Examining the Pedagogical Implications of a Secondary Teacher's Understanding of Angle Measure

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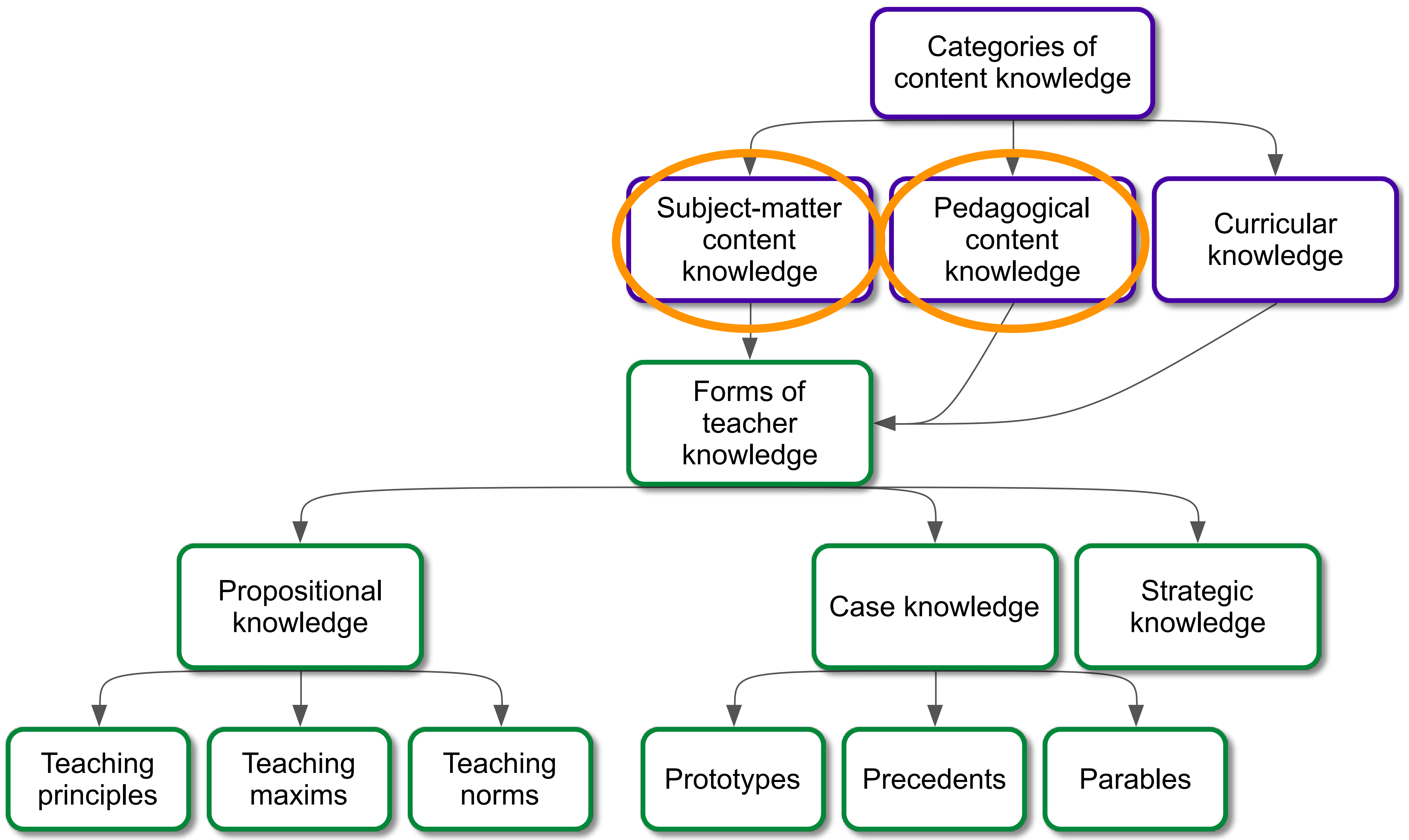
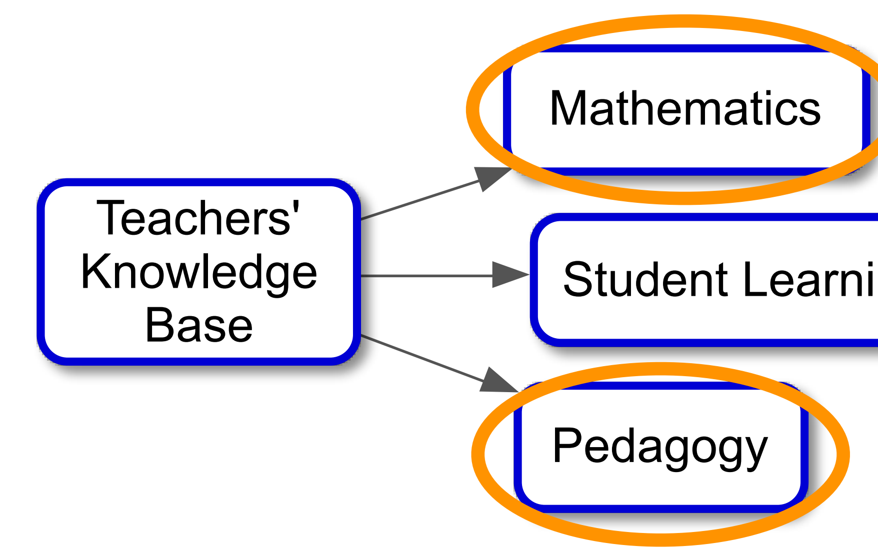
Arizona State University

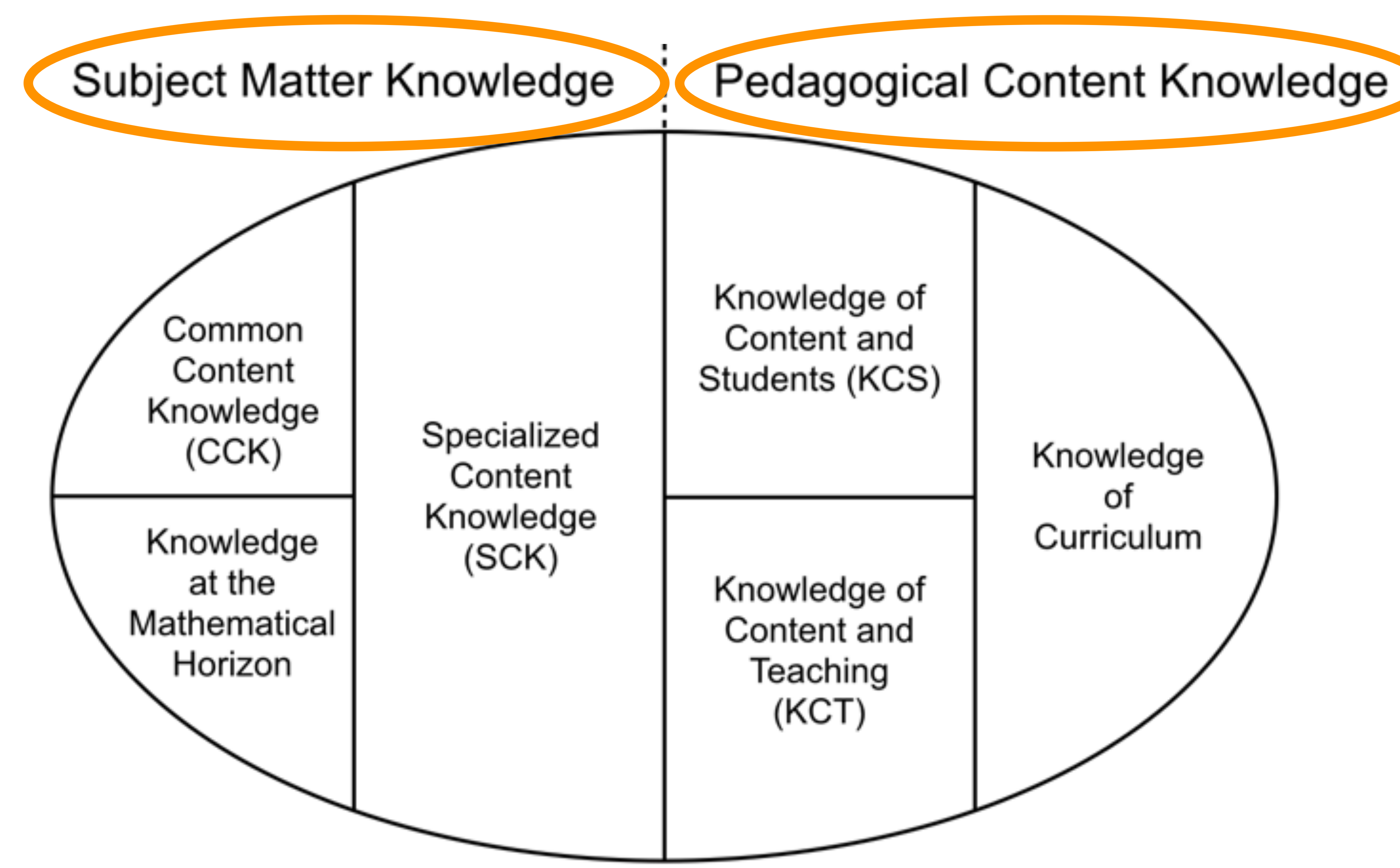
MKT Literature



Shulman (1986, 1987)



Harel (2008)



Hill, Ball, & Schilling (2008)

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Research Question

What are the pedagogical implications of a secondary mathematics teacher's understanding of angle measure?

Methodology

Participant (David): Taught Honors Algebra II and AP Calculus.

Experimental:

- Series of eight task-based clinical interviews.
- Videos of classroom teaching (38 sessions).

Analytical:

- Grounded theory (Strauss & Corbin, 1990).
- Generative approach for analyzing clinical interviews (Clement, 2000).

Theoretical Framework

- Piaget's Genetic Epistemology
  - Equilibration (scheme, assimilation, accommodation)
  - Abstraction (reflecting, reflected)

Learning from a Piagetian Perspective

- Learning is a constructive process that results from one's interaction with the environment.
- Learning involves a subject engaged in activity.
- Learning does not only occur by acting on the environment but also by reorganizing on a higher mental level actions coordinated at a lower level.

