- Knowing and Teaching Elementary Mathematics – American Mathematical Society
- [www.ams.org/notices/.../rev-howe.pdf‎](www.ams.org/notices/.../rev-howe.pdf);
- by L Ma - ‎Cited by 2513 -

Teacher Wang

I always spend more time on preparing a class than on teaching, sometimes three, even four times the latter.

I spend the time in studying the teaching materials; what is it that I am going to teach in this lesson? How should I introduce the topic? What concepts or skills have the students learned that I should draw on?

Is it a key piece on which other pieces of knowledge will build, or is it built on other knowledge? If it is a key piece of knowledge, how can I teach it so students grasp it solidly enough to support their later learning? If it is not a key piece, what is the concept or the procedure it is built on? How am I going to pull out that knowledge and make sure my students are aware of it and the relation between the old knowledge and the new topic?

What kind of review will my students need?

How should I present the topic step-by-step?

How will students respond after I raise a certain question?

Where should I explain it at length, and where should I leave it to students to learn it by themselves?

What are the topics that the students will learn which are built directly or indirectly on this topic? How can my lesson set a basis for their learning of the next topic, and for related topics that they will learn in their future? What do I expect the advanced students to learn from this lesson? What do I expect the slow students to learn? How can I reach these goals? etc.

In a word, one thing is to study whom you are teaching, the other thing is to study the knowledge you are teaching. If you can interweave the two things together nicely, you will succeed. We think about these two things over and over in studying teaching materials.

Believe me, it seems to be simple when I talk about it, but when you really do it, it is very complicated, subtle, and takes a lot of time. It is easy to be an elementary school teacher, but it is difficult to be a good elementary school teacher.

- Teacher response to question about a student error in 3 digit by 3 digit multiplication.

This should not be happening!

It means that 2 digit multiplication was taught incorrectly!

What teachers need

More than understanding mathematics in itself, teachers need to understand how mathematics develops through the years in children, and how to foster that growth.

It has been shown (D. Ball et al.) that mathematical knowledge for teaching (MKT) is a distinctive way of knowing mathematics, different from the knowledge of, say, a research mathematician.

Teaching Place Value

- Each stage in the progression

  $432 =$

  $400 + 30 + 2 =$

  $4\text{x}100 + 3\text{x}10 + 2\text{x}1 =$

  $4\text{x}(10\text{x}10) + 3\text{x}10 + 2\text{x}1 =$

  $4\text{x}10^2 + 3\text{x}10^1 + 2\text{x}10^0$

  represents a conceptual advance.

- Helping students master all stages
  should be an explicit goal
  of the elementary curriculum.

Stages for the Number Line

- 1. Pre-line: beginnings of length measurement – small whole number lengths measured by a unit; addition as combination (aka concatenation) of lengths, subtraction as comparison. Forming trains of 10-rods and 1-cubes to model base 10 notation.

- 2. Number ray: emphasize the metric nature – a number tells <u>distance from the end, as a multiple of the unit length</u>.

Stages, II

- 3. Use number ray with decade structure. Place trains on number ray for comparison. Add by having two number rays and sliding one appropriately. Observe that the far endpoint allows to read off the sum. Repeat to limited extent with 3-digit sums (using 100s rods).

- 4. Introduce fractions, featuring unit fractions. Observe that the whole number multiples of a fixed unit fraction form a set of evenly spaced points, similar to the whole numbers, but closer together.

- 5. Place multiples of distinct unit fractions on the same number line. Observe non-constancy of spacing. Demonstrate/discover resolution by embedding both systems of multiples in the system generated by the product (or LCM).

- 6. Add fractions using number ray. Stress parallel to whole number addition. Observe that the standard formula (treated in examples) describing the sum describes the results.
- 7. Introduce signed numbers, and correspondingly oriented segments. Extend number

Multiplication(?)

A. Interpret multiplication by whole numbers as dilation of the number line.
B. Interpret division by whole numbers as dilation of the number line.
C. Division by n = multiplication by 1/n.
D. Multiplication by m/n = multiplication by m follwed by multiplication by 1/n.

Multiplication, cont.

- E. Division by m/n = division by m followed by division by 1/n = multiplication by 1/m followed by multiplication by n = multiplication by n/m.
- F. Multiplication by –n = multiplication by n followed by reflection across 0 (= multiplication by -1).

What is a number?

- Is a number a count?
- Is 3 the equivalence class of all sets of cardinality 3?
- Is a number a ratio? (And is a ratio a number?)
- Is a number a point on the number line?

