Spheres

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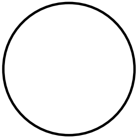
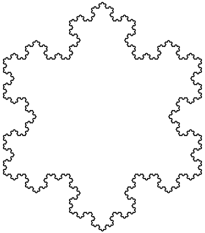
Examples of Spheres:

The **standard sphere** $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the locus

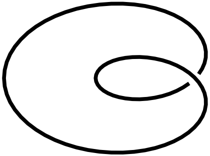
$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1 .$$



The standard 1-sphere $\mathbb{S}^1$.

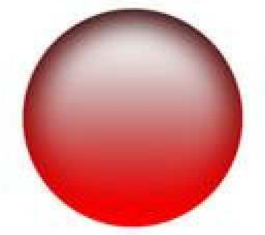


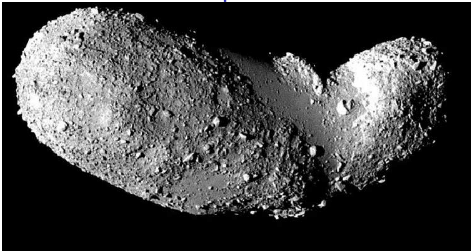
A **topological** 1-sphere.



A **smooth** 1-sphere.

Standard, Topological, and Smooth 2-spheres

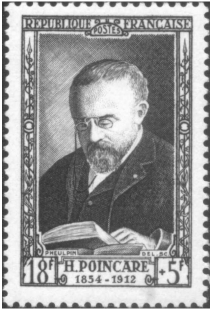




Asteroid Itokawa,
Japan Aerospace Agency

Dancing Bear by Anita Issaluk,
Chesterfield Inlet, Nunavut

Topological Characterization of Spheres



Poincaré's Question in 1904

(Oevre VI, p.498):

"Est-il possible que le groupe fondemental de V se réduise à la substitution identique, et que pourtant V ne soit pas simplement connexe?"

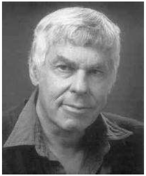
It took 100 years to find the answer:

Theorem GPH. *A closed n-dimensional manifold M^n is homeomorphic to $\mathbb{S}^n$ $\iff$ it has the same homotopy type as $\mathbb{S}^n$*
$\iff$ it has the same homology and fundamental group as $\mathbb{S}^n$
$\iff$ any proper subset can be shrunk to a point within M^n.

This is a compilation of work by many different people over 150 years!

For dimensions $n \leq 2$ it is classical. (Compare: Francis and Weeks, 1999.)

High Dimensional Cases.



Steve Smale made the first breakthrough in 1961, giving a proof for **smooth** n-manifolds with $n > 4$.

John Stallings and E. C. Zeeman, using a different method, proved this for **Piecewise Linear** manifolds with $n > 4$.

Max Newman and E. H. Connell modified the Stallings argument to cover all **topological** manifolds of dimension $n > 4$.

The case $n = 4$ is much harder.

Mike Freedman proved the 4-dimensional theorem in 1982, using wildly non-differentiable methods.

In fact, he classified all possible closed simply-connected topological 4-manifolds, using just two invariants:

- the quadratic form $x \mapsto x \cup x$, where

$$x \in H^2(M^4) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}, \qquad x \cup x \in H^4(M^4) \cong \mathbb{Z},$$

- and an invariant in $\mathbb{Z}/2$ which is zero when M^4 is smooth.

  (**Note:** *I will always use homology or cohomology with* **integer coefficients**.)

The hardest case: $n = 3$

Bill Thurston's **Geometrization Conjecture** suggested an effective description of all possible closed 3-manifolds.

Richard Hamilton introduced the **Ricci flow** method in an attempt to prove the Geometrization Conjecture.

Grisha Perelman managed to overcome all of the many difficulties with this method !

QED for Theorem GPH.

Suppose we translate Poincaré's question somewhat differently:

Consider a **smooth** manifold M^n, and ask whether it is **diffeomorphic** to the standard sphere $\mathbb{S}^n$.

