MAT 127 HW 18-19

1. Problems

1. Find the Maclaurin series of the following function. And find the degree two taylor polynomial centered at zero.
$$f(x) = (1-x)^{\frac{1}{5}}$$
2. Find the Maclaurin series of the following function. And find the degree two taylor polynomial centered at zero.
$$f(x) = (1-x^2)^{-\frac{1}{5}}$$
3. Find the Maclaurin series of the following function. And find the degree two taylor polynomial centered at zero.
$$f(x) = (1-3x)^{\frac{2}{5}}$$
4. Estimate the error of the degree two taylor polynomial centered at 0 on $[-1, 1]$ of
$$f(x) = e^{2x}.$$
5. Estimate the error of the degree two taylor polynomial centered at 0 on $[-\frac{\pi}{6}, \frac{\pi}{6}]$ of
$$f(x) = \sec(x).$$

2. Answer Key

1. Maclaurin series is
$$(1-x)^{\frac{1}{5}} = \sum_{n=0}^{\infty} \binom{\frac{1}{5}}{n} (-x)^n$$
and
$$T_2(x) = 1 - \frac{1}{5}x + \frac{-2}{25}x^2.$$
2. Maclaurin series is
$$(1-x^2)^{\frac{-1}{5}} = \sum_{n=0}^{\infty} \binom{\frac{-1}{5}}{n} (-x^2)^n$$
and
$$T_2(x) = 1 + \frac{1}{5}x^2 + \frac{2}{25}x^4.$$
3. Maclaurin series is
$$(1-2x)^{\frac{2}{5}} = \sum_{n=0}^{\infty} \binom{\frac{2}{5}}{n} (-2x)^n$$
and
$$T_2(x) = 1 + \frac{-4}{5}x + \frac{-3}{25}x^2.$$

4.
$$|R_2(x)| \leq \frac{8e^2}{6} = \frac{4e^2}{3}.$$

5.
$$|R_2(x)| \leq \frac{14}{9} \cdot \frac{\pi}{6} = \frac{14\pi}{54} = \frac{7\pi}{27}.$$

3. Solutions

1. Using the formula:
$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n.$$
We see that the Maclaurin series is
$$(1-x)^{\frac{1}{5}} = \sum_{n=0}^{\infty} \binom{\frac{1}{5}}{n} (-x)^n$$
and
$$T_2(x) = 1 - \frac{1}{5}x + \frac{-2}{25}x^2.$$

2. Using the formula:
$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n.$$
We see that the Maclaurin series is
$$(1-x^2)^{\frac{-1}{5}} = \sum_{n=0}^{\infty} \binom{\frac{-1}{5}}{n} (-x^2)^n$$
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3. Using the formula:
$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n.$$
We see that the Maclaurin series is
$$(1-2x)^{\frac{2}{5}} = \sum_{n=0}^{\infty} \binom{\frac{2}{5}}{n} (-2x)^n$$
and
$$T_2(x) = 1 + \frac{-4}{5}x + \frac{-3}{25}x^2.$$

4. Use the formula:
$$|R_2(x)| \leq \frac{\max_{[-1,1]} f'''(x)}{3!}|x|^3.$$
Note that $f(x) = e^{2x}$ and so $f'''(x) = 8e^{2x}$. And so the $\max_{[-1,1]} f'''(x) = 8e^2$. Also on $[-1,1]$ we have $|x| \leq 1$. So
$$|R_2(x)| \leq \frac{8e^2}{6} = \frac{4e^2}{3}.$$

5. Use the formula:
$$|R_2(x)| \le \frac{\max_{[-\frac{\pi}{6},\frac{\pi}{6}]} f'''(x)}{3!}|x|^3.$$
Note that $f(x) = \sec(x)$ and so
$$f'''(x) = 5\tan(x)\sec^3(x) + \tan^3(x)\sec(x).$$
As $\tan x$ is increasing on $[-\frac{\pi}{6}, \frac{\pi}{6}]$ and $\sec x$ is even and increasing on $[0, \frac{\pi}{6}]$ we see that
$$\max_{[-\frac{\pi}{6},\frac{\pi}{6}]} f'''(x) = f'''(1) = \frac{14}{3}.$$
Also on $[-\frac{\pi}{6}, \frac{\pi}{6}]$ we have $|x| \le \frac{\pi}{6}$. So
$$|R_2(x)| \le \frac{14}{9} \cdot \frac{\pi}{6} = \frac{14\pi}{54} = \frac{7\pi}{27}.$$