\right).$$

Injectivity of ∇_f



Fatou-Shishikura Inequality

- $\gamma(f)$ is the number of cycles of f, counting $\langle x\rangle$ with multiplicity

$$\gamma_{\langle x\rangle}(f) = \begin{cases} 0 & \text{if } \langle x\rangle \text{ is repelling or superattracting} \\ 1 & \text{if } \langle x\rangle \text{ is attracting or irrationally indifferent} \\ \nu & \text{if } \langle x\rangle \text{ is parabolic-repelling} \\ \nu+1 & \text{if } \langle x\rangle \text{ is parabolic-attracting} \\ & \qquad \text{or parabolic-indifferent} \end{cases}$$

- $\delta(f)$ is the number of infinite tails of postsingular orbits.

The inequality $\gamma(f) \le \delta(f)$ follows from a dimension comparison :

$$\begin{tikzcd}
\gamma(f) & \breve{\mathcal{D}}(X)^f \arrow[rr, hook] \arrow[dr, "\nabla_f"'] & & \mathcal{D}(X)^f \arrow[dl, "\nabla_f"] & \infty \\
\text{injective} & & \mathcal{Q}(X,P(f))/\nabla_f\mathcal{Q}(X,P(f)) & & \text{surjective ?} \\
& & \delta(f) & &
\end{tikzcd}$$

Herman Rings

Let $\hbar(f)$ be the number of Herman ring cycles.

- The injectivity of ∇_f persists in a larger space in which each Herman ring cycle contributes one additional dimension, corresponding to rotationally invariant discontinuities, and the above argument yields the improved inequality $\gamma(f) + \hbar(f) \leq \delta(f)$.
- The sharp inequality $\gamma(f) + 2\hbar(f) \leq \delta(f)$ is obtained after a supplementary argument involving a description of the infinitesimal quasiconformal deformation space, part of a larger discussion including results such as :
    - The canonical maps from quasiconformal deformation spaces to deformation spaces are immersions.
    - Finite type maps have no wandering domains.

From Deformation Space to Moduli Space

An element of $\mathrm{Def}_A^B(f)$ determines a conformal conjugacy class of finite type maps similar to f. In particular, for rational $f : \mathbb{P}^1 \to \mathbb{P}^1$, if $\varphi : (\mathbb{P}^1, B) \to \big(\mathbb{P}^1, \varphi(B)\big)$ represents $\tau \in \mathrm{Def}_A^B(f)$, then there are

- a unique $\psi : (\mathbb{P}^1, A) \to \big(\mathbb{P}^1, \psi(A)\big)$ representing $p(\tau) = \sigma_f(\tau)$, agreeing with φ on A and
- a unique rational map F_φ

such that the following diagram commutes :

$$\begin{tikzcd}
(\mathbb{P}^1, A) \arrow[r, "\psi"] \arrow[d, "f"'] & \big(\mathbb{P}^1, \psi(A)\big) \arrow[d, "F_\varphi"] \\
(\mathbb{P}^1, B) \arrow[r, "\varphi"] & \big(\mathbb{P}^1, \varphi(B)\big)
\end{tikzcd}$$

Moreover, if φ and $\hat{\varphi}$ represent the same point in $\mathrm{Def}_A^B(f)$, then F_φ and $F_{\hat{\varphi}}$ are conjugate by the unique projective transformation M such that $\hat{\varphi}_{|B} = M \circ \varphi_{|B}$.

From Deformation Space to Moduli Space

There is an induced map $\mathrm{Def}_A^B(f) \to \mathbf{rat}_D$ whose derivative at $\bullet$ is the connecting homomorphism for the diagram :

$$
\begin{tikzcd}
 & & & T_\bullet \mathrm{Def}_A^B(f) \arrow[d] & \\
0 \arrow[r] & H^0(\mathbb{P}^1, \Theta) \arrow[r] \arrow[d, "I-f^*"] & H^0(\mathbb{P}^1, \Theta/\Theta_{-B}) \arrow[r] \arrow[d, "I-f^*"] & H^1(\mathbb{P}^1, \Theta_{-B}) \arrow[r] \arrow[d, "I-f^*"] & 0 \\
0 \arrow[r] & H^0(\mathbb{P}^1, \Theta_{\Gamma(f)}) \arrow[r] \arrow[d] & H^0(\mathbb{P}^1, \Theta_{\Gamma(f)}/\Theta_{-A}) \arrow[r] & H^1(\mathbb{P}^1, \Theta_{-A}) \arrow[r] & 0 \\
 & T_{[f]}\mathbf{rat}_D & & &
\end{tikzcd}
$$

Here $\Gamma(f)$ is the (multi)set of critical points.