Episode 8: Improper integrals of type II

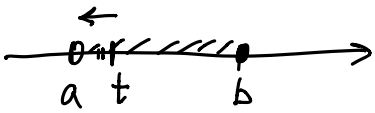
Type I $\int_a^{\infty}$, $\int_{-\infty}^{b}$, $\int_{-\infty}^{\infty}$ inf. limits

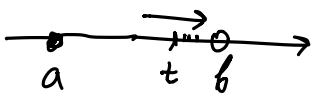
Type II $\int_a^b f(x)\,dx$ where f has an inf. discontinuity in $[a,b]$

Type II

$$\int_a^b f(x)\,dx$$

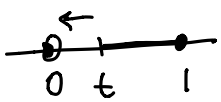
- f has inf. disc. at a: $\lim_{t\to a^+} \int_t^b f(x)\,dx$
- f has inf. disc. at b: $\lim_{t\to b^-} \int_a^t f(x)\,dx$
- f has inf. disc. at $c\in(a,b)$: $\int_a^b = \int_a^c + \int_c^b$

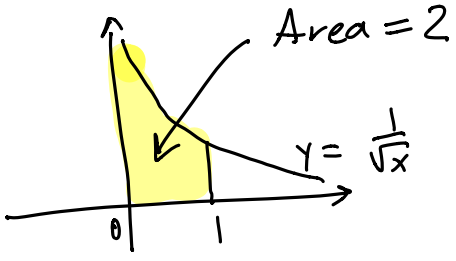




Ex. 1

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \left(2\sqrt{x}\right)\Big|_t^1 = \lim_{t\to 0^+} 2\left(\sqrt{1}-\sqrt{t}\right) = 2$$





Another way of writing:

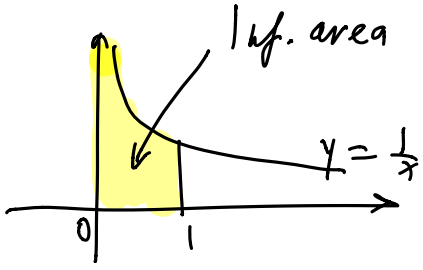
$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_{0^+}^1 = 2$$

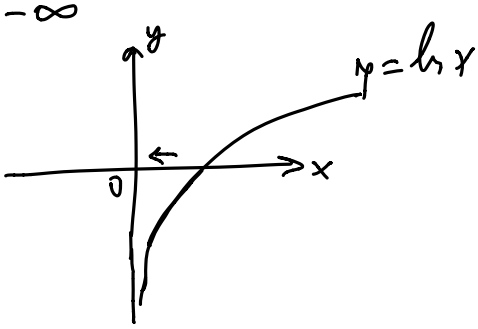
Ex. 2

$$\int_0^1 \frac{dx}{x} = \ln|x|\Big|_{0^+}^{1} = \underbrace{\ln 1}_{0} - \lim_{x\to 0^+} \ln|x| = -(-\infty) = \infty$$

diverges to ∞

$\lim_{x\to 0^+} \ln|x| = -\infty$

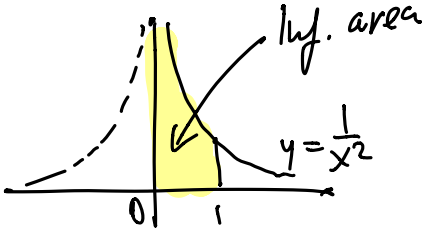




Ex. 3

$$\int_0^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_{0^+}^{1} = -(1-\infty) = \infty$$

diverges to ∞



$\lim_{x\to 0^+} \frac{1}{x} = \infty$

In general,

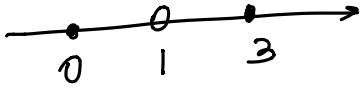
p-integral $\int_0^1 \frac{dx}{x^p}$

- $p < 1$ → conv. to $\frac{1}{1-p}$
- $p \geq 1$ → div. to ∞

Ex. 4

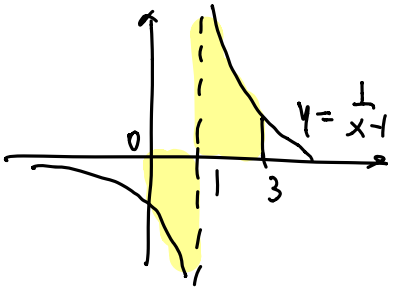
$$\int_0^3 \frac{dx}{x-1}$$

$f(x) = \frac{1}{x-1}$ has an inf. disc. at $x = 1 \in (0,3)$



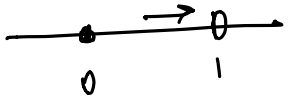
$$\int_0^3 = \underbrace{\int_0^1}_{\text{div. to } -\infty} + \int_1^3$$

div.

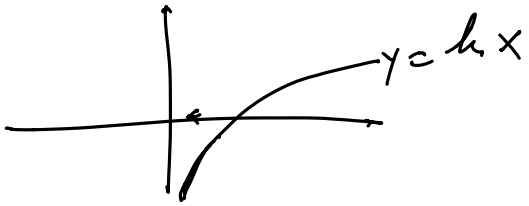


$$\int_0^1 \frac{dx}{x-1} = \ln|x-1| \Big|_0^{1^-} = -\infty - \underbrace{\ln|0-1|}_{0} = -\infty$$

div. to $-\infty$



$$\lim_{x\to 1^-} \ln|x-1| = -\infty$$



Comparison test for improper integrals

(to estimate conv / div. of integrals)

$-\infty \le a < b \le \infty$

$f(x) \le g(x)$ on (a,b)

$\int_a^b g(x)\,dx$ conv.

$\Rightarrow \int_a^b f(x)\,dx$ conv.

$f(x) \le g(x)$ on (a,b)

$\int_a^b f(x)\,dx$ div.

$\Rightarrow \int_a^b g(x)\,dx$ div.

Ex. 5 Does $\int_1^{\infty} \frac{x^2}{x^5+1}\,dx$ conv. or div. ?

thoughts: $\frac{x^2}{x^5+1} \sim \frac{1}{x^3}$ as $x\to\infty$

$\int_1^{\infty} \frac{dx}{x^3}$ conv. as p-int. with $p=3>1$

formal writing:

$$\underbrace{\frac{x^2}{x^5+1}}_{f} \leq \frac{x^2}{x^5} = \underbrace{\frac{1}{x^3}}_{g}$$

$x^5+1 > x^5$

$$\int_1^\infty \frac{dx}{x^3} \text{ conv. as } p\text{-int. with } p=3>1$$

$$\Rightarrow \int_1^\infty \frac{x^2\,dx}{x^5+1} \quad \text{conv. by comparison test}$$