Lecture 15

Linear Equations

Equation and its solutions

Recall that an **equation** is an equality between two algebraic expressions.

The variables in equation are called **unknowns**.

For example, $3x + 1 = 7$ is an equation with one unknown x.

To solve an equation means to find **all** its solutions,
that is all the values of the variables which **satisfy** the equation.
In other words, to find all values of the unknowns
which turn the equation into a true numerical equality.

For example, $x = 2$ is a solution of the equation $3x + 1 = 7$, since it satisfies the equation:

$$3 \cdot 2 + 1 = 7 .$$

Equivalent equations

Some equations are easy.

Example. $x = 2$ is an equation. But it looks like a solution, and it is a **solution** for itself!

Often, more complicated equations are replaced by simpler equations which have the same solutions.

If two equations have the same solutions,

that is if
any solution of the first equation is a solution of the second one
and vice versa:
each solution of the second equation is a solution of the first one
then we call the equations **equivalent**,
and write the equivalence sign " $\iff$ " between them, like this:

$$x + 1 = 3 \iff x = 2 .$$

How to **transform** an equation into an equivalent equation?

To this end, we will use two **elementary** transformations.

Add the same to both sides

> Any equation is equivalent to the equation obtained from it by **adding** the same expression to **both** sides.

Example 1. Consider the equation $x - 1 = 2$. If we add 1 to both sides, then we get an equavalent equation:

$$x - 1 = 2 \iff x - 1 + 1 = 2 + 1 \iff \boxed{x = 3}$$

Example 2. $5 - x = 0 \iff 5 - x + x = 0 + x \iff 5 = x \iff \boxed{x = 5}$

Example 3.

$$5 - x = 2 \iff 5 - x + (x - 2) = 2 + (x - 2) \iff$$
$$5 - x + x - 2 = 2 + x - 2 \iff 3 = x \iff \boxed{x = 3}$$

Similarly, **subtracting** the same expression from both sides of an equation gives rise to an equivalent equation:

$$x + 2 = 6 \iff x + 2 - 2 = 6 - 2 \iff \boxed{x = 4}$$

Example

$2x - 1 = 5 + x \xrightarrow{\text{subtract } x} 2x - 1 - x = 5 + x - x$

simplify ↓

$x - 1 = 5 \xrightarrow{\text{add } 1} x - 1 + 1 = 5 + 1$

simplify ↓

$x = 6$

These transformations are written as follows:

$$2x - 1 = 5 + x \iff 2x - 1 - x = 5 + x - x \iff$$
$$x - 1 = 5 \iff x - 1 + 1 = 5 + 1 \iff \boxed{x = 6}$$

Fast track

There is a trick that may help you to operate **more efficiently** with equations.

The subtraction of x from both sides of the equation $2x-1=5+x$, namely

$$2x-1=5+x \iff 2x-1-x=5+x-x \iff x-1=5$$

is equivalent to relocation x from the right hand side (RHS) of the equation to the left hand side (LHS) with the **opposite** sign:

$$2x-1=5+\overset{-}{x} \iff 2x-x-1=5 \iff x-1=5$$

Look how **fast** we can solve the equation:

$$2x-1=5+\overset{-}{x} \iff x\overset{+}{-1}=5 \iff x=6\,.$$

Multiply both sides by the same non-zero number

Any equation is equivalent to the equation obtained from it by **multiplying** both sides by the same **non-zero** number.

Example 1. $\frac{x}{2}=3 \iff \frac{x}{2}\cdot 2=3\cdot 2 \iff \boxed{x=6}$

Example 2. $3x=5 \iff 3x\cdot\frac{1}{3}=5\cdot\frac{1}{3} \iff \boxed{x=\frac{5}{3}}$

Similarly, **dividing** both sides of an equation by the same **non-zero** number gives rise to an equivalent equation:

$$2x=8 \iff \frac{2x}{2}=\frac{8}{2} \iff \boxed{x=4}$$

Adding the same expression to both sides of an equation and **multiplying** both sides by the the same non-zero number are called **elementary transformations** of the equation.

Example of elementary transformations

See how a sequence of **elementary transformations** brings an equation to a simple equivalent equation, which is the solution.

Example. Solve the equation $7x - 5 = 2x + 1$.

Solution. $7x - 5 = 2x + 1$

Move $2x$ to the LHS: $7x - 2x - 5 = 1$

Simplify: $5x - 5 = 1$

Move -5 to the RHS: $5x = 1 + 5$

Simplify: $5x = 6$

Divide by 5: $\boxed{x = \frac{6}{5}}$

Linear equations

An equation is called **linear**, if both its sides are polynomials of degree ≤ 1.

