

Episode 41. Binomial series

MacLaurin series for $f(x) = (1+x)^k$ $k \in \mathbb{R}$
 $k = N$ (pos. integer) $(1+x)^N$ pol. of deg. N

$$k = -2 \quad \frac{1}{(1+x)^2}$$

$$k = \frac{1}{3} \quad \sqrt[3]{1+x}$$

and so on

$$M.S. = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

at $x=0$

$$f(x) = (1+x)^k$$

$$f'(x) = k(1+x)^{k-1}$$

$$f''(x) = k(k-1)(1+x)^{k-2}$$

$$f^{(n)}(x) = k(k-1)(k-2)\dots(k-n+1)x^{k-n}$$

$$f(0) = (1+0)^k = 1$$

$$f'(0) = k(1+0)^{k-1} = k$$

$$f''(0) = k(k-1)$$

$$f^{(n)}(0) = k(k-1)\dots(k-n+1)$$

$$(1+x)^k \stackrel{\text{binomial expansion}}{=} 1 + \frac{k}{1!} x + \frac{k(k-1)}{2!} x^2 + \dots + \frac{k(k-1)\dots(k-n+1)}{n!} x^n + \dots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!} x^n, \quad |x| < 1$$

binomial series

Why binomial?

Binomial th (from algebra)

Let \underline{k} be a pos. integer $\binom{k}{1}$ $\binom{k}{2}$ $\binom{k}{n}$

$$(1+x)^k = 1 \cdot x^0 + \frac{\underline{k}}{1!} x + \frac{\underline{k(k-1)}}{2!} x^2 + \dots + \frac{\underline{k(k-1)(k-2)\dots 1}}{n!} x^n$$

binomial coeff.

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{k(k-1)\dots(k-n+1)}{n!} \quad n=0, 1, 2, \dots k$$

Ex. Find MacLaurin pol. of $y = 3$ for

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

Sol. $(1+x)^k$

$$f(x) = \frac{1}{\sqrt{2x+3}} = \frac{1}{\sqrt{3}(1+\frac{2x}{3})} = \frac{1}{\sqrt{3}} \left(1 + \frac{2x}{3}\right)^{-\frac{1}{2}}$$

$$(1+t)^k = 1 + \frac{\underline{k}}{1!} t + \frac{\underline{k(k-1)}}{2!} t^2 + \frac{\underline{k(k-1)(k-2)}}{3!} t^3 + \dots$$

binomial
exp.

$$t = \frac{2x}{3} \rightarrow \text{Substitute}$$

$$\left(1 + \frac{2x}{3}\right)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})\left(\frac{2x}{3}\right) + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{2} \left(\frac{2x}{3}\right)^2 +$$

$$-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2) \frac{\left(\frac{2x}{3}\right)^3}{6} + \dots =$$

$$= 1 - \frac{x^3}{3} + \frac{x^2}{2} - \frac{5x^3}{54}$$

$$T_3(x) = \frac{1}{\sqrt{3}} \left(1 - \frac{x^3}{3} + \frac{x^2}{2} - \frac{5x^3}{54} \right)$$