Lecture 14

Equalities, Identities and Equations

Equalities

An (algebraic) **equality** consists of two algebraic expressions connected by the equality sign "$=$".

For example,
$$x^2 - 3x + 1 = x + 2\,,$$
$$1 + 1 = 2\,,$$
$$0 = 1\,,$$
$$a + b = b + a\,,$$
$$(x-y)^2 = x^2 - 2xy + y^2\,.$$

An algebraic equality with a variable becomes a numerical one if we **evaluate** the expressions on both sides of the equality at some number.

For example, if we substitute $x = 1$ into both sides of the equality $x^2 = x$, it turns into a numerical equality $1^2 = 1$, which is true.

If we substitute $x = -1$, then we get $(-1)^2 = -1$, which is false.

True or false

equalities

- numerical: $1 + 1 = 2$
- with variables: $a(b+c) = ab + ac$

numerical equalities

- true: $1 + 4 = 2 + 3$
- false: $1 + 4 = 2 + 5$

equalities **with variables**

- always true (**identities**): $x^2 - y^2 = (x-y)(x+y)$
- always false (**contradictions**): $x = x + 1$
- true or false depending on values of variables: $x + 2 = 3$

Identities

Here are some important identities that we have learned:

$a+b=b+a$ (commutativity of addition)

$a(b+c)=ab+bc$ (distributive law)

$x^n \cdot x^m = x^{n+m}$ (multiplication rule for powers)

$x^2-y^2=(x-y)(x+y)$ (difference of squares)

$(x+y)^2=x^2+2xy+y^2$ (short multiplication)

Proving identities

A **typical** problem about an identity is to **prove** it.

That is, to prove that the equality is true for all values of the variables.

Example. Prove that $(x+1)^3=x^3+3x^2+3x+1$ for all values of x.

Solution. Work on the left hand side:

$$\begin{aligned}(x+1)^3=(x+1)(x+1)^2 &= (x+1)(x^2+2x+1)\\ &= x^3+2x^2+x+x^2+2x+1\\ &= x^3+3x^2+3x+1,\end{aligned}$$

which is the right hand side of the identity.

Therefore,
$(x+1)^3=x^3+3x^2+3x+1$ for all values of x, and the identity is proven.

Equation and its solution

Often we use the word "**equation**" instead of "equality".
This happens when we are interested to **find** the values of variables at which the equality turns to a **true numerical equality**.

A **solution** of an equation with a single variable is the value of the variable which turns the equation into a true numerical equality.

Example. Consider the equation $x + 2 = 3x$. At $x = 1$, the equation turns into a **true** numerical equality:

$$1 + 2 = 3 \cdot 1 .$$

If we substitute $x = 0$, then the equation turns into a **false** numerical equality:

$$0 + 2 = 3 \cdot 0 .$$

Therefore, $x = 1$ is a solution of the equation $x + 2 = 3x$, while $x = 0$ is not a solution.

All the solutions

It may happen that an equation has **no** solutions.

For example, the equation $0 \cdot x = 1$ has no solution, since $0 \cdot x \neq 1$ no matter what x is.

Some equations have **infinitely** many solutions.

For example, the equation $0 \cdot x = 0$ has infinitely many solutions. **Any** number is a solution.

To solve an equation means to find **all** its solutions, that is to find all values of the variable which turn the equation into a **true** numerical equality.

The variable in the equation is called **unknown**.
To solve an equation means to make this unknown known.

Several unknowns

An equation may have **several** unknowns.

For example, $x + 2y = 7$ is an equation with two unknowns x and y.
The equation turns into a **true** numerical equality if we plug in $x = 1$ and $y = 3$:

$$1 + 2 \cdot 3 = 7\,.$$

Plugging in $x = 1$ and $y = 2$ results into a **false** equality:

$$1 + 2 \cdot 2 = 7\,.$$

A **solution** of such equation is a **pair** of numbers which turns the equation into a true numerical equality. For example, the pair $x = 1$ and $y = 3$ is a solution.

Another solution is $x = -1$, $y = 4$. Indeed:

$$(-1) + 2 \cdot 4 = 7\,.$$

As we will learn later, equations like this have **infinitely many** solutions.

Summary

In this lecture, we have learned

- ☑ what an **equality** is
- ☑ that there are **numerical** equalities and equalities **with variables**
- ☑ what an **identity** is
- ☑ what a **contradiction** is
- ☑ what an **equation** is
- ☑ what a **solution** of an equation is
- ☑ what it means **to solve** an equation