Learning from a Piagetian Perspective

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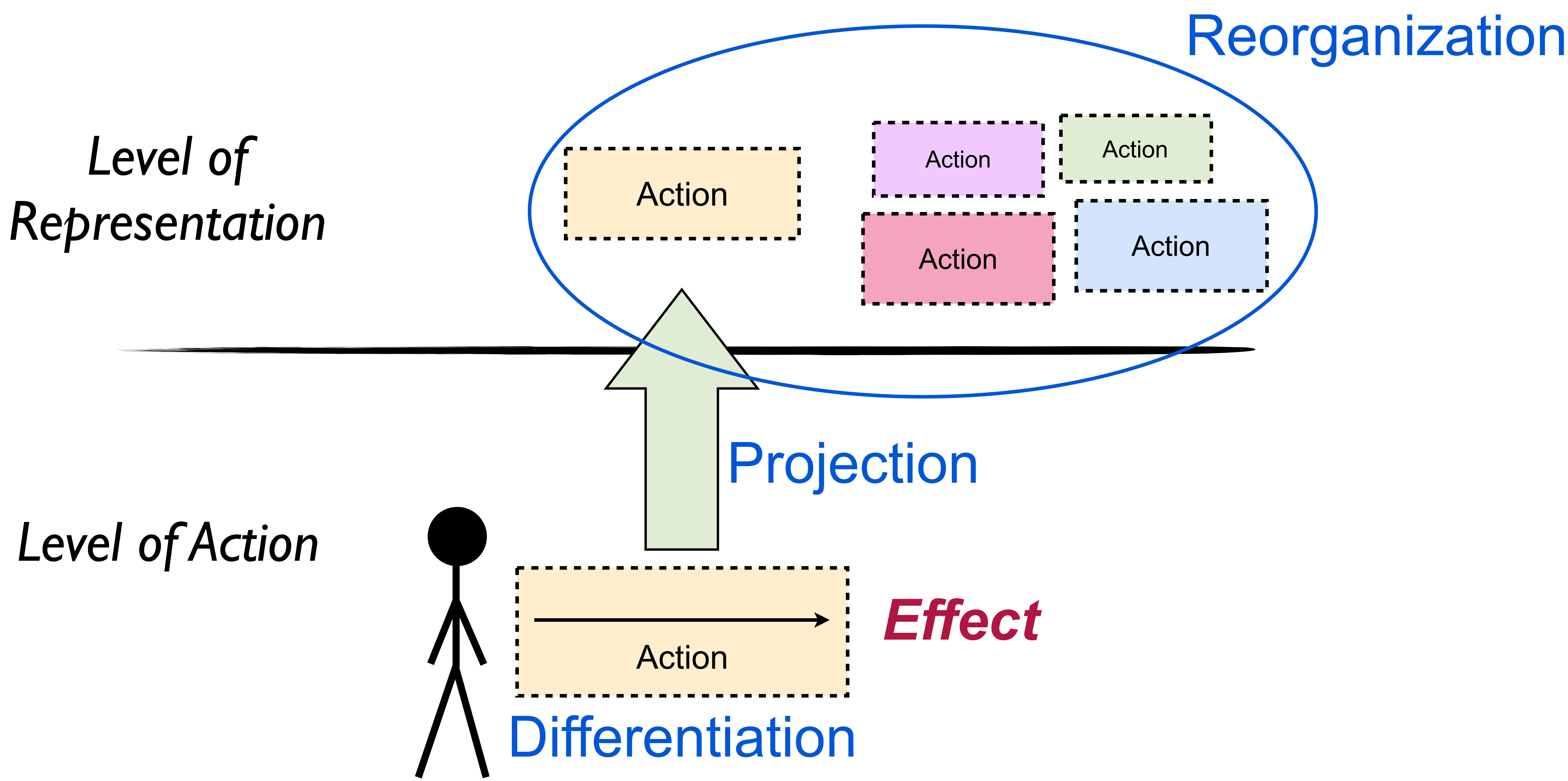
"all action—that is to say, all movement, all thought, or all emotion—responds to a need" (Piaget, 1967, p. 6).

Piagetian Abstraction

Piaget proposed abstraction as the functional mechanism of knowledge development.

- Empirical abstraction
- Pseudo-empirical abstraction
- Reflecting abstraction
- Reflected abstraction

Reflecting Abstraction



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Reflected Abstraction

Reflected abstraction involves operating on the actions that result from prior reflecting abstractions at the level of representation, which results in a coherence of actions and operations accompanied by conscious awareness.

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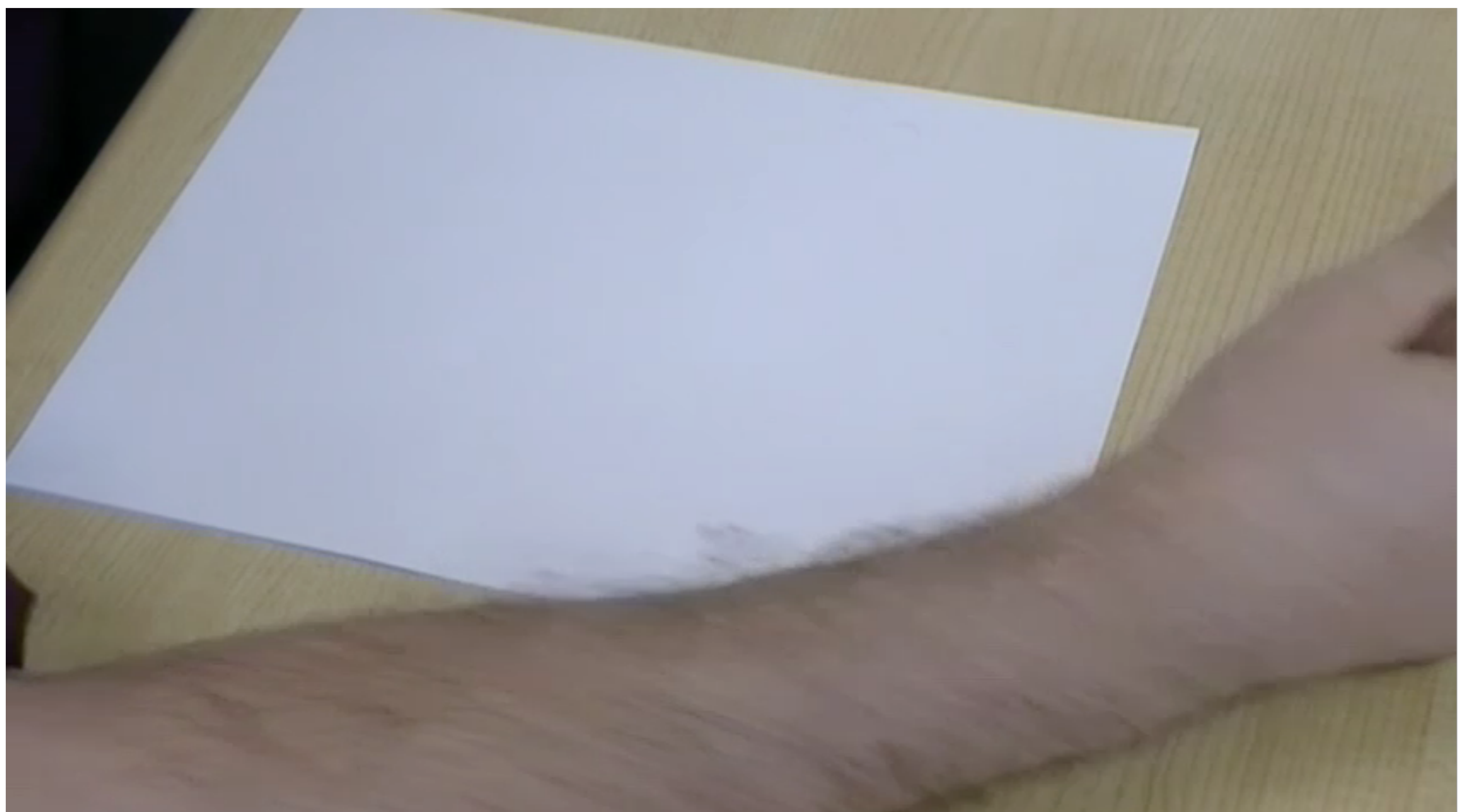
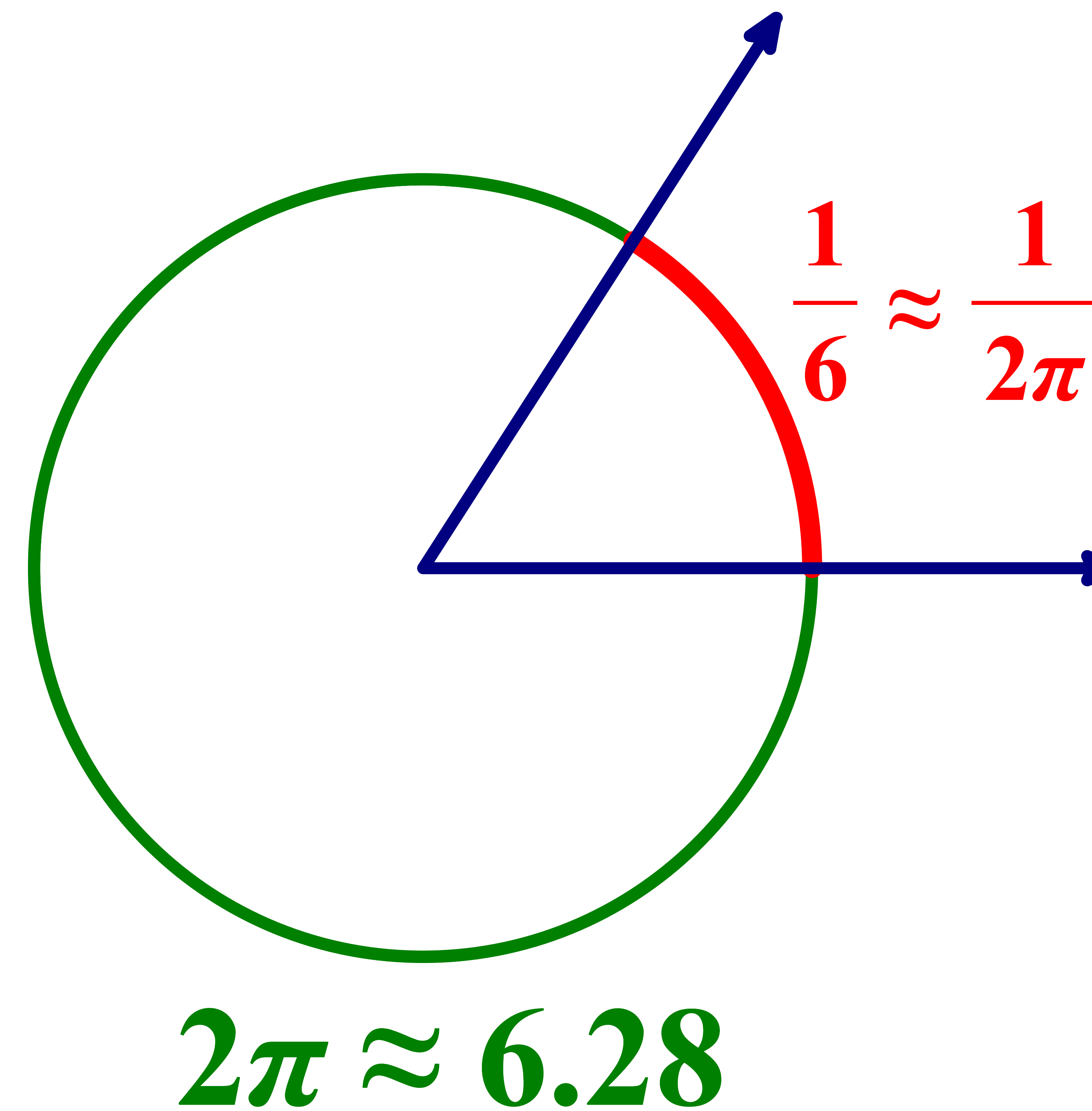
Results

- David possessed two complementary but conceptually distinct ways of understanding what it means to measure an angle in radians.
- David had not coordinated these two ways of understanding (i.e., schemes) into a coherent structure.
- This led to unfocused instruction.
- I engaged David in experiences that promoted reflected abstraction, which allowed him to coordinate his ways of understanding and bring them into conscious awareness.

David’s two ways of understanding angle measure in radians.

