Episode 22: Newton's law of cooling

The rate at which a hot body cools is proportional to the excess of its temperature above the temperature of its surrounding medium.

Let $y(t)$ be temp. at time t

$$\frac{dy}{dt} = \underbrace{k}_{\text{cooling coeff.}\ k<0}(y - y_{sur})$$

Problem. If a cup of coffee cools from 80°C to 50°C in 5 minutes in a 20°C room, how much longer will it take to cool to 40°C?

$7.9 - 5 = 2.9 \approx \underline{3 \text{ min}}$

$$\underset{t=0}{80°C} \xrightarrow{5\text{ min}} \underset{t=5}{50°C} \xrightarrow{?} 40°C \quad t = ? \quad t = 7.9$$

Let $y(t)$ be the temp at time t.

Given $\begin{cases} y(0) = 80°C & \text{initial cond.} \\ y(5) = 50°C & \text{extra cond.} \end{cases}$

$$\frac{dy}{dt} = k(y-20) \quad \text{(By Newton's law of cooling)}$$

$$\frac{dy}{y-20} = k\,dt$$

$$\ln|y-20| = kt + C_1$$

$$y - 20 = Ce^{kt}$$

$$\boxed{y(t) = 20 + Ce^{kt}}$$

$$80 = y(\underset{\underset{t}{\uparrow}}{0}) = 20 + Ce^{0} \Rightarrow C = 60$$

$$y(t) = 20 + 60e^{kt}$$

$$50 = y(\underset{\underset{t}{\uparrow}}{5}) = 20 + 60e^{5k} \Rightarrow 30 = 60e^{5k}$$

$$\frac{1}{2} = e^{5k}$$

$$\ln\frac{1}{2} = 5k$$

$$k = \frac{\ln 1/2}{5} \quad (\approx -0.14)$$

$$-\frac{\ln 2}{5}$$

$$\boxed{y(t) = 20 + 60\, e^{-\frac{\ln 2}{5} t}}$$

Find t s.t. $y(t) = 40$

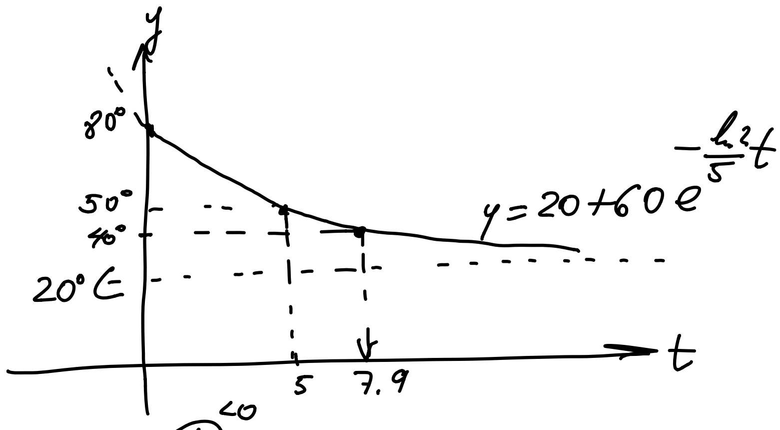
$$40 = 20 + 60\, e^{-\frac{\ln 2}{5} t}$$

$$20 = 60\, e^{-\frac{\ln 2}{5} t}$$

$$\frac{1}{3} = e^{-\frac{\ln 2}{5} t}$$

$$\underbrace{\ln \frac{1}{3}}_{-\ln 3} = -\frac{\ln 2}{5} t$$

$$\boxed{t = 5 \frac{\ln 3}{\ln 2}} \approx 7.9$$



$e^{\overset{<0}{\left(-\frac{\ln 2}{5}\right)} t} \xrightarrow[t\to\infty]{} 0$ and $y(t) \xrightarrow[t\to\infty]{} 20$

$7.9 - 5 = 2.9 \approx 3$ min more to cool down from 50°C to 40°C