We might try to use the following:

Lemma. *Any homeomorphism* $f : M^n \to \mathbb{S}^n$ *can be uniformly approximated by a smooth map* $M^n \to \mathbb{S}^n$.

Question: Can a homeomorphism between smooth manifolds always be approximated by a diffeomorphism?

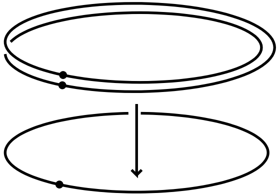
The answer is **No** !

Sphere Bundles over Spheres

In the middle 1950s, I was completely stunned by an apparent contradiction in mathematics.

Consider 3-sphere bundles over the 4-sphere:

$$\begin{array}{ccc} \mathbb{S}^3 & \subset & M^7 \\ & & \downarrow \\ & & \mathbb{S}^4\,. \end{array}$$

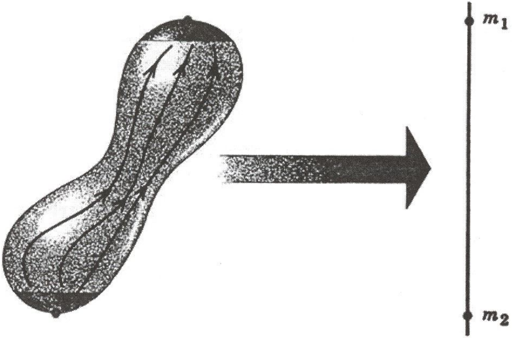


I found examples where M^7 was a sphere by a topological argument; but couldn't be by a differentiable argument.

The only way out of this apparent contradiction was to assume that M^7 was homeomorphic to $\mathbb{S}^7$, but not diffeomorphic to $\mathbb{S}^7$.

To understand such examples, we need methods for **proving homeomorphism**, and for **disproving diffeomorphism**.

Proving Homeomorphism: George Reeb's Criterion



Theorem: *Let M^n be a smooth closed manifold. If there is a Morse function $M^n \to \mathbb{R}$ with only two critical points, then M is a topological n-sphere.*

We want to prove that certain $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ are not diffeomorphic to $\mathbb{S}^7$.

The proof will be based on a linear equation

$$45\ \sigma(M^8) \;=\; 7\, p_2\langle M^8\rangle \;-\; p_1^{\,2}\langle M^8\rangle\,.$$

relating three different integer invariants for a **smooth** closed oriented 8-manifold.

I Must Answer Three Questions:

- What are these invariants?
- How does one prove such a relation between them?
- What does this have to do with 7-dimensional manifolds?

- For any closed oriented $4k$-dimensional manifold we can form the **signature** $\sigma(M^{4k})$ of the quadratic form

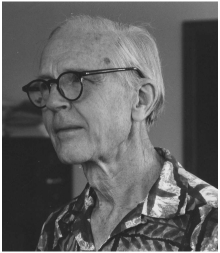
$$x \mapsto x^2 = x \cup x \quad \text{from} \quad H^{2k}(M^{4k};\, \mathbb{Z}) \quad \text{to} \quad H^{4k}(M^{4k};\, \mathbb{Z}) \cong \mathbb{Z}\,.$$

Simply diagonalize this form over the real numbers, and count the number of positive diagonal entries minus the number of negative ones.

This is an integer valued topological invariant.

- The two numbers $p_2\langle M^8\rangle$ and $p_1^2\langle M^8\rangle$ are integer invariants called **Pontrjagin numbers.**

Their description will require several steps.



Hassler Whitney showed that any smooth M^n has an essentially unique embedding $M^n \xrightarrow{\subset} \mathbb{R}^L$ provided that the dimension L is large enough ($L > 2n+1$).

Hermann Grassmann studied the manifold $G_n(\mathbb{R}^L)$ consisting of all n-dimensional planes through the origin in $\mathbb{R}^L$.

