Lecture 18

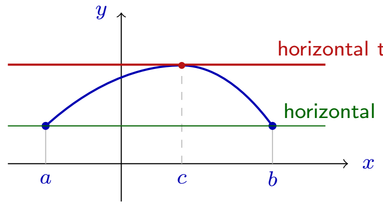
Mean Value Theorem

Objectives

In this lecture we discuss the statement, a proof and corollaries of the **Mean Value Theorem**.

Rolle's theorem

Theorem. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is number $c \in (a, b)$ such that $f'(c) = 0$.



Proof. By the Extreme Value Theorem (see Lecture 17), there exist points $c, d \in [a, b]$ at which f attains its maximum and its minimum values.

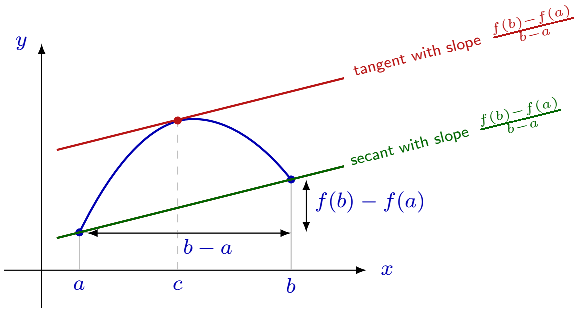
If the maximum and minimum values are both attained at the **endpoints** a, b, then the maximum and minimum values are equal, so f is constant on $[a, b]$ and $f'(c) = 0$ for all $c \in (a, b)$.

If one of the extreme values is attained at an **interior** point c or d, then, by Fermat's theorem (see Lecture 17), $f'(c) = 0$ or $f'(d) = 0$. □

The Mean Value Theorem (Lagrange's theorem)

Theorem. If f is continuous on $[a,b]$ and differentiable on (a,b),

then there exists a number $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.



Remark. Rolle's theorem is a special case of the Mean Value Theorem (MVT).

Indeed, if $f(a)=f(b)$, then, by MVT, $f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{0}{b-a} = 0$

for some $c \in (a,b)$.

Proof of the Mean Value Theorem

Proof. Consider a new function $g(x) = f(x) - \dfrac{f(b)-f(a)}{b-a}(x-a)$.

It is continuous on $[a,b]$ and differentiable on (a,b)

(since both functions f and $(x-a)$ are continuous)

and

$g(a) = f(a)$,

$g(b) = f(b) - \dfrac{f(b)-f(a)}{b-a}(b-a) = f(a)$.

Therefore, we may apply Rolle's theorem to g

and conclude that there exists some $c \in (a,b)$ such that $g'(c)=0$, that is

$0 = g'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a} \iff f'(c) = \dfrac{f(b)-f(a)}{b-a}$. $\square$

Corollaries of the Mean Value Theorem

Corollary 1. If $f'(x) = 0$ for all x in an interval, then f is constant on that interval.

Proof. Take any two distinct points a, b in the interval.

By the Mean Value Theorem, there exists some $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Since $f'(x) = 0$ for all x in the interval, we obtain $f(a) = f(b)$.

So f takes the same value at any two points in the interval,

that is, f is constant on the interval. □

Corollary 2. If $f'(x) = g'(x)$ for all x on some interval, then $f = g + C$.

Proof. $(f - g)' = f' - g' = 0$. So by Corollary 1,
$f - g = C$ for some constant C. Therefore, $f = g + C$. □

Using the MVT to prove inequalities

Example. Prove that $\tan x > x$ for all $x \in \left(0, \frac{\pi}{2}\right)$.

Solution. Take any $x \in \left(0, \frac{\pi}{2}\right)$ and consider the function $f(x) = \tan x$ on the interval $[0, x]$.

f is differentiable on $[0, x]$, therefore we may apply
the Mean Value Theorem to the function f and the interval $[0, x]$.

By the MVT, there exists a point $x_0 \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(x_0) \iff \frac{\tan x}{x} = (\tan x)'\Big|_{x = x_0} \iff \frac{\tan x}{x} = \frac{1}{\cos^2 x_0}.$$

Since $0 < x_0 < \frac{\pi}{2}$, then $\cos^2 x_0 < 1$. So $\frac{1}{\cos^2 x_0} > 1$. This gives us

$\frac{\tan x}{x} = \frac{1}{\cos^2 x_0} > 1$, that is, $\tan x > x$, as required.

Summary

In this lecture we learned

- Rolle's theorem
- the Mean Value Theorem (Lagrange's theorem)

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Comprehension checkpoint

- State Rolles' theorem and give its graphical interpretation.
- State the Mean Value Theorem and give its graphical interpretation.

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