- Is a number something that obeys the rules of arithmetic?

- Why do we believe that $\sqrt{2}$ exists?
- What about $\sqrt{\sqrt{2}}$?
- Why does .99999999999999 . . . .
- =1 ?

- Why can’t we divide by 0?

- Why can we divide by 2?

TEACHING FUNCTIONS

Problems and Prospects

A Powerful Idea

- Generality
  - any number of variables
  - no numbers required
    (geometric transformations,
    truth values, modular arithmetic, . . .

- Precision
  - domain, range, unicity of values

- Versatility
  - composition
  - structural uses
    - rules for operations,
    - types of algebraic structure

Problematics

What are functions?

- Is a table a function?
- Is a reflection, rotation, translation a function?
- And what about glide reflections?
- Is addition a function? Subtraction? Multiplication? Division?

What can we do with functions?

- Can we compose them?
- Can we invert them?
- Can we add them?
- Can we multiply them?

- Domain

  Does a function start with a formula?

  Or with a domain?

- What is the domain of

1	2	3	4	5
3	5	7	9	11

?

$$X \quad \rightarrow \quad 2x+1$$

- As a function with domain and range = R,

it is onto. This is important: we can invert

it and so solve. Also, it belongs to the group of affine transformations of R.

- As a function with domain Z (of N), it is <u>not</u> onto. This is important: being in the image characterizes odd numbers.

What is my rule?

n	1	2	3	4	5
R(n)	1	2	4	8	16

Number of Regions in a Disk, made by connecting n points

$${}_{n-1}C_4 + {}_{n-1}C_3 + {}_{n-1}C_2 + {}_{n-1}C_1 + {}_{n-1}C_0$$

1	**2**	**3**	**4**	**5**	**6**	**7**
1	2	4	8	16	31	57

What is the domain of

$$\sqrt{(-(x^2+y^2)^3+(4x-17)(x^2+y^2)^2+(68x-16)(x^2+y^2)+64x)}$$

- The function $xy = 10$.

- What about

$a_3 10^3 + a_2 10^2 + a_1 10^1 + a^0 10^0 \rightarrow a_3 + a_2 + a_1 + a_0$?

(comes up in divisibility rules

("casting out nines").)

Principle of Example Sufficiency

- When introducing a concept, give enough examples to provide a rounded conception. And continue to use them.

- Counterexamples:
- 1) an equilateral triangle with horizontal base, as representative of “triangle”.
- 2) use of “=“ to mean “compute now”.

CCSSM on Functions

Grade 6:

Expressions and Equations:

Represent and analyze quantitative relationships between dependent and independent variables.

- Includes formulas, graphing, tables.

- No mention of ``function”.

Grade 7

- Ratios and Proportional Relationships.

  - Graphs, equations/formulas.

  - No “functions”.

- Expressions and Equations:

- Use properties of operations to generate equivalent expressions.

    (No definition of equivalent.)

    - No “functions”.

Grade 8

Functions:

- Define, evaluate and compare functions.
- Use functions to model relationships between quantities.

- 1. Understand a function as a rule that assigns to each input exactly one output. Definition of graph of function.
- 2. Compare properties of functions represented in different ways. E.g., rate of change.
- 3. Know that linear functions have straight line graphs, and vice versa.

  Give examples of non-linear functions

- Modeling relationships:

  - work with linear relationships.

  - select graphs that reflect verbal descriptions.

High School - Functions

- Interpreting Functions.
- Building Functions.
- Linear, Quadratic and Exponential Functions.
- Trigonometric Functions.

Interpreting Functions

- Understand the concept of function.
  (“domain” and “range” are mentioned.)
- Interpret functions in a context.

- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship.

(“compose” is mentioned for STEM intending.)

- Build new functions from existing function.

(“inverse” functions, exp and log are mentioned for STEM intending.)

- CCSSM F-BF3: Identify the effect on a graph of the effect of

    adding a constant to a function,

  or multiplying a function by a constant,

  or adding a constant to a variable,

  or multiplying a variable by a constant.

- No mention of combining these, or the general affine substitution. Yet this idea could unify many topics in the high school curriculum.

- No attempt to connect this with the idea of composition.

- We meet functions of several variables, e.g.,

$$V = lwh$$

for the volume of a rectangular box (“cuboid”).

But there is no serious effort to analyze such functions, even to the extent of noting the effect of uniform scaling.