Let $\mathbf{G}_n$ be the limit as $L \to \infty$,

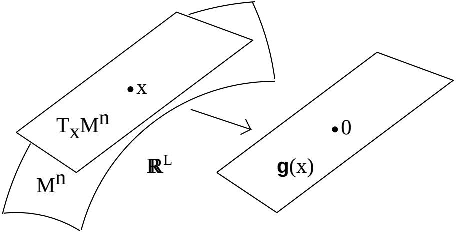
$$G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+2}) \subset \cdots \subset \mathbf{G}_n\,.$$

The (Generalized) Gauss Map

For a smooth manifold $M^n \subset \mathbb{R}^L$, the **"Gauss map"**

$$\mathbf{g} = \mathbf{g}_{M^n} : M^n \to G_n(\mathbb{R}^L) \subset \mathbf{G}_n$$

sends each $x \in M^n$ to the tangent n-plane $T_x M^n$, translated to the origin.



The Characteristic Homology Class

Every closed oriented M^n has a **fundamental homology class**

$$\mu \in H_n(M^n)\,.$$

For any smooth $M^n \subset \mathbb{R}^{n+L}$, the Gauss map $\mathbf{g} : M^n \to \mathbf{G}_n$ induces a homomorphism

$$\mathbf{g}_* : H_n(M^n) \to H_n(\mathbf{G}_n)\,.$$

If M^n is oriented, then the fundamental homology class

$\mu \in H_n(M^n)$ maps to a "**characteristic homology class**"

$$\langle M^n \rangle = \mathbf{g}_*(\mu) \in H_n(\mathbf{G}_n)\,.$$

Pontrjagin Numbers

Lev Pontrjagin introduced what we would now describe as cohomology classes

$$p_i \in H^{4i}(G_n)\,.$$

Modulo elements of finite order, these generate the cohomology ring $H^*(G_n)$.

Consider sequences

$$1 \;\le\; i_1 \;\le\; i_2 \;\le\; \cdots \;\le\; i_h \qquad \text{with} \qquad \sum_1^h i_j \;=\; k$$

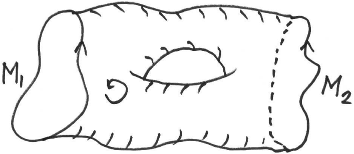
so that $p_{i_1} p_{i_2} \cdots p_{i_h} \;\in\; H^{4k}(\mathbf{G}_n)$.

Taking $n = 4k$, we can evaluate each such product on the characteristic homology class $\langle M^{4k}\rangle \in H_{4k}(\mathbf{G}_{4k})$.

This yields an integer $p_{i_1} p_{i_2} \cdots p_{i_h}\langle M^{4k}\rangle$ called a

Pontrjagin number.

René Thom's Cobordism Theory



Two closed oriented n-manifolds are **oriented cobordant** if their disjoint union, suitably oriented, is the boundary of a compact oriented $(n+1)$-manifold.

Theorem (mostly due to Thom). The characteristic homology class $\langle M^n \rangle \in H_n(\mathbf{G}_n)$ is a **complete cobordism invariant:**

M_1 and M_2 are cobordant if and only if $\langle M_1^n \rangle = \langle M_2^n \rangle$.

(Proved by Thom up to elements of finite order. C. T. C. Wall took care of 2-primary elements; Sergei Novikov and I took care of elements of odd order.)

The set of all cobordism classes of smooth oriented closed n-manifolds forms an **abelian group** Ω_n, with the disjoint union as sum operation.

Corollary. The correspondence

$$\text{(cobordism class of } M^n) \quad \mapsto \quad \langle M^n \rangle \in H_n(\mathbf{G}_n)$$

embeds Ω_n as a subgroup of finite index

$$\Omega_n \xrightarrow{\subset} H_n(\mathbf{G}_n)\,.$$

The Signature Formula

Lemma (Thom). If $n = 4k$, then the signature of the quadratic form

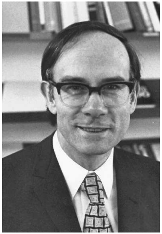
$$x \mapsto x^2 = x \cup x \quad \text{from} \quad H^{2k}(M^{4k}) \quad \text{to} \quad H^{4k}(M^{4k}) \xrightarrow{\cdot\mu} \mathbb{Z}$$

is a cobordism invariant; yielding a homomorphism $\sigma : \Omega_{4k} \to \mathbb{Z}$.