WoU 1: Angle measure as a comparison of subtended arc length and circumference





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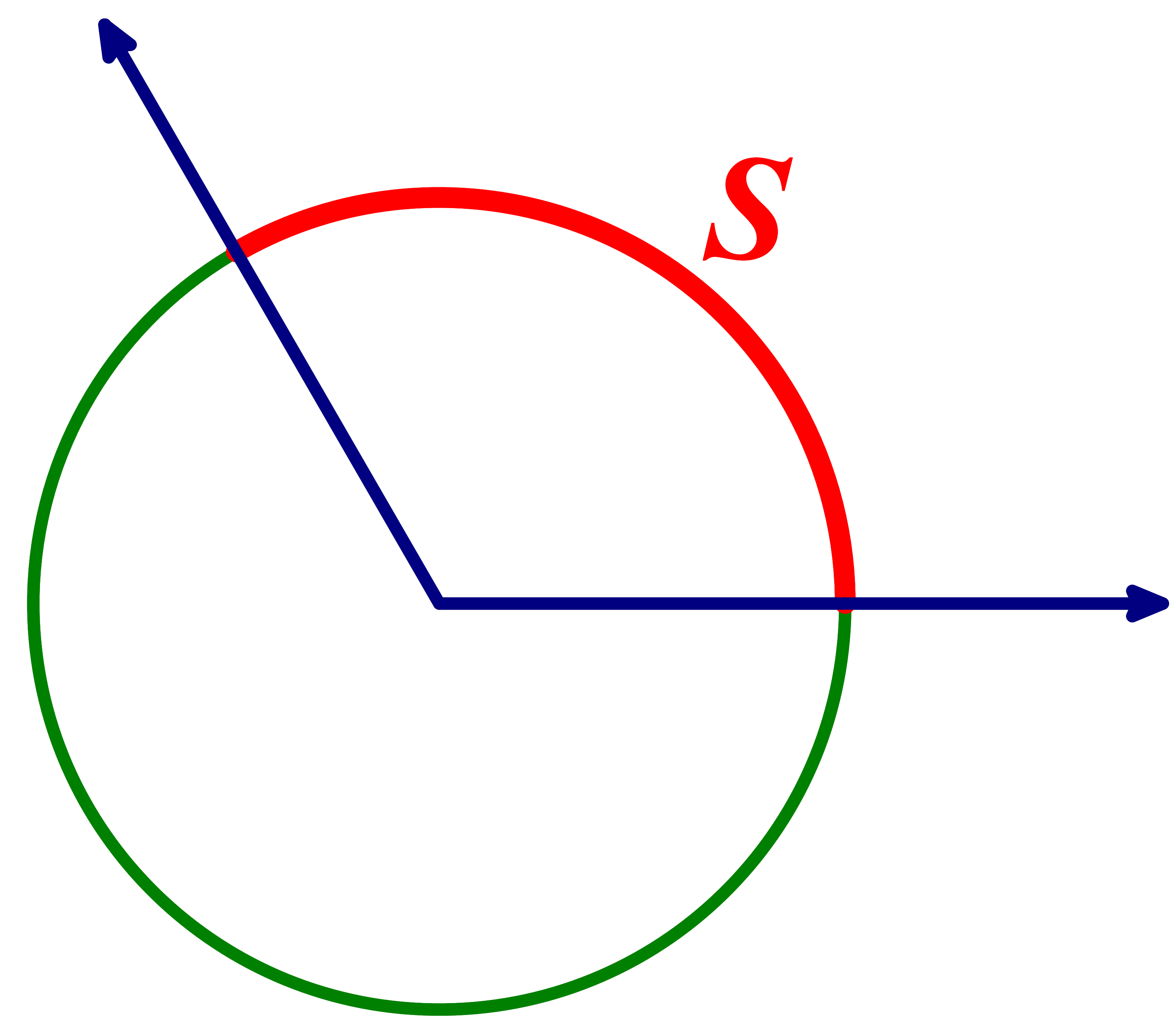
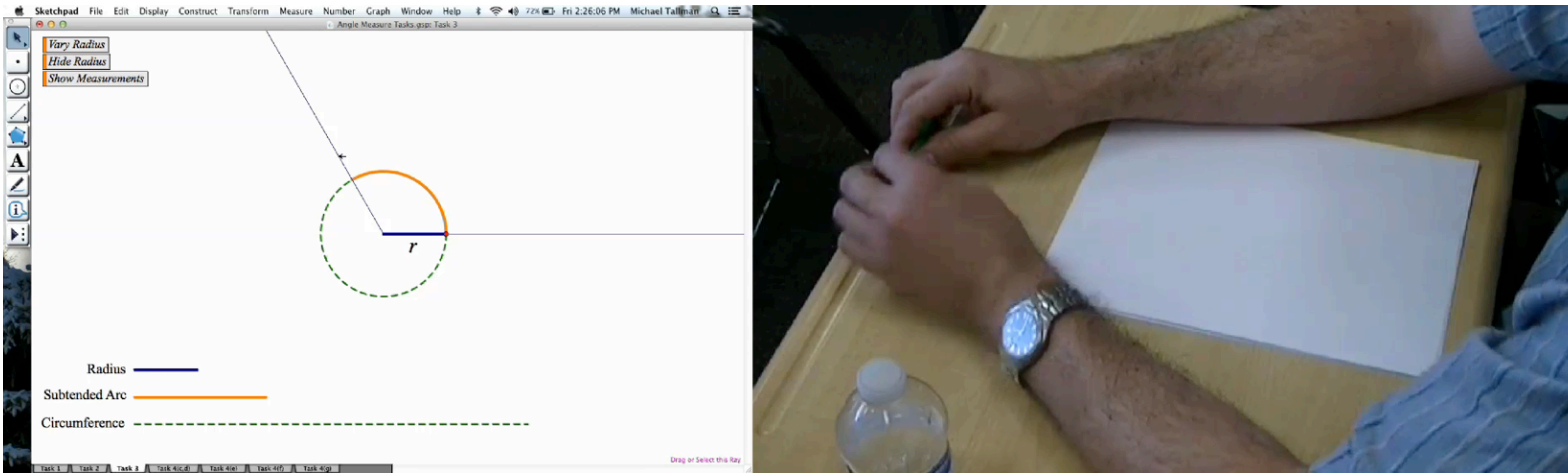
WoU 1: Angle measure as a comparison of subtended arc length and circumference

Q: What does it mean to say that an angle has a measure of 2.1 radians?



$S \approx {}^{1}/_{3}(2\pi)$

WoU 1: Angle measure as a comparison of subtended arc length and circumference



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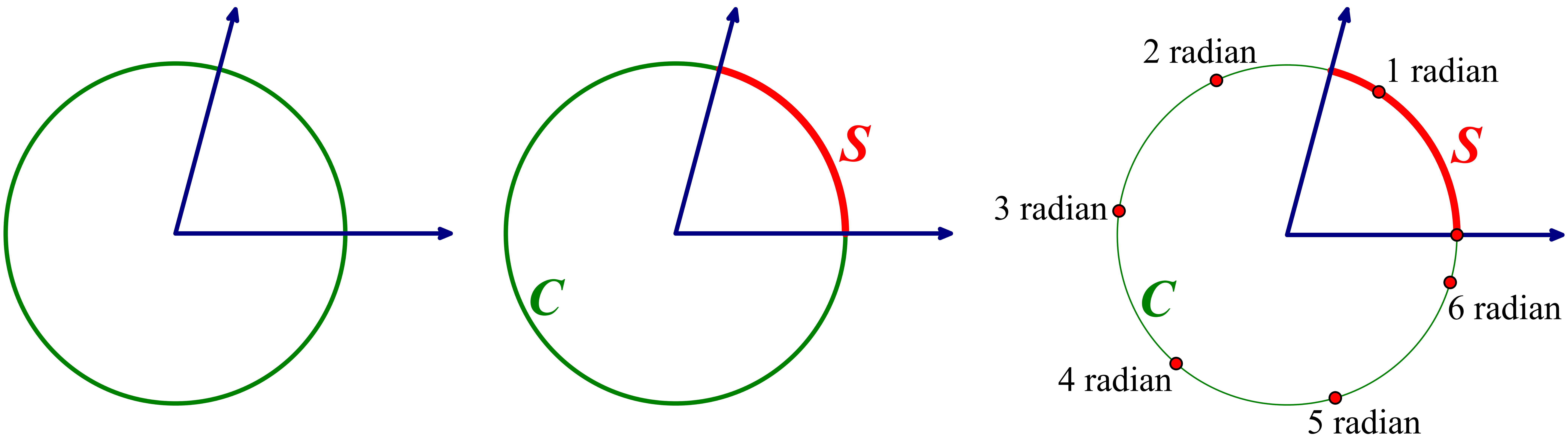
Summary of *WoU 1*

