1

Compute the following definite integrals:

1. $\int_{\pi}^{2\pi} (\sin(x) + 1)\ dx$

2. $\int_{0}^{5} (x^4 + 4x + 3)\ dx$

2

Find the area under the curve $g(x) = \cos(x+1) + x$ between $x = \pi - 1$ and $x = 2\pi - 1$.

3

Define a function:

$$f_a(x) = \int_a^{x^2} t^2\ dt$$

for some constant value a. Compute $f_a'(x)$. Does this depend on the value of a?

4

Let $R(t)$ denote the approximate number of cars traveling on a given highway as a function of time t in hours, and suppose $R(t)$ is given by the equation:

$$R(t) = 1000(t + \sqrt{t})$$

Determine the approximate average, to the nearest whole number, of cars on the highway over the first 3 hours.

5

Determine an interval over which the function $h(s) = e^{-s}$ has an average value of $\frac{1}{4\ln(2)}$.

Answer Key

1. (i) $\pi - 2$ (ii) 690.
2. $\frac{1}{2}(3\pi^2 - 2\pi)$.
3. The derivative is $f_a'(x) = 2x^5$, which is independent of a.
4. 7964.
5. $(\ln(2), \ln(4))$.

Solutions

1. Using the Fundamental Theorem of Calculus, we compute:

$$\int_{\pi}^{2\pi} (\sin(x) + 1)\ dx = (-\cos(x) + x)\Big|_{\pi}^{2\pi} = -1 + 2\pi - (1 + \pi) = \pi - 2$$

Likewise, using the Fundamental Theorem of Calculus, we compute:

$$\int_{0}^{5} (x^4 + 4x + 3)\ dx = (\frac{1}{5}x^5 + 2x^2 + 3x)\Big|_{0}^{5} = 5^4 + 2 \cdot 5^2 + 15 = 690$$

2. The area is given by the following definite integral, which we compute using the Fundamental Theorem of Calculus:

$$\int_{\pi-1}^{2\pi-1} (\cos(x+1) + x)\ dx = (\sin(x+1) + \frac{1}{2}x^2)\Big|_{\pi-1}^{2\pi-1} = (0 + \frac{1}{2}(2\pi-1)^2) - (0 + \frac{1}{2}(\pi-1)^2)$$

which simplifies to given an area of $\frac{1}{2}(3\pi^2 - 2\pi)$.

3. We can compute $f_a(x)$ using the Fundamental Theorem of Calculus:

$$f_a(x) = \frac{1}{3}t^3\Big|_{a}^{x^2} = \frac{1}{3}(x^6 - a^3)$$

Taking the derivative, the constant term in a^3 goes away and we find:

$$f_a'(x) = 2x^5$$

which does not depend on a.

4. The average value over the first three hours is given by:

$$\int_0^3 R(t)\ dt = 1000 \int_0^3 (t + \sqrt{t})\ dt = (500t^2 + \frac{2000}{3}t^{3/2})\Big|_0^3 = 500 \cdot 3^2 + \frac{2000}{3} \cdot 3^{3/2} = 4500 + 2000\sqrt{3}$$

The nearest whole number to $4500 + 2000\sqrt{3}$ is 7964, so the approximate average number of cars on the highway over the first three hours was 7964.

5. The average value of the function $h(s)$ over an interval (a, b) is given by the integral:

$$H(s) = \frac{1}{b-a}\int_a^b e^{-s}\ ds = \frac{-e^{-s}}{b-a}\Big|_a^b = \frac{e^{-a} - e^{-b}}{b-a}$$

If we let $a = \ln(2)$ and $b = \ln(4)$, we obtain:

$$H(s) = \frac{e^{-\ln(2)} - e^{-\ln(4)}}{\ln(4) - \ln(2)} = \frac{1/2 - 1/4}{\ln(4/2)} = \frac{1}{4\ln(2)}$$

as desired. Hence, the interval $(\ln(2), \ln(4))$ suffices for the problem.