Corollary. The signature of M^{4k} can be expressed as a linear combination of Pontrjagin numbers, with **rational** coefficients.

$$\sigma(M^{4k}) = \sum a(i_1, \ldots, i_h)\, p_{i_1} \cdots p_{i_h} \langle M^{4k} \rangle ,$$

to be summed over all $0 < i_1 \le i_2 \le \cdots \le i_h$ with sum k.



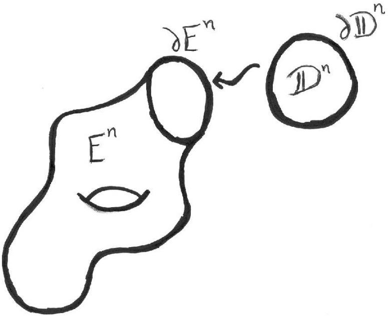
Hirzebruch computed these rational coefficients in terms of Bernoulli numbers

From 8-Manifolds to Exotic 7-Spheres.

Let E^n be a smooth compact n-manifold, bounded by a smooth manifold homeomorphic to $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$.

Choosing a homeomorphism $f : \mathbb{S}^{n-1} \to \partial E^n$, we can paste $\mathbb{D}^n$ onto E^n to obtain a closed topological manifold

$$M^n \;=\; E^n \cup_f \mathbb{D}^n\,.$$



If f is a diffeomorphism, then M^n can be made into a smooth manifold.

The Obstruction to Smoothness.

Now consider the case $n = 8$.

The signature of $M^8 = E^8 \cup_f \mathbb{D}^8$ can be computed from the cohomology of the pair $(E^8, \partial E^8)$.

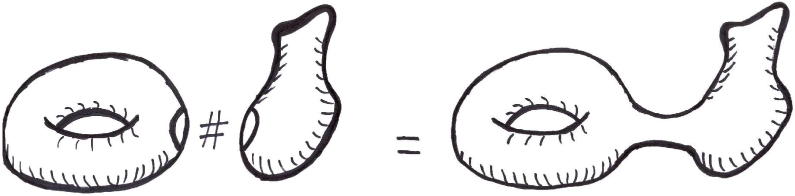
Similarly, the Pontrjagin number $p_1^2\langle M^8\rangle$ can be computed from knowledge of E^8 as a smooth manifold.

We can then solve for

$$p_2\langle M^8\rangle = \frac{45\,\sigma(M^8) + p_1^2\langle M^8\rangle}{7}.$$

Whenever this quotient is not an integer, we have proved that ∂E^8 cannot be diffeomorphic to $\mathbb{S}^7$.

If M_1 and M_2 are smooth, oriented, connected n-manifolds, then the **connected sum** $M_1 \# M_2$ is a new smooth, oriented, connected n-manifold.



This operation is well defined up to orientation preserving diffeomorphism. Thus we obtain a commutative, associative semigroup $\mathscr{M}_n$ of oriented diffeomorphism classes; with the class of $\mathbb{S}^n$ as identity element, $M^n \# \mathbb{S}^n \cong M^n$.

A Test for Invertibility

Lemma (Barry Mazur).

(1) M^n is invertible $(M^n \# N^n \cong \mathbb{S}^n)$

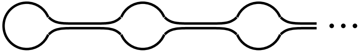
$\Leftrightarrow$ **(2)** $M^n \setminus \{\text{point}\} \xrightarrow{\subset} \mathbb{S}^n$

$\Leftrightarrow$ **(3)** $M^n \setminus \{\text{point}\} \cong \mathbb{R}^n$

$\Rightarrow$ **(4)** M^n is a topological sphere.

Proof that **(1)** $\Longrightarrow$ **(3)**,
using "**infinite connected sums**".