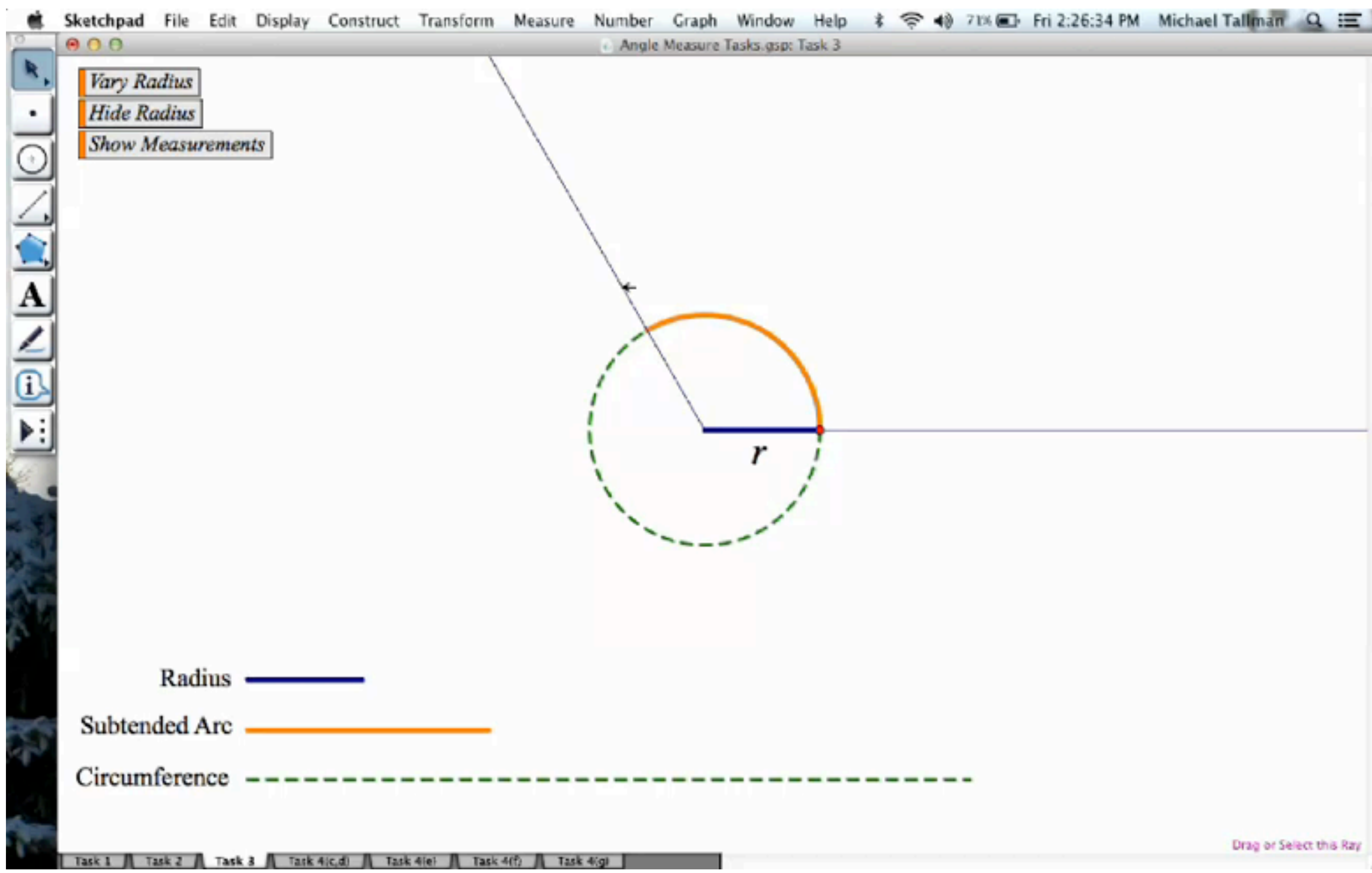


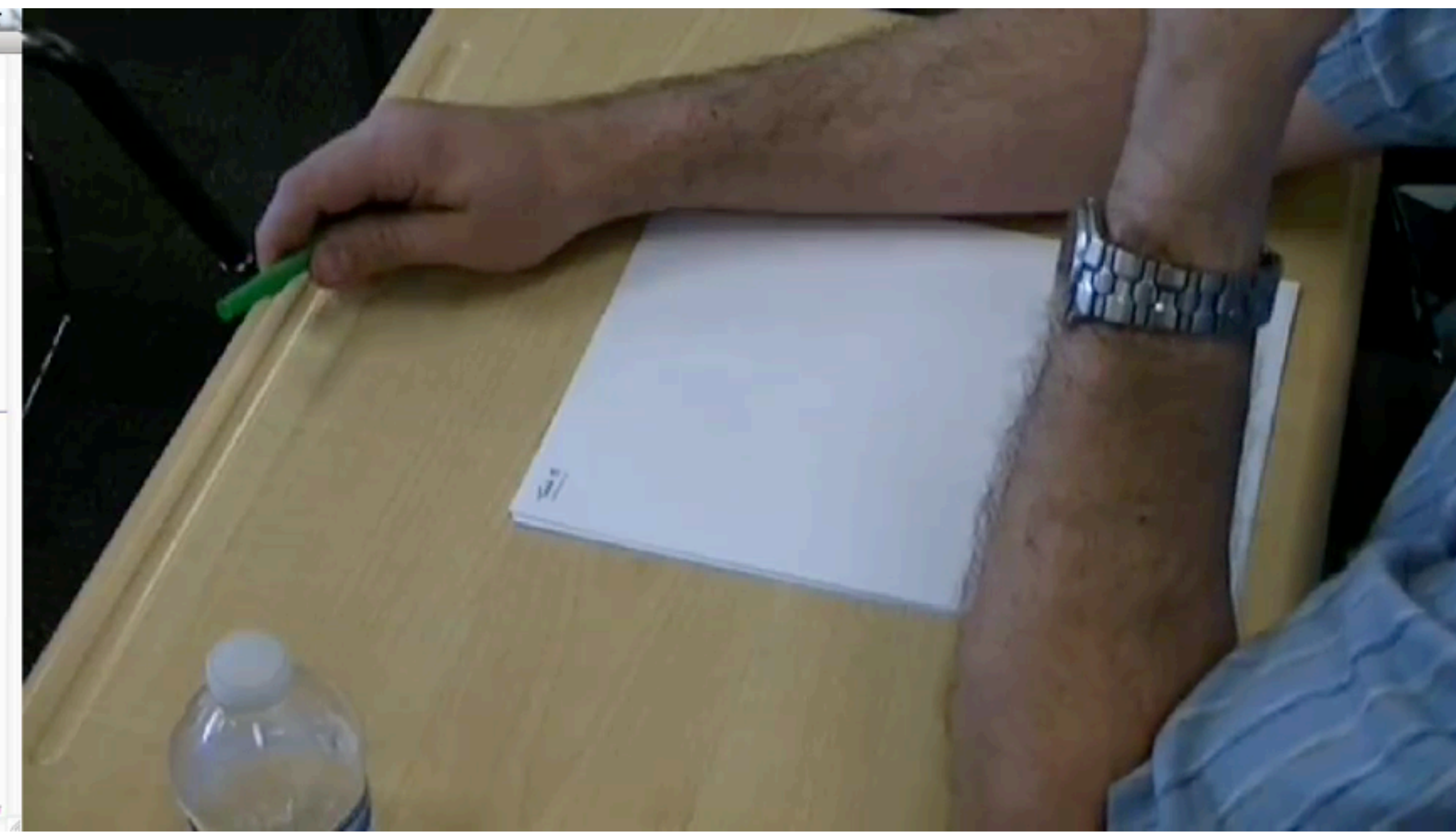
The angle subtends $\frac{S}{C}$ of the circle's circumference.

The angle subtends $\frac{S}{C}$ of the number of radians "in a circle."

$$\text{Angle Measure} = \left(\frac{S}{C}\right) \cdot 2\pi \text{ radians.}$$

WoU 2: Angle measure as a comparison of subtended arc length and a unit of measure





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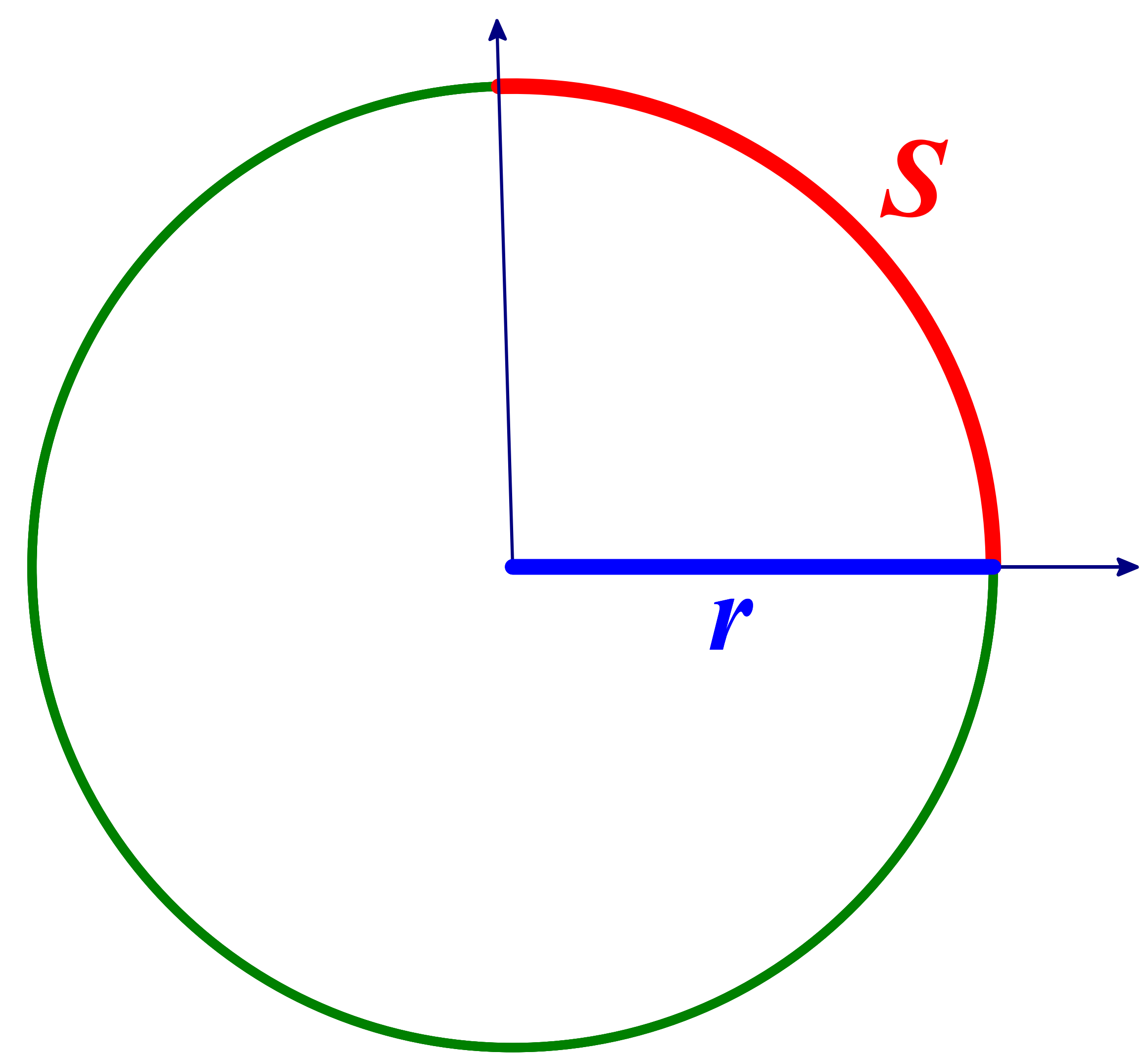
WoU 2: Angle measure as a comparison of subtended arc length and a unit of measure

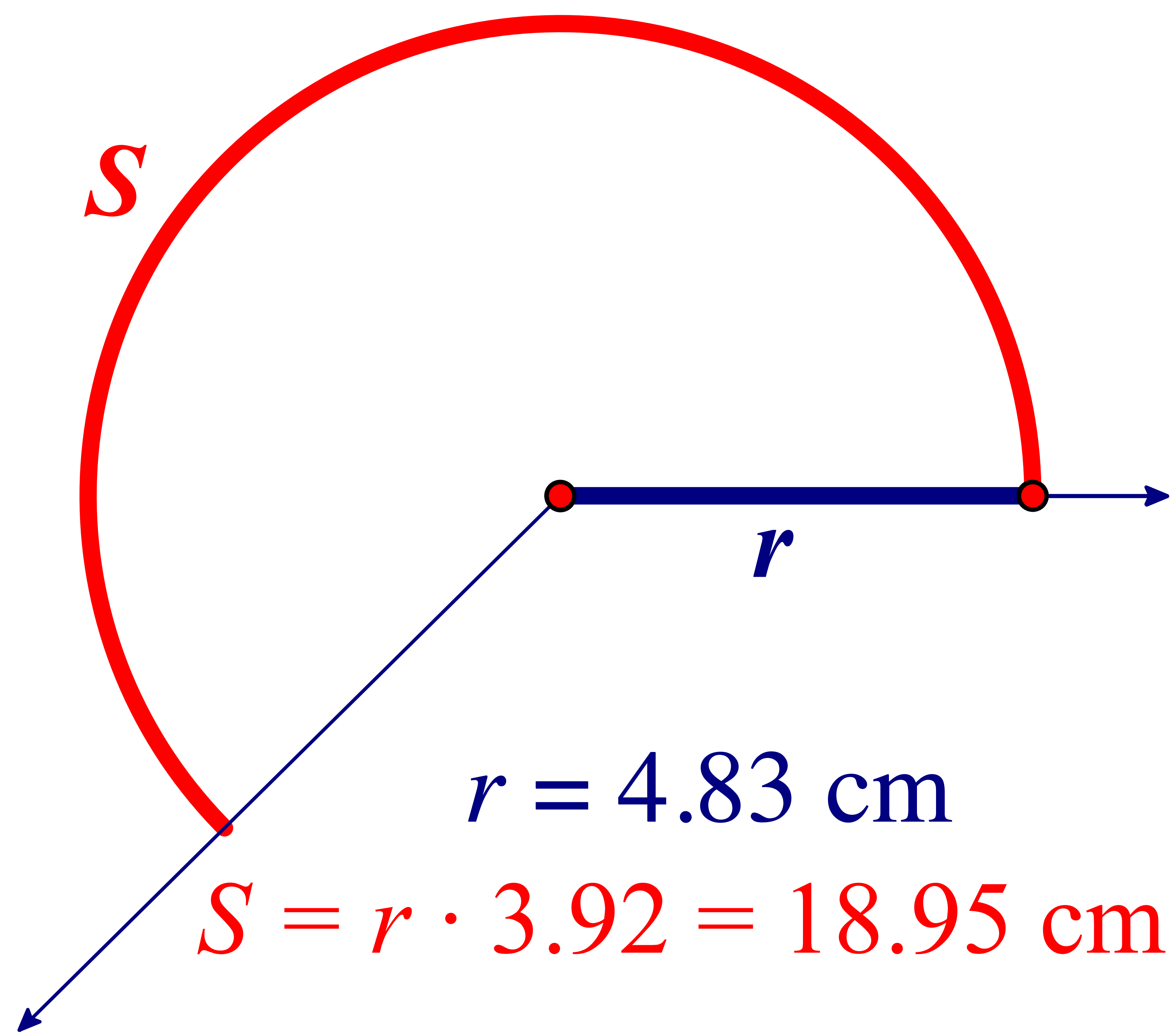


$r = 5.43$ cm

$S = r \cdot 1.6 = 8.69$ cm

Q: Using *Geometer's Sketchpad* construct an angle with a measure of 1.6 radians.