For example, $3(x - 2) + 4 = \frac{2}{3}(5x + 1) + x$ is a linear equation,

$x^2 + 2 = x$ is not.

A polynomial of degree ≤ 1 is called a **linear expression**.
Both sides of a linear equation are linear expressions.

By a sequence of elementary transformations, any linear equation can be transformed to an equation of the form $ax = b$ where a and b are some numbers and x is an unknown.

To do this, that is, to bring the equation to the form $ax = b$,

- simplify (if needed) both sides of the equation,
- collect all terms involving the unknown on one side of the equation, and all numbers on the other side,
- simplify the equation again.

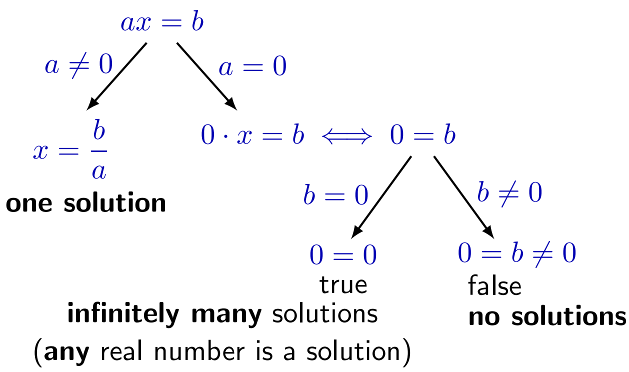
Example

Solve the equation $\frac{3}{2}(x-1) = \frac{x}{3} + 1$.

Multiply by 6:	$6 \cdot \frac{3}{2}(x-1) = 6\left(\frac{x}{3}+1\right)$	to get rid of fractions
Simplify LHS:	$9(x-1) = 6\left(\frac{x}{3}+1\right)$	
Distribute:	$9x - 9 = 2x + 6$	
Move $2x$ to LHS:	$7x - 9 = 6$	
Move -9 to RHS:	$7x = 15$ ⟵	equation in the form $ax = b$
Divide by 7:	$x = \frac{15}{7}$ ⟵	solution

Number of solutions of a linear equation

How many solutions may a linear equation $ax = b$ have? It depends on the numbers a and b.



A linear equation with one unknown may have
either **one** solution, or **no solutions**, or **infinitely many** solutions.

Examples of linear equations

Example 1. Solve the equation $2(x-3) = 2x+1$.

Solution. Distribute: $2x - 6 = 2x + 1$

Move $2x$ from RHS to LHS: $2x - 2x - 6 = 1$

Simplify: $-6 = 1 \longleftarrow$ **false** numerical equality

Answer. The equation has **no solutions**.

Example 2. Solve the equation $2 - x = \frac{1}{3}(6-3x)$.

Solution. Multiply both sides by 3: $3 \cdot (2-x) = 3 \cdot \frac{1}{3}(6-3x)$

Simplify: $6 - 3x = 6 - 3x$

Add $3x$ to both sides: $6 = 6 \longleftarrow$ **true** numerical equality

Answer. The equation is an **identity**.

Any number is a solution.

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How to check if solution is correct

While solving an equation, we can make mistakes.

There is a opportunity **to check** if the number obtained is a solution.

For this, **plug in** the number into the equation and check if the obtained numerical equality is **true**.

Example. Solve the equation $2x - 1 = 3(2x+1)$ and check your solution by substitution.

Solution. $2x - 1 = 3(2x+1) \iff 2x - 1 = 6x + 3 \iff -1 = 4x + 3 \iff$

$$-4 = 4x \iff -1 = x \iff \boxed{x = -1}$$

Check (substitute $x = -1$ into the original equation):

$$2 \cdot (-1) - 1 \stackrel{?}{=} 3(2 \cdot (-1) + 1)$$

$$-2 - 1 \stackrel{?}{=} 3(-2 + 1)$$

$$-3 \stackrel{?}{=} 3(-1)$$

$$-3 = -3 \ \checkmark$$

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Summary

In this lecture, we have learned

- ☑ which equations are called **equivalent**
- ☑ that there are elementary **transformations** of equations:
  - adding the same expression to both sides
  - multiplying both sides by the same non-zero number
- ☑ how to solve equations **efficiently**
- ☑ what a **linear** equation is
- ☑ how to solve a linear equation
- ☑ how many solutions a linear equation may have:
  - one
  - infinitely many
  - no solutions
- ☑ how **to check** a solution by substitution into the original equation