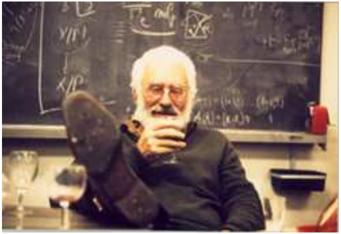
First consider the sum $\mathbb{S}^n \# \mathbb{S}^n \# \mathbb{S}^n \# \cdots \cong \mathbb{R}^n$



$$\begin{array}{ccccc} & (M\#N)\#(M\#N)\#\cdots & \cong & \mathbb{S}^n\#\mathbb{S}^n\#\cdots & \cong & \mathbb{R}^n \\ \cong & M\#(N\#M)\#(N\#M)\#\cdots & \cong & M\#\mathbb{R}^n & \cong & M\setminus\{\text{point}\}\,. \end{array}$$

Thus **(1)** $\Longrightarrow$ **(3)**. The proof that (**3**) $\Longrightarrow$ (**2**) $\Longrightarrow$ (**1**) is not difficult, so this proves the Lemma.

Work with Michel Kervaire



Let $\mathscr{S}_n \subset \mathscr{M}_n$ be the sub-semigroup of smooth oriented manifolds homeomorphic to $\mathbb{S}^n$. For example

$$\mathscr{S}_1 = \mathscr{S}_2 = \mathscr{S}_3 = 0\,.$$

Theorem. This semigroup $\mathscr{S}_n$ is a finite abelian group for $n > 4$, with

$$\mathscr{S}_5 = \mathscr{S}_6 = 0\,,$$

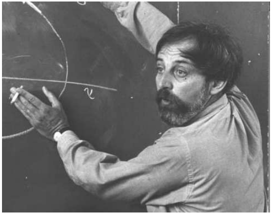
but:

$\mathscr{S}_7$	$\mathscr{S}_8$	$\mathscr{S}_9$	$\mathscr{S}_{10}$	$\mathscr{S}_{11}$	$\mathscr{S}_{12}$	$\mathscr{S}_{13}$	$\cdots$
$\mathbb{Z}/28$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/6$	$\mathbb{Z}/992$	0	$\mathbb{Z}/3$	$\cdots$

Three Necessary Ingredients for our Work

Witold Hurewicz introduced higher homotopy groups.

Jean-Pierre Serre developed the algebraic machinery needed to compute these groups

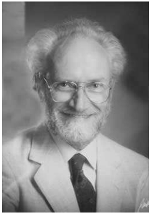


Raoul Bott computed the homotopy groups of the infinite rotation group **SO**.

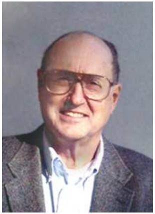
Further Developments by Many People.

Frank Adams

Greg Brumfiel

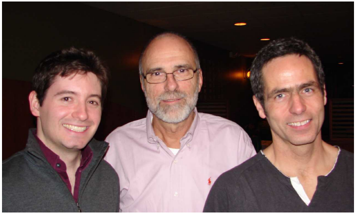


Bill Browder



Mark Mahowald

and for the latest news:



Mike Hill, Doug Ravenel and Mike Hopkins

The group $\mathscr{S}_n$ is now completely known for $n \leq 64$,

EXCEPT FOR THE CASE $n = 4$!

Dimensional Four: The Big Mystery.

Theorem (Simon Donaldson). If M^4 is smooth, simply-connected, with positive definite quadratic form, then the quadratic form can be diagonalized $\implies$ M^4 is homeomorphic to a connected sum

$$\mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2 .$$

But there are many unimodular quadratic forms which cannot be diagonalized; hence there are many **topological** manifolds which cannot be given any differentiable structure.

The combination of Donaldson's methods and Freedman's methods had amazing consequences:

Manifolds Homeomorphic to $\mathbb{R}^4$.

Theorem (Cliff Taubes). There are uncountably many distinct diffeomorphism classes of smooth manifolds homeomorphic to $\mathbb{R}^4$.

By way of contrast:

Theorem (Stallings + Munkres + Hirsch). If $n \neq 4$, then any smooth manifold homeomorphic to $\mathbb{R}^n$ must actually be diffeomorphic to $\mathbb{R}^n$. $\Longrightarrow$ $\mathscr{S}_n$ is a group for $n \neq 4$.

But the semigroup $\mathscr{S}_4$ is completely unknown:

Is it trivial?

Is it finite?

Is it a group?

? ? ? ? ?