Lecture 31

Quadratic Inequalities

Quadratic inequalities 2
Visualization 3
Geometric solution 4
Geometric solution 5
What if $a < 0$? 6
Example 1 7
Example 2 8
Example 3 9
Example 4 10
Example 4 11
Summary 12

Quadratic inequalities

We will solve inequalities of the following types:

$ax^2 + bx + c \geq 0$, $ax^2 + bx + c > 0$, $ax^2 + bx + c \leq 0$, $ax^2 + bx + c < 0$,

where $a \neq 0$, b , c are given coefficients, and x is unknown.

For example, $x^2 + 5x - 6 \leq 0$ is a quadratic inequality.
Here $a = 1$, $b = 5$, $c = -6$.

The coefficient a is **not** zero, otherwise the inequality would be not quadratic, but rather linear.

What does it mean to solve inequality?

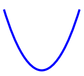
It means to find **all** the values of unknown x for which the inequality holds true.

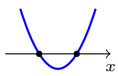
Visualization

Let us draw a picture illustrating a quadratic inequality.

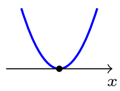
We know that the equation $y = ax^2 + bx + c$ defines a parabola,
and know how to draw this parabola.

If $a > 0$, then the parabola opens upward:

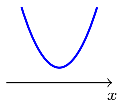




two x-intercepts

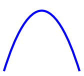


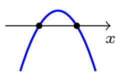
one x-intercept



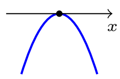
no x-intercepts

If $a < 0$, then the parabola opens downward:

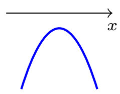




two x-intercepts



one x-intercept



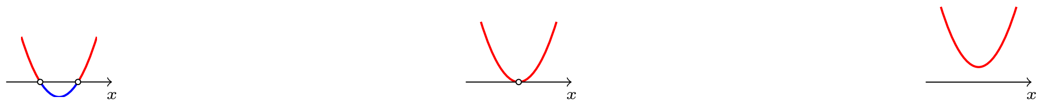
no x-intercepts

Geometric solution

Let us solve the inequality $ax^2 + bx + c > 0$ in the case when $a > 0$.

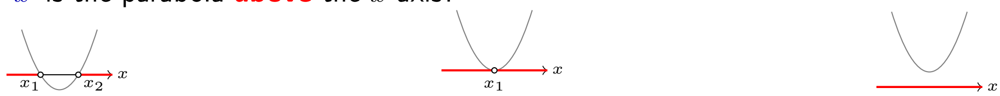
Let $y = ax^2 + bx + c$. Then $ax^2 + bx + c > 0 \iff y > 0$.

Thus, to solve the inequality $ax^2 + bx + c > 0$, we need to find where the parabola $y = ax^2 + bx + c$ is **above** the x-axis.



two x-intercepts | one x-intercept | no x-intercepts

For which x is the parabola **above** the x-axis?



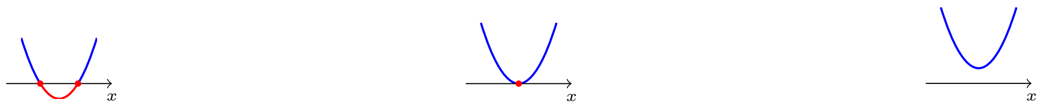
$x \in (-\infty, x_1) \cup (x_2, \infty)$ | $x \in (-\infty, x_1) \cup (x_1, \infty)$ | $x \in (-\infty, \infty)$

Geometric solution

Now let us solve the inequality $ax^2 + bx + c \leq 0$ again in the case when $a > 0$.

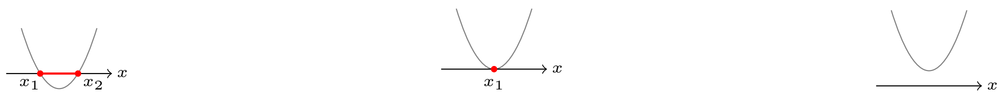
Let $y = ax^2 + bx + c$. Then $ax^2 + bx + c \leq 0 \iff y \leq 0$.

Thus, to solve the inequality $ax^2 + bx + c \leq 0$, we need to find where the parabola $y = ax^2 + bx + c$ is **below or on** the x-axis.



two x-intercepts | one x-intercept | no x-intercepts

For which x is the parabola **below or on** the x-axis?



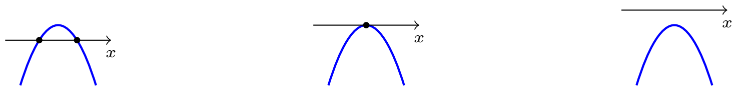
$x \in [x_1, x_2]$ | $x = x_1$ | no solution

What if $a < 0$?

