MAT131 Homework 31-32

Problems

1. Compute the following definite integrals:

   (a) $\displaystyle\int_1^2 \frac{6}{x} + \frac{\sqrt[3]{x^2}}{2} - \frac{1}{2x^3}\, dx$

   (b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(x) - 6\csc(x)\cot(x)\; dx$

2. Find the derivatives of the following functions:

   (a) $\displaystyle f(x) = \int_0^{e^{3x}} \frac{1}{t^4+t^2+1} dt$

   (b) $\displaystyle g(x) = \int_{x^{\ 3}}^{x} \cos^4(t) - \sin^2(t)\; dt$

3. Consider
$$f(x) = \int_1^{x^2} 3t - 12\, dt.$$

   (a) Find the extrema of f.

   (b) Find the equation of the tangent line to f at $x = 1$.

   (c) Find the inflection points of f.

4. Determine the area between $f(x) = 6x - x^2$ and the x-axis.

5. Determine the area between $f(x) = x^2 + 2x + 4$ and $g(x) = 3x + 6$

6. Determine the area between $f(x) = x^2$, $g(x) = x^3$, $x = 0$ and $x = 2$

Answer Key

1. (a) $\displaystyle\int_1^2 \frac{6}{x} + \frac{1}{2}x^{\frac{2}{3}} - \frac{1}{2}x^{-3}\,dx = 6\ln(2) + \frac{3(4)^{\frac{1}{3}}}{5} - \frac{39}{80}$

   (b) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(x) - 6\csc(x)\cot(x)\;dx = \frac{14}{\sqrt{3}} - 12$

2. (a) $\displaystyle f'(x) = \frac{3e^{3x}}{\left(e^{3x}\right)^4 + \left(e^{3x}\right)^2 + 1} = \frac{3e^{3x}}{e^{12x} + e^{6x} + 1}$

   (b) $g'(x) = \cos^4(x) - \sin^2(x) - \left(\cos^4(x^3) - \sin^2(x^3)\right)\left(3x^2\right)$

3. (a) There are minima at $x = \pm 2$. There is a maximum at $x = 0$.

   (b) $y = -18x + 18$

   (c) The points of inflection occur at $x = \pm\dfrac{2\sqrt{3}}{3}$

4. 180

5. $\dfrac{9}{2}$

6. $\dfrac{3}{2}$

Solutions

1. (a) First, notice that we can rewrite

$$\int_1^2 \frac{1}{6x} + \frac{\sqrt[3]{x^2}}{2} - \frac{1}{2x^3}\,dx = \int_1^2 \frac{1}{6x} + \frac{1}{2}x^{\frac{2}{3}} - \frac{1}{2}x^{-3}\,dx$$

The power rule for integration implies that

$$\begin{aligned}\int_1^2 \frac{6}{x} + \frac{1}{2}x^{\frac{2}{3}} - \frac{1}{2}x^{-3}\,dx &= \left(6\ln(x) + \frac{3}{10}x^{\frac{5}{3}} + \frac{1}{4}x^{-2}\right)\Big|_1^2 \\ &= \left(6\ln(2) + \frac{3}{10}2^{\frac{5}{3}} + \frac{1}{4}(2)^{-2}\right) - \left(6\ln(1) + \frac{3}{10}1^{\frac{5}{3}} + \frac{1}{4}(1)^{-2}\right) \\ &= 6\ln(2) + \frac{3(4)^{\frac{1}{3}}}{5} - \frac{39}{80}\end{aligned}$$

(b) We have

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2(x) - 6\csc(x)\cot(x)\;dx &= \tan(x) + 6\csc(x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \tan\left(\frac{\pi}{3}\right) + 6\csc\left(\frac{\pi}{3}\right) - \left(\tan\left(\frac{\pi}{6}\right) + 6\csc\left(\frac{\pi}{6}\right)\right) \\ &= \sqrt{3} + \frac{12}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}} + 12\right) \\ &= \frac{14}{\sqrt{3}} - 12\end{aligned}$$

2. (a) By the first Fundamental Theorem of Calculus and the chain rule,

$$f'(x) = \frac{3e^{3x}}{(e^{3x})^4 + (e^{3x})^2 + 1} = \frac{3e^{3x}}{e^{12x} + e^{6x} + 1}$$

(b) First, notice

$$\begin{aligned}g(x) &= \int_0^x \cos^4(t) - \sin^2(t)\,dt + \int_{x^3}^0 \cos^4(t) - \sin^2(t)\,dt \\ &= \int_0^x \cos^4(t) - \sin^2(t)\,dt - \int_0^{x^3} \cos^4(t) - \sin^2(t)\,dt\end{aligned}$$

It follows from first Fundamental Theorem of Calculus and the chain rule that

$$g'(x) = \cos^4(x) - \sin^2(x) - \left(\cos^4(x^3) - \sin^2(x^3)\right)\left(3x^2\right).$$

3. (a) By the first Fundamental Theorem of Calculus and the chain rule,

$$f'(x) = \left(3x^2 - 12\right)(2x).$$

It follows that $f'(x) = 0$ when $x = -2, 0, 2$. The first derivative test implies that f has a minimum at $x = -2$ and $x = 2$ and f has a maximum at $x = 0$.

(b) Since $f'(1) = (3(1)^2 - 12)(2(1)) = -18$ and

$$f(1) = \int_1^1 3t - 12\,dt = 0,$$

the point-slope formula implies that the equation of the tangent line is given by

$$y = y - 0 = -18(x-1) = -18x + 18.$$

(c) Recall that

$$f'(x) = \left(3x^2 - 12\right)(2x) = 6x^3 - 24x.$$

It follows that

$$f''(x) = 18x^2 - 24.$$

It follows that the inflection points occur at $x = \pm\dfrac{2\sqrt{3}}{3}$.

4. Since $6x - x^2 = x(6-x) = 0$ when $x = 0$ and $x = 6$, the area between $f(x)$ and the x-axis is given by

$$\int_0^6 6x - x^2\,dx = \left(3x^2 + \frac{1}{3}x^3\right)\Big|_0^6 = 3(6)^2 + \frac{6^3}{3} = 180$$

5. The curves $f(x)$ and $g(x)$ intersect when

$$x^2 + 2x + 4 = 3x + 6.$$

Simplifying gives

$$x^2 - x - 2 = 0.$$

It follows that $f(x)$ and $g(x)$ intersect at $x = -1$ and $x = 2$. It follows that the area between f and g is given by

$$\begin{aligned}\int_{-1}^2 3x + 6 - (x^2 + 2x + 4)\,dx &= \int_{-1}^2 -x^2 + x + 2\,dx \\ &= \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right)\Big|_{-1}^2 \\ &= \left(-\frac{2^3}{3} + \frac{2^2}{2} + 2(2)\right) - \left(-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1)\right) \\ &= \frac{9}{2}.\end{aligned}$$

6. Notice that $x^2 = x^3$ when

$$x0 = x^3 - x^2 = x^2(x-1).$$

It follows that $f(x)$ and $g(x)$ intersect when $x = 0$ and $x = 1$. It follows that the area between f and g is given by

$$\begin{aligned}
\int_0^1 x^2 - x^3 \, dx + \int_1^2 x^3 - x^2 \, dx &= \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_0^1 + \left(\frac{1}{4}x^4 - \frac{1}{3}x^3\right)\Bigg|_1^2 \\
&= \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\left(\frac{(2)^4}{4} - \frac{(2)^3}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right) \\
&= \frac{3}{2}.
\end{aligned}$$