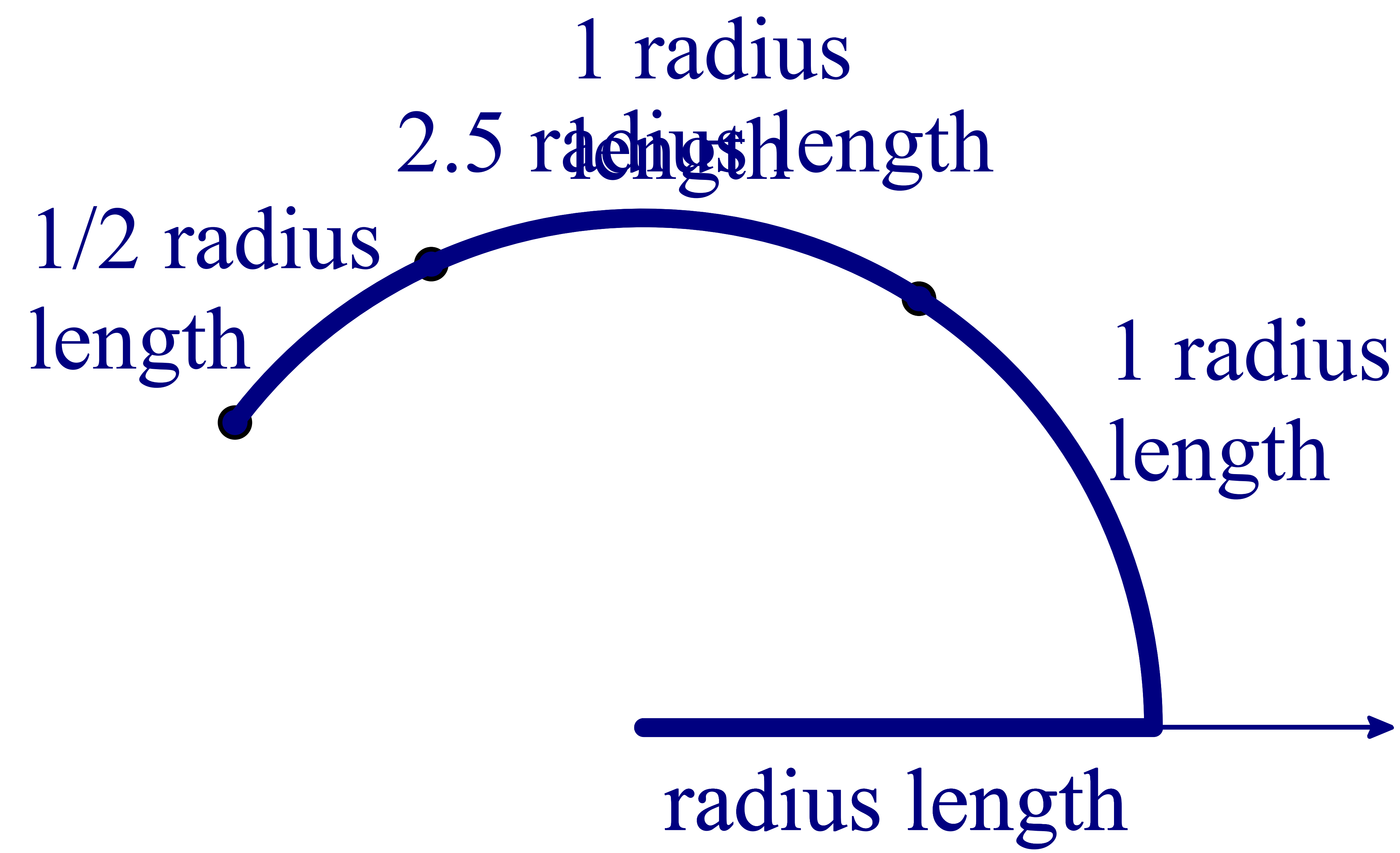
WoU 2: Angle measure as a comparison of subtended arc length and a unit of measure



Q: What does it mean to say that an angle has a measure of 3.92 radians?

WoU 2: Angle measure as a comparison of subtended arc length and a unit of measure

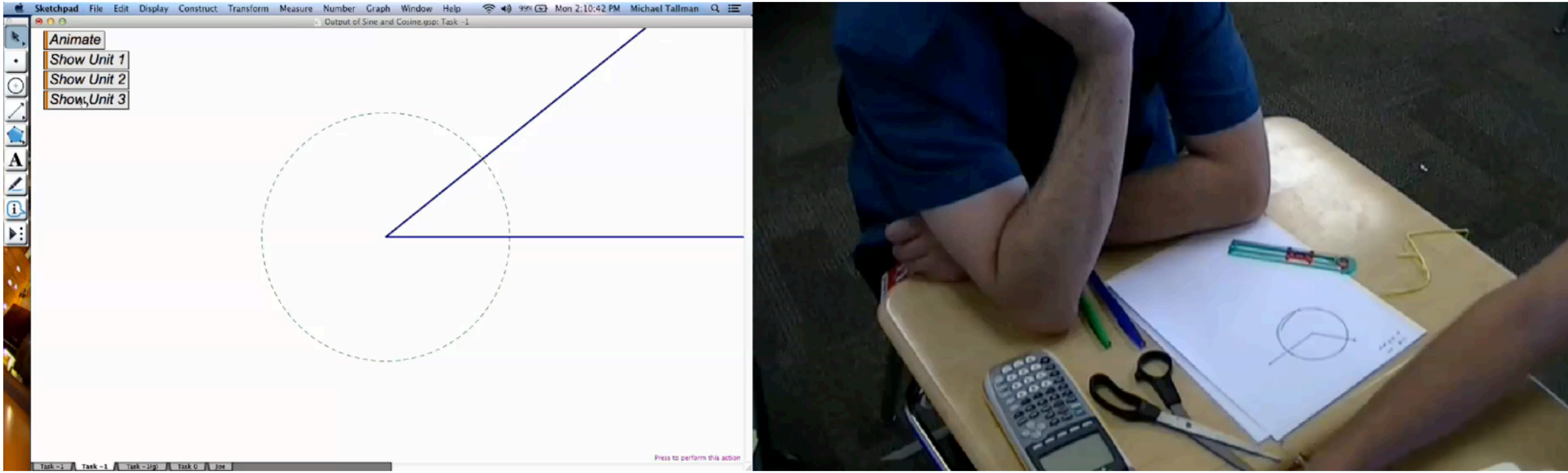
Q: Construct an angle with a measure of 2.5 radians.



Summary of *WoU 1* and *WoU 2*

WoU 1	***WoU 2***
Interpretation: An angle with a measure of n radians subtends $n/(2\pi)$ of the circumference of the circle centered at the vertex of the angle.	*Interpretation:* An angle with a measure of n radians subtends an arc that is n radius lengths, or n times as large as the radius of the subtended arc.
Measurement: The measure of an angle, θ, in radians is determined by the formula $\theta = \left(S/C\right) \cdot 2\pi$ where S represents the length of the subtended arc and C represents the length of the circumference.	*Measurement:* The measure of an angle, θ, in radians is determined by the formula $\theta = S/r$ where S represents the length of the subtended arc and r represents the length of the radius of the subtended arc.

David had not coordinated his two ways of understanding into a coherent cognitive scheme.



radius 1.4"

$S = 5.3"$

$$C = 2\pi(1.4)$$

$$\frac{S}{C} = \frac{5.3}{2\pi(1.4)} \approx \frac{5.3}{8.796} = 0.602 \text{ radians}$$

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David's work:

radius 1.4"

$s = 5.3$"

$C = 2\pi(1.4)$

$\frac{s}{C} = \frac{5.3}{2\pi(1.4)} \approx \frac{5.3}{8.796} = 0.602$ radians

$\frac{5.3}{1.4} = 3.785$ rad

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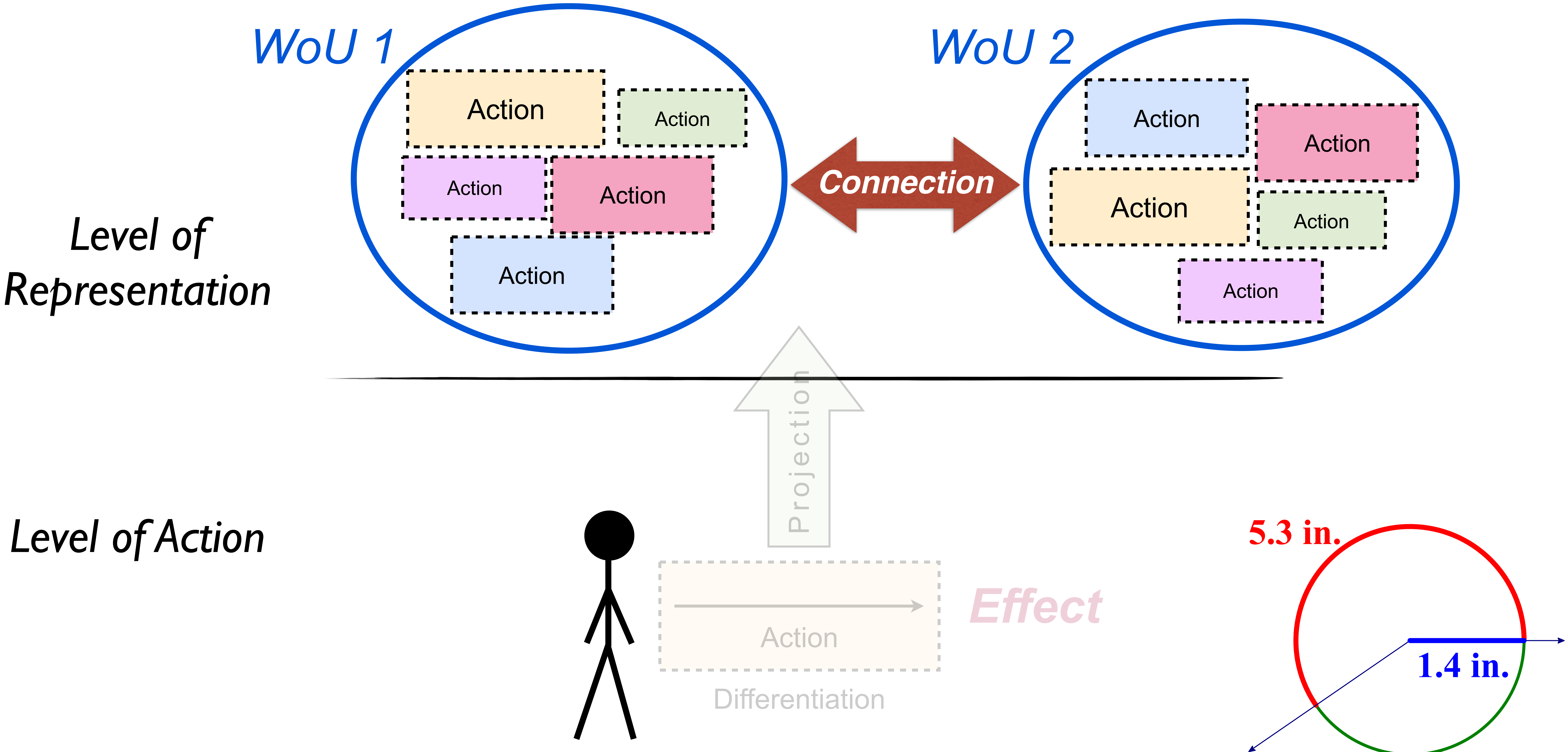
David's Angle Measure Schemes



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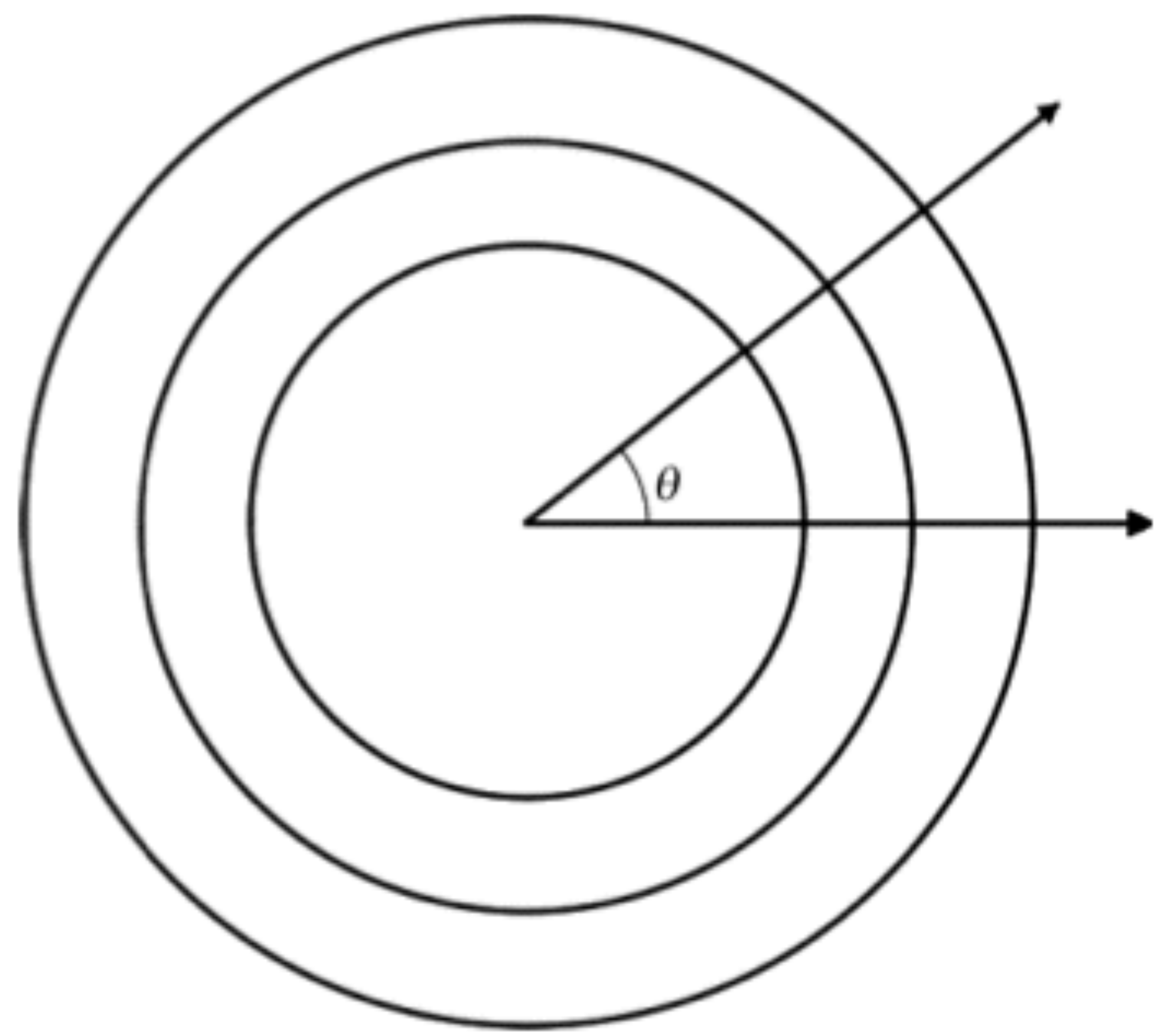
David's unfocused instruction.