We have a choice:

• **either** to solve the inequality using a parabola, as we did in the case $a > 0$,

Don't forget that the parabola $y = ax^2 + bx + c$ with $a < 0$ opens down:



• **or** multiply both sides of the inequality by -1, like

$$-3x^2 + x - 2 \geq 0 \iff 3x^2 - x + 2 \leq 0,$$

in order to make a-coefficient positive.

Don't forget to reverse the sign of inequality!

Example 1

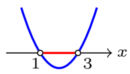
Solve the inequality $x^2 - 4x + 3 < 0$.

Solution. The parabola $y = x^2 - 4x + 3$ opens **upward**, since $a = 1 > 0$.

Determine the x-intercepts. They are the **roots** of the equation $x^2 - 4x + 3 = 0$.

$x^2 - 4x + 3 = 0 \iff (x-1)(x-3) = 0 \iff x_1 = 1,\ x_2 = 3$.

Therefore, the parabola looks as follows:



To solve the inequality $x^2 - 4x + 3 < 0$, we have to find **all** x for which the parabola is **below** the x-axis.

As we see, those x fill the interval $(1, 3)$.

The **answer** can be written in several ways:

$1 < x < 3$, or $x \in (1, 3)$, or simply $(1, 3)$.

Example 2

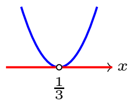
Solve the inequality $9x^2 - 6x + 1 > 0$.

Solution. The parabola $y = 9x^2 - 6x + 1$ opens **upward**, since $a = 9 > 0$.

Determine the x-intercepts. They are the **roots** of the equation $9x^2 - 6x + 1 = 0$.

$$9x^2 - 6x + 1 = 0 \iff (3x-1)^2 = 0 \iff x_1 = \frac{1}{3}.$$

Therefore, the parabola looks as follows:



To solve the inequality $9x^2 - 6x + 1 > 0$, we have to find **all** x for which the parabola is **above** the x-axis.

As we see, those x fill the whole line except the point $\frac{1}{3}$.

The **answer** can be written as $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$ or $\mathbb{R} \setminus \left\{\frac{1}{3}\right\}$.

Example 3

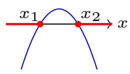
Solve the inequality $-x^2 + 3x - 1 \leq 0$.

Solution. The parabola $y = -x^2 + 3x - 1$ opens **downward**, since $a = -1 < 0$.

Determine the x-intercepts. They are the **roots** of the equation $-x^2 + 3x - 1 = 0$. Solve the equation:

$$-x^2 + 3x - 1 = 0 \iff x^2 - 3x + 1 = 0 \iff x_{1,2} = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

Therefore, the parabola looks as follows:



To solve the inequality $-x^2 + 3x - 1 \leq 0$, we have to find **all** x for which the parabola is **below or on** the x-axis.

Answer: $x \in \left(-\infty, \frac{3-\sqrt{5}}{2}\right] \cup \left[\frac{3+\sqrt{5}}{2}, \infty\right)$.

Example 4

Solve the inequality $-x^2 - x - 1 > 0$.

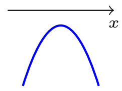
Solution. Alternative 1. The parabola $y = -x^2 - x - 1$ opens **downward**, since $a = -1 < 0$.

Determine the x-intercepts. They are the **roots** of the equation $-x^2 - x - 1 = 0$.

$-x^2 - x - 1 = 0 \iff x^2 + x + 1 = 0 \iff$

$x_{1,2} = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2}$. No real roots!

Therefore, the parabola looks as follows:



To solve the inequality $-x^2 - x - 1 > 0$, we have to find **all** x for which the parabola is **above** the x-axis.

As we see, there are no such x. **Answer:** no solutions.

10 / 12

Example 4

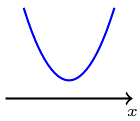
Let us solve the inequality $-x^2 - x - 1 > 0$ in a different way.

Alternative 2. $-x^2 - x - 1 > 0 \iff x^2 + x + 1 < 0$.

Instead of solving $-x^2 - x - 1 > 0$, we will solve an equivalent inequality $x^2 + x + 1 < 0$.

The parabola $y = x^2 + x + 1$ opens **upward** since $a = 1 > 0$,
and has **no** x-intercepts, since the discriminant $b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 1 = -3$ is negative.

Therefore, the parabola is situated above the x-axis:



To solve the inequality $x^2 + x + 1 < 0$ means to find all values of x for which the parabola is below the x-axis.

But there are **no** such x. **Answer:** the inequality has **no** solutions.

11 / 12

Summary

In this lecture, we have learned

- ☑ what a **quadratic inequality** is
- ☑ what it means to **solve** a quadratic inequality
- ☑ how to **visualize** a quadratic inequality by a **parabola**
- ☑ how to solve a quadratic inequality
  in terms of the **leading coefficient** and the **roots**
- ☑ how to write down the **answer**