1

Using the definition of the derivative, compute the derivatives of the following functions:

1. $f(x) = x + 2$.

2. $g(x) = x^2 + 4x$.

2

Determine the region where the function $f(x) = x^3 - x^2$ is decreasing.

3

Find the slope of the tangent line to the curve $g(x) = \frac{3}{x-4}$ at the point $(1, -1)$.

4

Is the point $(0, 1)$ a maximum or a minimum or neither for the function $f(x) = 1 - x^2$? Justify your answer.

5

Show that the function $f(x) = x^3 + 3x$ is always increasing.

Answer Key

1. (i) $f'(x) = 1$, (ii) $g'(x) = 2x + 4$.

2. The function $f(x)$ is decreasing in the region (0,2/3).

3. The slope of the tangent line at the given point is $g'(1) = -1/3$.

4. A maximum.

5. The derivative is $f'(x) = 3x^2 + 3 > 0$, which is always positive.

Solutions

1. Using the definition of the derivative, we compute:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h+2) - (x+2)}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

Using the definition of the derivative, we compute:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 + 4(x+h) - (x^2 + 4x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 4x + 4h - x^2 - 4x}{h} \\ &= \lim_{h\to 0} \frac{h(2x + 4 + h)}{h} \\ &= \lim_{h\to 0} (2x + 4 + h) \\ &= 2x + 4 \end{aligned}$$

The technique here was the expand, cancel terms, and then factor out h so that the denominator could be cancelled and the limit could be calculated.

2. The function is decreasing when its derivative is negative, so we must compute the derivative. Using the definition of the derivative, we compute:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - (x+h)^2 - (x^3 - x^2)}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^2 - 2xh - h^2 - x^3 + x^2}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 - 2x + 3xh)}{h} \\ &= \lim_{h\to 0} (3x^2 - 2x + 3xh) \\ &= 3x^2 - 2x \end{aligned}$$

Now, we need to check when $3x^2 - 2x < 0$. We observe that $3x^2 - 2x = x(3x - 2)$, so this is negative when $x < 0$ and $3x - 2 > 0$ and when $x > 0$ and $3x - 2 < 0$. Note that $3x - 2 > 0$ when $x > 2/3$, so we cannot have $x < 0$ and $3x - 2 > 0$ as then we would have $x < 0$ and $x > 2/3$. However, $3x - 2 < 0$ when $x < 2/3$, so we have $x > 0$ and $3x - 2 < 0$ precisely when $x > 0$ and $x < 2/3$. In other words, $f(x)$ is decreasing in the region $(0, 2/3)$.

3. The slope of the tangent line to $g(x)$ at the point $(1, -1)$ is by definition the derivative $g'(x)$ at the point $x = 1$. Using the definition of the derivative, we compute:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{(x+h)-4} - \frac{3}{x-4}}{h} \\ &= \lim_{h\to 0} \frac{\frac{3(x-4)-3(x+h-4)}{(x+h-4)(x-4)}}{h} \\ &= \lim_{h\to 0} \frac{-3h}{h(x+h-4)(x-4)} \\ &= \lim_{h\to 0} \frac{-3}{(x+h-4)(x-4)} \\ &= \frac{-3}{(x-4)^2} \end{aligned}$$

Here, we found a common denominator and then factored to cancel the h in the denominator. Then, we evaluated the limit using the limit laws for rational functions. It remains to compute $g'(1) = -3/(-3)^2 = -3/9 = -1/3$.

4. First, we compute the derivative, using the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1-(x+h)^2 - (1-x^2)}{h} \\ &= \lim_{h\to 0} \frac{1 - x^2 - 2xh - h^2 - 1 + x^2}{h} \\ &= \lim_{h\to 0} \frac{h(-2x-h)}{h} \\ &= \lim_{h\to 0} (-2x-h) \\ &= -2x \end{aligned}$$

The critical point is when $f'(x) = -2x = 0$, so when $x = 0$. This is precisely the point $(0, f(0)) = (0, 1)$. When $x < 0$, $f'(x) > 0$ and when $x > 0$, $f'(x) < 0$, so f is increasing up to the critical point from the left and decreasing to the right away from the critical point. Therefore, the critical point $(0, 1)$ is a maximum.

5. The function is increasing when the derivative is positive. Using the definition of the derivative, we compute:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 + 3(x+h) - (x^3+3x)}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h - x^3 - 3x}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 + 3 + 3xh + h^2)}{h} \\ &= \lim_{h\to 0} (3x^2 + 3 + 3xh + h^2) \\ &= 3x^2 + 3 \end{aligned}$$

Now, $f'(x) = 3x^2 + 3 = 3(x^2 + 1)$ and the function $x^2 + 1 > 0$ is always positive, so $f'(x) > 0$ is everywhere positive. Therefore, the function $f(x)$ is everywhere increasing.