Homework

All the problems are to be solved using the method of separation of variables.

1. Compute the general solution to the differential equation $y' = x^2 y - 2y$.
2. Compute the particular solution to the differential equation $y' = y + 2$ with initial condition $y(0) = 0$.
3. Compute the general solution and particular solution to the differential equation $y' = x^3 e^y$ with initial condition $y(0) = 1$.
4. Compute the particular solution to the differential equation $y' = e^{2x} y^4$ with initial condition $y(0) = 1$.
5. Compute the general solution to the differential equation $y' = 2y(\sec^2 x - 1)$.
6. Compute the general solution to the differential equation $yxy' = (x^2 - 1)$.

Solutions

1. Compute the general solution to the differential equation $y' = x^2y - 2y$.

   Solution: We can factor the right side of the equation to get $y' = y(x^2-2)$. Expressing y' as dy/dx, when we separate variables we get $\int \frac{dy}{y} = \int x^2 - 2dx$. Integrating this, we obtain $\ln|y| = x^3/3 - 2x + C_1$. Exponentiating, we get $|y| = e^{x^3/3-2x+C_1}$. We can bring the constant C_1 as the constant $C = e^{C_1}$ and this lets us remove the absolute value bars around y. So we have $y = Ce^{x^3/3-2x}$.

2. Compute the particular solution to the differential equation $y' = y + 2$ with initial condition $y(0) = 0$.

   Solution: Rewriting this equation with dy/dx notation, we have $dy/dx = y + 2$. Separating variables, we get $\frac{dy}{y+2} = dx$. Integrating both sides, we get $\ln|y+2| = x + C_1$. Exponentiating, we get $|y+2| = e^{x+C_1}$ which can be rewritten as $y + 2 = Ce^x$. Plugging in the initial condition, we get $0 + 2 = Ce^0 = C$. Hence, $C = 2$. Thus, our particular solution is $y = -2 + 2e^x$.

3. Compute the general solution and particular solution to the differential equation $y' = x^3e^y$ with initial condition $y(0) = 1$.

   Solution: Rewriting this equation with dy/dx notation, we have $dy/dx = x^3e^y$. Separating variables, we get $\frac{dy}{e^y} = x^3dx$. To continue, we want to take the integral of both sides, as follows:

   $$\begin{aligned}
   \int e^{-y}dy &= \int x^3dx \\
   -e^{-y} &= x^4/4 + C_1 \\
   e^{-y} &= -x^4/4 + C_2 \\
   -y &= \ln(-x^4/4 + C_2) \\
   y &= \ln(-x^4/4 + C_2)
   \end{aligned}$$

   Plugging in the initial condition $y(0) = 1$, we have $1 = \ln(C_2)$. Hence, $C_2 = 0$. Thus, the general solution is $y = \ln(-x^4/4 + C)$ and the particular solution is $y = \ln(-x^4/4)$.

4. Compute the particular solution to the differential equation $y' = e^{2x}y^4$ with initial condition $y(0) = 1$.

   Solution: Rewriting this equation with dy/dx notation, we have $dy/dx =$

$e^{2x}y^4$. The computation then follows as:

$$\frac{dy}{y^4} = e^{2x}dx$$
$$\int y^{-4}dy = \int e^{2x}dx$$
$$y^{-3}/(-3) = e^{2x}/2 + C_1$$
$$y^{-3} = (-3/2)e^{2x} + C_2$$
$$y = ((-3/2)e^{2x} + C_2)^{-1/3}$$

Plugging in the initial condition $y(0) = 1$, we get $1 = ((-3/2)e^0 + C_2)^(-1/3)$. Thus $1 = 1^{-3} = ((-3/2) + C_2)$. Hence, $C = 5/2$. So the particular solution is $y = ((-3/2)e^{2x} + 5/2)^{-1/3}$.

5. Compute the general solution to the differential equation $y' = 2y(\sec^2 x - 1)$.

   Solution: Rewriting this equation with dy/dx notation, we have $dy/dx = 2y(\sec^2 x - 1)$. Separating variables, we get $\dfrac{dy}{y} = 2(sec^2x - 1)dx$. The computation then follows as:

$$\int \frac{dy}{y} = 2\int \sec^2 x - 1dx$$
$$\ln|y| = 2(\tan x - x + C_1)$$
$$|y| = e^{2(\tan x - x + C_1)}$$
$$y = Ce^{2\tan x - 2x}$$

   And we are done.

6. Compute the general solution to the differential equation $yxy' = (x^2 - 1)$.

   Solution: Rewriting this equation with dy/dx notation, we have $yx\frac{dy}{dx} = (x^2 - 1)$. Separating variables, we get $ydy = \dfrac{x^2 - 1}{x}dx$. The computation then follows as:

$$\int ydy = \int x - (1/x)dx$$
$$y^2/2 = x^2/2 - \ln|x| + C_1$$
$$y^2 = x^2 - 2\ln|x| + C$$

   And we are done.

Answer Key

1. Compute the general solution to the differential equation $y' = x^2y - 2y$.

   $y = Ce^{x^3/3-2x}$.

2. Compute the particular solution to the differential equation $y' = y + 2$ with initial condition $y(0) = 0$.

   $y = -2 + 2e^x$

3. Compute the general solution and particular solution to the differential equation $y' = x^3e^y$ with initial condition $y(0) = 1$.

   General solution: $y = \ln(-x^4/4 + C)$ Particular solution: $y = \ln(-x^4/4)$

4. Compute the particular solution to the differential equation $y' = e^{2x}y^4$ with initial condition $y(0) = 1$.

   $y = ((-3/2)e^{2x} + 5/2)^{-1/3}$

5. Compute the general solution to the differential equation $y' = 2y(\sec^2 x - 1)$.

   $y = Ce^{2\tan x - 2x}$

6. Compute the general solution to the differential equation $yxy' = (x^2 - 1)$.

   $y^2 = x^2 - 2\ln|x| + C$