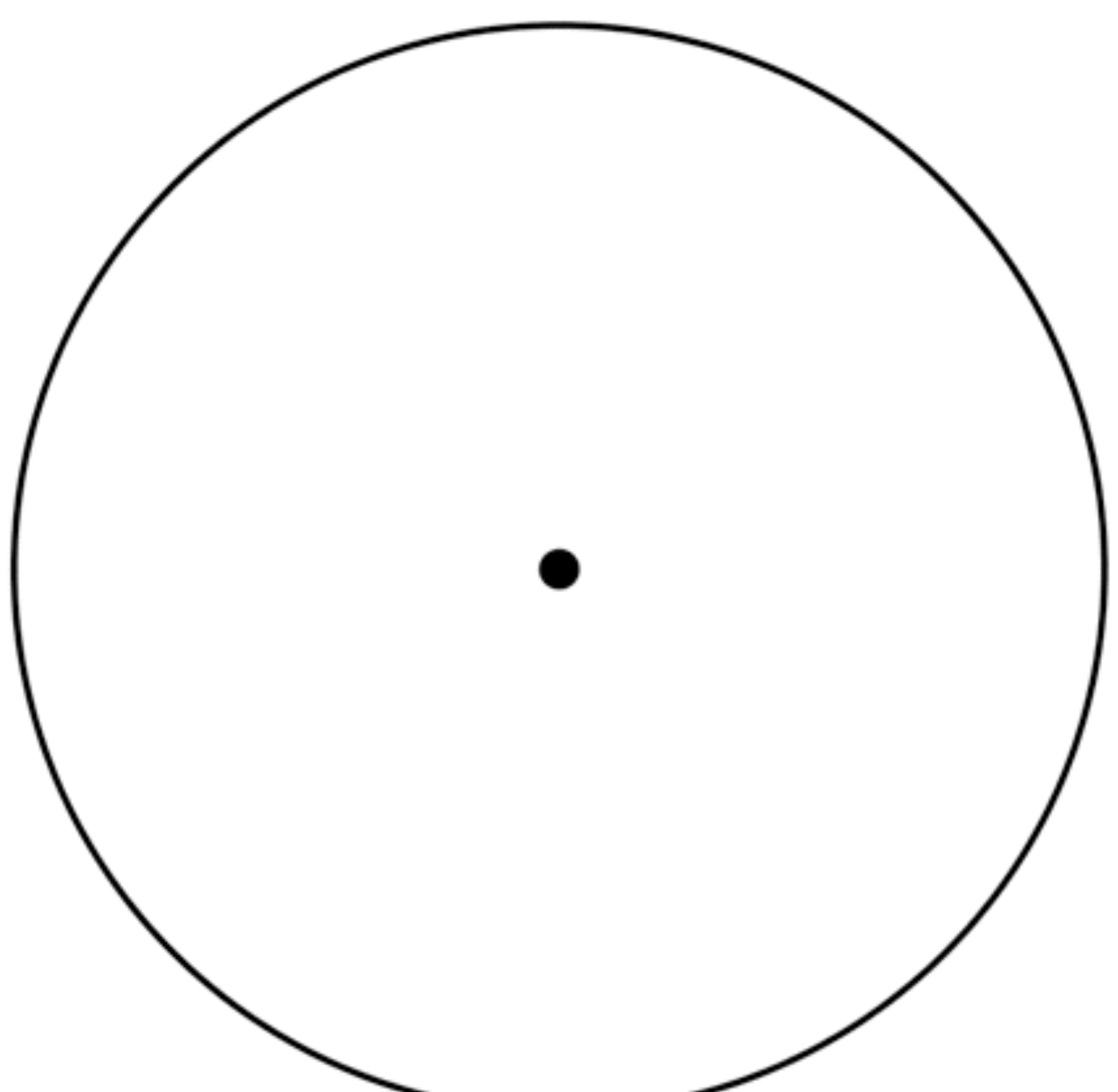
10. Given that an angle measures $\theta = 0.45$ radians, determine the length of each arc subtended by the angle if the circles have radius lengths of 2 inches, 2.4 inches, and 2.9 inches. (*Diagram is not drawn to scale.*)



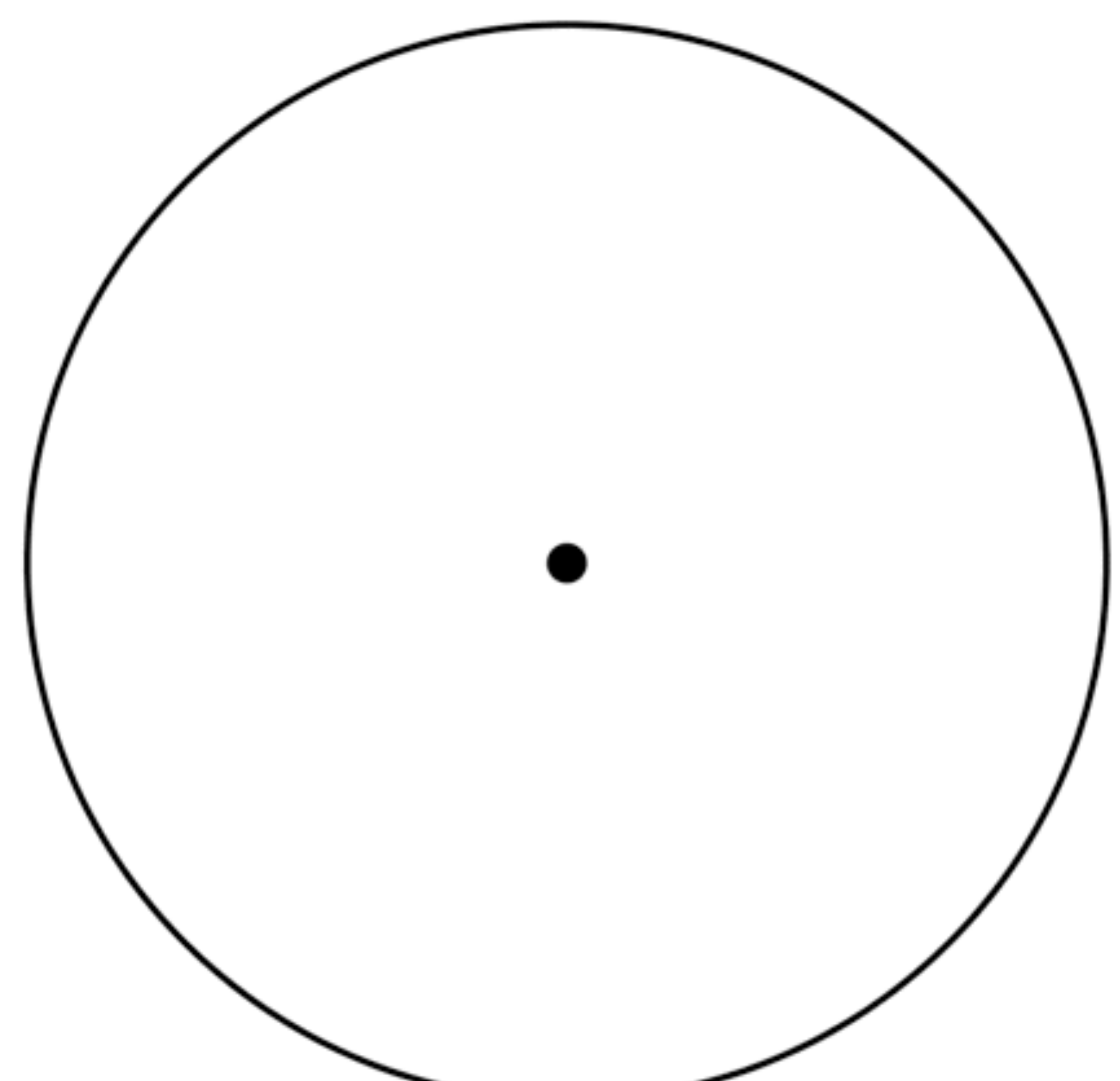
$$\text{Arc Length} = \frac{0.45}{2\pi} \cdot 2\pi r$$

4. For each angle measure given, do the following.
   i) Determine what portion of a circle's circumference (centered at the vertex) will be subtended by an angle with the given measure. (*Give your answer as a percentage*.)
   ii) Use the given circle and your answer to part (i) to sketch an angle with the given measure. (*Your answer will be approximate – do not use a protractor or other tools to assist you*.)
   iii) Use a piece of string the length of the circle's radius to measure the length of the subtended arc in *radius lengths*. How does this compare to the angle measure given in radians?

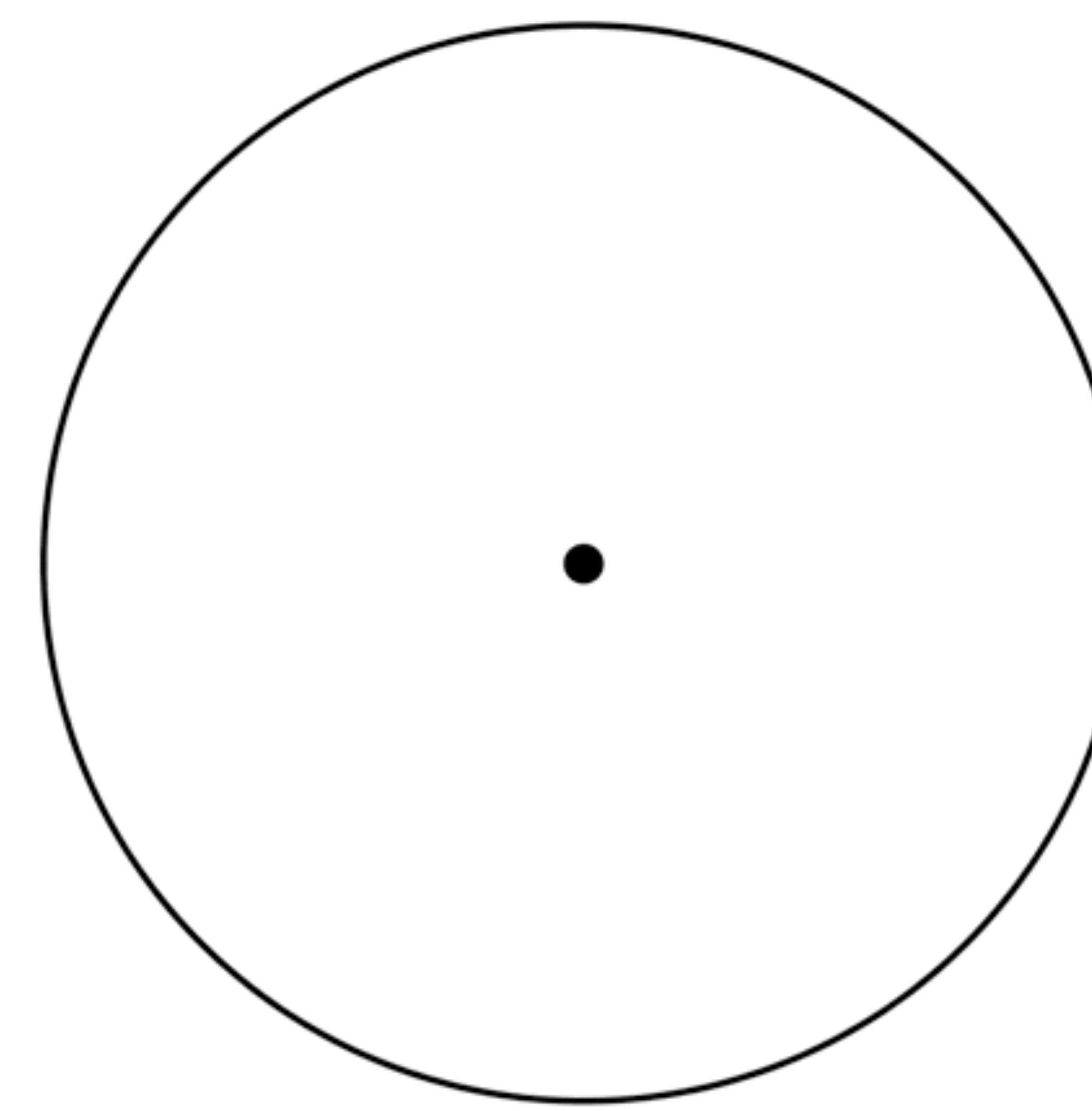
a. 3 radians



b. 5.5 radians

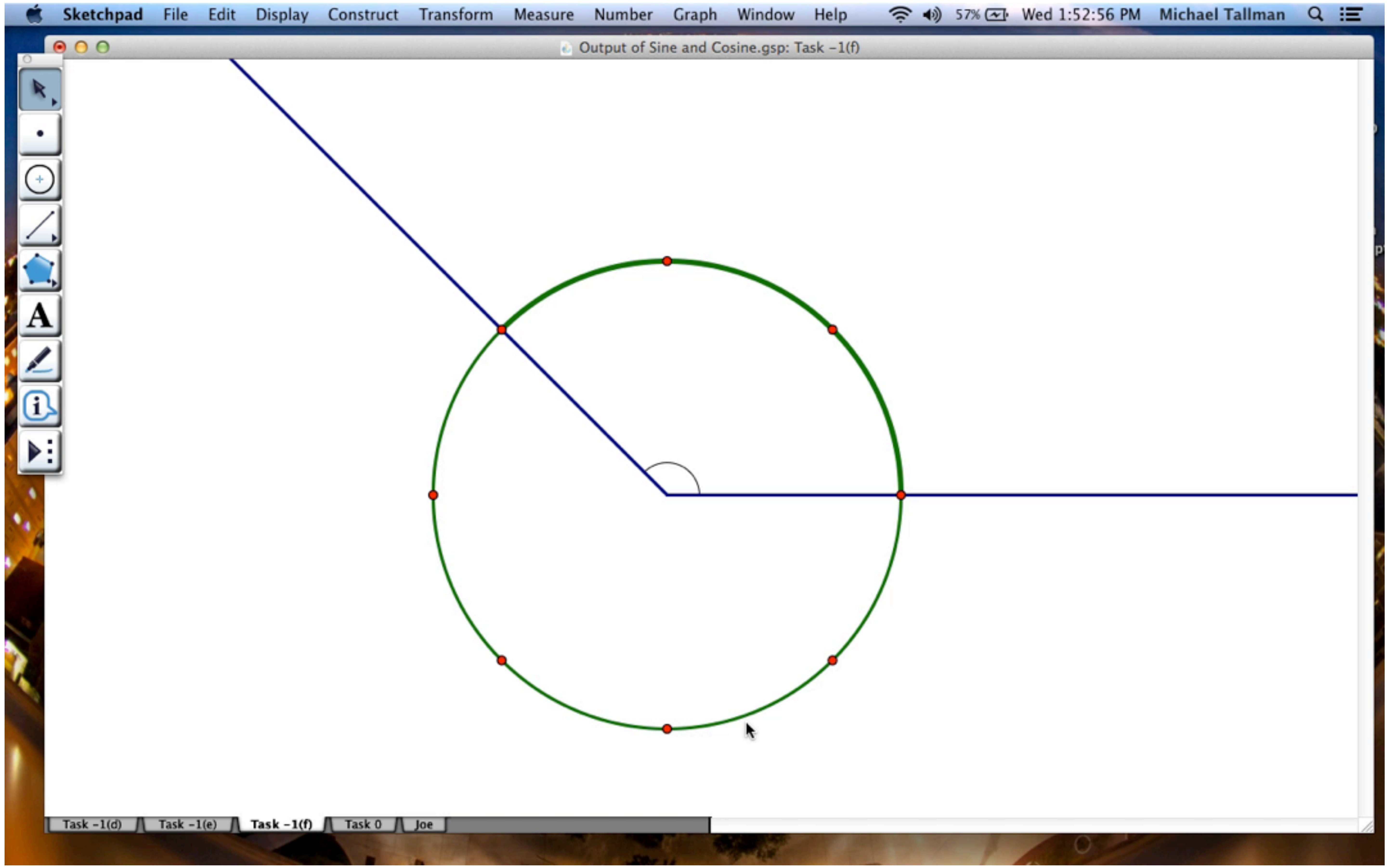


c. $\frac{1}{2}\pi$ radians



Engendering reflected abstraction allowed David to coordinate his two ways of understanding angle measure.

Engendering Reflected Abstraction



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Engendering Reflected Abstraction

Q: Courtney claims that measuring an angle in radians means measuring the arc length that the angle subtends in units of the radius of the circle centered at the vertex of the angle. Rebecca says that to measure an angle in radians, you take the length of the arc that the angle subtends divided by the length of the circumference of the circle centered at the vertex of the angle and then multiply this ratio by 2π. Are they both correct?

Engendering Reflected Abstraction

radius 1.4"

$C = 2\pi(1.4)$

$s = 5.3"$

$\frac{s}{C} = \frac{5.3}{2\pi(1.4)} \approx \frac{5.3}{8.796} = 0.602$ radians

Rebecca's claim.

$\frac{5.3}{1.4} = 3.785$ rad

Courtney's claim.

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Summary

- David possessed two complementary but conceptually distinct ways of understanding what it means to measure an angle in radians.
- David had not coordinated these two ways of understanding (i.e., schemes) into a coherent structure.
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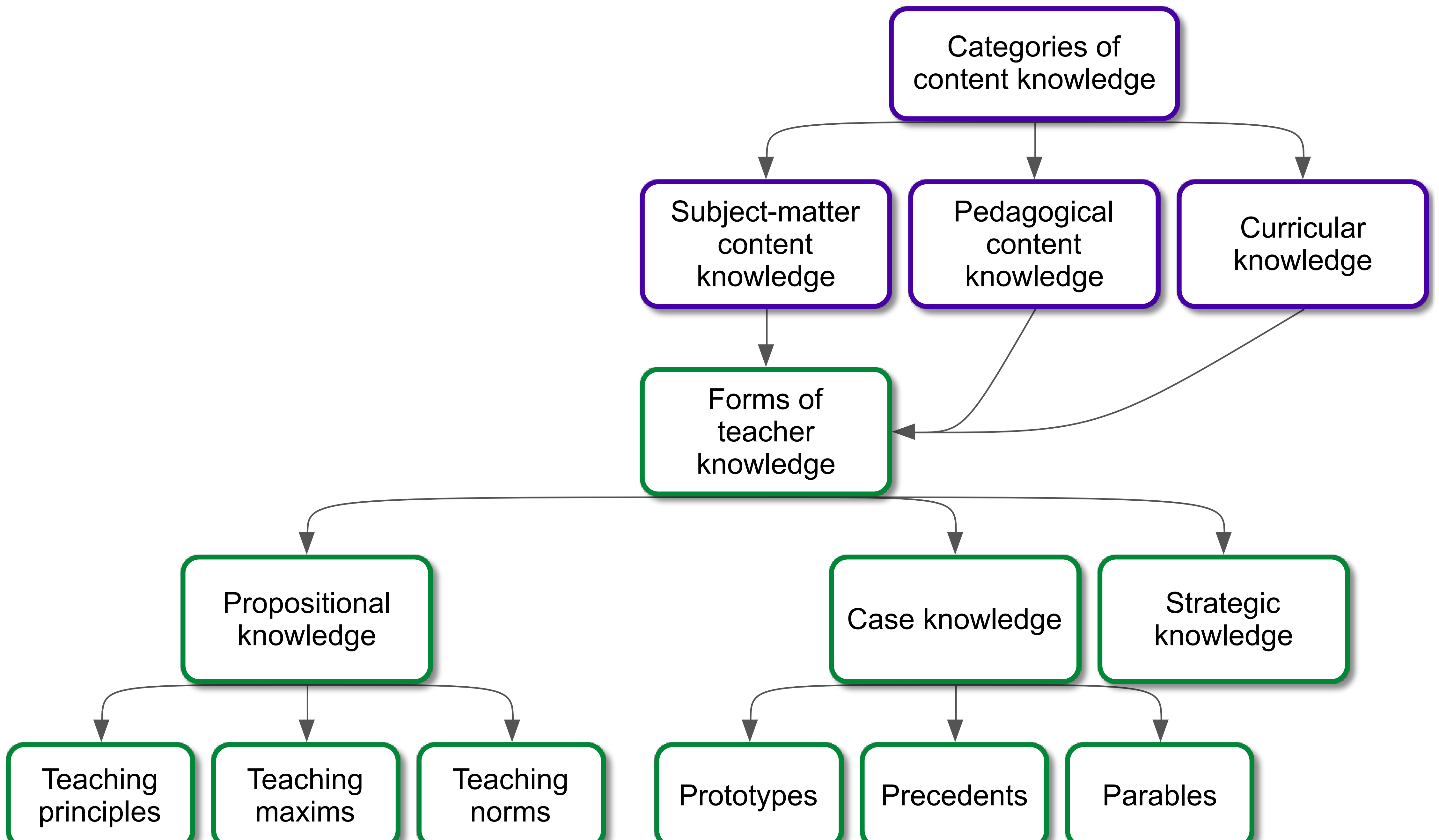
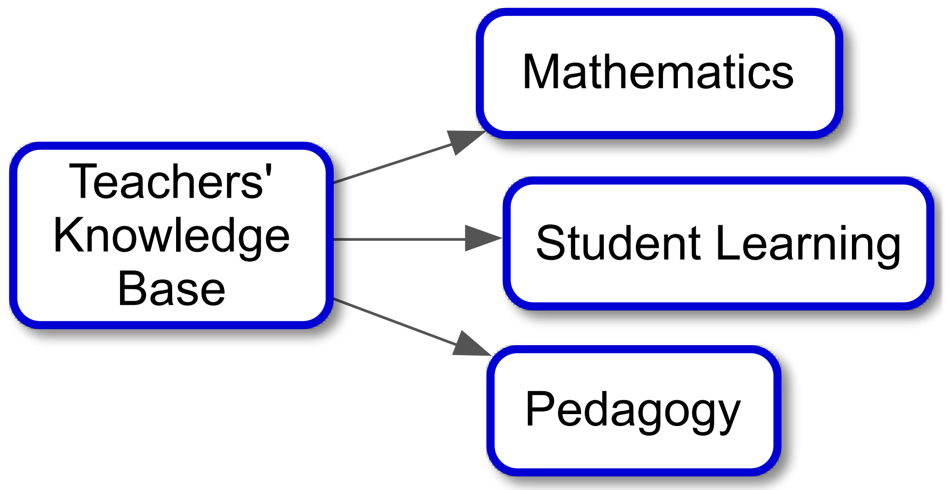
Discussion/Implications

- David's unawareness of the mental activity involved in the process of students' constructing a mature understanding of a mathematical idea made him unable to strategically employ his mathematical knowledge to support students' learning.
- This result suggests that the mathematical knowledge required for effective teaching involves more than powerful understandings of mathematical ideas; it involves an awareness of the mental actions and operations that constitute these understandings.

Discussion/Implications



Shulman (1986, 1987)



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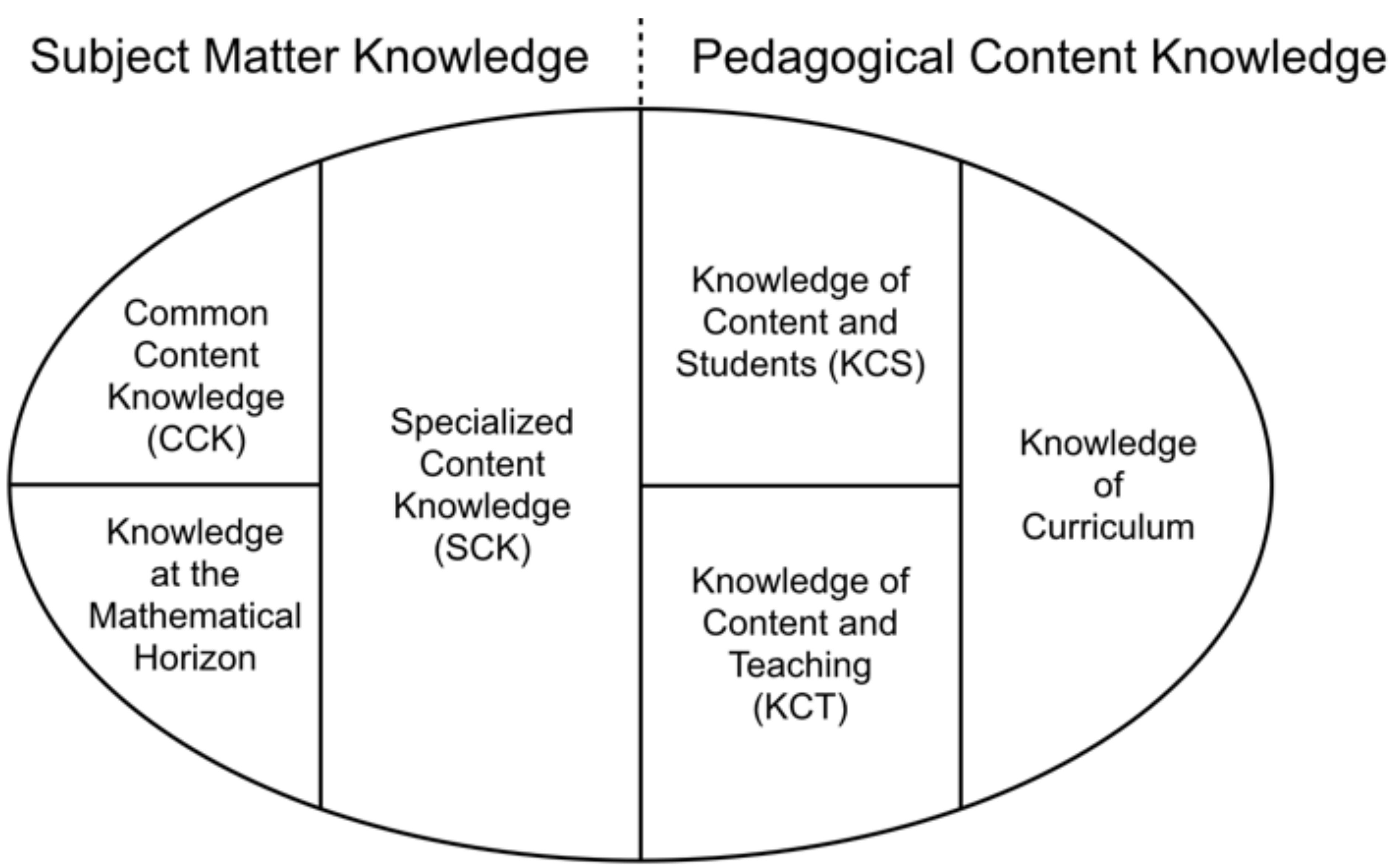

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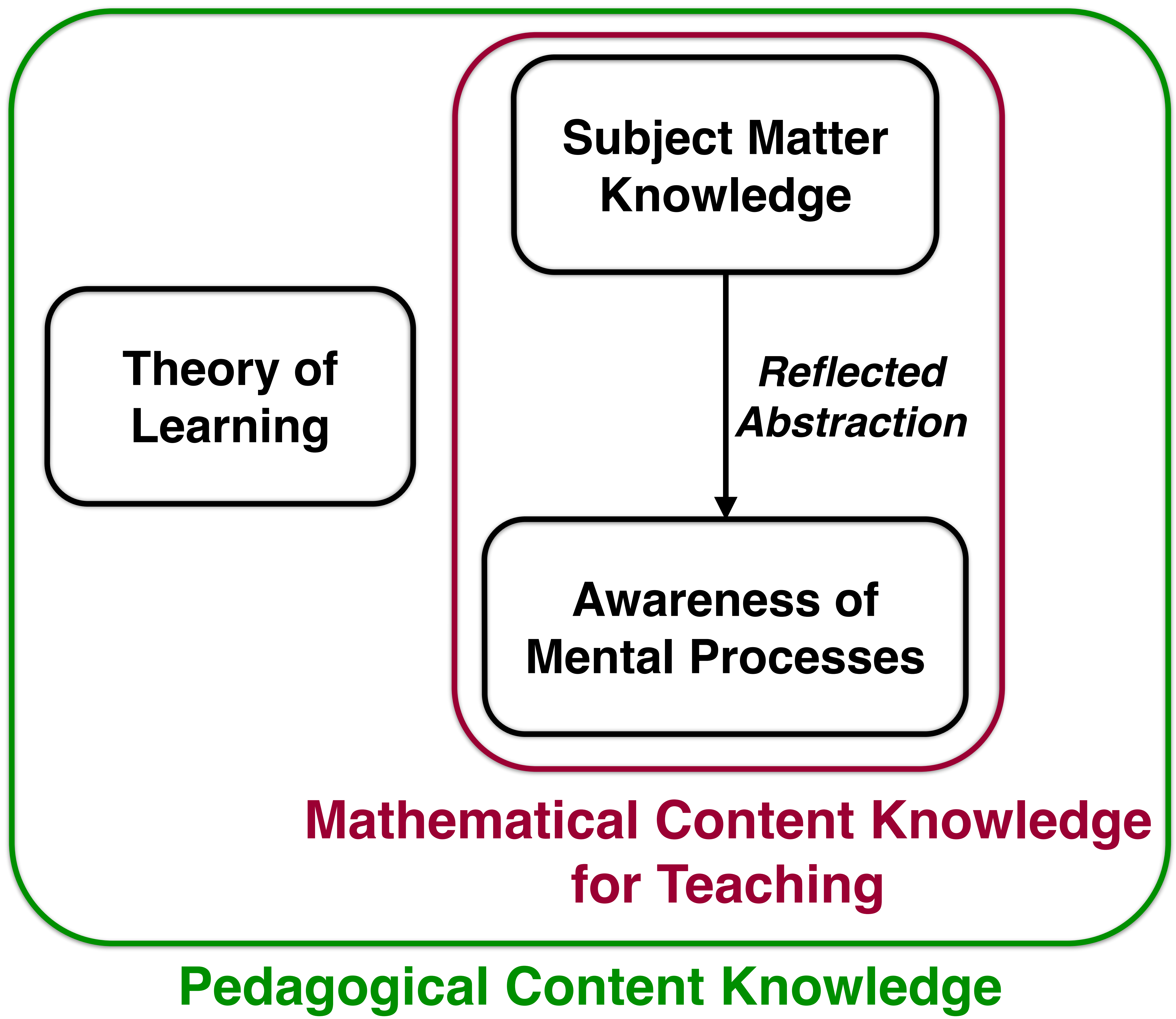
Model of Pedagogical Content Knowledge



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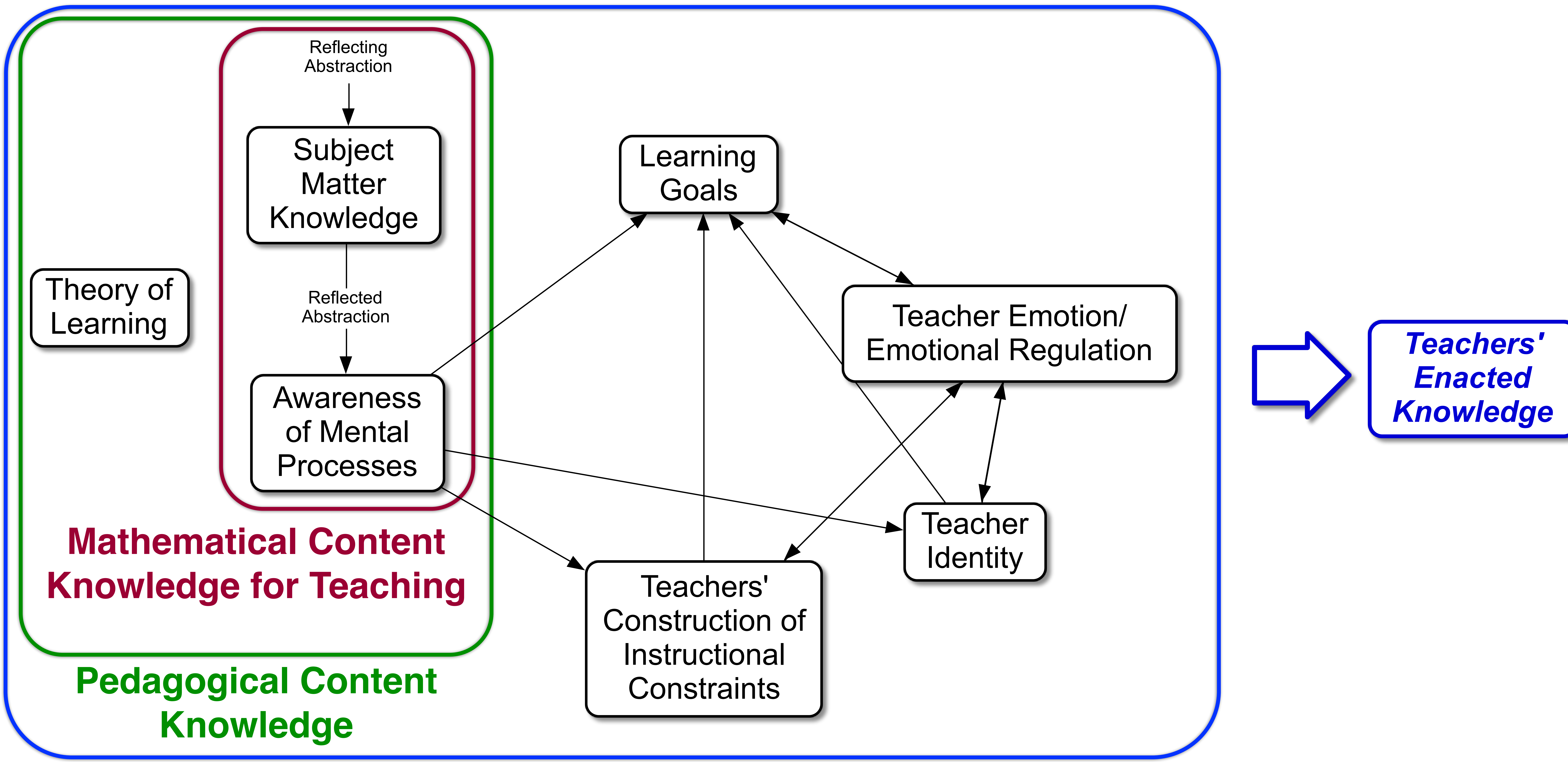
Model of Mathematics Teacher Knowledge



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